# Compositio Mathematica

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## Endoscopic groups and packets of non-tempered representations

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## 0. Introduction

Let **G** be a connected reductive algebraic group defined over $\mathbb{R}$, and let $G = \mathbf{G}(\mathbb{R})$ be its points over $\mathbb{R}$. Work of Shelstad [11, 12, 13, 14] has established several cases of functoriality with respect to L-groups of the tempered spectra of such groups $G$. In [2], Arthur has conjectured analogues that should be true for a wide class of non-tempered but conjecturally unitary representations. All of this, of course, was motivated by considerations involving the trace formula [10]. This paper describes the extent to which we have verified these conjectures for a certain class of derived functor modules (which are known, [19, 20] to be unitary). These results were announced in [1].

For most of this paper we assume, for simplicity of statement, that $G$ has relative discrete series. (This assumption is only a convenience, and is removed at the end of the paper.) In this introduction, for ultimate simplicity, we assume further that $G$ is connected.

In the Langlands classification of irreducible admissible representations, they (or their equivalence classes) are partitioned into finite sets, called *L*-packets. For example, the set of discrete series representations with a given infinitesimal (and central) character forms an *L*-packet. All of the representations in an *L*-packet are tempered or else none of them are: if they all are, then the *L*-packet is said to be tempered. Trace formula considerations have brought to the fore the notions of stability and *L*-indistinguishability. Two irreducible representations are *L*-indistinguishable if they occur in the same *L*-packet. If $\pi$ is an irreducible representation, let $\Theta_\pi$ denote its character. A stable distribution is any element of the closure of the space spanned by all distributions of the form $\Sigma_{\pi \in \Pi} \Theta_\pi$ for $\Pi$ any tempered *L*-packet. This is a natural definition to make for many reasons, for

example, as an analytic function, such a distribution is invariant under conjugation by elements of $\mathbf{G}(\mathbb{C})$ (when this makes sense) and not merely $G$. Such distributions can be transferred to inner forms of $G$ (via the matching of stable orbital integrals, [11, 1]), whereas unstable distributions cannot be. Furthermore, the vector space spanned by $\{\Theta_\pi : \pi \in \Pi\}$ is spanned by $\{\Theta_0, \ldots, \Theta_n\}$ where $\Theta_0 = \Sigma_{\pi \in \Pi} \Theta_\pi$ is stable (for $G$); and each $\Theta_i$, $i \geqslant 1$, is the transfer via $L$-functoriality to $G$ of a stable tempered distribution on a smaller, endoscopic, group $H_i$.

For non-tempered $L$-packets all of the above fail. In [2], Arthur has conjectured that in certain cases, a non-tempered $L$-packet $\Pi$ can be enlarged to a finite set of irreducible unitary representations $\hat{\Pi} \supseteq \Pi$, and that this enlarged packet will enjoy stability and tranfer via functoriality, analogous to Shelstad's results for tempered representations. Let $^L G = {}^L G^0 \rtimes W_{\mathbf{R}}$ be the $L$-group of $G$ (see section 1, or [4]). Special unipotent representations (see [3]) are associated in a natural way to unipotent orbits in $^L G^0$, and are conjecturally unitary. They come in enlarged packets etc. Further enlarged packets should be produced by cohomological induction from these unipotent representations on proper subgroups of $G$. This paper verifies all but one of Arthur's conjectures for certain enlarged packets of this type. The remaining conjecture, inversion, vis. that $\{\Theta_0, \ldots, \Theta_n\}$ spans $\langle \pi | \pi \in \hat{\Pi} \rangle$ holds in many cases, but seems to fail in many interesting cases – for example $\mathbf{Sp}(2)$. We do not treat inversion in this paper.

The enlarged packets which we will study are described as follows, at least under the simplifying assumption that $G$ is connected. Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a $\theta$-stable parabolic subalgebra of $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbf{R}} \mathbb{C}$, where $\mathfrak{g}_0$ is the Lie algebra of $G$ and $\theta$ is the Cartan involution of $\mathfrak{g}$ [16]. Let $L$ be the stabilizer of $\mathfrak{q}$ in $G$, then $L$ is the stabilizer of a compact torus, so $L$ is also a connected reductive group. Let $T \subseteq L$ be a compact Cartan subgroup of $G$. If $\pi$ is a one dimensional representation of $L$, set $\mathscr{R}_\mathfrak{q}(\pi) = \mathscr{R}_\mathfrak{q}^{1/2 \dim(\mathfrak{l}/\mathfrak{l} \cap \mathfrak{t})}(\pi)$ as in [16], p. 344. We assume $\pi$ unitary and make additional technical hypotheses on $\lambda \in \mathfrak{t}^*$, the highest weight of $\pi$, to guarantee, writing $A(\lambda) = \mathscr{R}_\mathfrak{q}(\pi)$, that $A(\lambda)$ is an irreducible unitary representation (cf. 2.10). Since $L$ is connected, $\pi$ is determined by $\lambda$. There is an action of the Weyl group $W = W(\mathfrak{g}, \mathfrak{t})$ on the datum $(\mathfrak{q}, L, \lambda)$, denoted $w \cdot (\mathfrak{q}, L, \lambda) = (\mathfrak{q}_w, L_w, w\lambda)$. Then $A(w\lambda)$ is also irreducible and unitary, meaning $\mathscr{R}_{\mathfrak{q}_w}(w\lambda)$, and we define $\hat{\Pi}_\lambda = \{A(w\lambda): w \in W\}$. Our main results are that $\hat{\Pi}_\lambda$ satisfies Arthur's conjectures (except for inversion). We state this more precisely below.

An important special case, and the only one in which every $A(w\lambda)$ is tempered, is obtained when $L = T$. Then $\hat{\Pi}_\lambda$ is just an $L$-packet of discrete series. In general, $A(\lambda)$ is in the discrete series if and only if $L$ is compact.

The sets $\hat{\Pi}_\lambda$ are not disjoint, for example $\hat{\Pi}_\lambda$ may contain some non-tempered and some discrete series representations.

The set $\hat{\Pi} = \hat{\Pi}_\lambda$ satisfies the following properties. It contains an ordinary $L$-packet. Let $S = W(G, T)\backslash W(\mathfrak{g}, \mathfrak{t})/W(\mathfrak{l}, \mathfrak{t})$ (cf. 2.9). Then $A(w\lambda)$ depends only on the coset of $w$ in $S$, and we may write $\hat{\Pi} = \{A(w\lambda): w \in S\}$. For $w \in S$ we will define an integer $\gamma(w)$ in 2.12. Identifying a representation with its character, we have

THEOREM (2.13). $\Sigma_{w \in S} (-1)^{\gamma(w)} A(w\lambda)$ *is stable.*

Let **H** be an endoscopic group for **G** which contains a group isomorphic to **L**, called **L**. The definitions in this introduction of enlarged packets only make sense, so far, for connected groups, so assume $H$ is connected. (This is a simplifying assumption which will be removed in the body of the paper.) Let $\mathrm{Tran}_H^G$ denote the transfer via $L$-functoriality of stable tempered distributions on $H$ to tempered distributions on $G$ [14]. In section 4 we extend this to non-tempered stable distributions, at first formally. Given $(\mathfrak{q}', L, \lambda')$ a datum for $H$, we have then $\hat{\Pi}' = \{A(w\lambda'): w \in S'\}$, where $S'$ is defined in an analogous way to $S$, an enlarged packet for $H$. We give a formula, 2.20, giving $\lambda$ as a function of the given $\lambda'$ such that

THEOREM 2.21. $\mathrm{Tran}_H^G \Sigma_{w \in S'} (-1)^{\gamma'(w)} A(w\lambda') = \varepsilon \Sigma_{w \in S} (-1)^{\gamma(w)} \kappa(w) A(w\lambda)$ *where $\gamma'$ is defined analogously to $\gamma$, $\varepsilon = \pm 1$, and $\kappa: S \to \{\pm 1\}$ is part of the endoscopic datum for H.*

Although we do not make full use of it, these results are most naturally stated in the language of $L$-groups. We discuss this without proofs in Section 3. Let $W_\mathbb{R}$ be the Weil group of $\mathbb{R}$. An ordinary $L$-packet $\Pi_\phi$ is associated to an admissible homomorphism $\phi: W_\mathbb{R} \to {}^L G$. Arthur considers homomorphisms factoring through ${}^L PGL_2(\mathbb{R})$, that is,

$$\phi: W_\mathbb{R} \xrightarrow{\psi} {}^L PGL_2(\mathbb{R}) \xrightarrow{\tilde{\phi}} {}^L G.$$

Here $\psi$ corresponds to the trivial representation of $PGL_2(\mathbb{R})$. Arthur's conjectures seek enlarged packets $\hat{\Pi}_\phi \supseteq \Pi_\phi$ satisfying stability and transfer-via-$L$-functoriality properties.

Now ${}^L PGL_2(\mathbb{R}) \approx SL_2(\mathbb{C}) \times W_\mathbb{R}$ and $\tilde{\phi}|_{SL_2(\mathbb{C})}$ corresponds (via the Jacobson-Morozov Theorem) to a unipotent orbit of ${}^L G^0$. Suppose $\tilde{\phi}(\mathbb{C}^*) \subseteq \mathrm{center} \, ({}^L G^0) \times W_\mathbb{R}$. If $G$ is quasisplit then $\phi$ is admissible and certain $\pi \in \Pi_\phi$ are said to be special unipotent representations. In particular if $\tilde{\phi}|_{SL_2(\mathbb{C})}$ corresponds to the principal unipotent orbit of ${}^L G_0$, then

$\hat{\Pi}_\phi = \Pi_\phi = \{\pi\}$, where $\pi$ is a single one-dimensional representation. If $G$ is not quasisplit we define one-dimensional $\pi$ in section 2 (where $L$ plays the role of $G$).

We verify Arthur's conjectures for $\phi$ of the following form. Given $L \subseteq G$ as above, assume that one can embed $^L L \to {}^L G$ (although we omit the proof of this). We assume

a. $\phi$ factors through $^L L$, that is, $\phi$: $W_{\mathbf{R}} \xrightarrow{\phi^L} {}^L L \longrightarrow {}^L G$.
b. $\phi^L$ defines a one-dimensional representation $\pi$ of $L$ by the preceeding discussion; i.e.

$$\phi^L: W_{\mathbf{R}} \longrightarrow {}^L PGL_2(\mathbb{R}) \xrightarrow{\tilde{\phi}^L} {}^L L$$

and $\tilde{\phi}^L|_{SL_2(\mathbb{C})}$ corresponds to the principal unipotent orbit of $^L L^0$.

(Condition a is analogous to the condition, cf. [4], that $\phi$ factor through $^L M \subseteq {}^L G$, $M$ a real Levi factor.)

Given $\phi$ we construct $\mathfrak{q}$, $L$, $\lambda = \pi|_T$ and $A(\lambda)$ (cf. Section 3). Let $\hat{\Pi}_\phi = \{A(w\lambda)\}$. That $\hat{\Pi}_\phi$ satisfies Arthur's conjectures is the content of theorems 2.13 and 2.21. The endoscopic groups $H$ we consider are those satisfying $^L L \subseteq {}^L H \subseteq {}^L G$. This condition precludes in many cases any inversion theorem for $\hat{\Pi}_\phi$ (cf. [2], 1.3.5): there may fail to be enough endoscopic groups $H$ to separate the characters of $\hat{\Pi}_\phi$. See Section 3. It is of course not the case that the operation of transfer preserves the class of derived functor modules to which we confine our attention in this paper. The two hypotheses, 2.16 and "$\zeta$-admissibility" (in the discussion preceding 2.21) are introduced to allow us to so confine our attention.

In the special case where $G$ and $H$ are connected, the conclusions of Theorems 2.13 and 2.21 can be verified, and constitute theorems 7.2 and 7.12. Here, their proofs proceed by expressing $A(\lambda)$ in terms of coherent continuation of discrete series representations (cf. 7.7). Since coherent continuation preserves stability (cf. 6.1) Theorem 7.2 follows from Shelstad's results on discrete series $L$-packets. This theorem was known to Zuckerman in 1978. Furthermore transfer commutes with coherent continuation (cf. 6.5) and Theorem 7.12 similarly follows.

A large portion of this paper is involved with the technicalities due to disconnected groups. First of all if $G$ is not connected $L_w$ may fail to be connected, so the one-dimensional representation $\pi_w$ of $L_w$ may fail to be determined by $w\lambda = \pi_w|_T$. We will later choose $\pi_w$ consistently (following Lemma 2.5), and write $A(w\lambda, \pi_w) = \mathcal{R}_{\mathfrak{q}_w}(\pi_w)$.

Furthermore Zuckerman's character formula for $A(\lambda, \pi)$ may fail; we obtain a weaker formula for $\Sigma_i A(\lambda, \pi_i)$ where each $\pi_i|_T = \lambda$. To distinguish

these components we used in addition a character formula for $A(\lambda)$, due to one of the authors [7], valid for disconnected groups. This formula describes $A(\lambda)$ in terms of standard (induced from real parabolic) representations, instead of coherent continuations of standard representations. A combination of the two techniques proves the Theorem in general. (Section 9.)

The paper is organized as follows. Section 1 recalls the material on $L$-groups which we shall need. In Section 2 we state without proof the stability (2.13) and transfer (2.22) properties of $A(w\lambda, \pi_w)$. This is done largely independently of $L$-group considerations. In Section 3 this is restated in term of $L$-groups, material which we do not use but include for the sake of completeness, and which allows comparison with Arthur's original formulation. Section 4 defines transfer in the necessary generality. In Section 5 we discuss results of [13] on the transfer of discrete series representations (5.6 5.8) which, applied to Levi subgroups, yields the transfer of standard modules (5.10). The role of coherent continuation is discussed in Section 6. We prove the weak versions (which, for connected groups, are equivalent to 2.13 and 2.21) of the main results, in Section 7. Section 8 uses the resolution of $A(w\lambda, \pi_w)$ in terms of standard modules, valid for disconnected groups, to prove Theorem 2.13 and nearly prove Theorem 2.21. This is completed in Section 9 by a combination of the two preceding sections.

We then indicate the relation of this formal definition of *Tran* to the matching of orbital integrals (this is routine). Finally, we show how to remove the assumption that $G$ possesses (relative) discrete series.

# 1. *L*-groups

We collect some material concerning $L$-groups which we will use repeatedly. Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over $\mathbb{R}$, $G = \mathbf{G}(\mathbb{R})$. We follow [9], see also [4] and [15]. Let $\sigma$ be the non-trivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$; $\sigma$ acts on $\mathbf{G}$ and various objects associated to $\mathbf{G}$. This action is denoted $\sigma_G$ or simply $\sigma$ if there is no danger of confusion. Let $\mathbf{G}^*$ be a quasisplit inner form of $\mathbf{G}$, $\psi: \mathbf{G} \to \mathbf{G}^*$ an inner twist: $\psi$ is an isomorphism (over $\mathbb{C}$) such that $\sigma_G(\psi^{-1})\psi$ is inner. Fix $\mathbf{T}^*$ a Cartan subgroup of $\mathbf{G}^*$ and $\mathbf{B}^*$ a Borel subgroup containing $\mathbf{T}^*$, both defined over $\mathbb{R}$. If $\mathbf{T}$ is a Cartan subgroup of $\mathbf{G}$, $X^*(\mathbf{T})$ (respectively, $X_*(\mathbf{T})$) is the lattice of rational characters (respectively one-parameter subgroups) of $\mathbf{T}$, and $\Delta(\mathbf{G}, \mathbf{T}) \subseteq X^*(\mathbf{T})$ (resp. $\Delta^\vee(\mathbf{G}, \mathbf{T}) \subseteq X_*(\mathbf{T})$) is the set of roots (resp. coroots) of $\mathbf{T}$ in $\mathbf{G}$. Also let $\Lambda$ (resp. $\Lambda^\vee$) be the root lattice (resp. coroot lattice) of $\mathbf{T}$ in $\mathbf{G}$. There is a pairing $\langle \, , \, \rangle$ between $X^*(\mathbf{T})$ and $X_*(\mathbf{T})$, and also between $\Lambda$ and $\Lambda^\vee$. The triple $(\mathbf{G}^*, \mathbf{B}^*, \mathbf{T}^*)$ defines a datum $(X^*(\mathbf{T}^*), \Delta, X_*(\mathbf{T}^*), \Delta^\vee)$ where

$\Delta = \Delta(\mathbf{B^*}, \mathbf{T^*})$. Conversely, such a datum defines a triple $(\mathbf{G}, \mathbf{B}, \mathbf{T})$ defined over $\mathbb{C}$. Let $({}^{L}G^0, {}^{L}B^0, {}^{L}T^0)$ be defined by $(X_*(\mathbf{T^*}), \Delta^{\vee}, X^*(\mathbf{T^*}), \Delta)$. Thus $X^*({}^{L}T^0) = X_*(\mathbf{T^*})$, $X_*({}^{L}T^0) = X^*(\mathbf{T^*})$, etc. Now $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on this datum, hence on ${}^{L}G^0$. Let $W_{\mathbb{R}}$ be the Weil group of $\mathbb{R}$: $W_{\mathbb{R}} = \mathbb{C}^* \cup \tau\mathbb{C}^*$, $\tau^2 = -1$, $\tau z\tau^{-1} = \bar{z}$, and there is a surjection $W_{\mathbb{R}} \longrightarrow\!\!\!\!\rightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Via this map let ${}^{L}G = {}^{L}G^0 \rtimes W_{\mathbb{R}}$. Note that ${}^{L}G$ (but not its isomorphism class) depends on the choice of $(\mathbf{G^*}, \mathbf{B^*}, \mathbf{T^*})$ and $\psi$.

