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Root systems and hypergeometric functions I

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1. Introduction

Let $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ be coordinates on $\mathbb{R}^{2n}$. Consider the dynamical systems with Hamiltonians

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + g\sum_{i<j}(q_i - q_j)^{-2} \tag{1.1}$$

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + g\sum_{i<j}\sin^{-2}(q_i - q_j). \tag{1.2}$$

The first system (1.1) describes n points on the line under the influence of a force which is inverse proportional to the cube power of their distances. Using the identity $\Sigma_{-\infty}^{\infty}(x - k\pi)^{-2} = \sin^{-2}x$ one can view the second system (1.2) as the periodic analogue of (1.1) on the circle. Moser [Mo] observed that both systems admit a Lax representation, and he concluded that the systems are completely integrable with integrals of an algebraic nature.

There is some evidence that the natural generalizations of systems (1.1) and (1.2) in the context of finite reflection groups and root systems respectively remain algebraically completely integrable. For the series B_n, C_n and D_n partial results are obtained in [OP], and for the dihedral group $I_2(n)$ this is shown in [Hec 1]. However, general proofs do not seem available.

The Schrödinger operators associated with systems (1.1) and (1.2) have the form

$$S = -\frac{1}{2}\sum_{i=1}^{n}\partial(q_i)^2 + g\sum_{i<j}(q_i - q_j)^{-2} \tag{1.3}$$

$$S = -\frac{1}{2}\sum_{i=1}^{n}\partial(q_i)^2 + g\sum_{i<j}\sin^{-2}(q_i - q_j). \tag{1.4}$$

By analogy between classical and quantum mechanics one might hope for a complete commuting set of differential operators containing the given Schrödinger operator S, and operators in this commuting algebra should be of an algebraic nature. For $n = 3$ this was shown by Koornwinder [K] in his thesis, and for abitrary n by Sekiguchi [S], Débiard [Deb] and Macdonald [Mac]. In this paper we study the simultaneous spectral resolution of this commuting algebra of differential operators in the periodic case (1.4). In fact we study this problem in the context of an arbitrary root system, the previous case (1.4) being of type A_n.

If G/K is a Riemannian symmetric space the radial parts for the action of K of the invariant differential operators form such a commuting algebra, and the eigenfunctions are the spherical functions. The goal of this paper is to generalize this theory to the case where the root multiplicities of the restricted root system of G/K are allowed to be arbitrary complex numbers.

This paper is organized as follows. In Section 2 we introduce some notation and make a conjecture on the precise form of the commuting algebra of differential operators. Next we discuss in which cases the conjecture is known. Any character of the commuting algebra of differential operators gives rise to a system of partial differential equations. In Section 3 we rewrite this system as a first order matrix system, and conclude that the system is holonomic. Moreover the system has regular (even simple) singularities at infinity. Here regular and simple singularities is meant in the sense of Deligne [Del 1], [Mal]. In Section 4 we discuss the one dimensional case which amounts to the theory of the hypergeometric function. In Section 5 we describe the fundamental group of the complement of the discriminant. In Section 6 we explicitly determine the monodromy of the system of partial differential equations, and from this conclude the existence of the multivariable hypergeometric function.

2. Commuting differential operators

Let E be a Euclidean space with inner product $(.\,,.)$. For $\alpha \in E$ with $\alpha \neq 0$ put $\alpha^\vee = 2(\alpha, \alpha)^{-1}\alpha$ and denote $r_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha$ for the orthogonal reflection in the hyperplane perpendicular to α. Let $R \subset E$ be a possibly non reduced root system with $\mathrm{rank}(R) = \dim(E) = n$. The Weyl group W is generated by the reflections r_α for $\alpha \in R$. Let $P = \{\lambda \in E; (\lambda, \alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in R\}$ denote the weight lattice of R, and $Q \subset P$ the root lattice of R. Put

$$R_0 = \{\alpha \in R; \alpha \notin 2R\} \tag{2.1}$$

$$R^0 = \{\alpha \in R; 2\alpha \notin R\} \tag{2.2}$$

for the indivisible roots and the inmultiplicable roots respectively. Both R_0 and R^0 are reduced root systems. Clearly Q is the root lattice of R_0 and P is the weight lattice of R^0.

Put $\mathfrak{a} = E^*$, $\mathfrak{t} = iE^*$, $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t} = E^* \otimes_{\mathbb{R}} \mathbb{C}$. Let H denote the complex torus with Lie algebra $\mathfrak{h}$ and character lattice $\hat{H} = P$. Write exp: $\mathfrak{h} \to H$ for the exponential map and log: $H \to \mathfrak{h}$ for the multi valued inverse. For $\lambda \in P$ and $h \in H$ the exponential $h^\lambda = e^{\lambda(\log h)}$ is a well defined single valued function. An exponential polynomial on H is an expression of the form $\Sigma\, a_\lambda h^\lambda$ with $a_\lambda \in \mathbb{C}$ and the sum taken over a finite set in P. We have a decomposition $H = A.T$ with $A = \exp(\mathfrak{a})$ the split form and $T = \exp(\mathfrak{t})$ the compact form of H. Since exp: $\mathfrak{a} \to A$ is a bijection with inverse log: $A \to \mathfrak{a}$ the exponential $a^\lambda = e^{\lambda(\log a)}$ is a well defined single valued function on A for all $\lambda \in \mathfrak{h}^*$.

Fix a system of positive roots R_+ in R. Let P_+ (resp. P_-) $= \{\lambda \in P; (\lambda, \alpha^\vee) \in \mathbb{Z}_+$ (resp. $\mathbb{Z}_-$) for all $\alpha \in R_+\}$ be the corresponding set of dominant (resp. anti dominant) integral weights. If $\{\lambda_1, \ldots, \lambda_n\}$ are the fundamental weights in P_+ then we put

$$\sigma = \lambda_1 + \cdots + \lambda_n = \frac{1}{2} \sum_{\alpha \in R^0_+} \alpha, \tag{2.3}$$

where $R^0_+ = R^0 \cap R_+$ and $R_{0,+} = R_0 \cap R_+$. The Weyl denominator is by definition

$$\Delta(h) = \sum_{w \in W} \varepsilon(w) h^{w\sigma} = h^\sigma \prod_{\alpha \in R^0_+} (1 - h^{-\alpha}), \tag{2.4}$$

where as usual $\varepsilon(w) = \det(w)$ is the sign character. Clearly $\Delta^w = \varepsilon(w)\Delta$ is anti invariant, and each anti invariant exponential polynomial is divisible by Δ. The fundamental invariant exponential polynomials are of the form

$$z_j = \sum_{w \in W/W^{-\lambda_j}} h^{-w\lambda_j}. \tag{2.5}$$

Here $W^\lambda = \{w \in W; w.\lambda = \lambda\}$ denotes the stabilizer of $\lambda \in \mathfrak{h}^*$.

It is easy to see [Bou, p. 188] that the algebra of all invariant exponential polynomials is equal to $\mathbb{C}[z_1, \ldots, z_n]$. In particular we can view Δ^2 as an element of $\mathbb{C}[z_1, \ldots, z_n]$, and the locus $\Delta^2 = 0$ in $\mathbb{C}^n$ is called the discriminant.

We have a natural isomorphism $\mathbb{C}[\mathfrak{h}^*] \simeq \mathfrak{U}(\mathfrak{h})$ of the polynomial algebra on $\mathfrak{h}^*$ and the symmetric algebra on $\mathfrak{h}$, denoted by $p \to \partial(p)$. Here we think of $p \in \mathbb{C}[\mathfrak{h}^*]$ as a polynomial function on $\mathfrak{h}^*$ and of $\partial(p) \in \mathfrak{U}(\mathfrak{h})$ as a constant coefficient differential operator on H.

