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## Principal polarizations of Prym–Tjurin varieties

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### Introduction

Jacobians of curves and Prym varieties of curves with an involution without fixed points are the best studied principally polarized abelian varieties. Tjurin introduced in [13] a very natural generalization of Prym varieties. Instead of considering automorphisms of curves  $C$ , he focused on symmetric correspondences  $D \in \text{Div}(C \times C)$ , which induce endomorphisms of the Jacobians  $i: J(C) \rightarrow J(C)$  satisfying a quadratic equation  $(i - 1)(i + q - 1) = 0$  for some interger  $q \geq 2$ . The generalized Prym (or Prym–Tjurin) variety is defined by  $P(C, i) = (1 - i)J(C)$  (see Section 1 for details). This situation appears in the study of curves of “lines” on simply connected threefolds whose Kodaira dimension is equal to  $-\infty$ . In this case the incidence correspondence usually satisfies the condition above, and the Prym–Tjurin variety is isomorphic to the intermediate Jacobian [3, 7, 11, 13].

The starting point in the study of Prym varieties is the remark that if  $\sigma: X \rightarrow X$  is an involution without fixed points and  $\Theta$  is a divisor on  $J(X)$  inducing the canonical polarization, then  $\Theta \cdot P(X, \sigma) \equiv 2\Xi$  where  $\Xi$  induces a principal polarization on  $P(X, \sigma)$ . Bloch and Murre were the first to study the problem: when does an equality

$$\Theta \cdot P(C, i) \equiv q\Xi \tag{*}$$

hold for some divisor  $\Xi$  inducing a principal polarization on the Prym–Tjurin variety  $P(C, i)$ . They considered the case that  $C$  is a curve of “lines” on some threefold and found a criterion in terms of the Abel-Jacobi map, which is defined only in this case [3]. Recently G. Welters proved that every principally polarized Abelian variety is isomorphic to some principally polarized Prym–Tjurin variety [16]. This result suggests a natural approach to the understanding of the geometry of principally polarized abelian varieties, namely a study of particular correspondences on algebraic curves satisfying the conditions above.

The main result of our paper (Theorem 3.1a) is that (\*) holds if the correspondence is effective and without fixed points (i.e.,  $\text{Supp}(D) \cap \Delta = \emptyset$  where  $\Delta \subset C \times C$  is the diagonal). This result is used in an essential way in the calculation of the intermediate Jacobians of threefolds with a pencil of Del Pezzo surfaces [7]. Another result (Theorem 3.1b, c) is that for any theta divisor  $\Xi$  (defined above on  $P(C, i)$  up to translation) there exists a theta divisor  $\Theta \subset J(C)$  such that  $\Xi$  is contained in the locus of singular points of multiplicity  $q$  in  $\Theta$  and  $\Theta \cdot P(C, i) = q\Xi$ . Using this we obtain more precise information about  $\Xi$  (Theorem 5.1). We also prove a criterion characterizing Prym–Tjurin varieties, which generalizes Masiewicki’s criterion for Prym varieties [14] and is analogous to Matsusaka’s criterion for Jacobi varieties [15].

Finally we give two examples of Prym–Tjurin varieties and discuss their connection with the intermediate Jacobians of threefolds with a pencil of Del Pezzo surfaces. The first one is a generalization of a construction of Recillas [12]; in this case  $P(C, i)$  is isomorphic to a certain Jacobian. The second one is closely related to the tetragonal construction of Donagi and its generalization by Beauville [2, 5]. Here  $P(C, i)$  is isomorphic to a certain Prym variety.

Throughout the paper we assume that the ground field  $k$  is algebraically closed. Restrictions on  $\text{char}(k)$  will be later imposed.

**Notation and conventions**

$X^*$  – dual variety of an Abelian variety  $X$ ;  ${}^1h: Y^* \rightarrow X^*$  – dual homomorphism of  $h: X \rightarrow Y$ ;  $n_x: X \rightarrow X$  – the homomorphism  $n_x(y) = ny$ ;  $t_u: X \rightarrow X$  – the translation map  $t_u(y) = y + u$ ;  $\Theta_{-u} = \Theta - u = t_u^* \Theta$  – translate of a divisor  $\Theta \in \text{Div}(X)$  by a point  $-u$ ;  $\Theta_u^-$  – divisor equal to  $((-1)^* \Theta) + u$ ;  $\lambda_\Theta: X \rightarrow X^*$  – polarization induced by a divisor  $\Theta$ ,  $\lambda_\Theta(u) = \text{Cl}(\Theta_{-u} - \Theta)$ ;  $T_x Y$  – Zariski tangent space to  $Y$  at  $x \in Y$ ;  $\sim$  – linear equivalence;  $\equiv$  – algebraic equivalence;  $\equiv_{\text{num}}$  – numerical equivalence;  $\text{Pic } C$  – group of linear equivalence classes of divisors on a curve  $C$ ;  ${}^1D$  – the pull-back of  $D \in \text{Div}(C_1 \times C_2)$  by the map  $\sigma: C_2 \times C_1 \rightarrow C_1 \times C_2$ , where  $\sigma(x, y) = (y, x)$ ;  $D(z)$  – the linear equivalence class of divisors on  $C_2$  equal to  $\pi_{2*}(\pi_1^*(z) \cdot D)$ , where  $z$  is a linear equivalence class on a curve  $C_1$ .