We need to pass between objects for $G$ and for ${}^{L}G^0$. This is accomplished (non-canonically) as follows ([14]). Let $\eta$ be a pseudo-diagonalization ($p$-$d$.) of $T = \mathbf{T}(\mathbb{R})$: $\eta$ is an isomorphism $\mathbf{T} \to \mathbf{T^*}$ of the form $\mathrm{Ad}(g) \circ \psi$ ($g \in \mathbf{G^*}$). This induces isomorphisms also denoted $\eta$:

$$X^*(\mathbf{T}) \overset{\eta}{\approx} X_*({}^{L}T^0)$$

$$X_*(\mathbf{T}) \overset{\eta}{\approx} X^*({}^{L}T^0) \tag{1.1}$$

$$W(\mathbf{G}, \mathbf{T}) \overset{\eta}{\approx} W({}^{L}G^0, {}^{L}T^0).$$

Here $W(\mathbf{G}, \mathbf{T})$ is the Weyl group of $\mathbf{T}$ in $\mathbf{G}$, etc.

Let $D$ be any Cartan subgroup of $G$, with Galois action $\sigma$. Then the characters of $D$ are given as follows (cf. [13], Section 4.1). Suppose $\mu, \lambda \in X^*(\mathbf{D}) \otimes \mathbb{C}$ satisfy $\frac{1}{2}(\mu - \sigma\mu) + (\lambda + \sigma\lambda) \in X^*(\mathbf{D})$ or both $(\mu - \sigma\mu) \in X^*(\mathbf{D})$ and $\frac{1}{2}(\mu - \sigma\mu) - (\lambda + \sigma\lambda) \in X^*(\mathbf{D})$. Define

$$\chi(\mu, \lambda)(e^X) = e^{1/2\langle \mu, X + \sigma\bar{X}\rangle + \langle \lambda, X - \sigma\bar{X}\rangle} \quad (X \in X_*(\mathbf{D}) \otimes \mathbb{C} \approx \mathfrak{d}) \ (e^X \in D).$$

Then $\chi(\mu, \lambda) = \chi(\mu', \lambda')$ if and only if $\mu = \mu'$ and $\lambda = \lambda' + \delta + (\beta - \sigma\beta)$ for some $\delta \in X^*(\mathbf{D})$, $\beta \in X^*(\mathbf{D}) \otimes \mathbb{C}$. In particular, if $D$ is compact, $\sigma = -1$ and $\hat{D} \approx X^*(\mathbf{D})$. Write $D = D^0F$ where $F = \{e^{\pi iX} | X \in X_*(\mathbf{D}), \sigma X = X\}$. Note that $\chi(\mu, \lambda)|_{D^0}$ is independent of $\lambda$ and $\chi(\mu, \lambda)|_F$ is independent of $\mu$. Let $M$ be the centralizer of the maximal $\mathbb{R}$-split torus of $D$; then $W(\mathbf{M}, \mathbf{D})$ acts on $\hat{D}$ and $w\chi(\mu, \lambda) = \chi(w\mu, \lambda)$.

## 2. Definitions, results, and some lemmas

We make some definitions and state the main results on stability and transfer properties of $\{A(w\lambda, \pi_w)\}$, Theorems 2.13 and 2.21. Here we remain as algebraic as possible. In the next section we give a reformulation in terms of $L$-homomorphisms and Arthur's conjecture [2]. Some lemmas are included in this section as well.

Let $G$ be a reductive group over $\mathbb{R}$ satisfying the conditions of section 1. Thus $\mathbf{G}$ is a connected reductive linear algebraic group defined over $\mathbb{R}$, $G = \mathbf{G}(\mathbb{R})$. Furthermore, we assume $G$ contains a compact Cartan subgroup $T$ which we fix. We fix $K$ a maximal compact subgroup, $K \supseteq T$, and its Cartan involution, $\theta$. Let $\mathfrak{g}_0$ be the Lie algebra of $G$, $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$. We work in the category of $(\mathfrak{g}, K)$ modules [16].

Let $L$ be the stabilizer in $G$ of a compact torus. After conjugating by $G$ we assume $L$ is the stabilizer under the coadjoint action of $\lambda_0 \in X^*(\mathbf{T}) \otimes \mathbb{C} \approx \mathfrak{t}^* \subseteq \mathfrak{g}^*$ (embedded via the Killing form). We write $L = L(\lambda_0)$; $L$ satisfies our hypotheses, $T \subseteq L$, and $L$ is connected if $G$ is.

Let $\mathfrak{q}$ be a $\theta$-stable parabolic subalgebra of $\mathfrak{g}$ [16] satisfying $L = \mathrm{Stab}_G(\mathfrak{q})$ (stabilizer of $\mathfrak{q}$ in $G$). Write $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ (Levi decomposition). For $\alpha \in \mathfrak{t}^*$ a root of $\mathfrak{t}$ in $\mathfrak{g}$ write $\mathfrak{g}^\alpha$ for the corresponding root space. There exists $\lambda_0 \in \mathfrak{t}^*$ such that $\mathfrak{l} = \mathfrak{t} \oplus \Sigma_{\langle \lambda_0, \alpha^\vee \rangle = 0} \mathfrak{g}^\alpha$, $\mathfrak{u} = \Sigma_{\langle \lambda_0, \alpha^\vee \rangle > 0} \mathfrak{g}^\alpha$. We write $\mathfrak{q} = \mathfrak{q}(\lambda_0)$. Let $\pi$ be an $(\mathfrak{l}, L \cap K)$ module. Then for $i = 0, 1, 2, \ldots,$

$$\mathscr{R}_{\mathfrak{q}}^i(\pi) \text{ is defined in [16], definition 6.3.1.} \tag{2.1}$$

This is a $(\mathfrak{g}, K)$ module. We always assume $i = (1/2) \dim (\mathfrak{k}/\mathfrak{l} \cap \mathfrak{k})$, and define $\mathscr{R}_{\mathfrak{q}}(\pi) = \mathscr{R}_{\mathfrak{q}}^{(1/2) \dim (\mathfrak{k}/\mathfrak{l} \cap \mathfrak{k})}(\pi)$.

Identify the infinitesimal character of a representation of $L$ or $G$ with (the Weyl group orbit of) an element in $\mathfrak{t}^*$ via the Harish-Chandra homomorphisms. If $V \subseteq \mathfrak{g}$ is a sum of weight spaces $\mathfrak{g}^\alpha$ for $\mathfrak{t}$, write $\Delta(V) = \{\alpha | \mathfrak{g}^\alpha \subseteq V\}$. If $V$ satisfies $\alpha \in \Delta(V) \Rightarrow -\alpha \notin \Delta(V)$ write $\varrho(V) = 1/2 \Sigma_{\alpha \in \Delta(V)} \alpha$. If $\pi$ has infinitesimal character $\gamma$, $\mathscr{R}_{\mathfrak{q}}(\pi)$ has infinitesimal character $\gamma + \varrho(\mathfrak{u})$.

Let $\pi$ be a one-dimensional representation of $L$, and $\lambda = \pi|_T$. Thus $\lambda$ determines $\pi$ if $L$ is connected.

2.2. DEFINITION: $A(\lambda, \pi) = \mathscr{R}_{\mathfrak{q}}(\pi)$.
Choose $\Delta_{\mathfrak{l}}^+$ a system of positive roots for $\Delta(\mathfrak{l})$. Let $\varrho(\mathfrak{l}) = 1/2 \Sigma_{\alpha \in \Delta_{\mathfrak{l}}^+} \alpha$, $\varrho_{\mathfrak{q}} = \varrho(\mathfrak{l}) + \varrho(\mathfrak{u})$. Then $A(\lambda, \pi)$ has infinitesimal character $\lambda + \varrho_{\mathfrak{q}}$.

Now $W(\mathbf{G}, \mathbf{T})$ acts on this data as follows. Since $T$ is compact $W(\mathbf{G}, \mathbf{T})$ acts on $T$ and $\hat{T}$. We identify $\Delta(\mathbf{G}, \mathbf{T})$ and $\Delta(\mathfrak{g}, \mathfrak{t})$, also $W(\mathbf{G}, \mathbf{T})$ and $W(\mathfrak{g}, \mathfrak{t})$. For $w \in W(\mathbf{G}, \mathbf{T})$, let

$$\mathfrak{q}_w = \mathfrak{q}(w\lambda_0) = \mathfrak{l}_w \oplus \mathfrak{u}_w,$$

$$\mathfrak{l}_w = \mathfrak{t} \oplus \sum_{\alpha \in \Delta(\mathfrak{l})} \mathfrak{g}^{w\alpha},$$

$$\mathfrak{u}_w = \sum_{\alpha \in \Delta(\mathfrak{u})} \mathfrak{g}^{w\alpha},$$

$$L_w = L(w\lambda_0) = \mathrm{stab}_G(\mathfrak{q}_w). \tag{2.3}$$

If $\pi_w$ is any one-dimensional representation of $L_w$ satisfying $\pi_w|_T = w\lambda$, we put

$$A(w\lambda, \pi_w) = \mathcal{R}_{q_w}(\pi_w) \tag{2.4}$$

For given $\lambda$ and $\pi$ as above we will later define $\pi_w$ a one-dimensional representation of $L_w$ satisfying $\pi_w|_T = w\lambda$, for any $w \in W(\mathbf{G}, \mathbf{T})$. Before we can do this we need the following lemma; it relates the various $L_w$ to each other and shows how a one dimensional representation may be defined by a character of a maximally split Cartan subgroup.

2.5. LEMMA:
1. *All the $L_w$ are inner forms of each other. Furthermore, if $G$ is quasisplit then there exists $w \in W(\mathbf{G}, \mathbf{T})$ such that $L_w$ is quasisplit.*
2. *Let $T_s$ be a maximally split Cartan subgroup of $G$, and let $\gamma = \chi(\mu, \nu) \in \hat{T}_s$ (cf. Section 1). Assume $\langle \mu, \delta \rangle = 0$ for all $\delta \in \Delta(\mathbf{G}, \mathbf{T}_s)$, and that $\langle \nu, \varepsilon \rangle = 0$ for all $\varepsilon \in \Delta(\mathbf{G}, \mathbf{T}_s)$ such that $\sigma\varepsilon = \varepsilon$. Then there exists a unique one-dimensional representation $\pi$ of $G$ such that $\pi|_{T_s} \simeq \gamma$.*
3. *Let $\gamma$, $\pi$ be as in 2. Let $T' \subseteq G$ be any Cartan subgroup of $G$ and let $g \in \mathbf{G}$ satisfy $\mathrm{ad}(g)$: $\mathbf{T}_s \to \mathbf{T}'$. Then*

$$\pi|_{T'} = \chi(\mathrm{ad}(g)\mu, \mathrm{ad}(g)\nu).$$

4. *Let $\pi$ be any finite dimensional representation of $G$, and let $T' \subseteq G$ be a Cartan subgroup. Suppose $g \in \mathbf{G}$ and $x \in T'$ are such that $\mathrm{ad}(g)x \in \mathbf{T}'$. Then $\pi(x) = \pi(gxg^{-1})$. That is, finite dimensional representations are stable.*

*Proof:* For 1, we first note that the $L_w$ are real forms which possess Cartan subgroups which are isomorphic over $\mathbb{R}$, e.g., the compact Cartan. So they are inner forms. Next we assume $G$ is quasisplit and begin to reduce the result to a scholium about root systems. For a group to be quasisplit it must be the case that for $D$ any Cartan subgroup, not all the imaginary roots are compact (for the image, under a Cayley transform, of the maximally split torus of a quasisplit group necessarily has a non-compact imaginary root). Suppose no $L_w$ is quasisplit. Then let $D \subseteq L(w\lambda)$ be of maximal split rank among all Cartans of all $L_w$. (Replacing $\lambda$ by $w^{-1}\lambda$ if necessary, we assume $D \subseteq L(\lambda)$.) Then $D$ has all imaginary roots compact, and in particular has $\alpha$, a compact imaginary root in $\Delta(\mathbf{L}(\lambda), \mathbf{D})$. We may assume that $D = \mathrm{Cay}(T) = s_1 \cdot \cdots \cdot s_n T$ where $s_i$ is a Cayley transform through the root $\alpha_i$ of $\Delta(\mathbf{L}(\lambda), \mathbf{T})$ and that $\{\alpha_i\}$ is a strongly orthogonal set of noncompact imaginary roots.

Now it suffices to show that there exists $w \in W(\mathbf{G}, \mathbf{T})$ such that a) $w\alpha_i$ is noncompact imaginary in $\Delta(\mathbf{L}_w, \mathbf{T})$ for all $i$, and b) if $w$ is regarded as an element of $W(\mathbf{G}, \mathbf{D})$ then $w\alpha$ is noncompact imaginary in $\Delta(\mathbf{L}_w, \mathbf{D}')$ where $\mathbf{D}'$ is the Cayler transform of $\mathbf{T}$ through the roots $\{w\alpha_i\}$. For condition a) ensures that the definition of $D'$ makes sense and that we may take $w\lambda$ for $\lambda$ and $D'$ for $D$ in the preceding paragraph. But condition b) ensures that we can Cayley transform $D'$ through $w\alpha$, a contradiction.

The sub-root-systems of $\Delta(\mathbf{G}, \mathbf{T})$ and $\Delta(\mathbf{L}(\lambda), \mathbf{T})$ which consist of all roots orthogonal to $\{\alpha_i\}$ correspond to some groups $G'$ and $L'(\lambda')$ with all hypotheses holding again. By induction on dim $G$ we are then reduced to the case dim $G' = $ dim $G$. In this case $\{\alpha_i\}$ is empty so condition a) is vacuous.

So we are reduced to the following scholium: if $G$ is quasisplit and $\alpha$ is any compact imaginary root in $\Delta(\mathbf{G}, \mathbf{T})$, then $W(\mathbf{G}, \mathbf{T}) \cdot \alpha$ does not consist solely of compact roots.

This is clear if $\mathfrak{g}$ is simply laced. The remaining cases are reduced to the rank two case, which is clear. We thank D. Garfinkle for this proof, which we omit.

For 2, we first note that the assumption implies that $\gamma|_{T_s \cap G_{der}}$ is trivial. But $G = T_s G_{der}$, so 2 is immediate.

For 3, let $T' = F'T'^0$ and $T_s = F_s T_s^0$ as in Section 1. We may assume $g$ is a Cayley transform. Then $\text{ad}g(\mu)$, $\text{ad}g(\nu)$ define a quasicharacter of $T'$ which agrees with $\pi$ on: the center of $G$, $T'^0$, and $F' \subseteq F_s$.