Suppose we have given complex numbers k_α for each $\alpha \in R$ such that $k_{w\alpha} = k_\alpha$ for all $\alpha \in R$ and $w \in W$. If m denotes the number of Weyl group orbits of roots in R, then $k = (k_\alpha)_{\alpha \in R}$ lies in a parameter space $K \simeq \mathbb{C}^m$. Fix an orthonormal basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{a}$, and consider the differential operator

$$L = L(k) = \sum_{j=1}^{n} \partial(X_j)^2 - \sum_{\alpha \in R_+} k_\alpha (1 + h^\alpha)(1 - h^\alpha)^{-1} \partial(X_\alpha). \tag{2.6}$$

Here $X_\alpha \in \mathfrak{a}$ is defined by $\beta(X_\alpha) = (\beta, \alpha)$ for all $\alpha, \beta \in \mathfrak{a}^*$. Introduce also

$$\varrho = \varrho(k) = \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha. \tag{2.7}$$

Clearly L is a Weyl group invariant differential operator with coefficients analytic on $H^{\mathrm{reg}} = \{h \in H; \Delta(h) \neq 0\}$. The operator $\Delta \circ L$ maps invariant exponential polynomials on H to anti invariant polynomials, and since the latter are divisible by Δ we obtain the following lemma.

LEMMA 2.1: *The operator L leaves the space of Weyl group invariant exponential polynomials on H invariant, i.e. the transform of L under the map $h \in H \to z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ is a differential operator in the Weyl algebra $\mathbb{A}_n = \mathbb{C}[z_1, \ldots, z_n, \partial_1, \ldots, \partial_n]$.*

The following proposition gives the relation between the operator L given by (2.6) and the Schrödinger operator S given by (1.4).

PROPOSITION 2.2: *Consider the function $\delta(a)$ on A^{reg} defined by*

$$a = \exp X \in A \to \delta(k, a) = \prod_{\alpha \in R_+} \left| 2 \sinh \frac{\alpha(X)}{2} \right|^{2k\alpha}. \tag{2.8}$$

Then on A^{reg} we have

$$\delta^{1/2} \circ (L + (\varrho, \varrho)) \circ \delta^{-1/2} = \sum_{j=1}^{n} \partial(X_j)^2$$

$$+ \sum_{\alpha \in R_+} \tfrac{1}{4} k_\alpha (1 - k_\alpha - 2k_{2\alpha})(\alpha, \alpha) \sinh^{-2} \frac{\alpha(X)}{2}. \tag{2.9}$$

Proof: It is sufficient to prove the formula on $A_+ = \exp \mathfrak{a}_+$ with $\mathfrak{a}_+ = \{X \in \mathfrak{a};\ \alpha(X) > 0 \text{ for all } \alpha \in R_+\}$. Clearly we have

$$\delta^{-1/2} \circ \partial(X_j) \circ \delta^{1/2} = \partial(X_j) + \tfrac{1}{2}\partial(X_j)(\log \delta)$$

$$\delta^{-1/2} \circ \partial(X_j)^2 \circ \delta^{1/2} = \partial(X_j)^2 + \partial(X_j)(\log \delta)\partial(X_j) + \delta^{-1/2}\partial(X_j)^2(\delta^{1/2})$$

and if we write $\square = \Sigma_j\, \partial(X_j)^2$ we get

$$\sum_j \partial(X_j)(\log \delta)\partial(X_j) = \sum_\alpha k_\alpha \coth \frac{\alpha(X)}{2}\, \partial(X_\alpha) \tag{2.10}$$

$$\delta^{-1/2} \square\, (\delta^{1/2}) = \sum_\alpha \left\{ -\tfrac{1}{4}k_\alpha(\alpha, \alpha) \sinh^{-2} \frac{\alpha(X)}{2} \right\} \tag{2.11}$$

$$+ \sum_{\alpha,\beta} \left\{ \tfrac{1}{4}k_\alpha k_\beta(\alpha, \beta) \coth \frac{\alpha(X)}{2} \coth \frac{\beta(X)}{2} \right\}.$$

First observe the that the right hand side of (2.10) is precisely the first order term of L in (2.6). We rewrite the right hand side of (2.11) as

$$(\varrho, \varrho) + \sum_\alpha \tfrac{1}{4}k_\alpha(k_\alpha + 2k_{2\alpha} - 1)(\alpha, \alpha) \sinh^{-2}\left(\frac{\alpha(X)}{2}\right)$$

$$+ \sum_{\alpha,\beta}{}' \tfrac{1}{4}k_\alpha k_\beta(\alpha, \beta) \left\{ \coth \frac{\alpha(X)}{2} \coth \frac{\beta(X)}{2} \right\}.$$

where $\Sigma'_{\alpha,\beta}$ denotes the sum over $\alpha, \beta \in R_+ \times R_+$ with α, β no multiples of each other. The formula (2.9) follows if we show that the term $\Sigma'_{\alpha,\beta} \ldots$ vanishes identically. Making various specializations for k_α in (2.11) and using the Weyl denominator formula this can be proved. Q.E.D.

COROLLARY 2.3: *For $k_\alpha \geqslant 0$ the operator L on $C_c^\infty(A)^W$ is symmetric with respect to the measure $\delta(a)da$, where da is Haar measure on A.*

REMARK 2.4: If G/K is a Riemannian symmetric space of non compact type and A a maximal split torus for G/K, then the radial part of the Laplacian on G/K with respect to the action of K on G/K is a Weyl group invariant differential operator on A of the form (2.6). However, our formulas differ from the usual ones [Hel 2, p. 186 Thm 5.8 and p. 267 Prop 3.9] in the following way: Our α is twice the usual one, and the root multiplicity $m_\alpha = 2k_\alpha$.

Introduce a partial ordering $\geqslant$ on $\mathfrak{h}^*$ by

$$\lambda \geqslant \mu \text{ if and only if } \lambda - \mu = \sum_j k_j\alpha_j \text{ with } k_j \in \mathbb{Z}_+. \tag{2.12}$$

Here $\{\alpha_1, \ldots, \alpha_n\}$ is a basis of simple roots for R_+ (or $R_{0,+}$) and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.

Consider the algebra $\mathscr{R}$ of functions on H^{reg} generated by the functions

$$h \in H^{\mathrm{reg}} \to (1 - h^\alpha)^{-1} \tag{2.13}$$

for all $\alpha \in R_+$. Since $(1 - h^{-\alpha})^{-1} = 1 - (1 - h^\alpha)^{-1}$ the Weyl group acts on $\mathscr{R}$. Denote by $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$ the set of differential operators on H^{reg} with coefficients in $\mathscr{R}$. Clearly $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$ is an algebra. The operator L given bij (2.6) lies in $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$. Any differential operator $P \in \mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$ has a convergent asymptotic expansion on $A_- = \exp \mathfrak{a}_-$ with $\mathfrak{a}_- = \{X \in \mathfrak{a}; \alpha(X) < 0$ for all $\alpha \in R_+\}$ of the form

$$P = \sum_{\mu \geqslant 0} h^\mu \partial(p_\mu), \quad h \in A_-.T \tag{2.14}$$

with $\partial(p_\mu) \in \mathfrak{U}(\mathfrak{h})$ by expanding the coefficient $(1 - h^\alpha)^{-1} = 1 + h^\alpha + h^{2\alpha} + \ldots$ for all $\alpha \in R_+$. In particular for the operator L this becomes

$$L = \sum_{j=1}^n \partial(X_j)^2 - 2\partial(X_\varrho) - 2\sum_{\alpha \in R_+} k_\alpha \sum_{j=1}^\infty h^{j\alpha}\partial(X_\alpha). \tag{2.15}$$

Definition 2.5: The algebra homomorphism $\gamma = \gamma(k)$: $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h}) \to \mathbb{C}[\mathfrak{h}^*]$ is defined by

$$\gamma: P = \sum_{\mu \geqslant 0} h^\mu \partial(p_\mu) \to \{\lambda \to p_0(\lambda + \varrho)\}. \tag{2.16}$$

Remark 2.6: It is easy to show that the map $P \to P' = \delta^{1/2} \circ P \circ \delta^{-1/2}$ is an algebra isomorphism on $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$. Here δ is the weight function defined by (2.8). Clearly $\delta(h)^{-1/2}$ has a convergent expansion on $A_-.T$ of the form

$$\delta(h)^{-1/2} = h^\varrho \sum_{\mu \geqslant 0} d_\mu h^\mu \tag{2.17}$$

with $d_\mu \in \mathbb{C}$ and $d_0 = 1$. Suppose $P = \Sigma_{\mu \geq 0} h^\mu \partial(p_\mu)$ and $P' = \Sigma_{\mu \geq 0} h^\mu \partial(p'_\mu)$ according to (2.14). Then it follows from (2.17) that $p'_0(\lambda) = p_0(\lambda + \varrho)$. Hence $\gamma(P)$ is the leading constant coefficient term of the operator P' at infinity in A_-.