Let  $z$  be a divisor on  $C_1$ . Let  $u_1, u_2$  be divisors and  $v_1, v_2$  be linear equivalence classes of divisors on  $C_2$ . The equality  $u_1 + v_1 = u_2 + v_2 + D(z)$  means  $\text{Cl}(u_1) + v_1 = \text{Cl}(u_2) + v_2 + D(\text{Cl}(z))$ .

If  $L$  is a divisor on  $C$ , then  $H^0(C, L)$  is the same as  $H^0(C, \mathcal{L}_C(L))$ . If  $\mathcal{L} = \text{Cl}(L)$ , then  $h^0(C, \mathcal{L})$  is the same as  $\dim_k H^0(C, \mathcal{L}_C(L))$ .

The expression “Let  $\Theta$  be a theta divisor on  $X$ ” means that we have already defined a principal polarization on  $X$  and  $\Theta$  is one of the effective divisors which induce it.

**1. Definition and some properties of Prym-Tjurin varieties**

(1.1) Let  $C$  be a nonsingular (possibly reducible) curve,  $C = \bigcup_{s=1}^n C_s$ ,  $C_s$  its irreducible components. The Jacobian of  $C$ ,  $J = J(C)$ , is isomorphic to the product  $J(C_1) \times \cdots \times J(C_n)$ . We shall identify  $J_s = J(C_s)$  with the subvariety  $0 \times \cdots \times 0 \times J_s \times 0 \times \cdots \times 0$  of  $J$ . Let  $D \in \text{Div}(C \times C) = \bigoplus_{s,t=1}^n \text{Div}(C_s \times C_t)$ ,  $D = \sum_{s,t=1}^n D_{st}$ . The correspondences  $D_{st}$  define endomorphisms  $i_{st}: J_s \rightarrow J_t$  by  $i_{st}(z) = D_{st}(z)$ . The matrix  $(i_{st})$  defines an endomorphism  $i: J \rightarrow J$  as follows: if  $z = (z_1, \dots, z_n)$ , then  $i(z)_t = \sum_{s=1}^n i_{st}(z_s)$ .

(1.2) *Assumption I.* Let the endomorphism  $i$  satisfy the equation

$$(i - 1)(i + q - 1) = i^2 + (q - 2)i + (q - 1) = 0$$

where  $q$  is an integer  $\geq 2$  such that  $\text{char}(k)$  does not divide  $q$ .

(1.3) **DEFINITION.** The Abelian variety

$$P(C, i) = (1 - i)J(C)$$

is called Prym-Tjurin variety.

Endomorphisms satisfying (1.2) were studied by Tjurin [13], Bloch and Murre [3] and the author [6]. We shall briefly recall some definitions and results of [3]. We introduce the Abelian subvariety  $B = (i + q - 1)J$  and denote  $P(C, i)$  by  $P$ . The following properties are proved in [3]:  $B + P = J$ ;  $B \cap P \subset J_q$  (points of order  $q$ );  $i|_B = 1_B$ ;  $i|_P = (1 - q)1_P$ : Denote by  $p_s: J \rightarrow J_s$  the projection. Let  $\Theta_s$  be a theta divisor on  $J_s$ , which determines the canonical polarization of  $J_s$  ( $\Theta_s$  is defined up to translation). The divisor  $\Theta = \sum_{s=1}^n p_s^* \Theta_s$  induces the canonical principal polarization of  $J$ . We introduce the maps  $\tau: B \times P \rightarrow J, j_b: B \rightarrow J, j_p: P \rightarrow J$ , where  $\tau(x, y) = x + y$ , and  $j_b, j_p$  are the embeddings. The pull-back of the polarization  $\lambda_\Theta$  by  $\tau, j_b$  and  $j_p$  is defined by

$$\lambda_{\tau^* \Theta} = \tau^* \circ \lambda_\Theta \circ \tau: B \times P \rightarrow B^* \times P^*,$$

$$\lambda_{j_b^* \Theta} = j_b^* \circ \lambda_\Theta \circ j_b: B \rightarrow B^*,$$

$$\lambda_{j_p^* \Theta} = j_p^* \circ \lambda_\Theta \circ j_p: P \rightarrow P^*.$$

Recall that the Rosati anti-involution of  $\text{End}(J)$  is defined by  $\sigma \mapsto \sigma' = \lambda_{\Theta}^{-1} \circ {}^t\sigma \circ \lambda_{\Theta}$  [10].

(1.4) *Assumption II.* For any  $s, t, 1 \leq s, t \leq n$  there exist divisors  $a_{st} \in \text{Div}(C_s), b_{st} \in \text{Div}(C_t)$  such that

$${}^tD_{ts} \sim D_{st} + a_{st} \times C_t + C_s \times b_{st}$$

PROPOSITION 1.5. *Assume  $i$  is an endomorphism of  $J$  satisfying (1.2). Then there exist divisors  $D_{st} \in \text{Div}(C_s \times C_t), s, t = 1, \dots, n$  which induce  $i$  as in (1.1) and the following conditions are equivalent:*

- i) (1.4)
- ii)  $\tau^*\Theta \equiv j_b^*\Theta \times P + B \times j_p^*\Theta$  (cf. [3])
- iii)  $(1 - i)' \circ (i + q - 1) = 0$
- iv)  $i = i'$ .