Four follows from the analogue of the B.-G.-G. resolution of a finite dimensional representation by stable sums of standard modules ([7], [8]), see also Section 8 of this paper.                    Q.E.D

We now define $\pi_w$. Assume first that $G$ is quasisplit. Varying our given $\lambda$ within its Weyl group orbit if necessary, we assume $L = L(\lambda)$ is quasisplit as well. We are given $\pi$ a one-dimensional representation of $L$ such that $\pi|_T = \lambda$. For any $w \in W(\mathbf{G}, \mathbf{T})$, choose $T_w \subseteq L_w$ a maximally split Cartan subgroup of $L_w$ and $g \in \mathbf{G}$ representing $w^{-1}$. Then there exists $\ell \in L$ such that $\text{ad}(\ell g)|_{\mathbf{T}_w}$ is defined over $\mathbb{R}$; let $\mathbf{T}'_w = \text{ad}(\ell g)\mathbf{T}_w \subseteq L$. Define $\pi_w(t) = \pi(\text{ad}(\ell g)t)$ for any $t \in T_w$. Then the conditions of Lemma 2.5.2, hold, and so $\pi_w$ extends uniquely to a one-dimensional representation, also denoted $\pi_w$, of $L_w$. By part three of this lemma, $\pi_w$ depends only on $\lambda$, $\pi$ and $w$. (Note that $\lambda$ must be chosen such that $L(\lambda)$ is quasisplit.) Furthermore, $\pi_w|_T = w\lambda$. To complete the definition, we have only to deal with the case where $G$ is not quasisplit.

If $G$ is not quasisplit let $\psi: \mathbf{G} \to \mathbf{G}^*$ be an inner twist. After conjugating by $g \in \mathbf{G}^*$ we assume $\psi|_{\mathbf{T}}$ is defined over $\mathbb{R}$, let $\mathbf{T}' = \psi(\mathbf{T})$. Given $\mathbf{L} \subseteq \mathbf{G}$, let $\mathbf{L}' = \psi(\mathbf{L})$; as above we assume $L'$ is quasisplit. Given $\pi'$ a one-dimensional

representation of $L'$, define $\pi$ a one-dimensional representation of $L$ as above. That is, choose $\ell \in L'$ such that $\mathrm{ad}(\ell) \circ \psi|_{T_s}$ is defined over $\mathbb{R}$ for $T_s$ a maximally split Cartan subgroup of $L$. Let $T'_s$ be the image of $T_s$. Define $\pi$ by specifying $\pi|_{T_s}$:

$$\pi(x) = \pi'(\mathrm{ad}(\ell)\psi(x)), \quad x \in T_s.$$

Similarly define $\pi_w$ for $L_w$. Then $\{(L_w, \pi_w)\}$ depends only on $(L', \pi')$; if $\lambda = \pi|_T$ then $\pi_w|_T = w\lambda$.

For later use we note the following. Suppose $D \subseteq L_w$ is a Cartan subgroup of $L_w$, $g \in G$ represents $w^{-1}$, and $\mathrm{ad}(g^{-1})|_D$ is defined over $\mathbb{R}$. Let $D'$ be the image of $D$ and further assume $D' \subseteq L$. Suppose $\pi|_{D'} = \chi(\mu, v)$, $\mu, v \in X^*(D) \otimes \mathbb{C}$.

2.6. LEMMA: $\pi_w|_D = \chi(\mathrm{ad}(g^{-1})\mu, \mathrm{ad}(g^{-1})v)$. *In particular suppose* $D \cap K \subseteq T$, $w \in W(M, D) \subseteq W(G, T)$ *where* $M$ *is the centralizer of the split part of* $D$. *Then* $\pi_w|_D = \chi(w\mu, v)$.

*Proof:* First suppose $G$ is quasisplit and $L$ is quasisplit. Choose $T_w \subseteq L_w$, $g$, $\ell$; $T_s$, $T'_w \subseteq L$ as in the definition of $\pi_w$. Choose $x \in L$, $\mathrm{ad}(x): T_s \to D'$, $y \in L$, $\mathrm{ad}(y): T_s \to T'_w$.

$$
\begin{array}{l}
L_w \xrightarrow{\;g\;} L \xrightarrow{\;\ell\;} L \\[4pt]
T_w \xrightarrow{\;\ell g \text{ def}/\mathbb{R}\;} T'_w \xleftarrow{\phantom{xx}} y. \\[4pt]
D \xrightarrow{\;g \text{ def}/\mathbb{R}\;} D' \\[4pt]
\qquad\qquad\quad T_s
\end{array}
\qquad (2.7)
$$

Suppose $\pi|_{T_s} = \chi(\varepsilon, \gamma)$. By Lemma 2.5

$$\mathrm{ad}(x)\varepsilon = \mu, \ \mathrm{ad}(x)\gamma = v + \delta + (\beta - \sigma\beta) \quad \text{some} \quad \begin{cases} \delta \in X_*(D') \\ \beta \in X_*(D') \otimes \mathbb{C} \end{cases}$$

$$(2.8)$$

(here $\sigma =$ Galois action of $D'$).

Thus by Lemma 2.5

$$\pi_w|_{T_w} = \chi(\mathrm{ad}(g^{-1}\ell^{-1}y)\varepsilon, \ \mathrm{ad}(g^{-1}\ell^{-1}y)\gamma),$$

and

$$\pi_w|_D = \chi(\mathrm{ad}((g^{-1}xy^{-1}\ell g)g^{-1}\ell^{-1}y))\varepsilon, \ \mathrm{ad}((g^{-1}xy^{-1}\ell g)g^{-1}\ell^{-1}y)\varepsilon);$$

$$= \chi(\mathrm{ad}(g^{-1})\mu, \ \mathrm{ad}(g^{-1})v)$$

by 2.8, using the fact that $\mathrm{ad}(g^{-1})|_{D'}$ is defined over $\mathbb{R}$, and Section 1. This completes the proof for $G$ quasisplit. The proof is similar for general $G$, we omit the details. The second part follows from the first, noting

$$\chi(w\mu, \ wv) = \chi(w\mu, \ v) \quad \text{for} \quad w \in W(\mathbf{M}, \mathbf{D}).$$

2.9. DEFINITION: $S = W(G, T)\backslash W(\mathbf{G}, \mathbf{T})/W(\mathbf{L}, \mathbf{T})$.
Now Vogan and Wallach have shown (cf. [16], [19] Theorem 1.3, [20]):

2.10. LEMMA: *Assume that the real part of* $\langle \alpha, \ \lambda + \varrho \rangle \geqslant 0$ *for all* $\alpha \in \Delta(\mathfrak{u}, \mathfrak{h})$. *Then*:

1. $A(w\lambda, \pi_w)$ *is irreducible, unitary, with infinitesimal character* $\lambda + \varrho$.
2. $A(w\lambda, \pi_w) = A(w'\lambda, \pi_{w'})$ *if and only if*
   $W(G, T)w \ W(\mathbf{L}, \mathbf{T}) = W(G, T)w' \ W(\mathbf{L}, \mathbf{T})$.

*Thus it makes sense to write* $A(w\lambda, \pi_w)$, $w \in S$.

2.11. DEFINITION: $\hat{\Pi} = \{A(w\lambda, \pi_w)|w \in S\}$, an "enlarged packet for $G$".

2.12. DEFINITION: For $w \in W(\mathbf{G}, \mathbf{T})$, $\gamma(w) = (1/2) \dim (L_w/L_w \cap K)$. This depends on $L$, and is well defined on $S$.

2.13. THEOREM: $\Sigma_{w \in S} (-1)^{\gamma(w)} A(w\lambda, \pi_w)$ *is stable. (Recall we identify* $A(w\lambda, \pi_w)$ *with its character.)*

This is proved in Section 8.
   A special case of 2.13 is $L = T$. Then $\hat{\Pi}$ is an ordinary discrete series $L$-packet. In this case $\gamma(w) = 0$, and this reduces to [14]. We note that $A(w\lambda, \pi_w)$ is a discrete series representation if and only if $L_w$ is compact. If $L$ is compact and non-abelian, $\hat{\Pi}$ contains some discrete series representations and some non-tempered representations.
   We turn now to transfer for $\hat{\Pi}$. We follow [14], see also ([12], Section 5). Recall $\Lambda^\vee \subseteq X_*(\mathbf{T})$ is the coroot lattice of $\mathbf{T}$ in $\mathbf{G}$; $T$ is compact so

$\sigma_T = -1$. Let $\kappa \in \varepsilon(T)^{\wedge}$, the characters of

$$\varepsilon(T) = \Lambda^{\vee}/(\Lambda^{\vee} \cap 2X_*(\mathbf{T})). \tag{2.14}$$

Note $\kappa$ takes values $\pm 1$. We restrict ourselves to these $\kappa$ because the character of $A(w\lambda, \pi_w)$ has support on $T$, hence the endoscopic groups $H$ we construct must "share" $T$ with $G$.

So define an endoscopic group, denoted $H(T, \kappa)$, attached to $(T, \kappa)$ as in ([14], Section 2.3). Since $T$ is fixed we write $H^{\kappa} = H(T, \kappa)$. At the same time we make some choices which will be necessary later. Thus we have the following.

*2.15*:
(a) $\mathbf{H}$ is a quasisplit group satisfying our assumptions.
(b) $\mathbf{H} \supseteq \mathbf{B}_H \supseteq \mathbf{T}_H$ are chosen Borel and Cartan subgroups respectively.
(c) We assume $\mathbf{T}_H = \mathbf{T}^*$ as complex groups (with possibly different Galois actions); so

$${}^L T_H^0 = {}^L T^0.$$

Also

$${}^L H^0 = (\text{centralizer in } {}^L G^0 \text{ of } s)^0, \text{ for some } s \in {}^L T^0.$$

(d) ${}^L H = {}^L H^0 \rtimes W_{\mathbb{R}}$. Here the action of $W_{\mathbb{R}}$ is given as follows. Let $\tau(t) = t^{-1}$ for $t \in {}^L T^0$. Then $\sigma_H: {}^L H^0 \to {}^L H^0$ is defined uniquely by $\sigma_H|_{L_{T0}} = w \circ \tau$, for some $w \in W({}^L H^0, {}^L T^0)$ and $\sigma_H({}^L B_H^0) = {}^L B_H^0$. (Note that $\tau$ equals the transfer of $\sigma_T$ to ${}^L T^0$.)
(e) Choose $g \in \mathbf{G}^*$ such that $\mathbf{T}_H = \mathbf{T}_* \xrightarrow{\text{ad } g} \mathbf{G}^*$ is defined over $\mathbb{R}$. Let $\mathbf{T}_N$ be the image. By modifying $\psi$ we assume ${}^L H$ is in standard position with respect to $T_N$ (cf. [13]).
(f) We choose a framework of Cartan subgroups (cf. [13]). That is, choose
    (i) $\{T_0', T_1', \ldots, T_N' = T_H\}$ representatives for all conjugacy classes of Cartan subgroups of $H$. We assume $T_i'$ is standard with respect to $T_N'$, i.e. $S(T_i') \subseteq S(T_N')$ where $S(\ )$ denotes maximal $\mathbb{R}$-split subtorus. We assume $T_0'$ is compact and write it $T_c'$.
    (ii) $\{T_0, \ldots, T_N\}$ standard (with respect to $T^*$) Cartan subgroups of $G^*$. Write $T_c = T_0$.
    (iii) Set $\begin{cases} M_j' = \text{centralizer in } H \text{ of } S(T_j') \\ M_j = \text{centralizer in } G^* \text{ of } S(T_j) \end{cases}$. Choose $\psi_j: T_j' \xrightarrow[\text{ad}(m_j')]{} T_H = T^* \xrightarrow[\text{ad}(m_j^{-1})]{} T_j$ isomorphisms/$\mathbb{R}$, for some $m_j' \in M_j'$, $m_j \in M_j$.
    (iv) If for some $g \in \mathbf{G}^*$, $\psi^{-1} \circ \text{ad}(g)|_{T_j}$ is defined over $\mathbb{R}$ choose $T_j^G \subseteq G$:

$$\psi_j^G = \psi^{-1} \circ \text{ad}(g): \quad T_j \longrightarrow T_j^G$$

defined over $\mathbb{R}$. We say $T'_j$ originates in $G$. We assume $T^G_0$ is our given compact Cartan subgroup $G$, and write $T^G_c = T^G_0$. Let $\sigma_j$ be the Galois action of $T'_j$, $T_j$ or $T^G_j$.

See ([13], Section 2.3) for a proof that all of the above choices are possible. We impose one additional assumption (2.16). All such choices will ultimately be irrelevant.

(g) We choose a transfer $\mathrm{Tran}^G_H(\ )$ of stable distributions on $H$ to $G$:

   (i) We assume an admissible embedding $\zeta\colon {}^L H \longrightarrow {}^L G$ exists and has been chosen. Following [13], such an embedding is defined by $\mu^*$, $\lambda^* \in X^*(\mathbf{T}^*) \otimes \mathbb{C}$, and written $\zeta(\mu^*, \lambda^*)$. We write $\mu^*$, $\lambda^* \in X^*(\mathbf{T}_n) \otimes \mathbb{C}$ also for $\mathrm{ad}(m_n^{-1})(\mu^*)$, $\mathrm{ad}(m_n^{-1})(\lambda^*)$. If the embedding doesn't exist, our results are still valid whenever the results of [14] hold.

   (ii) Choose a set of transfer factors [14]. This amounts to the following: for all pairs $(D, \eta)$, $D$ a Cartan subgroup, $\eta$ a p.d. of $D$, satisfying $(D, \eta) \in \mathbf{T}_H(G)$ ([14], Section 31.) we are given $\varepsilon(D, \eta) = \pm 1$. Conditions on $\varepsilon(D, \eta)$ imply there are only two choices for $\{\varepsilon(D, \eta)\}$.

Given the choices in i) and ii). $\mathrm{Tran}^G_H(\ )$ of stable (tempered) distributions is defined [12]. We extend this to non-tempered distributions in Section 4. Let $\hat{\Pi}' = \{A(w\lambda', \pi'_w)\}$ be an enlarged packet for $H$. Here $\lambda'_0 \in X^*(\mathbf{T}'_c) \otimes \mathbb{C}$, $\mathfrak{q}' = \mathfrak{q}(\lambda'_0)$, $\pi' \in \hat{L}'$ etc. have been chosen for $H$. Since $H$ is quasisplit we may and do assume $L' = L(\lambda'_0)$ is quasisplit.

Given $\hat{\Pi}'$ and an admissible embedding ${}^L H \longrightarrow {}^L G$ (cf. 2.15 g) we define $\hat{\Pi}$ for $G$. First assume $G = G^*$ is quasisplit. Let $\psi_0\colon T'_c \longrightarrow T_c$ be as in 2.15 f. Let $\lambda_0 = \psi_0(\lambda'_0) \in X^*(\mathbf{T}_c) \otimes \mathbb{C}$, $L = L(\lambda_0)$. We assume

2.16. $\mathbf{L}$ is isomorphic to $\mathbf{L}'$ (over $\mathbb{C}$).

   That is we require

$$\psi_0(\Delta(\mathbf{L}', \mathbf{T}'_c)) = \Delta(\mathbf{L}, \mathbf{T}_c). \tag{2.17}$$

Without this restriction, the transfer may not be a linear combination of modules induced (cohomologically) from one-dimensional representations of a $\theta$-stable parabolic, e.g., let $G = Sp(2, \mathbb{R})$, $H = SO(2, 2)$, and consider the trivial representation of $H$, i.e., $\mathfrak{q} = \mathfrak{l} = \mathfrak{h}$.

Note that then $\mathbf{L}'$ and $\mathbf{L}$ contain Cartan subgroups isomorphic over $\mathbb{R}(T'_c \approx T_c)$. It follows that $L$ and $L'$ are inner forms. Since $G$ is quasisplit we assume (using Lemma 2.5) (by modifying $\psi_0$)that $\mathbf{L}$ is quasisplit (hence $L' \approx L$).
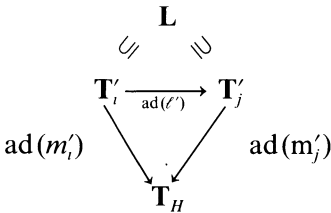
Let $\mathfrak{q} = \mathfrak{q}(\lambda_0)$. To define $\hat{\Pi}$ we need only define $\pi$, a one-dimensional

representation of $L$; then $\hat{\Pi}$ for $G$ is defined as in 2.11 (recall this uses first 2.5 for $G$ quasisplit and then an inner twist for general $G$). We proceed to define $\pi$.