LEMMA 2.7: *For* $P = \Sigma_{\mu \geq 0} h^\mu \partial(p_\mu)$ *in* $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$ *we have* $[L, P] = L \circ P - P \circ L = 0$ *if and only if the polynomials* $p_\mu(\lambda) \in \mathbb{C}[\mathfrak{h}^*]$ *satisfy the recurrence relations*:

$$(2\lambda - 2\varrho + \mu, \mu)p_\mu(\lambda)$$

$$= 2 \sum_{\alpha \in R_+} k_\alpha \sum_{j=1}^{\infty} \{(\lambda + \mu - j\alpha, \alpha)p_{\mu - j\alpha}(\lambda) - (\lambda, \alpha)P_{\mu - j\alpha}(\lambda + j\alpha)\}. \tag{2.18}$$

Proof: A formal computation, left to the reader. Q.E.D.

COROLLARY 2.8: *Write* $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})^L$ *for the algebra of all differential operators* P *in* $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$ *with* $[L, P] = 0$. *Then* γ: $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})^L \to \mathbb{C}[\mathfrak{h}^*]$ *is an injective algebra homomorphism. In particular* $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})^L$ *is a commutative algebra. For* $P = \Sigma_{\mu \geq 0} h^\mu \partial(p_\mu)$ *in* $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})^L$ *we have* $\deg(p_\mu) < \deg(p_0) = \deg(\gamma(P))$ *for* $\mu > 0$.

PROPOSITION 2.9: *If* $P \in \mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})^L$ *is Weyl group invariant, then* $\gamma(P) \in \mathbb{C}[\mathfrak{h}^*]^W$.

Proof: Fix a simple root $\alpha \in R_{0,+}$, and denote by $R_\alpha = R \cap \{\mathbb{Z}\alpha\}$, $R_{\alpha,+} = R_\alpha \cap R_+$. Let $\mathscr{R}_\alpha$ be the algebra of functions generated $h \to (1 - h^\beta)^{-1}$ for $\beta \in R_{\alpha,+}$. The map $(1 - h^\gamma)^{-1} \to 1$ for $\gamma \in R_+ \backslash R_{\alpha,+}$ induces an algebra homomorphism $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h}) \to \mathscr{R}_\alpha \otimes \mathfrak{U}(\mathfrak{h})$ denoted by $P \to P_\alpha$. Hence the relation $[L, P] = 0$ for $P \in \mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$ implies $[L_\alpha, P_\alpha] = 0$. Essentially this is a commutation equation for ordinary differential operators. Using $\gamma(P) = \gamma(P_\alpha)$ a one variable computation shows $\gamma(P)(r_\alpha\lambda) = \gamma(P)(\lambda)$. Q.E.D.

CONJECTURE 2.10: (*Surjectivety of the Harish–Chandra homomorphism*). *Write* $\mathbb{D} = \mathbb{D}(k)$ *for the algebra of all Weyl group invariant differential operators in* $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$ *which commute with the operator* $L = L(k)$ *given by* (2.6). *Then the map*

$$\gamma: \mathbb{D} \to \mathbb{C}[\mathfrak{h}^*]^W \tag{2.19}$$

defined by (2.16) *is an isomorphism onto for all* $k \in K$.

REMARK 2.11: The conjecture has been proved in several cases. First of all, if the numbers $2k_\alpha$ are root multiplicities of the restricted root system of a symmetric space G/K [Hel 1: p. 532 Table VI] then the surjectivety of γ: $\mathbb{D} \to \mathbb{C}[\mathfrak{h}^*]^W$ follows from the commutative diagram

$$\begin{tikzcd}
\mathbb{D}(G/K) \arrow[rr, "\text{Harishrish–Chandra}", "\text{isomorphism}"'] \arrow[dr, "\text{radial part}"'] & & \mathbb{C}[\mathfrak{h}^*]^W \\
& \mathbb{D} \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

Here $\mathbb{D}(G/K)$ is the algebra of invariant differential operators on G/K. For the details we refer to Harish–Chandra [Ha] or Helgason [Hel 2]. However by Lemma 2.7 we see that the set of all $k \in K \cong \mathbb{C}^m$ for which $\gamma(k)$: $\mathbb{D}(k) \to \mathbb{C}[\mathfrak{h}^*]^W$ is surjective is Zariski closed in K. But for R of type B_n or BC_n there are $\mathbb{Z}_+$-parameter families of symmetric spaces. Hence the conjecture is also true for the line in K containing this $\mathbb{Z}_+$. One can refine this argument, for A_2 as follows. For the surjectivety of γ in this case it is sufficient to prove that any cubic invariant in $\mathbb{C}[\mathfrak{h}^*]^W$ is in the image of γ. Since γ is surjective for at least 5 different values of $k(k = 0$ for trivial reasons, and $k = \frac{1}{2}, 1, 2, 4$ corresponding to the symmetric spaces SL(3)/SO(3), SL(3) $\times$ SL(3)/SL(3), SL(6)/Sp(6), E_6/F_4 respectively) we conclude from a degree count in k in equations (2.18) that any cubic invariant in $\mathbb{C}[\mathfrak{h}^*]^W$ is in the image of γ for all $k \in \mathbb{C}$. This proves Conjecture 2.10 for R of type A_2 and all $k \in \mathbb{C}$.

Another case in which the conjecture has been proved is for rank of R equal 2 and all $k \in K$. For R of type A_2 and BC_2 this has been done by Koornwinder [K], and for R of type G_2 by Opdam [Op]. In the latter paper a survey on the methods for rank of R equal 2 is given.

Finally Sekiguchi [Se] and Debiard [Deb] gave formulas for a generating set of $\mathbb{D}$ for R of type A_n. However both papers do not give complete proofs. We are grateful to Prof. I.G. Macdonald for giving us a complete proof of these formulas [Mac].

REMARK 2.12: Everything which follows in this paper holds under the assumption that Conjecture 2.10 is true.

DEFINITION 2.13: The system of differential equations on H^{reg} of the form

$$P\phi = \gamma(P)(\lambda)\phi \quad P \in \mathbb{D}, \quad \lambda \in \mathfrak{h}^* \tag{2.20}$$

is called the system of hypergeometric (partial) differential equations.

Definition 2.14: Given a parameter $k = (k_\alpha)_{\alpha \in R}$ in K the associated parameter $k' = (k'_\alpha)_{\alpha \in R}$ is defined by $k'_\alpha = 1 - k_\alpha$ for $\alpha \in R^0$ and $k'_\alpha = -k_\alpha$ for $\alpha \in R \setminus R^0$.

Clearly $k'' = k$ for all k in K. If $\varrho = \varrho(k)$ and $\varrho' = \varrho(k')$ then $\varrho + \varrho' = \sigma$ where σ is defined by (2.3). Moreover if $\delta(k, a)$ denotes the weight function defined by (2.8), then we get $\delta(k, a)^{1/2}\delta(k', a)^{1/2} = \Delta(a)$ where $\Delta(a)$ is the Weyl denominator for R^0 defined by (2.4). As a corollary of Proposition 2.2. we have

Corollary 2.15: *The expression*

$$\delta(k)^{1/2} \circ (L(k) + (\varrho(k), \varrho(k))) \circ \delta(k)^{-1/2}$$

is invariant under the substitution $k \to k'$.

Corollary 2.16: *Conjecture* 2.10 *is true for the parameter* $k \in K$ *if and only if it is true for the associated parameter* $k' \in K$.

3. Holonomic systems and simple singularities

In this section we make a first study of the system (2.20) of hypergeometric differential equations. For this we need the concept of harmonic polynomials.

Definition 3.1: An element $\partial(q) \in \mathfrak{U}(\mathfrak{h})$ is called harmonic if $\partial(q)(p) \equiv 0$ for all $p \in \mathbb{C}[\mathfrak{h}]^W$ with $p(0) = 0$.

The harmonics in $\mathfrak{U}(\mathfrak{h})$ are denoted by $\mathfrak{H}(\mathfrak{h})$. The dimension d of $\mathfrak{H}(\mathfrak{h})$ is equal to the order $|W|$ of the Weyl group W. A well known result of Chevalley says that

$$\mathfrak{U}(\mathfrak{h}) \simeq \mathfrak{H}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{h})^W. \tag{3.1}$$

For $\lambda \in \mathfrak{h}^*$ and $k \in K$ we write

$$I(\lambda) = I(\lambda, k) = \{P \in \mathbb{D}; \gamma(P)(\lambda) = 0\} \tag{3.2}$$

so that the hypergeometric equations (2.20) get the form $P\phi = 0$, $P \in I(\lambda, k)$.