*Proof.* The endomorphism  $i$  is given by a matrix  $(i_{st})$  of homomorphisms  $i_{st}: J(C_s) \rightarrow J(C_t)$ . The first statement of (1.5) and the equivalence i)  $\Leftrightarrow$  iv) follow from the well known facts that any homomorphism between two Jacobians of complete, irreducible, nonsingular curves  $h: J(X) \rightarrow J(Y)$  is induced by some correspondence  $Z \in \text{Div}(X \times Y)$ , that  $Z'$  and  $Z$  induce the same homomorphism  $h$  if and only if  $Z' \sim Z + a \times Y + X \times b$  for some divisors  $a \in \text{Div}(X), b \in \text{Div}(Y)$  and that the dual homomorphism  ${}^t h: J(Y) \rightarrow J(X)$  is induced by  ${}^t Z$  (see [9] pages 155, 126, 127).

ii)  $\Leftrightarrow$  iii): We have the decomposition

$$\lambda_{\tau^*\Theta} = \left\| \begin{array}{cc} \lambda_{j_b^*\Theta} & \gamma \\ \delta & \lambda_{j_p^*\Theta} \end{array} \right\|$$

Now, ii) is equivalent to  $\gamma = 0 = \delta$  which in turn is equivalent to  $(i + q - 1)' \circ (1 - i) = (1 - i)' \circ (i + q - 1) = 0$

iv)  $\Leftrightarrow$  iii): This is obvious.

iii)  $\Leftrightarrow$  iv) (the argument that follows is suggested to the author by the referee): The equality  $(1 - i)'(i + q - 1) = 0$  implies  $1 - i' = 0$  on  $B$ . Condition iii) implies  $(i + q - 1)'(1 - i) = 0$ , so  $i' + q - 1 = 0$  on  $P$ . Thus  $i' = 1$  on  $B$  and  $i' = 1 - q$  on  $P$ , therefore  $i' = i$ . Q.E.D.

PROPOSITION 1.6. *Let  $D \in \text{Div}(C \times C)$  satisfy (1.2) and (1.4). Then there exist principally polarized Abelian varieties  $(P_0, \Xi), (X, \Theta_X)$  and*

homomorphisms  $\sigma, j, \mu, \sigma_1, j_1, \mu_1$

$$\begin{array}{ccccc}
 X & \xleftarrow{\sigma_1} & J & \xleftarrow{\sigma} & P_0 \\
 \mu_1 \downarrow & & \parallel & & \downarrow \mu \\
 B & \xleftarrow{j_b} & J & \xleftarrow{j_p} & P \\
 & i+q-1 & & 1-i & 
 \end{array}$$

satisfying the following properties:

- a)  $j^*\Theta \equiv q\Xi$  and  $j_1^*\Theta \equiv q\Theta_X$
- b) the following equalities hold:

$$j_p \circ \mu = j; j \circ \sigma = 1 - i; \sigma \circ j = q_{P_0}; \sigma \circ i = (1 - q)\sigma;$$

$$j_b \circ \mu_1 = j_1; j_1 \circ \sigma_1 = i + q - 1; \sigma_1 \circ j_1 = q_X; \sigma_1 \circ i = \sigma_1.$$

REMARK. The choice of  $P_0$  and  $X$  is not unique in general (see (1.6.4)).

*Proof of (1.6).* We shall prove the existence of  $P_0, \Xi, \sigma, j$  and  $\mu$ . The part concerning  $X$  can be proved by similar arguments.

We first sketch the proof in case  $k = \mathbb{C}$ . Let  $J = V_J/\Lambda_J, P = V_P/\Lambda_P$  and let  $E_P$  be the restriction on  $V_P$  of the canonical Riemann form of  $J(C)$  (i.e. that induced by the intersection form  $-(\ , \ ) : H_1(C, \mathbb{Z}) \times H_1(C, \mathbb{Z}) \rightarrow \mathbb{Z}$ ). Let  $\Lambda_P^*$  be the dual lattice of  $\Lambda_P$ . Then  $q\Lambda_P^* \subset \Lambda_P$ . The existence of  $P_0, j, \mu$  satisfying (1.6) is equivalent to the existence of a lattice  $\Lambda_0$  such that  $\Lambda_0 \subset \Lambda_P$  and the restriction of  $E' = (1/q)E_P$  on  $\Lambda_0$  is unimodular. These lattices are in one-to-one correspondence with those for which  $q\Lambda_P^* \subset \Lambda_0 \subset \Lambda_P$  and  $\Lambda_0/q\Lambda_P^*$  is a maximal isotropic subgroup of  $\Lambda_P/q\Lambda_P^*$  with respect to the skew-symmetric bihomomorphism  $\exp(2\pi i E')$ . The homomorphism  $\sigma$  is that induced by  $1 - i_*$ . This is well-defined since  $(1 - i_*)\Lambda_J \subset q\Lambda_P^*$ .