For this (and for later use) it is convenient to impose the following condition on the framework of Cartan subgroups.

2.18.
 (i) Assume $T_j' \subseteq L'$ if $T_j'$ is conjugate within $H$ to a Cartan subgroup of $L'$.
(ii) If $T_i'$, $T_j' \subseteq L'$, assume $m_i'^{-1} m_j' \in L'$. Thus the diagram

$$
\begin{array}{c}
\mathbf{L} \\
\Swarrow \quad \Searrow \\
\mathbf{T}_i' \xrightarrow{\mathrm{ad}(\ell')} \mathbf{T}_j' \\
\mathrm{ad}(m_i') \searrow \quad \swarrow \mathrm{ad}(m_j') \\
\mathbf{T}_H
\end{array}
$$

commutes, where $\ell' = m_i'^{-1} m_j' \in L'$.
(iii) Similarly for $L$: assume $T_j \subseteq L$ if $T_j$ is $G^*$ conjugate to a Cartan subgroup of $L$; assume $m_i^{-1} m_j \in L$ if $T_i$, $T_j \subseteq L$.

It is easy to see (cf. [13], Section 2.3) such choices are possible. Now choose $T' = T_n' \subseteq L'$ a maximally split Cartan subgroup of $L'$. Choose $\Delta_{l'}'^+$ a system of positive roots for $\Delta_{l'}' = \Delta(l', t')$. Let

$$
\Delta_{u'}' = \mathrm{ad}(m_n'^{-1} m_0')(\Delta(u', t_c')), \quad \Delta'^+ = \Delta_{l'}'^+ \cup \Delta_{u'}'.
$$

Let

$$
\varrho' = \varrho_{l'}' + \varrho_{u'}' \in X^*(\mathbf{T}_n') \otimes \mathbb{C}
$$

as usual. Let

$$
\Delta_l^+ = \psi_n(\Delta_l'^+), \quad \Delta_u = \mathrm{ad}(m_n^{-1} m_0)(\Delta(u, t_c));
$$

and let $\Delta^+ = \Delta_l^+ \cup \Delta_u$, $\varrho = \varrho_l + \varrho_u$.

Given any $\theta$-stable Cartan subgroup $D$ of $G$ with Galois action $\sigma$ we will have occasion to use the following (cf. [16], Section 6.7.1). Given $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{d})$, let $B \subseteq \Delta^+$ be a set of complex roots satisfying

$$
B \cap \theta B = \varnothing, \quad B \cup \theta B = \{\alpha \in \Delta^+ \mid \sigma\alpha \neq \pm\alpha, \, -\sigma\alpha \in \Delta^+\}.
$$

Let $\varrho(B, \Delta^+) = \frac{1}{2} \Sigma_{\alpha \in B} \alpha$. Note that $2\varrho(B, \Delta^+)|_{D \cap K}$ is independent of the choice of $B$, and is a character of $D \cap K$.

Suppose $\pi'|_{T_n'} = \chi(\mu', \nu')$, $\mu'$, $\nu' \in X_*(\mathbf{T}') \otimes \mathbb{C}$ as in section 1. We carry notation from $T_n'$ to $T_n$ via $\psi_n$ without change in notation: e.g. write $\pi'|_{T_n}$ for $\psi_n(\pi'|_{T_n'})$.

2.19. DEFINITION: The one-dimensional representation $\pi$ of $L$ is defined by

$$\pi|_{T_n} = \chi(\mu' + \mu^* + \varrho' - \varrho, \nu' + \lambda^* + \varrho'(B, \Delta'^+) - \varrho(B, \Delta^+)).$$

$$(2.20)$$

The reason for this definition, and that in fact this *does* define a one-dimensional representation of $L$, will be seen in Section 8 (Lemma 8.13). It is then straightforward to see $\pi$ is independent of the choice of $B \subseteq \Delta'^+$, $\Delta'^+$, and $\{(L_w, \pi_w)\}$ depends only on $(L', \pi')$ and $\zeta$.

We recapitulate and state the theorem on lifting.

Let $H = H^\kappa$ be an endoscopic group containing a compact Cartan subgroup $T_c'$. Given $\lambda_0' \in \mathfrak{t}_c^*$ let $L' = L(\lambda_0')$, $\mathfrak{q}' = \mathfrak{l}' \oplus \mathfrak{u}'$. We assume $L'$ is quasisplit. Let $\pi'$ be a one-dimensional representation of $L'$; $\lambda' = \pi'|_{\mathfrak{t}_c'}$. Then $\hat{\Pi}' = \{A(w\lambda', \pi_w')\}$ is defined, depending only on $(L', \pi')$. We let $\lambda_0 = \psi_0(\lambda_0') \in \mathfrak{t}_c^*$; we assume $L(\lambda_0) \approx L(\lambda_0')$. Then given an admissible embedding $\zeta: {}^L H \to {}^L G$, we obtain $\hat{\Pi} = \{A(w\lambda, \pi_w)\}$ for $G^*$, depending only on $\hat{\Pi}'$ and $\zeta$. We impose a compatibility condition on $\lambda'$ and $\zeta$ to ensure that the representations $A(w\lambda, \pi_w)$ are irreducible. We require that $\lambda'$ be what we shall call "$\zeta$-admissible", that is, $\lambda'$ satisfies the hypothesis of 2.10 (relative to $\mathfrak{u}'$) and $\lambda$ satisfies the same hypothesis (relative to $\mathfrak{u}$). Via $\psi$: $\mathbf{G} \to \mathbf{G}^*$ we obtain $\hat{\Pi} = \{A(w\lambda, \pi_w)\}$ for $G$ depending only on $\hat{\Pi}$ and $\zeta$. Given transfer factors, $\mathrm{Tran}_H^G(\ )$ is defined. This was called "$\mathrm{Lift}_H^G(\ )$" in [14]. This is the "setting of Theorem 2.21". We define $\varepsilon_\kappa = \pm 1$ in 5.5.

2.21. THEOREM: $\mathrm{Tran}_H^G(\Sigma_{w \in S'} (-1)^{\gamma'(w)} A(w\lambda', \pi_w')) = \varepsilon_\kappa \Sigma_{w \in S} (-1)^{\gamma(w)} \times \kappa(w) A(w\lambda, \pi_w)$.

## 3. *L*-group formulation

In this section we interpret the main theorems in terms of $L$-homomorphisms. We do not use this in the sequel and so are very brief. This is Arthur's formulation of his conjectures [2]. Given $L \subseteq G$ as in Section 2, fix a pseudo-diagonalization $\eta$ of $\mathbf{T}$. Then ${}^L L^0$ embeds in ${}^L G^0$: ${}^L L^0$ is generated by ${}^L T^0$ and $\{\eta(\alpha) | \alpha \in \Delta(\mathbf{L}, \mathbf{T})\}$. Write $\varepsilon^0: {}^L L^0 \to {}^L G^0$. A homomorphism ${}^L G_1 \to {}^L G_2$ is said to be admissible if it commutes with projection to $W_{\mathbb{R}}$.

3.1. LEMMA $\varepsilon^0$ *extends to an admissible embedding* $\varepsilon\colon {}^L L \to {}^L G$.

We omit the proof of this lemma, but note that it uses the congruences of [13] to produce the embedding, and the results of Section 8 to solve the congruences (lemma 8.13).

Fix $\varepsilon$. Given $\phi\colon W_{\mathbf{R}} \to {}^L G$, assume:

3.2. Image $(\phi) \subseteq {}^L L$, and

3.3. $W_{\mathbf{R}} \xrightarrow{\phi^L} {}^L L$ corresponds to a one-dimensional unipotent representation as in the introduction.

That is, $\phi$ factors:

$$
W_{\mathbf{R}} \xrightarrow{\;\psi\;} SL_2(\mathbf{C}) \times W_{\mathbf{R}} \xrightarrow{\;\tilde{\phi}^L\;} {}^L L \xrightarrow{\;\varepsilon\;} {}^L G \tag{3.4}
$$
$$
\phi^L
$$

where

$$
\psi(z) \;=\; \begin{pmatrix} |z|^{1/2} & 0 \\ 0 & |z|^{-1/2} \end{pmatrix} \times z, \quad \psi(\tau) \;=\; 1 \times \tau,
$$

$\tilde{\phi}^L(\mathbf{C}^*) \subseteq$ center $({}^L L^0) \rtimes W_{\mathbf{R}}$ and $\tilde{\phi}^L|_{SL_2(\mathbf{C})}$ corresponds to the principal unipotent orbit of ${}^L L^0$. Let $C_\phi = $ centralizer of $\tilde{\phi}(SL_2(\mathbf{C}) \times W_{\mathbf{R}})$ in ${}^L G^0$, $\mathscr{C}_\phi = C_\phi/C_\phi^0 Z({}^L G^0)^{W_{\mathbf{R}}}$.

We consider the conjectures of [2] for such $\phi$. We construct $\hat{\Pi}_\phi$ as follows. Let $L^*$ be the quasisplit inner form of $L$, so $\phi^L$ is admissible for $L^*$, and $\Pi_{\phi^L} = \{\pi^*\}$, $\pi^*$ a one-dimensional representation of $L^*$. If $L$ is not quasisplit $\pi$ is obtained from $\pi^*$ by the procedure following 2.5. Let $\lambda = \pi|_T$. We suppose there exists q $\supseteq$ l such that 2.10 is satisfied. Thus $A(\lambda, \pi)$ is defined. This applies to all $L_w$ (since ${}^L L_w \approx {}^L L$), and the above construction yields $A(w\lambda, \pi_w)$. Define $\hat{\Pi}_\phi = \{A(w\lambda, \pi_w)\}$. Conjecture 1.3.2 of [2] holds for $\hat{\Pi}_\phi$ by [19] or [20], and Theorem 2.13 verifies conjecture 1.3.3 (ii) in this case.

As in [14] §5.2 there is a map $\hat{\Pi}_\phi \to \mathscr{C}_\phi^{\wedge}$ which may however fail to be an injection. See the discussion below of inversion.

We note without proof that if $L$ is quasisplit the $L$-packet $\Pi$ containing $A(\lambda, \pi)$ is contained in $\hat{\Pi}_\phi$. In particular if $G$ is quasisplit, $L$ is quasisplit $\phi$ is admissible and $\hat{\Pi}_\phi \supseteq \Pi_\phi$. More generally if $L$ is the "most quasisplit" among $\{L_w\}$ then $\hat{\Pi}_\phi \supseteq \Pi$ for $\Pi$ the $L$-packet containing $A(\lambda, \pi)$.

Now given $x \in \mathscr{C}_\phi$ we define an endoscopic group $H = H_x$ as in [14]. Then

$H$ satisfies ${}^{L}L^{0} \subseteq {}^{L}H^{0}$. Fix $\zeta: {}^{L}H \to {}^{L}G$; then there exists $\phi'$ satisfying 3.2 and 3.3 (for $H$) which lifts to $\phi$:

$$
\begin{array}{ccc}
& & {}^{L}G \\
& \nearrow{\phi} & \uparrow{\zeta} \\
W_{\mathbf{R}} & \xrightarrow{\phi'} & {}^{L}H
\end{array}
\tag{3.5}
$$

Applying the above procedure to $H$ we obtain $\hat{\Pi}_{\phi'} = \{A(w\lambda', \pi_{w'})\}$. Then Theorem 2.21 verifies conjecture 1.3.2 (iii).

We discuss inversion. Given $x \in \mathscr{C}_{\phi}$, $H = H_{x}$ as above, let $\Theta_{x}$ be the virtual character given by Theorem 2.21. Then inversion ([2], 1.3.5) requires $A(w\lambda, \pi_{w})$ may be expressed in terms of $\{\Theta_{x}\}_{x \in \mathscr{C}_{\phi}}$ i.e. $(\mathbf{Z}\langle\{\Theta_{x}\}\rangle = \mathbf{Z}\langle\{A(w\lambda, \pi_{w})\}\rangle)$. However $|\mathscr{C}_{\phi}| < |\hat{\Pi}_{\phi}| = |S|$ in some cases, so this may not be possible. The failure of [2], 1.3.5 arises as follows. Given $x$, $\kappa: S \to \pm 1$ as in Theorem 2.21 is obtained from $x$ as in [14], §5.2; write $\kappa = \kappa_{x}$. Now $\hat{\Pi}_{\phi} \xleftrightarrow{1-1} S$ and the pairing $\hat{\Pi}_{\phi} \times \mathscr{C}_{\phi} \to \pm 1$ is defined by $\langle w, x \rangle = \kappa_{x}(w)$. This defines the map $\hat{\Pi}_{\phi} \to \mathscr{C}_{\phi}^{\wedge}$ which may fail to be injective.

## 4. Pseudo *L*-packets and transfers

We collect some information we need later. Given $\phi: W_{\mathbf{R}} \longrightarrow {}^{L}G$ an admissible homomorphism, $\Pi_{\phi}$ its (ordinary) $L$-packet, let

$$
\chi_{\phi} = \sum_{\pi \in \Pi_{\phi}} \pi.
\tag{4.1}
$$

If $\phi$ is tempered this is stable ([11], lemma 5.2). If $\phi$ is not tempered this may fail. Now $\{\pi\}$ are obtained as subquotients of $\{\text{Ind}_{P}^{G}\pi \otimes 1 | \pi \in \Pi_{\phi}^{M}\}$. Here $P = MN$ is a cuspidal parabolic subgroup of $G$, $\phi$ factors through ${}^{L}M \subseteq {}^{L}G$, and $\Pi_{\phi}^{M}$ is an $L$-packet of discrete series representations. Following [5] we define:

4.2. DEFINITION: $\tilde{\Pi}_{\phi} = \{\text{Ind}_{MN}^{G}\pi \otimes 1 | \pi \in \Pi_{\phi}^{M}\}$ a "pseudo $L$-packet",

$$
\tilde{\chi}_{\phi} = \sum_{\pi \in \tilde{\Pi}_{\phi}} \pi.
$$

Thus $\tilde{\chi}_{\phi} = \chi_{\phi}$ if $\phi$ tempered.

We refer to ([2], definition 1.2.1), and [11], Section 5 and 5.1) for the definition of a stable distribution.

4.3. LEMMA: $\tilde{\chi}_\phi$ is stable.

*Proof:* The arguments of ([11], Lemmas 5.1 and 5.3) for tempered $\phi$ carry over exactly to $\tilde{\chi}_\phi$. The main point is that $\chi_\phi$, if $\phi$ is tempered, or more generally $\tilde{\chi}_\phi$, is a sum of *full* induced representations for which one has an explicit formula. We omit the details.

Let $H$ be an endoscopic group for $G$. Let $\mathscr{S}(\ )$ denote the Schwarz space of a group. Given the choices of 2.15 g the transfer from $\mathscr{S}(G)$ to $\mathscr{S}(H)$ is defined in [14] via matching of orbital integrals. Its inverse is the transfer, written $\mathrm{Tran}_H^G(\ )$, of stable tempered distribution on $H$ to tempered distributions on $G$. We seek to extend this to non-tempered distributions.

Unfortunately it is not known that $C_c^\infty(G) \to C_c^\infty(H)$ under the above map ($C_c^\infty = C^\infty$-functions with compact support). This precludes (for the moment) defining transfer of distributions as above. Alternatively, as in [2] we define $\mathrm{Tran}_H^G(\Theta)$ for $\Theta$ a stable virtual character without reference to orbital integrals by extending the main result of [14]. See below.

There are two other possibilities which are compatible. In [5] a transfer by matching is defined from $C_c^\infty(G)_{K\text{-finite}} \to C_c^\infty(H)_{K_H\text{-finite}}$. This is enough to define $\mathrm{Tran}_H^G(\Theta)$. Alternatively ([2], theorem 1.1.1) says $\{\Theta_\pi | \pi$ irreducible, tempered$\}$ is dense in the space of all invariant distributions. (A continuity argument is then needed to define $\mathrm{Tran}_H^G(\ )$ more generally.)