Proposition 3.2: *We have an isomorphism*

$$\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h}) \xrightarrow{\sim} \mathscr{R} \otimes \mathfrak{H}(\mathfrak{h}) \oplus \mathscr{R} \otimes \mathfrak{H}(\mathfrak{h}) . I(\lambda, k) \tag{3.3}$$

of left $\mathscr{R}$*-modules.*

Proof: To simplify the notation we write $\mathfrak{U}$, $\mathfrak{H}$, I instead of $\mathfrak{U}(\mathfrak{h})$, $\mathfrak{H}(\mathfrak{h})$, $I(\lambda, k)$. Put $\mathfrak{U}^r$ equal to the homogeneous elements in $\mathfrak{U}$ of degree r, and $\mathfrak{U}_r = \oplus_{j \leqslant r} \mathfrak{U}^j$ the elements of degree $\leqslant r$. Also let $\mathfrak{H}^r = \mathfrak{H} \cap \mathfrak{U}^r$, $\mathfrak{H}_r = \mathfrak{H} \cap \mathfrak{U}_r$ and $I_r = I \cap \mathscr{R} \otimes \mathfrak{U}_r$. We prove by induction on r that

$$\mathscr{R} \otimes \mathfrak{U}_r \simeq \mathscr{R} \otimes \mathfrak{H}_r \oplus \left\{ \sum_{j \geqslant 1} \mathscr{R} \otimes \mathfrak{H}_{r-j} \,.\, I_j \right\}.$$

The case $r = 0$ is clear since $I_0 = \{0\}$. According to (3.1) we can write a given $\partial(q) \in \mathfrak{U}^r$ as

$$\partial(q) = \sum_l \partial(q_l)\partial(p_l)$$

with $\partial(q_l) \in \mathfrak{H}^{r-r_l}$ and $\partial(p_l) \in \mathfrak{U}^{r_l}$ Weyl group invariants. By Corollary 2.8 and Conjecture 2.10 there exists $P_l \in I_{r_l}$ such that

$$P_l \equiv \partial(p_l) - p_l(\lambda) \bmod (\mathscr{R} \otimes \mathfrak{U}_{r_l - 1}).$$

Since $\partial(q_l)(\mathscr{R} \otimes \mathfrak{U}_{r_l - 1}) \subset \mathfrak{U}_{r-1}$ we get

$$\partial(q) \equiv \sum_l p_l(\lambda)\partial(q_l) + \sum_l \partial(q_l)P_l \bmod (\mathscr{R} \otimes \mathfrak{U}_{r-1})$$

and using the induction hypothesis we get

$$\mathscr{R} \otimes \mathfrak{U}_r \simeq \{\mathscr{R} \otimes \mathfrak{H}_r\} + \left\{ \sum_{j \geqslant 1} \mathscr{R} \otimes \mathfrak{H}_{r \;\; j} \circ I_j \right\}.$$

That this sum is in fact a direct sum follows by taking the r-th order symbol and using (3.1) and the induction hypothesis. Q.E.D.

COROLLARY 3.3: *Let $J(\lambda) = J(\lambda, k) = \mathscr{R} \otimes \mathfrak{U}(\mathfrak{h}) \,.\, I(\lambda, k)$ be the left ideal in $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$ generated by $I(\lambda, k)$. Then we have a direct sum decomposition*

$$\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h}) \simeq \mathscr{R} \otimes \mathfrak{H}(\mathfrak{h}) \oplus J(\lambda). \tag{3.4}$$

DEFINITION 3.4: Fix a basis $\{q_1, \ldots, q_d\}$ of homogeneous harmonics in $\mathbb{C}[\mathfrak{h}]$ such that $\deg(q_i) \leqslant \deg(q_{i+1})$ and $q_1 \equiv 1$. To each $P \in \mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$ we assign a matrix $A(P) \in \mathfrak{gl}(d, \mathscr{R})$ by the requirement

$$P \circ \partial(q_i) + \sum_{j=1}^{d} A_{ij}(P)\partial(q_j) \in J(\lambda). \tag{3.5}$$

PROPOSITION 3.5: *The map* A: $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h}) \to \mathfrak{gl}(d, \mathscr{R})$ *is a morphism of left* $\mathscr{R}$*-modules, and satisfies*

$$A(\partial(X)P) = \partial(X)(A(P)) - A(P)A(\partial(X)) \tag{3.6}$$

for all $X \in \mathfrak{h}$ *and* $P \in \mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$.

Proof: Clear using the Leibniz rule. Q.E.D.

COROLLARY 3.6: (*Frobenius integrability condition*). *For* X, $Y \in \mathfrak{h}$ *we have* $[\partial(X) + A(\partial(X)), \partial(Y) + A(\partial(Y))] = 0$.

Proof: For X, $Y \in \mathfrak{h}$ we have

$$\begin{aligned}&[\partial(X) + A(\partial(X)), \partial(Y) + A(\partial(Y))]\\ &= (\partial(X)(A(\partial(Y))) - A(\partial(Y))A(\partial(X))) - (\partial(Y)(A(\partial(X)))\\ &\quad + A(\partial(X))A(\partial(Y))) = A(\partial(X)\partial(Y)) - A(\partial(Y)\partial(X)) = 0.\end{aligned}$$

Q.E.D.

DEFINITION 3.7: The system of first order differential equations on H^{reg} of the form

$$(\partial(X) + A(\partial(X)))\Phi = 0 \quad \forall X \in \mathfrak{h}, \tag{3.7}$$

where $\Phi = (\phi_1, \ldots, \phi_d)^t$ is called the matrix form of the hypergeometric differential equations.

PROPOSITION 3.8: *If* ϕ *is a solution of* (2.20), *then* $\Phi = (\partial(q_1)\phi, \ldots, \partial(q_d)\phi)^t$ *is a solution of* (3.7). *Conversely if* $\Phi = (\phi_1, \ldots, \phi_d)^t$ *is a solution of* (3.7), *then* $\phi = \phi_1$ *is a solution of* (2.20) *and* $\phi_i = \partial(q_i)\phi_1$.

Proof: Suppose ϕ is a solution of (2.20), i.e. $P\phi = 0$ for all $P \in J(\lambda)$. If we put $\Phi = (\partial(q_1)\phi, \ldots, \partial(q_d)\phi)^t$ then it follows from (3.5) that $(P + A(P))\Phi = 0$ for all $P \in \mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$. In particular Φ is a solution of (3.7). Conversely suppose $\Phi = (\phi_1, \ldots, \phi_d)^t$ is a solution of (3.7). Using (3.6) and induction on the order of differential operators it is easy to see that $(P + A(P))\Phi = 0$ for all $P \in \mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$. Since $A_{1j}(P) = 0$ for all $P \in J(\lambda)$ we get $P\phi_1 = 0$ for all $P \in J(\lambda)$. Moreover $\phi_i = \partial(q_i)\phi_1$ because $A_{1j}(\partial(q_i)) = -\delta_{ij}$. Q.E.D.

COROLLARY 3.9 (*Holonomicity on H^{reg}*): *Locally on H^{reg} the solution space of* (2.20) *has dimension d* (*d equals the order $|W|$ of the Weyl group W*) *and consists of analytic functions.*

The central subgroup C of H is by definition

$$C = \{h \in H;\ h^{\alpha} = 1 \text{ for all } \alpha \in R\} \tag{3.8}$$

and the torus H/C with character lattice Q is called the adjoint torus. Clearly C is contained in $\{h \in H;\ wh = h \text{ for all } w \in W\}$, with equality if R is reduced, and we have an induced action of W on H/C. Let $\{\alpha_1, \ldots, \alpha_n\}$ be the basis of simple roots of R_+ (or $R_{0,+}$), and put $x_j = h^{\alpha_j}$ for $j = 1, \ldots, n$. The map

$$x = (x_1, \ldots, x_n)\colon H/C \to \mathbb{C}^n \tag{3.9}$$

is injective with image $\{x = (x_1, \ldots, x_n);\ x_j \neq 0 \text{ for all } j\}$. Hence (3.9) defines a partial compactification of H/C, and using the action of the Weyl group this can be extended to a global compactification of H/C. This global compactification is nothing but the toroidal compactification of H/C corresponding to the decomposition of $\mathfrak{a}^*$ into Weyl chambers, see e.g. [Da].