Let us consider now the general case. Let  $L \in j_P^* \text{Cl}(\Theta)$  and  $\varrho = \lambda_L : P \rightarrow P^*$  be the induced polarization. It is proved in [3] that  $\text{Ker } \varrho \subset P_q$ . Let us consider the map  $\varphi : P^* \rightarrow P$  defined as follows: if  $y \in P^*$  and  $y = \varrho(x)$ , then  $\varphi(y) = qx$ . This is well-defined since  $\text{Ker } \varrho \subset P_q$ .

(1.6.1) LEMMA. *The map  $\varphi$  is a regular homomorphism and  $\varphi \circ \varrho = q_P, \varrho \circ \varphi = q_{P^*}$ .*

*Proof.* Let  $\pi_1, \pi_2$  be the projections of  $P \times P^*$  onto the corresponding factors and  $\Gamma \subset P \times P^*$  be the graph of  $\varrho$ . Let  $\Gamma_1 = (q_P \times 1_{P^*})\Gamma$ . The

morphism  $\pi_2|_{\Gamma}: \Gamma_1 \rightarrow P^*$  is an isomorphism since  $\text{Ker } \varrho \subset P_q$ ,  $\pi_2|_{\Gamma}$  is separable and  $\pi_2|_{\Gamma} = (\pi_2|_{\Gamma_1}) \cdot (q_P \times 1_{P^*})$ . By definition

$$\varphi = \pi_1 \circ (\pi_2|_{\Gamma_1})^{-1}. \quad \text{Q.E.D.}$$

Let  $v: P' \rightarrow P$  be an isogeny and  $\varrho' = {}'v \circ \varrho \circ v$  be the induced polarization. If  $\text{Ker } \varrho' \subset P'_q$ , then one can define  $\varphi': P'^* \rightarrow P'$  as above.

(1.6.2) LEMMA. *Ker  $\varrho' \subset P'_q$  if and only if  $\varphi = v \circ h \circ {}'v$  for some homomorphism  $h: P'^* \rightarrow P'$ . Moreover  $h = \varphi'$ . There exists a principal polarization  $\lambda': P' \rightarrow P'^*$  such that  $\varrho' = q\lambda'$  if and only if  $\text{Ker } \varrho' \subset P'_q$  and  $\varphi'$  is an isomorphism.*

*Proof.* If  $\text{Ker } \varrho' \subset P'_q$ , then  $\varphi = v \circ \varphi' \circ {}'v$  by the definition of  $\varphi$ . Let  $\varphi = v \circ h \circ {}'v$ . We claim that  $h \circ \varrho' = q_P$ . Indeed

$$\varrho v h \varrho' = \varrho v h {}'v \varrho v = \varrho \varphi \varrho v = q \circ \varrho v = \varrho v q_P$$

Hence  $\text{Im}(h\varrho' - q_P) \subset \text{Ker } \varrho v$ , hence  $h\varrho' = q_P$ . It is clear now that  $\text{Ker } \varrho' \subset P'_q$ . To prove the last statement of (1.6.2) note that if  $\varphi'$  is an isomorphism we can put  $\lambda' = \varphi'^{-1}$ , and by definition  $\varrho' = q\lambda'$ . Hence  $\text{Ker } \varrho' = P'_q$ , so  $\lambda'$  is equal to some principal polarization [10].

(1.6.3) LEMMA. *There exists a divisor  $M$  on  $P^*$  such that  $\varphi = \lambda_M$  via the isomorphism  $P^{**} \cong P$ .*

*Proof.* Consider the divisor  $qL$  on  $P$ . We have  $\lambda_{qL} = q\lambda_L = \lambda_L \circ q_P = \lambda_L \circ \varphi \circ \lambda_L$  (1.6.1)  $= {}'\lambda_L \circ \varphi \circ \lambda_L$  via the identification  $P^{**} = P$  [10]. Consider  $K(qL) = \text{Ker } \lambda_{qL}$  and the skew-symmetric bihomomorphism  $e^{qL}: K(qL) \times K(qL) \rightarrow k^*$  (ibid. §23). The required divisor  $M$  exists if  $\text{Ker } \lambda_L \subset K(qL)$  is isotropic. If  $x, y \in \text{Ker } \lambda_L \subset P_q$ , then

$$e^{qL}(x, y) = e^L(x, qy) = 1 \quad (\text{ibid.}) \quad \text{Q.E.D.}$$

We can construct  $P_0, \mu, j$  in the following way. Choose  $H \subset K(M) = \text{Ker } \lambda_M$  to be a maximal isotropic subgroup with respect to  $e^M$ . Let  $P_0^* = P^*/H$  and  ${}'\mu: P^* \rightarrow P_0^*$  be the quotient map. Now, let  $P_0 = P_0^{**}, \mu$  be the dual of  ${}'\mu$  and  $j = j_p \circ \mu$ . It is proved in ([10], §23) that there exists a principal polarization  $\varphi_0: P_0^* \xrightarrow{\sim} P_0$  such that  $\varphi = \lambda_M = \mu \circ \varphi_0 \circ {}'\mu$ . From (1.6.2) it

follows that  $\varphi_0^{-1} = \lambda_{\Xi}$ , where  $\lambda_{\Xi}$  is a principal polarization of  $P_0$  and  $j^*\Theta = \mu^* \circ j_p^* \Theta = \mu^* L \equiv q\Xi$ . It remains to construct  $\sigma$ . Let  $\sigma' = 'j_p \circ \lambda_{\Theta}: J \rightarrow P^*$ .