Formally, let $\Theta$ be a stable virtual character, i.e., a finite sum with integral coefficients of characters of irreducible representations. Let $P = MAN$ be the Langlands decomposition of a cuspidal parabolic of $G$. Here $A$ is connected. For our purposes we make the following:

4.4. DEFINITION: A standard module $X(P, \pi, v)$ is $X(P, \pi, v) = \mathrm{Ind}_{MAN}^G \pi \otimes$ $e^v \otimes 1$ for $\pi$ a limit of a discrete series representation of $M$, $v \in \mathfrak{a}^*$, $e^v \in \hat{A}$, with $v$ chosen so that $X(P, \pi, v)$ is isomorphic with one of the representations of 6.5.2 [16].

Now $\Theta$ may be written as a sum of standard characters, i.e. characters of standard modules ([16], 6.6.7). So it is enough to define $\mathrm{Tran}_H^G(\Theta)$ for $\Theta$ a stable sum of standard modules. These are obtained precisely as sums of terms $\tilde{\chi}_\phi$; so it is enough to define $\mathrm{Tran}_H^G(\tilde{\chi}_\phi)$. For $\phi$ tempered, $\mathrm{Tran}_H^G(\tilde{\chi}_\phi) = \mathrm{Tran}_H^G(\chi_\phi)$ is already defined, and a main result of [14] is the following.

Let $\phi'$ be tempered, and let $\phi = \zeta \circ \phi'$. Let $MA$ be a Levi subgroup of $G$; then $^L(MA) \to {}^LG$ canonically. Suppose $^L(MA)$ is a minimal Levi

subgroup of $^L G$ satisfying $Im(\phi) \subseteq {}^L(MA)$. If $MA \subseteq G$ we say $\phi$ is relevant to $G$ (cf. [4]). Let $\Pi_\phi^{MA}$ be the $L$-packet of (relative) discrete series representations of $MA$ defined by $\phi$. Then $\Pi_\phi^{MA} = \{\pi_i \otimes e^\nu | \pi_i$ some discrete series representations of $M$ all having the same infinitesimal and central characters, $\nu \in i\mathfrak{a}_0^*\}$. Define $N$ as usual.

4.6. THEOREM: ([*14*], *theorem 4.1.1*): *Given $\phi'$ tempered,*

$$\mathrm{Tran}_H^G(\chi_{\phi'}) =$$

$$\begin{cases} \displaystyle\sum_{\pi \in \Pi_\phi} \varepsilon(\pi)\pi = \sum_i \varepsilon_i \, \mathrm{Ind}_{MAN}^G (\pi_i \otimes e^\nu \otimes 1) & \text{if } \phi \text{ is relevant to } G \\ 0 & \text{if } \phi \text{ is not relevant to } G. \end{cases}$$

*Here $\varepsilon(\pi) = \pm 1$, $\varepsilon_i = \pm 1$ are defined in* [14].

Now suppose $\phi'$ is not tempered, but satisfies all the other conditions of the preceding paragraph. Thus $\Pi_\phi^{MA} = \{\pi_i \otimes e^\nu | \pi_i$ some discrete series of $M$, $\nu \in \mathfrak{a}^*\}$. Now $\nu \notin i\mathfrak{a}_0^*$. Let $\mathbf{Im}(\nu)$ be the $i\mathfrak{a}_0^*$ component of $\nu$. Choose an admissible homomorphism $\phi_0'$ such that with $\phi_0 = \zeta \circ \phi_0'$, $\Pi_{\phi_0}^{MA} = \{\pi_i \otimes e^{\mathbf{Im}\nu} | \pi_i$ and $\nu$ as above$\}$. We define $\mathrm{Tran}_H^G(\tilde{\chi}_{\phi'})$ by deforming $\nu$ to $\mathbf{Im}(\nu)$ and using 4.6:

4.7. DEFINITION: Given $\phi'$, if $\phi = \zeta \circ \phi'$ is relevant to $G$, let $\Pi_\phi^{MA} = \{\pi_i \otimes e^\nu\}$ as above. Let

$$\mathrm{Tran}_H^G(\tilde{\chi}_{\phi'}) = \begin{cases} \displaystyle\sum_i \delta_i \, \mathrm{Ind}_{MAN}^G (\pi_i \otimes e^\nu \otimes 1) & \text{if } \phi \text{ is relevant to } G \\ 0 & \text{if } \phi \text{ is not relevant to } G. \end{cases}$$

The signs $\delta_i = \pm 1$ are defined as follows. By theorem 4.6, $\mathrm{Tran}_H^G(\chi_{\phi_0}) = \sum_i \varepsilon_i \, \mathrm{Ind}_{MAN}^G(\pi_i \otimes e^{\mathbf{Im}\nu} \otimes 1)$, $\varepsilon_i$ defined in [14]. Let $\delta_i = \varepsilon_i$.

4.8. LEMMA: *Lemma 4.2.4 of [14] holds in this context.*

*Proof:* This is immediate since both the right-hand side of 4.7 and the right and side of 4.2.4 of [17], are analytic in $\nu$.

Thus we could have used the cited lemma as a definition of $\mathrm{Tran}_H^G( \ )$. The next results follows exactly as in [14].

4.9. LEMMA: *If $\pi$ has infinitesimal character $\lambda \in X^*(T_H) \otimes \mathbb{C}$, then $\mathrm{Tran}_H^G(\pi)$ has infinitesimal character $\eta^{-1}(\lambda + \mu^*)$.*

## 5. Discrete series

All of our transfer problems reduce to transfer of discrete series representations. We formulate the necessary results of [14] in our terms. Similar results hold for limits of discrete series.

We parametrize the discrete series of $G$ as follows. Let $\Delta^+$ be any system of positive roots for $\Delta(G, T)$ with $\varrho = 1/2 \sum_{\alpha \in \Delta^+} \alpha \in X^*(T) \otimes \mathbb{C}$. Recall (Section 1) our canonical identification of $t$ with $X_*(T) \otimes \mathbb{C}$, $\hat{T}$ with $X^*(T)$, etc. Suppose $\lambda \in t^* \approx X^*(T) \otimes \mathbb{C}$ is regular, i.e. $L(\lambda) = T$. Let $q = q(\lambda) = t \oplus u(\lambda)$ ($u(\lambda) = \sum_{\langle \lambda, \alpha^\vee \rangle > 0} g^\alpha$ as usual). Let $\Delta_\lambda^+ = \Delta(u(\lambda))$, $\varrho_\lambda = 1/2 \sum_{\alpha \in \Delta_\lambda^+} \alpha$.

5.1. DEFINITION: Given $\lambda \in X^*(T) + \varrho$ regular, the discrete series representation $\pi(\lambda)$ is defined to be

$$\pi(\lambda) = \mathscr{R}_{q(\lambda)}(\chi(\lambda - \varrho_\lambda)),$$

and

$$\Theta_\lambda = \text{global character of } \pi(\lambda).$$

The infinitesimal character of $\pi(\lambda)$ is $\lambda$, $\pi(\lambda) \approx \pi(\lambda')$ if and only if $w\lambda = \lambda'$ for some $w \in W(G, T)$. Consider $\Theta_\lambda$ as a function on the regular elements $T_{\text{reg}}$ of $T$. Then if $e^h \in T_{\text{reg}}$, $h \in X_*(T) \otimes \mathbb{C}$,

$$\Theta_\lambda(e^h) = (-1)^{(1/2)\dim(G/K)} \frac{\displaystyle\sum_{w \in W(G,T)} \text{sgn}(w) \, e^{\langle w\lambda - \varrho_\lambda, h \rangle}}{\displaystyle\prod_{\alpha \in \Delta_\lambda^+} (1 - e^{\langle -\alpha, h \rangle})} \tag{5.2}$$

This follows from [16].

We compare this with the notation of [14]. Let $\eta$ be a *p-d.* of $T$. Given $\mu \in X_*({}^L T^0) \otimes \mathbb{C}$ regular, let $\Delta_\mu^+$ and $\varrho_\mu \in X_*({}^L T^0) \otimes \mathbb{C}$ be defined analogously to $\Delta_\lambda^+$ and $\varrho_\lambda$. Let $\varrho'$ be one half the sum of the positive coroots of ${}^L T^0$ in ${}^L G^0$ in any order.

5.3. DEFINITION: Given $\mu \in X_*({}^L T^0) + \varrho'$ regular, $\eta^{-1}(\mu) \in X^*(T) + \varrho$, so define $\pi(\mu) = \pi(\eta^{-1}(\mu))$.

Now the discrete series representations $\pi(w, \Delta_\mu^+)$ with characters $\Theta(w\mu, 0, w\Delta_\mu^+)$ are defined in ([14], page 406). From 5.2 we immediately obtain:

$$\pi(\mu) = \pi(1, \Delta_\mu^+), \quad \pi(w\mu) = \pi(w, \Delta_\mu^+). \tag{5.4}$$

These depend on $\eta$.

We describe transfers. Suppose we are given $H = H^\kappa$ and a transfer $\text{Tran}_H^G(\ )$. We change notation and let $\Delta^+ = \Delta^\vee({}^L B^0, {}^L T^0)$, $\Delta'^+ = \Delta^\vee({}^L B^0 \cap {}^L H^0, {}^L T^0)$, with corresponding $\varrho$ and $\varrho'$. Suppose $\gamma' \in X_*({}^L T^0) + \varrho'$ is regular. Recall (2.15 g ii) $\mu^*, \nu^*$ defining the embedding $\zeta$. By ([13], Theorem 3.4.1) if $\gamma = \gamma' + \mu^*$ then $\gamma \in X_*({}^L T^0) + \varrho$. Assuming $\gamma$ is regular these define discrete series representations of $H$ and $G$ respectively.

We may assume $\gamma'$ is $\Delta'^+$ dominant. Given such an element we define $\varepsilon_\kappa = \pm 1$ as follows. Let $w^*$ be the unique element of $W({}^L G^0, {}^L T^0)$ such that $w^* \gamma'$ is $\Delta^+$ dominant. Choose $\eta$ a *p-d.* satisfying $(T, \eta) \in T_H(G)$.

5.5. DEFINITION: $\varepsilon_\kappa = \varepsilon(T, \eta) \, \text{sgn}\,(w^*)$. This depends on our choices of $\eta$, $\{\varepsilon(T, \eta)\}$ and $\zeta$. Let $S = W(G, T)\backslash W(\mathbf{G}, \mathbf{T})$, $S' = W(H, T')\backslash W(\mathbf{H}, \mathbf{T}')$.

5.6. PROPOSITION: $\text{Tran}_H^G(\Sigma_{w \in S'} \, \pi(w\gamma')) = \varepsilon_\kappa \, \Sigma_{w \in S} \, \kappa(w)\pi(w\gamma)$.
This is merely a restatement of ([14], 4.1.1) in our terms. This is independent of $\eta$ ([14], 4.4.5).

Now suppose we are in the setting of Theorem 2.21. Then $d\lambda' + \varrho'$ is regular, $d\lambda + \varrho$ is regular, so $\pi(w(d\lambda' + \varrho'))$ and $\pi(w(d\lambda + \varrho))$ are defined.

5.7. COROLLARY: $\text{Tran}_H^G(\Sigma_{w \in S'} \, \pi(w(d\lambda' + \varrho'))) = \varepsilon_\kappa \, \Sigma_{w \in S} \, \kappa(w)\pi(w(d\lambda + \varrho))$.

We use this to transfer standard representations from $H$ to $G$. By [14], if $\zeta : {}^L H \to {}^L G$ is given, then for all $j$, $\zeta|_{L M_j'} : {}^L M_j' \to {}^L M_j$, and $\text{Tran}_{M_j'}^{M_j}(\ )$ is defined. Let $P_j' = M_j' N_j'$ be a parabolic subgroup of $H$, $\mathbf{P}_j' \supseteq \mathbf{B}_H$. Similarly let $P_j^* = M_j N_j \subseteq G^*$, $\mathbf{P}_j^* \supseteq \mathbf{B}^*$. If $T_j'$ originates in $G$ let $P_j = M_j^G N_j^G = \psi_j^G(P_j^*)$. Let $\Theta'$ be the sum of characters in a discrete series $L$-packet for $M_j'$.

5.8. PROPOSITION:

$$\text{Tran}_H^G(\text{Ind}_{P_j}^H(\Theta' \otimes 1))$$

$$= \begin{cases} \text{Ind}_{P_j}^G \, (\text{Tran}_{M_j'}^{M_j^G}(\Theta') \otimes 1) & \text{if } T_j' \text{ originates in } G \\ 0 & \text{if } T_j' \text{ does not originate in } G. \end{cases}$$

## 6. Coherent continuation

For later use we prove that transfer commutes with coherent continuation. This is used in ([14], Lemma 4.4.8).

Given $T \subseteq G$ ($T$ compact), $\Lambda \subseteq X^*(\mathbf{T})$ the root lattice of $\mathbf{G}$ with respect to $\mathbf{T}$, which is the lattice of differentials of weights of finite dimensional representations of the adjoint group of $\mathbf{G}$. Let $\xi \in X^*(\mathbf{T}) \otimes \mathbb{C}$, and suppose $\{\pi_{\xi+\lambda} | \lambda \in \Lambda\}$ is a coherent family of (virtual) $\mathbf{G}$-modules (cf. [16], definition 7.2.5).

**6.1. LEMMA:** *Suppose* $\pi = \pi_\xi$ *is stable. Then so is* $\pi_{\xi+\lambda}$ *for all* $\lambda \in \Lambda$.

*Proof:* There exists a sequence of virtual finite dimensional representations $\{F_i\}$ and projections $P_i$ onto infinitesimal characters $\lambda_i$ (cf. [21]) satisfying.

*6.2:* $\pi_{\xi+\lambda} = P_n(F_n \otimes (P_{n-1}(\ldots P_1(F_1 \otimes \pi))\ldots))$, see [6], proof of Lemma 3.39. Each step of the process preserves stability (cf. Lemma 2.5), proving the lemma.

Given $\xi \in X^*(\mathbf{T}) \otimes \mathbb{C}$ regular, $\pi$ a virtual module with regular infinitesimal character $\xi$, suppose $w \in W(\mathbf{G}, \mathbf{T})$ satisfies $w\xi = \xi + \lambda$, for some $\lambda \in \Lambda$. Let $W(\xi)$ be the set of such $w$. There exists a unique coherent family $\{\pi_{\xi+\lambda} | \lambda \in \Lambda\}$ satisfying $\pi_\xi = \pi$, so define

**6.3. DEFINITION:** $w_\xi \cdot \pi = \pi_{w^{-1}\xi}$ (cf. [16], 7.2.16).

This depends on $\xi$ (not just its $W(\mathbf{G}, \mathbf{T})$ orbit). This defines a group action on the space of virtual modules with infinitesimal character $\xi$, preserving the subspace of stable modules.

Let $H$ be an endoscopic group for $G$, with the choices of 2.15, so in particular $\mathrm{Tran}_H^G(\ )$ is defined.

Given $\pi'$ a stable virtual module for $H$, with regular infinitesimal character $\xi' \in X^*(\mathbf{T}') \otimes \mathbb{C}$, let $\{\pi'_{\xi'+\lambda'} | \lambda' \in \Lambda'\}$ be the coherent family satisfying $\pi'_{\xi'} = \pi'$. Let $\xi = \psi_0(\xi') + \mu^*$. Suppose $\lambda = \psi_0(\lambda') \in \Lambda$ for some $\lambda' \in \Lambda'$. Let $\tilde{\pi}_{\xi+\lambda} = \mathrm{Tran}_H^G(\pi'_{\xi'+\lambda'})$. Let $\pi = \tilde{\pi}_\xi = \mathrm{Tran}_H^G(\pi'_{\xi'})$.

**6.4. PROPOSITION:** $\tilde{\pi}_{\xi+\lambda} = \pi_{\xi+\lambda}$, *i.e.,* $\{\mathrm{Tran}_H^G(\pi'_{\xi'+\lambda'} | \lambda' \in \Lambda')\}$ *is (part of) a coherent family.*

*Proof:* This follows immediately from lemma 4.2.4 of [14], since, as can be seen by inspection, the character formula there for $\mathrm{Tran}_H^G$ preserves coherence. (Lemma 3.4.4 of [6] gives a useful characterisation of coherence, which we use here. Note that the proof of 4.3.2 of [14] uses the Schmid identities to prove an analogous form of coherent continuation to a wall.)