Let $\{Y_1, \ldots, Y_n\}$ be a basis of $\mathfrak{a}$ such that $\alpha_i(Y_j) = \delta_{ij}$. In the coordinates (3.9) the differentiation $\partial(Y_j)$ becomes $x_j\partial_j$ with $\partial_j = \partial/\partial x_j$. If we put $A_j = A(\partial(Y_j))$ for $j = 1, \ldots, n$ then the system (3.7) in the coordinates (3.9) takes the form

$$(x_j\partial_j + A_j)\Phi = 0 \quad j = 1, \ldots, n. \tag{3.10}$$

The matrix coefficients of A_j are power series in $x_1, \ldots, x_n$ which converge on the polydisc $D^n = \{x;\ |x_j| < 1, j = 1, \ldots, n\}$. A system of the form (3.10) is called a system with simple singularities. For these systems the classical method of series substitution can be applied. Using Proposition 3.8 we can work equally well with the system (2.20). Consider a series of the form

$$\phi(\mu, k; h) = \sum_{\nu \geqslant \mu} \Gamma_\nu(\mu, k)h^\nu \tag{3.11}$$

with coefficients $\Gamma_\nu(\mu, k) \in \mathbb{C}$ and $\Gamma_\mu(\mu, k) = 1$. The vector $\mu \in \mathfrak{h}^*$ is called the leading exponent of the series (3.11). The second order equation

$$L\phi = (\lambda - \varrho, \lambda + \varrho)\phi \tag{3.12}$$

in (2.20) plays a predominant role. Using the expansion (2.15) for L a formal computation shows that $\phi = \phi(\mu, k; h)$ is a solution of (3.12) if and only if the leading exponent μ satisfies

$$(\mu - \varrho, \mu - \varrho) = (\lambda, \lambda), \tag{3.13}$$

and the coefficients $\Gamma_\nu(\mu, k)$ satisfy the recurrence relations

$$\{(\mu - \varrho, \mu - \varrho) - (\nu - \varrho, \nu - \varrho)\}\Gamma_\nu(\mu, k)$$

$$= 2 \sum_{\alpha \in R_+} k_\alpha \sum_{j=1}^{\infty} (\nu - j\alpha, \alpha)\Gamma_{\nu - j\alpha}(\mu, k). \tag{3.14}$$

In case R is reduced and $k_\alpha = 1$ for all $\alpha \in R$ this is Freudenthal's recurrence formula for the weight multiplicities of a finite dimensional representation with lowest weight $\mu \in P_-$, see [FdV, Section 48]. More generally in the context of symmetric spaces such formulas were derived in [Ha, p. 271].

The recurrence relations (3.14) can be uniquely solved if $(\mu - \varrho, \mu - \varrho) - (\nu - \varrho, \nu - \varrho) \neq 0$ for all $\nu > \mu$, or equivalently

$$(\mu - \varrho, \kappa^\vee) + 1 \neq 0 \quad \text{for all } \kappa > 0. \tag{3.15}$$

One can show that for these μ the series (3.11) with $\Gamma_\nu(\mu, k)$ satisfying the recurrence relations (3.14) converges absolutely on A_- (Hel 2, p. 428]). Using expansions of the form (2.14) and the commutativety of $\mathbb{D}$ it follows that for μ satisfying (3.15) the series (3.11) is a solution of all the differential equations

$$P\phi(\mu, k; h) = \gamma(P)(\mu - \varrho)\phi(\mu, k; h) \quad \text{for } P \in \mathbb{D}. \tag{3.16}$$

Corollary 3.10: *The indicial equation for the leading exponents $\mu \in \mathfrak{h}^*$ of solutions of the form* (3.11) *of the system* (2.20) *of hypergeometric equations becomes*

$$\mu = w\lambda + \varrho \quad \textit{for } w \in W. \tag{3.17}$$

Proof: Indeed $\gamma(P)(\mu - \varrho) = \gamma(P)(\lambda)$ for all $P \in \mathbb{D}$ implies $\mu = w\lambda + \varrho$ for some $w \in W$. Q.E.D.

Corollary 3.11: *Suppose $\lambda \in \mathfrak{h}^*$ is regular* (*i.e.* $(\lambda, \alpha^\vee) \neq 0$ *for all* $\alpha \in R$) *and λ satisfies $(\lambda, \kappa^\vee) + 1 \neq 0$ for all $\kappa \in Q$, $\kappa \neq 0$. Then the functions $\phi(w\lambda + \varrho, k; a)$ with $w \in W$ are a basis for the solution space of* (2.20) *on A_-.*

LEMMA 3.12: *Suppose* $\mu \in P_-$ *satisfies* (3.15). *Then the coefficients* $\Gamma_\nu(\mu, k)$ *satisfy* $\Gamma_{w\nu}(\mu, k) = \Gamma_\nu(\mu, k)$ *for all* $w \in W$. *In particular* $\Gamma_\nu(\mu, k) = 0$ *unless* μ *lies in the set*

$$C(\mu) = \{\nu \in \mathfrak{h}^*;\ w\nu \geqslant \mu \text{ for all } w \in W\}. \tag{3.18}$$

Proof: By induction on the order $\leqq$ on the set $\{\nu \in \mathfrak{h}^*;\ \nu \geqslant \mu\}$, and left to the reader. Q.E.D.

DEFINITION 3.13: Suppose $\mu \in P_-$ satisfies (3.15). The Weyl group invariant exponential polynomials

$$\phi(\mu, k; h) = \sum_{\nu \in C(\mu)} \Gamma_\nu(\mu, k) h^\nu \tag{3.19}$$

are called (multivariable) Jacobi polynomials on H associated with the root system R.

COROLLARY 3.14: *The commutant* $\mathbb{D}$ *of* L *in the Weyl group invariant differential operators in* $\mathscr{R} \otimes \mathfrak{U}(\mathfrak{h})$ *transforms in the* $z = (z_1, \ldots, z_n)$ *coordinates on* $W \backslash H$, *defined by* (2.5), *into differential operators in the Weyl algebra* $\mathbb{A}_n = \mathbb{C}[z_1, \ldots, z_n, \partial_1, \ldots, \partial_n]$.

4. The ordinary hypergeometric equation

In this section we assume that R is a rank one root system. Assume $R_+^0 = \{\alpha\}$ in the notation of (2.2). Either $R = \{\pm\alpha\}$ is of type A_1 or $R = \{\pm\beta, \pm\alpha\}$ with $\alpha = 2\beta$ is of type BC_1.

Let $\beta \in \mathfrak{h}^*$ such that $\alpha = 2\beta$. Then $y = h^\beta$ is a coordinate on $H \cong \mathbb{C}^*$. The nontrivial Weyl group element acts by $y \to y^{-1}$ on H. Let $z = \frac{1}{2} - \frac{1}{4}(y + y^{-1}) = \frac{1}{4}y^{-1}(1 - y)^2$ be a coordinate on $W \backslash H$ in accordance with (2.5). Under the map $y \to z$ the negative chamber $A_- = \{y;\ 0 < y < 1\}$ is mapped bijectively onto the negative real axis $\{z;\ z < 0\}$, and the circle $T = \{y;\ |y| = 1\}$ is mapped onto the interval $\{z;\ 0 \leqslant z \leqslant 1\}$. The map $y \to z$ has branch points of order two at $H^{\text{sing}} = \{y;\ y = \pm 1\}$. See Fig. 1.

The inner product on $\mathfrak{h}$ and $\mathfrak{h}^*$ is normalized by $(\beta, \beta) = 1$. In the y coordinate the operator L defined by (2.6) takes the form

$$L = \theta^2 - \left(k_\beta \frac{1 + y}{1 - y} + 2k_\alpha \frac{1 + y^2}{1 - y^2}\right)\theta, \quad \theta = y\partial_y. \tag{4.1}$$

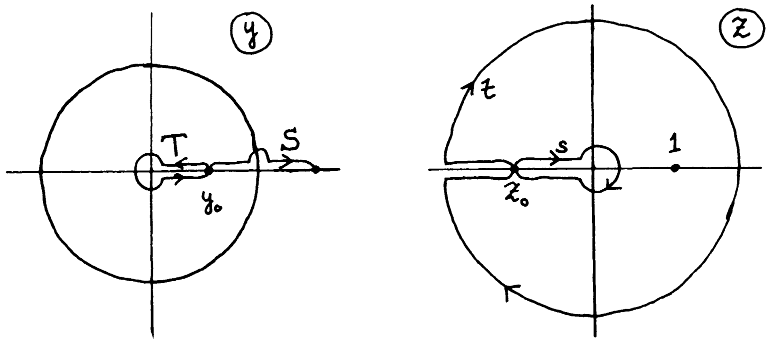

Fig. 1. The mapping $z = \frac{1}{2} - \frac{1}{4}(y + y^{-1})$.