*Claim.*  $\varphi \circ \sigma' = 1 - i$

We have:  $q\varphi\sigma' = q_{P^*} \circ \sigma' = q'j_p \lambda_{\Theta}$ ;

$$\begin{aligned} q(1 - i) &= 'j_p \lambda_{\Theta}(1 - i) = 'j_p '(1 - i)\lambda_{\Theta} \text{ (since } i' = i) \\ &= '(1 - i)j_p \lambda_{\Theta} = q'j_p \lambda_{\Theta}. \end{aligned}$$

Hence  $\text{Im}((1 - i) - \varphi \circ \sigma') \subset \text{Ker } q$ , hence  $1 - i = \varphi \circ \sigma'$ .

Put  $\sigma = \varphi_0' \mu \sigma': J \rightarrow P_0$ . Now, let us verify the properties stated in b). Using the claim we obtain  $j \circ \sigma = 1 - i$ . We have  $j(\sigma(B)) = (1 - i)(B) = 0$ , hence  $\sigma(B) = 0$  since  $j$  has a finite kernel. It follows that  $\sigma \circ i = (1 - q)\sigma$ . The map  $\sigma$  is epimorphic. Let  $y \in P_0$  and let  $y = \sigma(x)$  for some  $x \in P$ . Then

$$(\sigma \circ j)(y) = (\sigma \circ j \circ \sigma)(x) = (\sigma \circ (1 - i))(x) = q\sigma(x) = qy$$

We thus obtain the equality  $\sigma \circ j = q_{P_0}$ . Q.E.D.

(1.6.4) REMARKS. Lemma 1.6.2 show that if  $(P', \Xi')$  is a principally polarized Abelian variety and if  $\mu: P' \rightarrow P$  is a homomorphism such that  $\mu^*j_p^*\Theta \equiv q\Xi'$ , then up to isomorphism  $P'$  can be obtained from  $P^*$  by taking the quotient modulo a maximal isotropic subgroup of  $K(M)$  as it is described above. Moreover it is easily seen that homomorphisms  $\sigma: J \rightarrow P', j: P' \rightarrow J$  satisfying (1.6b) are uniquely defined by  $\mu$  and coincide up to isomorphism with those defined above for the quotients of  $P^*$ .

**PROPOSITION 1.7.** *Let  $\Xi$  and  $\Theta_x$  be theta divisors on  $P_0$  (resp.  $X$ ) inducing the principal polarizations introduced in (1.6). Then there exists a theta divisor  $\Theta$  on  $J$  such that*

$$q\Theta \sim \sigma_1^*\Theta_x + \sigma^*\Xi \tag{1}$$

*Proof.* Let  $\Theta'$  be a theta divisor on  $J$ . According to (1.6a)

$$\sigma_1^*j_1^*\Theta' + \sigma^*j^*\Theta' \equiv q(\sigma_1^*\Theta_x + \sigma^*\Xi)$$



The left-hand side is equal to  $(i + q - 1)^*\Theta' + (1 - i)^*\Theta'$ . Let  $\delta = (i + q - 1, 1 - i): J \rightarrow B \times P$ . Then

$$\begin{aligned} (i + q - 1)^*\Theta' + (1 - i)^*\Theta' &= \delta^*(j_b^*\Theta' \times P + B \times j_p^*\Theta') \\ &\equiv \delta^*\tau^*\Theta' \text{ (according to (1.5ii))} \\ &= q_j^*\Theta' \equiv q^2\Theta' \end{aligned}$$

This proves the equality  $q\Theta' \equiv \sigma_1^*\Theta_X + \sigma^*\Xi$ . The composed map  $q_{J^*} \circ \lambda_\Theta: J \rightarrow J^*$  is an epimorphism, hence every divisor on  $J$  which is algebraically equivalent to 0 is linearly equivalent to  $q\Theta'_{-x} - q\Theta'$  for some  $x \in J$ . Hence  $\sigma_1^*\Theta_X + \sigma^*\Xi - q\Theta' \sim q\Theta'_{-x} - q\Theta'$  for some  $x \in J$ . Let  $\Theta = \Theta'_{-x}$ .

Q.E.D.

## 2. The inversion theorem and its corollaries

(2.1) Let us fix the divisors  $\Theta$ ,  $\Theta_X$  and  $\Xi$  satisfying (1.7). It is clear that  $\Theta = \sum_{s=1}^n p_s^*\Theta_s$  where the  $\Theta_s$  are theta divisors on  $J_s$ . By definition  $\Theta_s = 0$  if  $g(C_s) = 0$  and  $\Theta_s = W_{g_s-1}(C_s) - v_s$  for some  $v_s \in \text{Pic}^{g_s-1}C_s$  if  $g(C_s) \geq 1$ , where  $W_{g_s-1}(C_s)$  denotes the subvariety of  $\text{Pic}^{g_s-1}C_s$  consisting of elements  $\mathcal{L}_s$  with  $h^0(C_s, \mathcal{L}_s) \geq 1$ . Let  $a_s \in C_s$ ,  $s = 1, \dots, n$  and  $\alpha: C \rightarrow J$  be the following map: if  $x \in C_s$ , then  $\alpha(x) = \text{Cl}(x - a_s) \in J_s \subset J$ . We shall denote  $\alpha|_{C_s}$  by  $\alpha_s$ . The next proposition follows easily from the definitions (cf. [9] p. 32).