Embed $W(\mathbf{H}, \mathbf{T}')$ in $W(\mathbf{G}, \mathbf{T})$ via $\psi_0$.

6.5. COROLLARY: *Let $\pi'$ be a stable module for $H$ with infinitesimal character $\xi'$. Then for $w \in W(\xi')$,*

$$\mathrm{Tran}_H^G(w_{\xi'} \cdot \pi') = w_\xi \cdot \mathrm{Tran}_H^G(\pi').$$

## 7. An intermediate result

We prove Theorems 7.2 and 7.12, which are weak versions of Theorems 2.13 and 2.21, respectively. In the special case where $G$ is connected (or, for 2.21, where both $G$ and $H$ are connected), these are equivalent, and 7.2 is due to Zuckerman. (In general, we will use 7.2 and 7.12 as intermediate results in the proof of 2.21.)

The principal tool here is a character formula due to Zuckerman for $A(w\lambda, \tilde{\pi}_w)$ (where $\tilde{\pi}_w$ may be a direct sum of one-dimensional representations). In this section we assume $G$ has a compact Cartan, but there are analogous results in the general case.

Let $\hat{\Pi} = \{A(w\lambda, \pi_w)\}$ be as in Section 2. Recall the definitions of $\lambda$, $L_w$, $q_w$, $\pi_w$, etc. In particular, $\pi_w|_T = w\lambda$. Here we consider all possible choices of $\pi_w$ satisfying $\pi_w|_T = w\lambda$. Given a group $G$ with compact Cartan subgroup $T$ let $G^\dagger = TG^0 = TG_{\mathrm{der}}^0$. Now $w\lambda$ defines a unique one-dimensional representation, also denoted $w\lambda$, of $G^\dagger$.

7.1. DEFINITION: $\tilde{A}(w\lambda) = \mathscr{R}_{q_w}(\mathrm{Ind}_{L_w^\dagger}^{L_w}(w\lambda))$. Note that $\mathrm{Ind}_{L_w^\dagger}^{L_w}(w\lambda)$ is a direct sum of one-dimensional representations.

Thus $\tilde{A}(w\lambda) = A(w\lambda, \pi_w)$ if $L_w^\dagger = L_w$ in which case $\pi_w$ is uniquely determined.

Suppose we are in the setting of Theorem 2.13. Let $n_w = |L_w/L_w^\dagger| = |W(L_w, T)/W(L_w^\dagger, T)|$.

7.2. THEOREM (*Zuckerman*): $\sum_{w \in S} (-1)^{\gamma(w)} 1/(n_w)\tilde{A}(w\lambda)$ *is stable.*

We prove a series of lemmas. Recall Definition 5.1 of the discrete series representation $\pi(\lambda)$. Suppose $\lambda \in \hat{T}$ defines a one-dimensional representation $\lambda$ of $G^\dagger$ (i.e., $\lambda|_{G_{\mathrm{der}}^0} = 1$). In this section only, to conserve notation, we identify $\lambda \in \hat{T}$ with its differential $d\lambda \in X^*(\mathbf{T})$. Then (with $\Delta^+$ and $\varrho$ as preceding Definition 5.1) if $\lambda + \varrho \in X^*(\mathbf{T}) + \varrho$ is regular, $\pi(\lambda + \varrho)$ is defined.

Let $\sigma \cdot$ be the coherent continuation action based at $\lambda + \varrho$ (cf. Section 6.)

7.3. LEMMA (*Zuckerman*):

$$\operatorname{Ind}_{G^{\dagger}}^{G}(\lambda) \ = \ \frac{(-1)^{(1/2)(\dim(G/K))}}{|W(G^{\dagger}, T)|} \sum_{\sigma \in W(\mathbf{G},\mathbf{T})} \operatorname{sgn}(\sigma)\sigma \cdot \pi(\lambda + \varrho). \tag{7.4}$$

The following proof is due to Vogan.

*Proof:* Let $\Theta_1$ (resp. $\Theta_2$) be the character of the left (resp. right) hand side of 7.4. A priori $\Theta_2$ is only a virtual character.

*Case 1:* $G = G_{\mathrm{der}}^{0}$. Then $\lambda = 1$ and $\Theta_1$ is the trivial caracter. The character formula for $\pi(\lambda + \varrho)$ is (5.2), $\sigma \cdot \pi(\lambda + \varrho)$ has the same formula with $\sigma(\lambda + \varrho)$ in place of $\lambda + \varrho$ but $\varrho_{\lambda + \varrho}$ and $\Delta_{\lambda + \varrho}^{+}$ left fixed:

$$\sigma \cdot \Theta_{\lambda + \varrho}(e^{h}) \ = \ (-1)^{(1/2)(\dim(G/K))} \frac{\displaystyle\sum_{w \in W(G, T)} \operatorname{sgn}(w) e^{\langle w\sigma(\lambda+\varrho)-\varrho, h\rangle}}{\displaystyle\prod_{\alpha \in \Delta_{\lambda}^{+}} (1 - e^{\langle -\alpha, h\rangle})}.$$

Summing over $\sigma \in W(\mathbf{G}, \mathbf{T})$ we immediately have $\Theta_1|_{T_{\mathrm{reg}}} = \Theta_2|_{T_{\mathrm{reg}}}$. Let $\alpha$ be any simple root of $\Delta^{+}$, $s_{\alpha}$ the reflection through $\alpha$; then $s_{\alpha} \cdot \Theta_2 = -\Theta_2$. A simple application of ([16], corollary 7.3.19) shows that if $X$ is an irreducible consituent of $\Theta_2$ then $\alpha \in \tau(X)$ for all simple roots $\alpha$ ([16], definition 7.3.8). This implies that each such $X$ is a finite dimensional representation; by its infinitesimal character we see $\Theta_2$ is a multiple of the trivial representation. Since $\Theta_2|_{T_{\mathrm{reg}}} = \Theta_1|_{T_{\mathrm{reg}}}$ we are done with this case.

*Case 2:* $G = G^{\dagger}$. Now $\Theta_1$ is one-dimensional, essentially a character of the center of $G$. The preceding argument applies with minor changes.

*Case 3:* Any $G$. Apply Case 2 to $G^{\dagger}$ and induce both sides to $G$. Note that $\operatorname{Ind}_{G^{\dagger}}^{G}(\sigma \cdot \pi(\lambda + \varrho)) = \sigma \cdot \operatorname{Ind}_{G^{\dagger}}^{G}(\pi(\lambda + \varrho))$ (by [16], proposition 0.4.7). Essentially by definition (of $\mathscr{R}_{\mathfrak{q}}$ or of the discrete series as in [11], p. 17), $\operatorname{Ind}_{G^{\dagger}}^{G}(\pi(\lambda + \varrho)) = \pi(\lambda + \varrho)$ for $G$, and we are done.

Given $\lambda \in \hat{T}$ defining a one-dimensional representation $\lambda$ of $L^{\dagger}$, $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ satisfying $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Delta(\mathfrak{u})$. Let $\Delta_{\mathfrak{l}}^{+}$ be any system of positive roots for $\Delta(\mathfrak{l})$, with corresponding $\varrho(\mathfrak{l})$, and $\varrho_{\mathfrak{q}} = \varrho(\mathfrak{l}) + \varrho(\mathfrak{u})$.

7.5. COROLLARY (*Zuckerman*):

$$\mathscr{R}_{\mathfrak{q}}(\operatorname{Ind}_{L^{\dagger}}^{L}(\lambda)) \ = \ \frac{(-1)^{(1/2)\dim(L/L \cap K)}}{|W(L^{\dagger}, T)|} \sum_{\sigma \in W(\mathbf{L}, \mathbf{T})} \operatorname{sgn}(\sigma)\sigma \cdot \pi(\lambda + \varrho_{\mathfrak{q}}). \tag{7.6}$$

(Here $\sigma \cdot$ is the coherent continuation action based at $\lambda + \varrho_q$). In particular, in the setting of Definition 7.1,

$$\tilde{A}(w\lambda) = \frac{(-1)^{\gamma(w)}}{|W(L_w^\dagger, T)|} \sum_{\sigma \in W(L_u, T)} \text{sgn } (\sigma)\sigma \cdot \pi(w(\lambda + \varrho_q)). \tag{7.7}$$

(Here $\sigma \cdot$ is the coherent continuation action based at $w(\lambda + \varrho_q)$).

*Proof:* Apply Lemma 7.3 to $L$, and apply $\mathscr{R}_q$ to both sides. We note (cf. [16] Propositions 7.4.1 and 7.4.3b):

$$\mathscr{R}_q(\sigma \cdot \pi(\lambda + \varrho(\mathfrak{l}))) = \sigma \cdot \mathscr{R}_q(\pi(\lambda + \varrho(\mathfrak{l}))). \tag{7.8}$$

Furthermore,

$$\mathscr{R}_q(\pi(\lambda + \varrho(\mathfrak{l}))) = \pi(\lambda + \varrho_q) \text{ (for } G). \tag{7.9}$$

This follows by induction by stages ([16], Corollary 6.3.10) upon writing $\pi(\lambda + \varrho(\mathfrak{l}))$ (for $L$) as $\mathscr{R}_\mathfrak{b}(\lambda)$ ($\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}_\mathfrak{l}$ a Borel subalgebra of $\mathfrak{l}$). This proves (7.6) and (7.7) follows immediately.

*Proof of Theorem 7.2:* By (7.7) we have

$$\sum_{w \in S} (-1)^{\gamma(w)} \frac{1}{n_w} \tilde{A}(w\lambda)$$

$$= \sum_{w \in S} (-1)^{\gamma(w)} \frac{1}{n_w} \left[ \frac{(-1)^{\gamma(w)}}{|W(L_w^\dagger, T)|} \sum_{\sigma \in W(L_u, T)} \text{sgn } (\sigma)\sigma_{w(\lambda+\varrho_q)} \cdot \pi(w(\lambda + \varrho_q)) \right]. \tag{7.10}$$

Now $\sigma_{w(\lambda+\varrho_q)} = \sigma'_{\lambda+\varrho_q}$ for $\sigma' = w^{-1}\sigma w \in W(\mathbf{L}, \mathbf{T})$. Thus (7.10) equals

$$\sum_{\sigma \in W(\mathbf{L}, \mathbf{T})} \text{sgn } (\sigma)\sigma_{\lambda+\varrho_q} \cdot \left[ \sum_{w \in S} \frac{1}{n_w |W(L_w^\dagger, T)|} \pi(w(\lambda + \varrho_q)) \right].$$

Replacing S by $W(\mathbf{G}, \mathbf{T})$ changes the constant to one independent of $w$, and we obtain

$$\frac{1}{|W(G,T)||W(\mathbf{L}, \mathbf{T})|} \sum_{\sigma \in W(\mathbf{L}, \mathbf{T})} \text{sgn } (\sigma)\sigma_{\lambda+\varrho_q} \cdot \left[ \sum_{w \in W(\mathbf{G}, \mathbf{T})} \pi(w(\lambda + \varrho_q)) \right]. \tag{7.11}$$

The term inside the brackets is a (multiple of) the sum of the elements in a discrete series $L$-packet, hence is stable ([11], lemma 5.2). The entire sum is stable by Lemma 6.3, proving the theorem.

Now consider transfers. Suppose we are in the setting of Theorem 2.21. Define $\tilde{A}(w\lambda)$, $\tilde{A}(w\lambda')$ by 7.1, and let $n'_w = |L'_w/L'^{\dagger}_w|$.

7.12  THEOREM:  $\mathrm{Tran}_H^G (\Sigma_{w \in S'} (-1)^{\gamma(w)} 1/n'_w \tilde{A}(w\lambda')) = \varepsilon_\kappa \Sigma_{w \in S} (-1)^{\gamma(w)} 1/n_w \kappa(w)\tilde{A}(w\lambda)$.

*Proof:* Write the sum on the left-hand side as in (7.11) with $H$ in place of $G$ to obtain:

$$\frac{1}{|W(H, T')||W(L', T')|} \times$$

$$\mathrm{Tran}_H^G \left( \sum_{\sigma \in W(L', T')} \mathrm{sgn}\,(\sigma)\sigma_{\lambda' + \varrho_{q'}} \cdot \left[ \sum_{w \in W(H, T')} \pi(w(\lambda' + \varrho_{q'})) \right] \right).$$

Now $\lambda + \varrho_q = \psi_0(\lambda' + \varrho_{q'}) + \mu^*$. Hence we use corollary 6.5 to commute $\mathrm{Tran}_H^G(\ )$ past $\sigma_{\lambda' + \varrho_{q'}}$. Then apply proposition 5.8 to obtain (transferring $\kappa$ to $W(G, T)$):

$$\frac{\varepsilon_\kappa}{|W(G, T)||W(L, T)|} \sum_{\sigma \in W(L, T)} \mathrm{sgn}\,(\sigma)\sigma_{\lambda + \varrho_q} \cdot \left[ \sum_{w \in W(G, T)} \kappa(w)\pi(w(\lambda + \varrho_q)) \right]$$

Working backward from (7.11) to (7.10) on $G$, with $\kappa(w)$ at each step, we get

$$\varepsilon_\kappa \sum_{w \in S} (-1)^{\gamma(w)} \frac{1}{n_w} \kappa(w)\tilde{A}(w\lambda)$$

proving the theorem.

Note that we can also push both sides of 7.12 to a wall and have equality.

## 8. Resolutions by standard modules

In this section we first prove Theorem 2.13 on the stability of $\Sigma_{w \in S} (-1)^{\gamma(w)} A(w\lambda, \pi_w)$. Next we consider Theorem 2.21. We obtain a partial result: the set of standard modules occurring on the left-hand side of 2.21 is contained in the set occurring on the right-hand side. One each side these

have multiplicity $\pm 1$. This is proved using the character formula of [7] which holds for disconnected groups. In the next section we combine this with the results of Section 7 and a simple counting argument to prove Theorem 2.21. In this section, the arguments go through, substantially, without the assumption of equal rank for $G$. For simplicity however, we retain this assumption.

We recall results of [7] on the resolution of $A(\lambda, \pi)$ via standard modules. In this section we wish to allow $T$ to take on new meaning, so to specify the compact Cartan subgroup we write $T_c$. Given $D$ any $\theta$-stable Cartan subgroup of $G$, write $D = TA$ (direct product): $T = D \cap K$, $A$ a vector group. It is important to note $A$ is *not* necessarily the maximally split algebraic subgroup of $D$ – for example $A$ is connected and may fail to be an algebraic group. Let $MA$ be the centralizer of $A$ in $G$. Now $MA$ is an algebraic group, by abuse of notation we write it as **M**.

Given $\Delta^+ = \Delta^+(\mathbf{G}, \mathbf{D})$ let $\varrho = \varrho(\Delta^+)$ and $\varrho(B, \Delta^+)$ be as preceeding 2.19, $\Delta_{\mathbf{M}}^+ = \Delta^+ \cap \Delta(\mathbf{M}, \mathbf{D})$, $\varrho_{\mathbf{M}} = \varrho(\Delta_{\mathbf{M}}^+)$. Here we change notation slightly from Section 5. If $\gamma \in \hat{T}$ and $d\gamma + \varrho_{\mathbf{M}}$ is dominant and regular for $\Delta_{\mathbf{M}}^+$, let

$$\pi_M(\gamma, \Delta_{\mathbf{M}}^+) = \mathscr{R}_{\mathfrak{b}}(\gamma), \tag{8.1}$$

a discrete series representation of $M$ with infinitesimal character $d\gamma + \varrho_{\mathbf{M}}$. Here $\mathfrak{b} = \mathfrak{t} \oplus \Sigma_{\alpha \in \Delta_{\mathbf{M}}^+} \mathfrak{m}^\alpha$.

Length of a standard module (having regular infinitesimal character) is defined in [16], 8.1.4. But see a correction in 12.1 of [18]. It is constant on pseudo $L$-packets. Such modules have unique submodules.