Clearly L is invariant under the substitution $y \to y^{-1}$, and in the z-coordinate we get

$$L = z(z-1)\partial_z^2 + \{(1 + k_\beta + 2k_\alpha)z - (\tfrac{1}{2} + k_\beta + k_\alpha)\}\partial_z. \tag{4.2}$$

The differential equation $L\phi = (\lambda - \varrho, \lambda + \varrho)\phi$ discussed in Section 3 has in the z-coordinate the form

$$z(1-z)\partial_z^2\phi + \{(\tfrac{1}{2} + k_\beta + k_\alpha) - (1 + k_\beta + 2k_\alpha)z\}\partial_z\phi + \{(\lambda, \lambda) - (\varrho, \varrho)\}\phi = 0. \tag{4.3}$$

A comparison with the ordinary hypergeometric equation

$$z(1-z)\partial_z^2\phi + \{c - (1 + a + b)z\}\partial_z\phi - ab\phi = 0 \tag{4.4}$$

yields the following relations for the parameters

$$a = (\lambda + \varrho, \alpha^\vee) \qquad b = (-\lambda + \varrho, \alpha^\vee) \qquad c = \tfrac{1}{2} + k_\beta + k_\alpha. \tag{4.5}$$

Substituting a series

$$\phi(\mu, k; y) = \sum_{n=0}^{\infty} \Gamma_n(\mu, k) y^{m+n} \tag{4.6}$$

of the form (3.11) with leading exponent $m = (\mu, \alpha^\vee)$ and leading coefficient $\Gamma_0(\mu, k) = 1$ into equation (4.3) gives the indicial equation (cf. (3.13) and (3.17))

$$\mu = \pm\lambda + \varrho. \tag{4.7}$$

Under the assumption that $a - b = 2(\lambda, \alpha^\vee)$ is not an integer the corresponding solutions have in the z coordinate the form

$$\begin{aligned} \phi(\lambda + \varrho, k; z) &= 2^{-2a}(-z)^{-a}F(a, 1 + a - c, 1 + a - b; z^{-1}) \\ \phi(-\lambda + \varrho, k; z) &= 2^{-2b}(-z)^{-b}F(b, 1 + b - c, 1 + b - a; z^{-1}) \end{aligned} \tag{4.8}$$

where $F(a, b, c; z)$ denotes the Gaussian hypergeometric function. Introduce the c-function $c(\lambda, k)$ by

$$c(\lambda, k) = \frac{2^{2a}\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)} \tag{4.9}$$

where a, b, c are given by (4.5). The following proposition is due to Kummer.

PROPOSITION 4.1: *Assume $a - b = 2(\lambda, \alpha^\vee) \notin \mathbb{Z}$ and $c \notin \{0, -1, -2, \ldots\}$. Analytic continuation along the negative real axis gives the relation*

$$F(a, b, c; z) = c(\lambda, k)\phi(\lambda + \varrho, k; z) + c(-\lambda, k)\phi(-\lambda + \varrho, k; z) \tag{4.10}$$

where $\phi(\pm\lambda + \varrho, k; z)$ are the solutions (4.8).

Proof: The proof is classical, see, e.g. [Er] or [WW]. Q.E.D.

PROPOSITION 4.2: *Assume $a - b = 2(\lambda, \alpha^\vee) \notin \mathbb{Z}$ and $c \notin \{2, 3, 4, \ldots\}$. Analytic continuation along the negative real axis gives the relation*

$$\begin{aligned} &2^{2(1-a-b)}(-z)^{1-c}(1 - z)^{c-a-b}F(1 - b, 1 - a, 2 - c; z) \\ &\quad = c(\lambda, k')\phi(\lambda + \varrho, k; z) + c(-\lambda, k')\phi(-\lambda + \varrho, k; z) \end{aligned} \tag{4.11}$$

where k' is the associated parameter according to Definition 2.14.

Proof: Using formulas (4.5) we get the following formulas for the associated parameters:

$$\begin{aligned} \lambda' &= \lambda, \varrho + \varrho' = \sigma \Leftrightarrow (\varrho + \varrho', \alpha^\vee) = 1, \\ a' &= (\lambda' + \varrho', \alpha^\vee) = 1 - b, \\ b' &= (-\lambda' + \varrho', \alpha^\vee) = 1 - a, \quad c' = 2 - c. \end{aligned}$$

Using (4.10) we get $F(a', b', c'; z) = c(\lambda', k')\phi(\lambda' + \varrho', k'; z) + c(-\lambda', k')\phi(-\lambda' + \varrho', k'; z)$. The factor in front on the left hand side of (4.11) is equal to $y^{a+b-1}(1 + 0(y))$ for $y \to 0$. Since $a + b - 1 = (2\varrho - \sigma, \alpha^\vee) = (\varrho - \varrho', \alpha^\vee)$ we see that the product of this factor and $\phi(\lambda' + \varrho', k'; z)$ $(\phi(-\lambda' + \varrho', k'; z))$ is equal to $\phi(-\lambda + \varrho, k; z)(\phi(\lambda + \varrho, k; z)$ respectively). Q.E.D.

REMARK 4.3: The condition $2(\lambda, \alpha^\vee) \notin \mathbb{Z}$ can be weakened to the condition $(\lambda, \gamma^\vee) \notin \mathbb{Z}$ for all $\gamma \in R$. Indeed for R of type BC_1 this is just the same condition since $(\lambda, \beta^\vee) = 2(\lambda, \alpha^\vee)$. However for R of type A_1 the series (4.6) satisfies $\Gamma_n = 0$ for n odd due to the fact that $k_\beta = 0$. Hence the series (4.6) can be regarded as a series in $x = y^2$, and the desired factor 2 is obtained. Using the duplication formula for the Γ-function it is easy to see that the apparent poles of the c-function for $b - a$ equal to an odd integer do cancel.

Fix a base point z_0 on the negative real axis. The fundamental group $\Pi_1(\mathbb{C}\backslash\{0, 1\}, z_0)$ is free on two generators s and t as indicated in Fig. 1. Analytic continuation of solutions of (4.3) along curves in $\mathbb{C}\backslash\{0, 1\}$ defines a representation

$$M(\lambda, k)\colon \Pi_1(\mathbb{C}\backslash\{0, 1\}, z_0\} \to GL(2, \mathbb{C}) \tag{4.12}$$

the so called monodromy representation of the hypergeometric equation (4.3). The importance of the monodromy representation in the study of the hypergeometric equation has been emphasized by Riemann.

PROPOSITION 4.4: *Assume* $(\lambda, \gamma^\vee) \notin \mathbb{Z}$ *for all* $\gamma \in R$. *Relative to the basis* (4.8) *of solutions of* (4.3) *along the negative real axis we have*

$$M(\lambda, k)(s) = C(\lambda, k)\begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i c}\end{pmatrix} C(\lambda, k)^{-1} \tag{4.13}$$

$$M(\lambda, k)(t) = \begin{pmatrix} e^{2\pi i a} & 0 \\ 0 & e^{2\pi i b}\end{pmatrix} \tag{4.14}$$

where the matrix $C(\lambda, k)$ *is given by*

$$C(\lambda, k) = \begin{pmatrix} c(\lambda, k) & c(\lambda, k') \\ c(-\lambda, k) & c(-\lambda, k') \end{pmatrix} \tag{4.15}$$

Proof: Clear by Proposition 4.1, Proposition 4.2 and Remark 4.3. Q.E.D.

REMARK 4.5: If $c \in \mathbb{Z}$ the matric $C(\lambda, k)$ defined by (4.15) is no longer well defined. However, formula (4.13) has an analytic continuation for $c \in \mathbb{Z}$, but the matric $M(\lambda, k)(s)$ will no longer be semisimple for $c \in \mathbb{Z}$.