**PROPOSITION 2.2.** *Let  $u \in J$  and  $\zeta_s$  be divisors of the classes  $\alpha_s^* \text{Cl}(\Theta_u^-)$ . Then  $\deg \zeta_s = g(C_s)$  and*

$$u = \sum_{s=1}^n (\zeta_s - a_s - v_s)$$

(2.3) The map  $j$  is a sum of the maps  $j_s = p_s \circ j: P_0 \rightarrow J_s$ . Let us denote by  $\phi: C \rightarrow P_0$  the composition  $\sigma \circ \alpha$  and by  $\phi_s$  the restriction of  $\phi$  on  $C_s$ . Note that if  $z \in J_s$ , then  $(j_s \circ \sigma)(z) = (1 - i_{ss})(z)$ .

**PROPOSITION 2.4.** *There exists elements  $\kappa_s, {}^t\kappa_s \in \text{Pic } C_s$ ,  $s = 1, \dots, n$ , depending only on  $\Xi$  such that the following holds. If  $e \in P_0$  and*

$\xi_s \in \phi_s^* \text{Cl}(\Xi_e^-)$ , then

$$j(e) = \sum_{s=1}^n (\xi_s + D_{ss}(a_s) - a_s - \kappa_s)$$

$$j(e) = \sum_{s=1}^n (\xi_s + {}^tD_{ss}(a_s) - a_s - {}^t\kappa_s)$$

*Proof.* Let  $e' \in P_0$  and  $qe' = e$ . Let  $\zeta_s \in \alpha_s^* \text{Cl}(\Theta_{j(e')}^-)$  and  $\xi'_s \in (\sigma_1 \circ \alpha_s)^* \text{Cl}(\Theta_{\bar{X}}^-)$ .

*Claim.*  $q\zeta_s \sim \xi'_s + \xi_s$

Denote by  $\beta_s$  the map  $t_{-j(e')} \circ \alpha_s: C_s \rightarrow J$ . If  $x \in C_s$  we have

$$(\sigma \circ \beta_s)(x) = \sigma(\alpha_s(x) - j(e')) = \phi_s(x) - e$$

$$(\sigma_1 \circ \beta_s)(x) = \sigma_1(\alpha_s(x) - j(e')) = (\sigma_1 \circ \alpha_s)(x).$$

To prove the claim pull back (1) by  $\beta_s \circ (-1_J)$ .

The 1.eq. classes of  $\xi'_s$  for  $s = 1, \dots, n$  do not depend on  $e$ . According to (2.2) and the claim

$$\begin{aligned} j(e) &= qj(e') = \sum_{s=1}^n (q\zeta_s - qa_s - qv_s) \\ &= \sum_{s=1}^n (\xi_s - \lambda_s(a_1, \dots, a_n)) \end{aligned} \tag{2}$$

where  $\lambda_s(a_1, \dots, a_n) \in \text{Pic } C_s, s = 1, \dots, n$ , and do not depend on  $e$ . Let us fix new points  $b_s \in C_s$  and denote by  $\psi_s: C_s \rightarrow P_0$  the map  $\psi_s(x) = \sigma(x - b_s)$ . Obviously  $\phi_s(x) = \psi_s(x) - \sigma(a_s - b_s)$  and

$$\text{Cl}(\xi_s) = \phi_s^* \text{Cl}(\Xi_e^-) = \psi_s^* \text{Cl}(\Xi_{e+\sigma(a_s-b_s)}^-)$$

The equality (2) tells us that

$$j_s(e + \sigma(a_s - b_s)) = \xi_s - \lambda_s(b_1, \dots, b_n) \tag{3}$$

Since  $(j_s \circ \sigma)(a_s - b_s) = (1 - i_{ss})(a_s - b_s)$  and  $i'_{ss} = i_{ss}$  we get

$$\begin{aligned} j_s(e) &= \xi_s + D_{ss}(a_s) - a_s - \kappa_s \\ &= \xi_s + {}'D_{ss}(a_s) - a_s - {}'\kappa_s \end{aligned} \tag{4}$$

for some elements  $\kappa_s, {}'\kappa_s \in \text{Pic } C_s$ . The last statement to be proved is that  $\kappa_s, {}'\kappa_s$  depend only on  $\Xi$ . From (4) follows that  $\kappa_s, {}'\kappa_s$  could depend only on  $\Xi, e$  and  $a_s$ . However the expression for  $\kappa_s, {}'\kappa_s$  which one gets from (3) shows that  $\kappa_s, {}'\kappa_s$  do not depend on  $e$  and  $a_s, s = 1, \dots, n$ . Q.E.D.

Let  $\Delta_s \subset C_s \times C_s$  be the diagonal,  $F_s = \text{Cl}(D_{ss}) \cdot \Delta_s$  and  $K_{C_s}$  be the canonical class of  $C_s$ .