**8.2. THEOREM:** [7] *If $A(\lambda, \pi)$ has regular infinitesimal character, then it has a resolution by standard modules* $\{\mathrm{Ind}_{MAN}^G(\pi_M((2\varrho(B, \Delta^+) \otimes \pi)|_T, \Delta_M^+) \otimes (\varrho \otimes \pi)|_A \otimes 1)\}$ *where*

1. *$D$ runs over all conjugacy classes of Cartan subgroups of $L$ (with $\mathrm{MAN}$ cuspidal; $N$ is chosen as in [7]).*
2. *$\Delta^+$ runs over all $\Delta^+ \supseteq \Delta(\mathfrak{u}, \mathfrak{d})$.*

*Furthermore, letting $\gamma = 1/2 \dim (L/L \cap K)$, there is an exact sequence*

$$0 \to A(\lambda, \pi) \to X^\gamma \to X^{\gamma-1} \to \cdots \to X^0 \to 0 \tag{8.3}$$

*where $X^j$ is the direct sum of those standard modules as above with length $j$.*

We rewrite this as

$$A(\lambda, \pi) = \sum_{j=0}^{\gamma} (-1)^{\gamma-j} X^j.$$

It follows from [7] that $X^\gamma$ is a standard module with $A(\lambda, \pi)$ as Langlands submodule. Similarly, write

$$A(w\lambda, \pi_w) = \sum_{j=0}^{\gamma(w)} (-1)^{\gamma(w)-j} X_w^j. \tag{8.5}$$

(Note that if $A(\lambda, \pi)$ is a push-to-a wall, we have a similar formula involving limits of discrete series. Then 6.4 allows us to reduce to the case of regular infinitesimal character.)

Now suppose $\pi|_D = \chi(\mu, v)$. Then it follows from the definitions that the pseudo $L$-packet containing the standard module

$$\mathrm{Ind}_{MAN}^G(\pi_M((2\varrho(B, \Delta^+) \otimes \chi(\mu, v))|_T, \Delta_\mathbf{M}^+) \otimes (\varrho \otimes \chi(\mu, v))|_A \otimes 1) \tag{8.6}$$

is

$$\{\mathrm{Ind}_{MAN}^G(\pi_M((2\varrho(B, z\Delta^+) \otimes \chi(z\mu, v))|_T, z\Delta_\mathbf{M}^+) \otimes (z\varrho \otimes \chi(z\mu, v))|_A \otimes 1)$$

$$\text{such that} \quad z \in W(\mathbf{M}, \mathbf{D})\}. \tag{8.7}$$

Recall Theorem 2.13 claims $\Sigma_{w \in S} (-1)^{\gamma(w)} A(w\lambda, \pi_w)$ is stable. This will follow from the next lemma.

8.8. LEMMA: *Given $A(w\lambda, \pi_w)$ and a standard module $X$ occurring in the resolution of $A(w\lambda, \pi_w)$, let $X'$ be any other standard module in the same pseudo $L$-packet as $X$. Then there exists $w' \in W(\mathbf{G}, \mathbf{T}_c)$, such that $X'$ occurs in the resolution of $A(w'\lambda, \pi_{w'})$. Furthermore, the double coset $W(\mathbf{G}, \mathbf{T}_c)w' W(\mathbf{L}, \mathbf{T}_c)$ is uniquely determined by this condition.*

*Proof:* Suppose $\pi_w|_D = \chi(\mu, v)$, so $X$ is given by 8.6 and $X'$ is given by 8.7. Conjugating by $G$ we assume $\mathbf{T} \subseteq \mathbf{T}_c$. Then $z \in W(\mathbf{G}, \mathbf{T}_c)$, take $w' = zw$. Then $D \subseteq L(w'\lambda)$. By Lemma 2.5 $\pi_{w'}|_D = \chi(z\mu, v)$ and the first part of the Lemma follows from Theorem 8.2 applied to $A(w'\lambda, \pi_{w'})$. The second part follows from Theorem 8.2 upon noting $w'\Delta^+ \supseteq \Delta(\mathfrak{u}_w, \mathfrak{d})$ implies $z = w'w^{-1} \in W(\mathbf{L}, \mathbf{T}_c)$.

8.9. COROLLARY: $\Sigma_{w \in S} (-1)^{\gamma(w)} A(w\lambda, \pi_w)$ *may be written as a sum of standard modules $\Sigma_i \varepsilon_i X_i$, $\varepsilon_i = \pm 1$, $X_i \neq X_j$ if $i \neq j$.*

*(If a module $\pi$ is written $\pi = \Sigma_{j=1}^n a_j X_j$, $a_j \neq 0$ we say "$X_j$ occurs in the resolution of $\pi$.")*

In fact, one sees by inspection of 8.6 that if $X_i = \mathrm{Ind}_{MAN}^G(\delta_i \otimes 1)$, and

$X_j = \text{Ind}_{MAN}^G(\delta_j \otimes 1)$ both occur in the sum of 8.9, then either $X_i$ and $X_j$ are $L$-indistinguishable or else the infinitesimal characters of $\delta_i$ and $\delta_j$ differ.

*Proof of Theorem 2.13:* Let $\Theta = \Sigma_{w \in S} (-1)^{\gamma(w)} A(w\lambda, \pi_w)$. By the preceding Lemma the standard modules occurring in the resolution of $\Theta$ have multiplicity $\pm 1$ and are a union of pseudo $L$-packets $\{\tilde{\Pi}_{\phi_i} | i = 1, \ldots, n\}$ (cf. definition 4.2). Recall $\tilde{\chi}_{\phi_i} = \Sigma_{\pi \in \tilde{\Pi}_{\phi_i}} \pi$. The sign with which a given standard module $X$ occurs is

$$(-1)^{\gamma(w)}(-1)^{\gamma(w)-\text{length}(X)} = (-1)^{\text{length}(X)} \tag{8.10}$$

which is constant on pseudo $L$-packets. Hence $\Theta = \Sigma_{i=1}^n \varepsilon_i \tilde{\chi}_{\phi_i}$, $\varepsilon_i = \pm 1$ which by Lemma 4.3 proves Theorem 2.13.

We turn now to a partial proof of Theorem 2.21 which will be completed in Section 9. Suppose we are in the setting of that theorem, in particular $\{A(w\lambda', \pi'_w)\}$ and $\{A(w\lambda, \pi_w)\}$ have been defined. Let $\Theta'$ be the term in parentheses on the left hand side of 2.21, i.e.,

$$\Theta' = \sum_{w \in S'} (-1)^{\gamma(w)} A(w\lambda', \pi'_w). \tag{8.11}$$

Let

$$\Theta = \text{Tran}_H^G(\Theta'). \tag{8.12}$$

First assume $G$ is quasisplit, $L$ is quasisplit. Recall $T'_n \subseteq L'$ and $T_n \subseteq L$, maximally split Cartan subgroups of $L$ and $L'$ have been chosen, and $\psi_n$: $T'_n \approx T_n$. Recall $\pi|_{T_n}$ is defined by 2.20. Let other notation be as in that definition.

8.13. LEMMA: *$\pi$ extends to a one-dimensional representation, also denoted $\pi$, of $L$.*

*Proof:* Write $T'_n = T'A'$, other notation as in 8.2; let

$$X' = \text{Ind}_{M'A'N'}^H(\pi_{M'}((2\varrho(B, \Delta'^+) \otimes \pi')|_{T'}, \Delta_M'^+) \otimes (\varrho' \otimes \pi')|_{A'} \otimes 1). \tag{8.14}$$

By Theorem 8.2 $X'$ occurs in the resolution of $\Theta'$ and has $A(\lambda', \pi')$ as Langlands submodule. By a direct calculation using Section 1, we see

$$2\varrho'(B, \Delta'^+)|_{T'} \otimes \varrho'|_{A'} = \chi(\varrho' - \varrho'_{M'}, \varrho'(B, \Delta'^+)). \tag{8.15}$$

Let $\chi'$ be the sum of the representations in the pseudo $L$-packet $\tilde{\Pi}$ containing $X'$. Let

$$\chi \;=\; \chi(\mu' + \mu^* + \varrho' - \varrho_M, \quad \nu' + \lambda^* + \varrho'(B, \Delta'^+)) \in T_n^{\wedge}. \qquad (8.16)$$

(here we have transferred data to $T_n$ via $\psi_n$). By [14] (here 4.7 and 5.7 but in the notation of 8.1), letting $T_n = TA$ etc.,

$$X \approx \operatorname{Ind}_{MAN}^{G}(\pi_M(\chi|_T, \Delta_M^+) \otimes \chi|_A \otimes 1) \qquad (8.17)$$

occurs in the resolution of $\operatorname{Tran}_H^G(\chi')$. By Theorem 7.12 there exists $\bar{\pi}$, a one-dimensional representation of $L$, and $w$ such that $X$ occurs in the resolution of $A(w\bar{\lambda}, \bar{\pi}_w)$ ($\bar{\lambda} = \bar{\pi}|_{T_c}$). Applying 8.2 to $A(w\bar{\lambda}, \bar{\pi}_w)$, we see that a standard module occurring in its resolution is of the form induced from the following representations of $MAN$:

$$\pi_M((2\varrho(B, \Delta^+) \otimes \bar{\pi}_w)|_T, \Delta_M^+) \otimes (\varrho \otimes \bar{\pi}_w)|_A \otimes 1,$$

for some positive system for $G$, $\Delta^+$, as in 8.2. Now applying the analogue of 8.15 (for $G$ and $\Delta^+$ instead of $H$ and $\Delta'^+$) to this, we have that the representation of $MAN$ is of the form $\pi_M((\bar{\pi}_w \otimes \chi(\varrho - \varrho_M, \varrho(B, \Delta^+)))|_T, \Delta_M^+) \otimes (\bar{\pi}_w \otimes \chi(\varrho - \varrho_M, \varrho(B, \Delta^+)))|_A \otimes 1$.

Assuming that $w = 1$ does not change the pseudo-$L$-packet of the standard module we are considering, and we know that $\operatorname{Tran}_H^G(\chi')$ is stable, so we may put $w = 1$. We may also take $\Delta^+$ to be related to $\Delta'^+$ as in the hypotheses of formula 2.20, since this will still satisfy the hypotheses on $\Delta^+$ given in Theorem 8.2; the point is that the $\mathbf{Z}/2\mathbf{Z}$ character data (a triple) containing $\Delta'^+$ has maximal length among all those in 8.2 for $H$, but $\operatorname{Tran}_H^G$ preserves the inequalities of lengths, so any triple including any $\Delta^+$ determined by 8.17 also has maximal length among all such data occurring in the (appropriate sum of) formulas 8.2 (for $G$) (corresponding to formula 7.12).

Comparing this representation of $MAN$ to the one occurring in the formula 8.17, and using 8.16, we see that, again up to the action of an irrelevant Weyl group element, we have

$$\bar{\pi}|_{T_n} \;=\; \chi(\mu' + \mu^* + \varrho' - \varrho, \nu' + \lambda^* + \varrho'(B, \Delta'^+) - \varrho(B, \Delta^+)), \qquad (8.18)$$

so by 2.20 $\pi|_{T_n} = \bar{\pi}|_{T_n}$, proving the Lemma.

Now let $X'$ be any standard module occurring in the resolution of $\Theta'$, i.e. in the resolution of $A(w\lambda', \pi_w')$ for some $w$. Let $\chi'$ be the sum of the modules in the pseudo $L$-packet containing $X'$. Let $X$ be any standard module

occurring in the resolution of $\operatorname{Tran}_H^G(\chi')$. As in the proof of the preceding lemma $X$ occurs in the resolution of $A(z\lambda, \bar{\pi}_z)$ for some $z$ and some one-dimensional representation $\bar{\pi}$ of $L$ satisfying $\bar{\pi}|_{L^0} \simeq \pi|_{L^0}$. The main result we need in Section 9 is

8.19. PROPOSITION: *X occurs in the resolution of $A(z\lambda, \pi_z)$.*

*Proof:* The argument is similar to the proof of Lemma 8.13 carried out on any Cartan subgroup of $L$. By Theorem 8.2 choose $D' = T'_j = T'A' \subseteq L$, $\Delta'^+$, etc. such that

$$X' \approx \operatorname{Ind}_{M'A'N'}^H (\pi_{M'}((2\varrho'(B, \Delta'^+) \otimes \pi'_w)|_{T'}, \Delta_{\mathbf{M}}'^+) \otimes (\varrho' \otimes \pi'_w)|_{A'} \otimes 1). \tag{8.20}$$

First assume $G$ is quasisplit. By Theorem 8.2 we may take $D = T_j = TA$ and write

$$X \approx \operatorname{Ind}_{MAN}^G (\pi_M((2\varrho(B, \Delta^+) \otimes \pi_z)|_T, \Delta_{\mathbf{M}}^+) \otimes (\varrho \otimes \pi_z)|_A \otimes 1). \tag{8.21}$$

It is enough to show

$$\bar{\pi}_z|_D \approx \pi_z|_D \tag{8.22}$$

where by the comments preceding the proposition we know

$$\bar{\pi}_z|_{D^0} \approx \pi_z|_{D^0}. \tag{8.23}$$

Suppose $\bar{\pi}_z|_D = \chi(\bar{\mu}, \bar{v})$, $\pi_z|_D = \chi(\mu, v)$. It is enough to show

$$v = \bar{v} + \delta + (\beta - \sigma\beta) \tag{8.24}$$

for some $\delta \in X_*(\mathbf{D})$, $\beta \in X_*(\mathbf{D}) \otimes \mathbf{C}$, $\sigma = \sigma_j$ the Galois action of $\mathbf{D}$. We abbreviate 8.24 by writing $v \equiv \bar{v}$.

First assume $w = z = 1$. Since $X$ occurs in the resolution of $\operatorname{Tran}_H^G(X')$ by 8.15, 8.20, 8.21 and [14]:

$$\bar{v} = \psi_j(v' + \varrho(B', \Delta'^+, T'_j)) - \varrho(B, \Delta^+, T_j) + v^* \tag{8.25}$$

where in the presence of several Cartan subgroups we write $\varrho(B, \Delta^+) = \varrho(B, \Delta, D)$ to specify $D$.

By the definition of $\pi$ (2.20):

$$v = \text{ad}\,(\ell)[\psi_n(\varrho'(B, \Delta'^+, T_n') + \text{ad}\,(\ell'^{-1})v') - \varrho(B, \Delta^+, T_n)] + v^* \tag{8.26}$$

where $T_n'$, $T_n$ are as in the proof of Lemma 8.10. Also $\ell' \in \mathbf{M}_n' \cap \mathbf{L}'$, $\text{ad}\,(\ell')$: $T_n' \to T_j'$, as in 2.18, $\text{ad}\,(\ell')$: $\Delta^+(\mathbf{H}, T_n') \to \Delta^+(\mathbf{H}, T_j')$ similarly $\ell$. Hence by the definition of $\psi_n$ (2.15f):

$$\text{ad}\,(\ell) \circ \psi_n = \psi_j \circ \text{ad}\,(\ell') \tag{8.27}$$

i.e. the following diagram commutes:

$$\tag{8.28}$$

Comparing 8.25 and 8.26 it is enough to show

$$\text{ad}\,(\ell)\varrho(B, \Delta^+, T_n) - \varrho(B, \Delta^+, D)$$

$$\equiv \psi_j\,(\text{ad}\,(\ell')\,\varrho(B, \Delta'^+, T_n') - \varrho(B, \Delta^+, D')). \tag{8.29}$$

Recall $\Delta^+ = \Delta_I^+ \cup \Delta_u$, $\Delta'^+ = \Delta_I'^+ \cup \Delta_{u'}'$. Write $\varrho(B, \Delta^+, T_n) = \varrho(B, \Delta_I^+, T_n) + \varrho(B, \Delta_u, T_n)$ etc. with the obvious notation. Then 8.29 is clear for the terms involving $\Delta_I^+$. Furthermore $\text{ad}\,(\ell)\varrho(B, \Delta_u, T_n) - \varrho(B, \Delta_u, D) \equiv 0$, $\text{ad}\,(\ell')\varrho(B, \Delta_{u'}', T_n') - \varrho(B, \Delta_{u'}', D') \equiv 0$ since if $\alpha \in \Delta_u$ and $\alpha$ is complex, then $\text{ad}\,(\ell)\alpha$ is either imaginary or complex (similarly for $\ell'$); we leave the details to the reader. This proves 8.29.

This completes the proof in case $G$ is quasisplit and $z = w = 1$. For general $z$, $w$ the proof is similar using Lemma 2.6. For general $G$ the same holds, incorporating $\psi$ at each step. We omit the details.

Recall $\Theta = \text{Tran}_H^G(\Theta')$.