5. The fundamental group of the complement of the discriminant

In this section we keep the notation of Sections 2 and 3. Recall that $\{\alpha_1, \ldots, \alpha_n\}$ is a basis of simple roots for $R_{0,+}$. Let $\{\beta_1, \ldots, \beta_n\}$ be a basis of simple roots for R^0_+ with the ordering $\beta_j = \alpha_j$ or $\beta_j = 2\alpha_j$. Let $\{\lambda_1, \ldots, \lambda_n\}$ be the corresponding set of fundamental weights for P_+, i.e. $(\lambda_j, \beta_k^\vee) = \delta_{jk}$. The coroot lattice $Q^\vee$ is the lattice generated by all coroots $\alpha^\vee \in R^\vee$. Using the linear isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ by means of $(.,.)$, the lattice $2\pi i Q^\vee$ can be identified with the unit lattice $U = \{X \in \mathfrak{h};\ \exp(X) = e\}$ of the complex torus H. If $\{Z_1, \ldots, Z_n\}$ is a basis of $\mathfrak{h}$ with $\lambda_j(Z_k) = 2\pi i\delta_{jk}$, then $\{Z_1, \ldots, Z_n\}$ is a $\mathbb{Z}$ basis for U. Clearly $Z_j \simeq 2\pi i\beta_j^\vee$. Denote by r_j the simple reflection in W corresponding to β_j.

Fix a base point $X_0 \in \mathfrak{a}_-$, and let $h_0 = \exp(X_0) \in A_- \subset H^{\mathrm{reg}}$. The image point z_0 of h_0 under the mapping (2.5) lies in the complement of the discriminant.

For Z in the unit lattice U consider the curve

$$T_Z(t) = \exp(X_0 + tZ), \quad 0 \leqslant t \leqslant 1. \tag{5.1}$$

In particular for $Z = Z_j$ we write T_j for this curve. Consider also curves

$$S_j(t) = \exp((1-t)X_0 + tr_jX_0 + \varepsilon(t)Z_j), \quad 0 \leqslant t \leqslant 1. \tag{5.2}$$

Here $\varepsilon\colon [0, 1] \to [0, \frac{1}{2})$ is continuous with $\varepsilon(0) = \varepsilon(1) = 0$ and $\varepsilon(\frac{1}{2}) > 0$. For example take $\varepsilon(t) = \frac{1}{4}\sin \pi t$. Denote by t_Z, t_j and s_j the images of T_Z, T_j and S_j respectively under the $z = (z_1, \ldots, z_n)$ mapping (2.5). Observe that t_z, t_j and s_j are closed curves with begin point z_0 in the complement of the discriminant. The corresponding equivalance classes in $\pi_1(\mathbb{C}^n \backslash \Delta^2(z) = 0, z_0)$ are also denoted by t_z, t_j, and s_j. Let m_{ij} be the order of the element $r_i r_j$ in W, and let $n_{ij} = (\beta_i, \beta_j^\vee)$ be the Cartan integers of R^0.

THEOREM 5.1: *The fundamental group $\pi_1(\mathbb{C}^n \backslash \Delta^2 = 0, z_0)$ of the complement of the discriminant is generated by $t_1, \ldots, t_n$ and $s_1, \ldots, s_n$. A complete set of relations is given by*

$$t_i t_j = t_j t_i \tag{5.3}$$

$$s_i s_j s_i \ldots = s_j s_i s_j \ldots, \quad i \neq j \text{ and } m_{ij} \text{ factors on both sides} \tag{5.4}$$

$$s_i t_j = t_j t_i^r s_i t_i^{-r}, \quad i \neq j \text{ and } n_{ij} = -2r \text{ even} \tag{5.5}$$

$$s_i t_j = t_j t_i^{r+1} s_i^{-1} t_i^{-r}, \quad i \neq j \text{ and } n_{ij} = -(2r+1) \text{ odd.} \tag{5.6}$$

REMARK 5.2: In this form the above theorem is due to van der Lek and Looyenga [vdL, p. 69]. Previously topological results on the complement of the discriminant have been obtained by Brieskorn [Br], Deligne [De 2] and Nguyên Viet Dûng [Ng].

The group described in the above theorem is called the extended Artin group of the root system R^0. Relations (5.4) are called the braid relations, and relations (5.5) and (5.6) are called the push relations. The following consequence of Theorem 5.1. is crucial for later applications.

COROLLARY 5.3: *If $r_j Z = Z$ for $Z \in U \simeq 2\pi i Q^\vee$ for some $j = 1, \ldots, n$, then $t_Z s_j = s_j t_Z$.*

Proof: Straight forward using (5.3), (5.5) and (5.6). Q.E.D.

6. The monodromy representation of the system of hypergeometric differential equations

In this section we explicitly compute the monodromy representation of the system of hypergeometric differential equations (2.20). By Corollary 3.9 and Corollary 3.14 this system can also be considered as a system of linear partial differential equations on $\mathbb{C}^n$ with polynomial coefficients and holonomic on the complement of the discriminant $\Delta^2(z) = 0$.

By Corollary 3.11 the functions

$$\phi(w\lambda + \varrho, k; h), \quad w \in W \tag{6.1}$$

form a basis for the solution space on A_- under the assumption $(\lambda, \alpha^\vee) \neq 0$ and $(\lambda, \kappa^\vee) \neq 1$ for all $\alpha \in R$, $\kappa \in Q$ and $\kappa \neq 0$.

DEFINITION 6.1: For $\alpha \in R^0$ define the function $c_\alpha(\lambda, k)$ by

$$c_\alpha(\lambda, k) = \frac{2^{2(\lambda,\alpha^\vee)+k_{\alpha/2}+2k_\alpha}\Gamma(\frac{1}{2} + k_{\alpha/2} + k_\alpha)\Gamma(-2(\lambda, \alpha^\vee))}{\Gamma(-(\lambda, \alpha^\vee) + \frac{1}{2}k_{\alpha/2} + k_\alpha)\Gamma(-(\lambda, \alpha^\vee) + \frac{1}{2}k_{\alpha/2} + \frac{1}{2})}. \tag{6.2}$$

DEFINITION 6.2: For $\beta \in R_0$ define the function $c_\beta(\lambda, k)$ by

$$c_\beta(\lambda, k) = \frac{2^{(\lambda,\beta^\vee)+k_\beta+2k_{2\beta}}\Gamma(\frac{1}{2} + k_\beta + k_{2\beta})\Gamma(-(\lambda, \beta^\vee))}{\Gamma(-\frac{1}{2}(\lambda, \beta^\vee) + \frac{1}{2}k_\beta + k_{2\beta})\Gamma(-\frac{1}{2}(\lambda, \beta^\vee) + \frac{1}{2}k_\beta + \frac{1}{2})}. \tag{6.3}$$

The ambiguity in the notation $c_\alpha(\lambda, k)$ for $\alpha \in R$ is removed by the following lemma.

LEMMA 6.3: *Suppose* $\alpha = \beta \in R^0 \cap R_0$. *Then* $c_\alpha(\lambda, k) = c_\beta(\lambda, k)$ *where* $c_\alpha(\lambda, k)$ *is defined by* (6.2) *and* $c_\beta(\lambda, k)$ *by* (6.3).

Proof: If $\alpha = \beta \in R^0 \cap R_0$, then $k_{\alpha/2} = k_{2\beta} = 0$. Hence we have

$$c_\alpha(\lambda, k) = \frac{2^{2(\lambda,\alpha^\vee)+2k_\alpha}\Gamma(\frac{1}{2} + k_\alpha)\Gamma(-2(\lambda, \alpha^\vee))}{\Gamma(-(\lambda, \alpha^\vee) + k_\alpha)\Gamma(-(\lambda, \alpha^\vee) + \frac{1}{2})}$$

and

$$c_\beta(\lambda, k) = \frac{2^{(\lambda,\beta^\vee)+k_\beta}\Gamma(\frac{1}{2} + k_\beta)\Gamma(-(\lambda, \beta^\vee))}{\Gamma(-\frac{1}{2}(\lambda, \beta^\vee) + \frac{1}{2}k_\beta)\Gamma(-\frac{1}{2}(\lambda, \beta^\vee) + \frac{1}{2}k_\beta + \frac{1}{2})}.$$

Using the duplication formula $\Gamma(2z) = 2^{2z-1}.\pi^{-1/2}.\Gamma(z)\Gamma(z + \frac{1}{2})$ we obtain the identity

$$c_\alpha(\lambda, k) = c_\beta(\lambda, k) = \frac{2^{2k_\alpha-1}.\Gamma(\frac{1}{2} + k_\alpha).\Gamma(-(\lambda, \alpha^\vee))}{\pi^{1/2}.\Gamma(-(\lambda, \alpha^\vee) + k_\alpha)}. \qquad \text{Q.E.D.}$$

Observe that for $\alpha = 2\beta \in R^0$, $\beta \in R_0$ we have $c_\alpha(\lambda, k) = c_\beta(\lambda, k)$ since $\beta^\vee = 2\alpha^\vee$.