**PROPOSITION 2.5.** *Let  $\Xi$  be a symmetric theta divisor on  $P_0$ . Then the elements  $\kappa_s, {}'\kappa_s$  defined in (2.4) satisfy the equalities  $\kappa_s + {}'\kappa_s = K_{C_s} + F_s, s = 1, \dots, n$ .*

*Proof.* A particular case of (2.5) is proved in [6]. We shall only sketch the proof, referring for details to [6]. We have defined maps  $j_s: P_0 \rightarrow J_s, \phi_s: C_s \rightarrow P_0$  such that  $j_s \circ \phi_s(x) = (x - a_s) - i_{ss}(x - a_s)$ . Let  $\Delta_s^*$  and  $D_{ss}^*$  denote the divisorial correspondences

$$\begin{aligned} \Delta_s^* &= \text{Cl}(\Delta_s - a_s \times C_s - C_s \times a_s) \\ D_{ss}^* &= \text{Cl}(D_{ss} - {}'D_{ss}(a_s) \times C_s - C_s \times D_{ss}(a_s)), \end{aligned}$$

$\delta_s$  be the divisorial correspondence of Poincaré on  $C_s \times J_s$  normalized by the condition  $\delta_s|_{a_s \times J_s} = 0$ , and  $\delta'_s = (id_{C_s} \times j_s)^* \delta_s$ . The following equality holds (ibid. p. 208)

$$(id_{C_s} \times \phi_s)^* \delta'_s = \Delta_s^* - D_{ss}^* \tag{5}$$

If  $m: P_0 \times P_0 \rightarrow P_0$  is the sum map,  $p_1, p_2$  are the projection maps and  $\Sigma = m^* \Xi - p_1^* \Xi - p_2^* \Xi$  then (ibid. p. 209)

$$(\phi_s \times 1_{P_0})^* \Sigma = -\delta'_s \tag{6}$$

Combining (5) and (6) we get

$$(\phi_s \times \phi_s)^* \Sigma = D_{ss}^* - \Delta_s^* \tag{7}$$

The equality  $\kappa_s + {}^t\kappa_s = K_{C_s} + F_s$  can be proved by restricting (7) on the diagonal  $\Delta_s$  (ibid. pp. 210, 211). We would only mention that the equalities

$$\kappa_s = \phi_s^* \text{Cl}(\Xi) + D_{ss}(a_s) - a_s, \quad {}^t\kappa_s = \phi_s^* \text{Cl}(\Xi) + {}^tD_{ss}(a_s) - a_s,$$

which have to be used, follow from (2.4) if we put  $e = 0$ . Q.E.D.

**COROLLARY 2.6.** *If  $D_{ss} \sim {}^tD_{ss}$ , then  $2\kappa_s = K_{C_s} + F_s$ . If in addition  $F_s = 0$ , then  $\kappa_s$  is a theta characteristic.*

### 3. The main theorem

**THEOREM 3.1.** *Let  $D = \sum_{s,t=1}^n D_{st} \in \text{Div}(C \times C)$  be a correspondence satisfying (1.2) and (1.4). Assume that for any  $s = 1, \dots, n$ : 1)  $D_{ss} \geq 0$ ; 2)  $\deg D_{ss}(x_s) = \deg {}^tD_{ss}(x_s)$  for any  $x_s \in C_s$ ; 3)  $D_{ss}$  is without fixed points (i.e.  $\text{Supp } D_{ss} \cap \Delta_s = \emptyset$ ). Then*

- a) *There exists a principal polarization  $\lambda_{\Xi}$  of  $P(C, i)$  such that the restriction on  $P(C, i)$  of the canonical polarization  $\lambda_{\Theta}$  of  $J(C)$  is equal to  $\lambda_{q\Xi}$  (equivalently  $j_p^* \Theta \equiv q\Xi$ ).*
- b) *Let  $\Xi$  be an arbitrary theta divisor inducing the principal polarization of  $P(C, i)$  obtained in a) Let  $\kappa_s$  be the elements defined in (2.4). Then  $\deg \kappa_s = g_s - 1$ , for any  $s = 1, \dots, n$ , and if*

$$\Theta = \sum_{s=1}^n p_s^*(W_{g_s-1}(C_s) - \kappa_s) \tag{8}$$

(cf. (2.1)) then  $P(C, i) \not\subset \text{Supp } \Theta$  and

$$\Theta \cdot P(C, i) = q\Xi$$

- c) *Assume  $\text{char}(k) = 0$ . If  $e \in P(C, i)$  and*

$$e = \sum_{s=1}^n (\mathcal{L}_s - \kappa_s),$$

where  $\mathcal{L}_s \in \text{Pic}^{g_s-1} C_s$ , then

$$e \notin \text{Supp } \Xi \Leftrightarrow \sum_{s=1}^n h^0(C_s, \mathcal{L}_s) = 0$$

$$e \in \text{Supp } \Xi \Leftrightarrow \sum_{s=1}^n h^0(C_s, \mathcal{L}_s) \geq q$$

*Proof.* First note that if we prove b), c) for some  $\Xi$  then obviously b), c) would hold for all the translations of  $\Xi$ . So in b), c) we can choose  $\Xi$  to be a symmetric theta divisor.