8.30. COROLLARY: *$X$ occurs with multiplicity $\pm 1$ in the resolution of $\Theta$.*

*Proof:* We first remark that 8.9 implies that any standard module occurs in the resolutions of the right hand side of Theorem 2.21 at most once. Furthermore, if $X$ does occur, by 2.13 so does $\tilde{\Pi}(X)$, by which we denote

the pseudo-$L$-packet containing $X$. Hence $\tilde{\Pi}(X)$ occurs at most once. Now write $X$ as in 8.21 and suppose $X$ occurs in $\Theta$. Then $X$ occurs in the right hand side of 2.21. But then, by 8.20, we must have

$$\operatorname{Tran}_{M'A'}^{MA} \Pi(\pi_{M'A'}(2\varrho'(B, \Delta'^+)|_T \otimes \pi'_w \otimes \varrho'|_{A'}, \Delta'^+_{\mathbf{M}}))$$

$$= \Pi(\pi_{MA}(2\varrho(B, \Delta^+)|_T \otimes \pi_z \otimes \varrho|_A, \Delta^+_{\mathbf{M}}))$$

(we regard this as a statement about the occurrence of certain characters in others, up to $\pm$ so that $\kappa$ is irrelevant), for some $M'$, $\Delta'^+$, and $\pi'_w$. Clearly $M'$ is uniquely determined by this condition, as is $\varrho'|_{A'}$. But the infinitesimal character $\Lambda'$ of $\pi_{M'A'}(2\varrho'(B, \Delta'^+)|_T \otimes \pi'_w \otimes \varrho'|_{A'}, \Delta'^+_{\mathbf{M}})$ is also fixed, and so, putting $\Lambda'_{\mathbf{im}}$ as the infinitesimal character of $\pi_{M'}(2\varrho'(B, \Delta'^+)|_{T'} \otimes \pi'_w|_{T'}, \Delta'^+_{\mathbf{M}})$, $\Lambda'_{\mathbf{im}}$ is also determined. But then 8.9 implies there is only one pseudo-$L$-packet occurring in the resolution of $\Theta'$ with $\Lambda'_{\mathbf{im}}$ and $\varrho'|_{A'}$ satisfying these conditions.

## 9. Proof of Theorem 2.21

In the setting of Theorem 2.21 we claim

$$\varepsilon_\kappa \sum_{w \in S} (-1)^{\bar{\gamma}(w)} \kappa(w) A(w\lambda, \pi_w) = \operatorname{Tran}_H^G \left( \sum_{w \in S'} (-1)^{\bar{\gamma}'(w)} A(w\lambda', \pi'_w) \right). \quad (9.1)$$

By Theorem 7.12 we are done if $G$ and $H$ are connected. In general by Section 8, the two sides of 9.1 are sums of the same standard modules, with multiplicities $\pm 1$. We use Section 7 to check the signs. The proof is just book-keeping: the signs are independent of whether $\pi_w$ or $\bar{\pi}_w$ occur in 9.1, provided $\pi_w|_{L_w^+} = \bar{\pi}_w|_{L_w^+}$; and Theorem 7.12 may be thought of as 9.1 summed over all such $\pi_w$.

Let $\pi' = \pi'_1$, the given one-dimensional representation of $L'$ ($L'$ is quasi-split and contained in $H$). Let $N = |L'/L'^+|$, and let $\pi_1^\dagger, \ldots, \pi_N^\dagger$ be the inequivalent irreducible representations of $L'$ satisfying $\pi_i^\dagger|_{L^+} \approx \pi'|_{L^+}$. Let $\{\pi'_{i,w} | w \in S'\}$, $\{\pi_{i,w} | w \in S\}$ be the corresponding representations of $\{L'_w | w \in S'\}$ and $\{L_w | w \in S\}$. The $\pi'_{i,w}$ are *not* recessarily all distinct (similarly $\pi_{i,w}$).

9.2. LEMMA: *Fix $w$*.

$$\sum_{i=1}^N \pi_{i,w} = \frac{N}{n_w} \operatorname{Ind}_{L_w^+}^{L_w}(w\lambda).$$

*Proof:* Suppose $\pi$ is a one-dimensional representation of $L_w$ satisfying $\pi|_{L_w^+} \approx w\lambda$. Then $\pi \approx \pi_{i,w}$ for some $i$. The lemma follows immediately.

Let

$$X_i = \varepsilon_\kappa \sum_{w \in S} (-1)^{\gamma(w)} \kappa(w) A(w\lambda, \pi_{i,w}) \tag{9.3}$$

$$X_i' = \sum_{w \in S'} (-1)^{\gamma'(w)} A(w\lambda', \pi_{i,w}'). \tag{9.4}$$

Then 9.1 is equivalent to

$$X_i = \mathrm{Tran}_H^G(X_i') \quad i = 1, \ldots, N. \tag{9.5}$$

9.6. LEMMA: $\sum_{i=1}^N X_i = \mathrm{Tran}_H^G \left( \sum_{i=1}^N X_i' \right)$.

*Proof:* This follows immediately from Theorem 7.12 and Lemma 9.2.

Write

$$X_i = \sum_{A,\pi,v} \varepsilon(A, \pi, v, i) \,\mathrm{Ind}_{MAN}^G(\pi \otimes e^v \otimes 1) \tag{9.7}$$

as in 4.4. Here $\varepsilon(A, \pi, v, i) = \pm 1$ or $0$ by Corollary 8.9. Let

$$\mathrm{Tran}_H^G(X_i') = \sum_{A,\pi,v} \delta(A, \pi, v, i) \,\mathrm{Ind}_{MAN}^G(\pi \otimes e^v \otimes 1), \; \delta(A, \pi, v, i) = \pm 1 \tag{9.8}$$

or $0$ by Corollary 8.30. Then 9.5 is equivalent to

$$\varepsilon(A, \pi, v, i) = \delta(A, \pi, v, i) \; \forall \, A, \pi, v, \quad i = 1, \ldots, N. \tag{9.9}$$

Fix $A, \pi, v$; write $\varepsilon_i = \varepsilon(A, \pi, v, i)$, $\delta_i = \delta(A, \pi, v, i)$. We know

$$\sum_{i=1}^N \varepsilon_i = \sum_{i=1}^N \delta_i \tag{9.10}$$

by Lemma 9.6,

$$\delta_i \neq 0 \text{ implies } \varepsilon_i \neq 0 \tag{9.11}$$

by Proposition 8.19,

$$\varepsilon_i \varepsilon_j \neq 0 \text{ implies } \varepsilon_i = \varepsilon_j \tag{9.12}$$

by 8.4.

That $\varepsilon_i = \delta_i$, $i = 1, \ldots, N$ follows by a simple counting argument, proving 9.9 and Theorem 2.21.

## 10. Some concluding remarks

For convenience we have assumed in the statements of theorems 2.13 and 2.21, that $G$ has possessed (relative) discrete series. We now remove this restriction. We do this by reduction to the equal rank case. In this section we also explain in Theorem 10.3 and Proposition 10.4, how to relate our formal definition of transfer, in 4.7, to the matching of orbital integrals. Transfer of tempered stable distributions is defined by duality to this matching (see 3.1.1 and 4.0.1 of [14]). We show that our formal definition for non-tempered distributions satisfies this same duality.

Let $T^c$ be a $\theta$-stable fundamental Cartan subgroup of $G$, we assume $H$ is a $(T^c, \kappa)$ group ([12]), so that $H$ shares $T^c$ with $G$. Let $T^c = TA$ be the decomposition with respect to $\theta$, and let $M = G^A$ and $M' = H^A$. Now $M$ and $M'$ satisfy the hypotheses of Sections 7 and 9, so we have available all the results of Section 9 with $M$ and $M'$ playing the roles of $G$ and $H$.

Let $W(\mathfrak{g}, \mathfrak{t}^c)^\theta$ denote those elements of the Weyl group which commute with $\theta$. Given $\lambda \in i\mathfrak{t}_0^*$, $L$, $\pi$, $\mathfrak{q}$ as usual, put

$$S = W(\mathfrak{l}, \mathfrak{t}^c)^\theta \backslash W(\mathfrak{g}, \mathfrak{t}^c)^\theta / W(G, T^c).$$

It is useful to note that, by lemmas of Vogan and Knapp (3.12 and 4.16 of [18]) we may represent $w \in S$ by $w \in W(\mathfrak{m}, \mathfrak{t}^c)$. Since $w$ commutes with $\theta$, $w\lambda \in i\mathfrak{t}_0^*$ and $L_w$ and $\mathfrak{q}_w$ are still $\theta$-stable. The proof of Lemma 2.5 need not be changed.

Since $w \in S$ can be represented by $w \in W(\mathfrak{m}, \mathfrak{t}^c)$, the enlarged packets for $G$ and for $M$ have the same cardinality.

The proofs of Section 8 hold now with only linguistic changes, except for Lemma 8.13 which relies on 7.12. But $\pi$, initially a character of $T_n$, extends to a one-dimensional representation of $L \cap M$ by Lemma 8.13, (applied to an appropriate packet for $M'$). By inspection of formula 2.20 we see that the differential of $\pi$ is orthogonal to all roots of $L$. Then $\pi$ is trivial on $L_{der}^0 \cap M$. Since $L = (L \cap M) \cdot L_{der}^0$, $\pi$ extends uniquely to a one-dimensional representation of $L$.

To finish the proof of 2.21 we need only prove 9.9 without using Section 7 (although a generalization of the results of 7 is possible). Now this is essentially checking that the sign with which a standard module occurs in the right hand side of 9.1 is the same as that with which it occurs in the left

hand side. Explicitly, if

$$\tilde{X}' = \mathrm{Ind}_{M_i N_i}^{H}(\sigma \otimes 1)$$

is the sum over a pseudo $L$-packet of standard modules occurring in the resolution of $\Theta'$ (notation 8.11) then

$$\tilde{X} = \mathrm{Tran}_{H}^{G} \tilde{X}' = \mathrm{Ind}_{M_i N_i}^{G} (\mathrm{Tran}_{M_i'}^{M_i} \sigma \otimes 1)$$

by 9 of [1] (or 4.7), and so

$$\tilde{X} = \mathrm{Ind}_{M_i N_i}^{G} \left( \sum_{w} \varepsilon_i(\pi_w^i) \pi_w \otimes 1 \right)$$

for $\{\pi_w^i\}$ an appropriate $L$-packet of discrete series for $M_i$ and $\varepsilon_i$ defined as in 4.6 for $\zeta: {}^L M_i' \to {}^L M_i$ the restriction of

$$\zeta: {}^L H \to {}^L G.$$

Hence the sign with which $X$, a typical standard module occurring in the resolution of $\Theta$, occurs in the right hand side of 9.1 is

$$(-1)^{\ell(X')} \varepsilon_i(\pi_w)$$

if $X = \mathrm{Ind}_{M_i N_i}^{G}(\pi_w \otimes 1)$, and in the left hand side,

$$(-1)^{\ell(X)} \varepsilon(\pi_w) = \varepsilon_\kappa (-1)^{\ell(X)} \kappa(w)$$

here $\ell(X')$ is the length of any standard module contained in the pseudo-$L$-packet $\tilde{X}'$, and $\ell(X)$ is the length of $X$ (which does not depend on $w$).

So we wish to prove

$$(-1)^{\ell(X')} \varepsilon_i(\pi_w) = (-1)^{\ell(X)} \varepsilon(\pi_w) \tag{10.1}$$

assuming 10.1's holding for $M'$ and $M$, i.e.

$$(-1)^{\ell(Y',M')} \varepsilon_i^M(\pi_w) = (-1)^{\ell(Y,M)} \varepsilon^M(\pi_w) \tag{10.2}$$

with the obvious notation, where $Y$ and $Y'$ are standard modules from analogous packets for $M$ and $M'$. (We may assume 10.2 holds since it

follows from 9.1 which was proved under the hypotheses that $G$ and $H$ possessed (relative) discrete series.)

We reduce 10.1 to 10.2 in two steps. First, we note that 3.4.2, 4.4.10, and the fourth and fifth displayed formulas on p. 398 of [14] relate $\varepsilon_\kappa$ for $\mathrm{Tran}_H^G$ and $\varepsilon_\kappa^M$ for $\mathrm{Tran}_{M'}^M$ by a constant independent of the pseudo-$L$-packet. This completes the first part of the reduction.

The second part of the reduction, which we only sketch, compares the lengths of standard modules in the respective resolutions of derived functor modules, between $G$ and $M$.

For example, if $G$ is complex, $A(\lambda)$ is stable, and endoscopy is essentially trivial: every module in the resolution of $A(\lambda)$ is of course stable. Then the main theorem of this paper is also trivial.

Calculations with the lengths of standard modules parametrized by Langlands data are more easily done in terms of the $\mathbf{Z}/2\mathbf{Z}$-character data parameterization of standard modules (2.2.3, 2.2.4, and 6.7.3 of [16]), which involve positive systems of roots with involution $\theta$. But then this calculation can be split into that involving imaginary roots and complex roots separately. So it is easily seen that, in general, if $X_M$ is a standard module for $M$ such that $X = \mathrm{Ind}_{MN}^G(X_M \otimes 1)$ occurs in the resolution of $A(\lambda, \pi)$, the data for those modules in the resolution, not accounted for by the calculation involving imaginary roots only, come from choices of positive root system differing from that of $X$ (or that induced from some $X_M^2$ in the block of $X_M$) only by complex roots of $L$. The differences in length also depend only on these complex roots of $L$. But these are shared with $M'$ and $H$ and play the same role there. We omit the details. This proves 10.1 and thus accomplishes the reduction to the equal rank case.

For the purpose of this paper, *Tran* has been formally extended from stable tempered distributions to non-tempered stable sums of standard modules by 4.7. We now connect this with the matching of orbital integrals. Assume we are in the setting of Theorem 2.2.1, and let $\Theta = Tran_H^G(\Theta')$, notation as in 8.11 and 8.12.

**THEOREM** 10.3: *If $f \in C_c^\infty(G)$ and $f' \in C_c^\infty(H)$ satisfy 3.1.1 of [14] (i.e. "match" orbital integrals) then*

$$\Theta(f) \;=\; \Theta'(f').$$

*Proof:* One can proceed from 4.8, and then argue to analytically continue 4.4.6 of [14] or use Harish-Chandra's transformation of descent as follows.

Write $\Theta = \Sigma \pm \mathrm{Ind}_{M_t N_t}^G(\sigma_i \otimes 1)$ for $\sigma_i$ a $\kappa$-unstable sum of tensor

products of discrete series with (possible non-unitary) characters of the split component of $M_i$, and $\Theta' = \Sigma \pm \text{Ind}_{M_i'N_i'}^{H}(\sigma_i' \otimes 1)$. By descent as in 3.4.2. of [14], we have $f_{M_i'}' \in C_c^{\infty}(M_i')$ and $f_{M_i} \in C_c^{\infty}(M_i)$ such that we have matching of orbital integrals for the transfer from $M_i'$ to $M_i$. A well known result of Harish-Chandra yields

$$\text{Ind}_{M_iN_i}^{G}(\sigma_i \otimes 1)\,(f) = \sigma_i(f_{M_i}).$$

But $(\text{Tran}_{M_i'}^{M_i}(\sigma_i'))(f_{M_i}) = \alpha_i(f_{M_i}')$ by duality of matching of orbital integrals to transfer.                                                                                     Q.E.D.

PROPOSITION 10.4: $\Theta$ *is the unique distribution satisfying theorem 10.3.*

*Proof:* Let $K$ be the maximal compact subgroup of $G$ compatible with $\theta$, and let $K_H$ be the analogous maximal compact subgroup of $H$. Let $C_c^{\infty}(G, K)$ denote the space of smooth, compactly supported, $K$-finite functions on $G$, similarly for $C_c^{\infty}(H, K_H)$. Clozel and Delorme have shown, [5], that if $f \in C_c^{\infty}(G, K)$ then there exists an $f' \in C_c^{\infty}(H, K_H)$ satisfying the hypotheses (matching of orbital integrals as in 3.1.1 of [14]). But $C_c^{\infty}(G, K)$ is dense in $C_c^{\infty}(G)$ so $\Theta$ must be unique.                                             Q.E.D.

## Acknowledgements

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