DEFINITION 6.4: The Harish–Chandra c-function is defined by

$$c(\lambda, k) = c_0 \prod_{\alpha \in R^0_+} c_\alpha(\lambda, k) = c_0 \prod_{\beta \in R_{0,+}} c_\beta(\lambda, k) \tag{6.4}$$

where the constant $c_0 = c_0(k)$ is chosen in such a way that $c(-\varrho, k) = 1$.

COROLLARY 6.5: *Suppose $R = R^0 = R_0$ is a reduced system. Then*

$$c(\lambda, k) = \frac{q((\varrho, \alpha^v))}{q(-(\lambda, \alpha^v))} \tag{6.5}$$

where $q(x)$ is the function

$$q(x) = \prod_{\alpha \in R_+} \frac{\Gamma(x + k_\alpha)}{\Gamma(x)}.$$

REMARK 6.6: The product formula for the c-function over $\beta \in R_{0,+}$ is the same as in Helgason [Hel 2, p. 477] apart from minor differences in notation. Helgason writes $i\lambda$ where we write λ. The minus sign comes from the fact that our expansions hold on A_- rather than A_+. The factor 2 between $\alpha^\vee = 2(\alpha, \alpha)^{-1}\alpha$ and $\alpha_0 = (\alpha, \alpha)^{-1}\alpha$ and between k_α and m_α is explained in Remark 2.4.

Relative to the basis (6.1) of the solution space of the hypergeometric equations we have the monodromy representation

$$M(\lambda, k)\colon \Pi_1(\mathbb{C}^n \backslash \{\Delta^2 = 0\}, z_0) \to GL(d, \mathbb{C}) \tag{6.6}$$

where d equals the order of the Weyl group W. In the notation of Section 5 the function $\phi(w\lambda + \varrho, k; h)$ is an eigenfunction for the monodromy of the curve t_Z with eigenvalue exp $(w\lambda + \varrho, Z)$. Here Z is in the unit lattice $U \simeq 2\pi i Q^v$ of H.

Fix a simple root $\alpha_j \in R_{0,+}$ and let r_j denote the corresponding simple reflection in W. Using Corollary 5.3 it is clear that the two dimensional space

$$\operatorname{span} \{\phi(w\lambda + \varrho, k; h), \phi(r_j w\lambda + \varrho, k; h)\} \tag{6.7}$$

is invariant under the monodromy of the curve s_j.

THEOREM 6.7: *Assume $(\lambda, \alpha^\vee) \neq 0$ and $(\lambda, \kappa^\vee) \neq 1$ for all $\alpha \in R$, $\kappa \in Q$, $\kappa \neq 0$. Relative to the basis $\{\phi(w\lambda + \varrho, k; h), \phi(r_j w\lambda + \varrho, k; h)\}$ of* (6.7) *the monodromy matrix of the curve s_j has the form*

$$M(\lambda, k)(s_j) = C_j(w\lambda, k)\begin{pmatrix}1 & 0\\ 0 & e^{2\pi i c_j}\end{pmatrix} C_j(w\lambda, k)^{-1} \tag{6.8}$$

where $c_j = \frac{1}{2} + k_{\alpha_j} + k_{2\alpha_j}$ and the matrix $C_j\,(w\lambda, k)$ is given by

$$C_j(w\lambda, k) = \begin{pmatrix} c_{\alpha_j}(w\lambda, k) & c_{\alpha_j}(w\lambda, k') \\ c_{\alpha_j}(r_j w\lambda, k) & c_{\alpha_j}(r_j w\lambda, k')\end{pmatrix}. \tag{6.9}$$

Here k' denotes the associated parameter (cf. Definition 2.14).

Proof: As explained in Section 3 the system has regular singularities along the coordinate planes $x_j = 0$ at infinity. Taking boundary values along such a hyperplane $x_j = 0$ gives a reduction to a lower rank root system. The essential point is that taking boundary values commutes with the action of the monodromy. For more details we refer to Section 3 of Heckman [Hec 2]. By induction on the rank of R the statement reduces to rank one, and one can apply Proposition 4.4. Q.E.D.

COROLLARY 6.8: *The monodromy matrices $M(\lambda, k)(s_j)$ satisfy the Hecke relations*

$$(M(\lambda, k)(s_j) - 1)(M(\lambda, k)(s_j) - q_j) = 0 \tag{6.10}$$

$$M(\lambda, k)(s_i)M(\lambda, k)(s_j)\ldots = M(\lambda, k)(s_j)M(\lambda, k)(s_i)\ldots \tag{6.11}$$

Here $q_j = e^{2\pi i c_j}$ with $c_j = \frac{1}{2} + k_{\alpha_j} + k_{2\alpha_j}$, and in (6.11) *we have m_{ij} factors on both sides.*

THEOREM 6.9: *Assume* $(\lambda, \alpha^\vee) \neq 0$ and $(\lambda, \kappa^\vee) \neq 1$ *for all $\alpha \in R$, $\kappa \in Q$, $\kappa \neq 0$. Assume $c_j = \frac{1}{2} + k_{\alpha_j} + k_{2\alpha_j} \notin \{0, -1, -2, \ldots\}$ for $j = 1, \ldots, n$. Let V be a Weyl group invariant tubular neighborhood of A in H, and $V^{\text{reg}} = V \cap H^{\text{reg}}$. Then the function*

$$F(\lambda, k; h) = \sum_{w \in W} c(w\lambda, k)\phi(w\lambda + \varrho, k; h) \tag{6.12}$$

which is a priori defined as an analytic function on A_-, has an analytic continuation to a single valued Weyl group invariant analytic function on V^{reg}.

As a solution of the hypergeometric equations (2.20) *on* V^{reg} *the function* $F(\lambda, k; h)$ *is up to a constant uniquely characterized by this property.*

Proof: Using Theorem 6.7 it follows that the function

$$c_{\alpha_j}(w\lambda, k)\phi(w\lambda + \varrho, k; h) + c_{\alpha_j}(r_j w\lambda, k)\phi(r_j w\lambda + \varrho, k; h)$$

is fixed under the monodromy $M(\lambda, k)(s_j)$ of s_j. Using that $(c_{\alpha_j}(w\lambda, k): c_{\alpha_j}(r_j w\lambda, k)) = (c(w\lambda, k): c(r_j w\lambda, k))$, which is immediate by Definition 6.4, the theorem follows. Q.E.D.

REMARK 6.10: The proof of the theorem uses in an essential way Conjecture 2.10. In the paper [Hec 2] the function $F(\lambda, k; h)$ will be constructed for an arbitrary root system and independently of Conjecture 2.10. The ingredients are a reduction to rank two, in which case Conjecture 2.10 has been proved [Op], and Deligne's version of the Riemann–Hilbert correspondence. As a by product the function $F(\lambda, k; h)$ turns out to be analytic on all of V. This is a reflection of the fact that the system (2.20) has regular singularities along the discriminant $\Delta^2 = 0$ as well.

By analogy with the group case the normalization for the c-function by $c(-\varrho, k) = 1$ is chosen. With this normalization one expects the following to be true.

CONJECTURE 6.11: $F(\lambda, k: e) = 1$.

Partial results for R of rank two in [Op] do confirm this.

REMARK 6.12: The c-function was first introduced in the group case by Harish–Chandra in his study of the spherical Plancherel formula [Ha]. The normalization $c(-\varrho, k) = 1$ follows in the group case from an integral formula for the c-function.

By an explicit evaluation of this integral Gindikin and Karpelevic proved the product formula for the c-function. The method followed here gives an independent proof of the Gindikin–Karpelevic product formula just using monodromy arguments.

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