The idea of the proof of a) is to prove first an analogue of b) for the principally polarized abelian variety  $P_0$ . So let  $\Xi$  be a symmetric theta divisor of  $P_0$ . The condition  $\text{Supp } D_s \cap \Delta_s = \emptyset$  implies  $F_s = \text{Cl}(D_{ss}) \cdot \Delta_s = 0$ . From (2.5) it follows that  $\kappa_s + {}^t\kappa_s = K_{C_s}$ . The assumption  $\deg D_{ss}(a_s) = \deg {}^tD_{ss}(a_s)$  gives  $\deg \kappa_s = \deg {}^t\kappa_s$ , hence  $\deg \kappa_s = g_s - 1$  for  $s = 1, \dots, n$ . Note that if for any  $s$   $\phi_s(C_s) \not\subset \text{Supp } \Xi_e$ , then in (2.4) we can choose  $\xi_s$  to be the effective divisors  $\phi_s^* \Xi_e$ . Let  $\Theta$  be the theta divisor defined by (8).

LEMMA 3.2.  $j^{-1}(\text{Supp } \Theta) \subset \text{Supp } \Xi$ .

*Proof.* Let  $e \in P_0 \setminus \text{Supp } \Xi$ . We have to prove that  $j(e) \notin \text{Supp } \Theta = \bigcup_{s=1}^n p_s^* \Theta_s$ , where  $\Theta_s = W_{g_s-1}(C_s) - \kappa_s$ . Let  $a_s \in C_s$ ,  $s = 1, \dots, n$ , be arbitrary points. They define maps  $\phi_s$  (2.3) and we claim that for any  $s$ ,  $\phi_s(C_s) \not\subset \text{Supp } \Xi_e$ . Indeed,  $\phi_s(a_s) = 0 \notin \text{Supp } \Xi_e$  since  $\Xi$  is symmetric. This argument shows moreover that  $a_s \notin \text{Supp } \xi_s$ , where  $\xi_s = \phi_s^* \Xi_e$ . According to (2.4)

$$j(e) = \sum_{s=1}^n (\xi_s + D_{ss}(a_s) - a_s - \kappa_s)$$

Let  $L_s = \xi_s + D_{ss}(a_s) - a_s$ . The divisors  $\xi_s$  and  $D_{ss}(a_s)$  are effective and we have mentioned that  $a_s \notin \text{Supp } (\xi_s + D_{ss}(a_s))$ . Hence  $h^0(C_s, L_s) = h^0(C_s, L_s + a_s) - 1$ . Suppose  $h^0(C_s, L_s) = r_s > 0$  for some  $s$ . We have  $\deg L_s = \deg \kappa_s = g_s - 1$ . By Riemann-Roch

$$h^0(C_s, L_s + a_s) - h^0(C_s, K_{C_s} - L_s - a_s) = 1$$

Hence  $h^0(C_s, K_{C_s} - L_s - a_s) = r_s$ . Therefore  $a_s$  is a base point of the linear system  $|K_{C_s} - L_s|$ . According to (2.4)  $\kappa_s$  is independent of  $a_s$ . From the equality  $j_s(e) = L_s - \kappa_s$  it follows that  $\text{Cl}(L_s)$  is independent of  $a_s$  as well. Hence  $|K_{C_s} - L_s|$  has infinitely many base points – a contradiction. Therefore  $h^0(C_s, L_s) = 0$  for any  $s = 1, \dots, n$ . Q.E.D.

LEMMA 3.3.  $j^*\Theta = q\Xi$ .

*Proof.* The divisor  $\Xi$  induces a principal polarization of  $P_0$ , hence if  $\Xi_\alpha$ ,  $\alpha = 1, \dots, m$ , are the irreducible components of  $\Xi$  then  $\Xi = \sum_{\alpha=1}^m \Xi_\alpha$  and  $\Xi_\alpha$  are linearly independent in  $\text{NS}(P_0) \otimes_{\mathbb{Z}} \mathbb{Q}$  [4]. We have proved in (3.2) that  $j^*\Theta = \sum_{\alpha=1}^m v_\alpha \Xi_\alpha$ , where  $v_\alpha \in \mathbb{Z}$  and  $v_\alpha \geq 0$ . From (1.6)  $j^*\Theta \equiv q\Xi = \sum_{\alpha=1}^m q\Xi_\alpha$ . Hence  $q = v_\alpha$  for all  $\alpha = 1, \dots, m$ .

*Proof of (3.1a, b).* By definition  $j = j_p \circ \mu$  (1.6). The divisor  $j_p^*\Theta$  is well-defined since  $j_p(P) = j(P_0) \not\subset \text{Supp } \Theta$  by (3.2). Let  $j_p^*\Theta = \sum_{\beta=1}^{m'} r_\beta \Xi'_\beta$  be the decomposition of  $j_p^*\Theta$ . The map  $\mu: P_0 \rightarrow P$  is étale ( $\text{char}(k) \nmid q$ ), hence for any  $\beta$ ,  $\mu^*\Xi'_\beta = \sum_{\gamma=1}^{m'_\beta} T_{\beta\gamma}$ , where  $T_{\beta\gamma}$ ,  $\gamma = 1, \dots, m'_\beta$ , are the irreducible components of  $\mu^*\Xi'_\beta$ . Therefore

$$j^*\Theta = \mu^*j_p^*\Theta = \sum_{\beta,\gamma} r_\beta T_{\beta\gamma}$$

where  $T_{\beta\gamma}$  are irreducible and  $T_{\beta\gamma} = T_{\beta'\gamma'}$  if and only if  $\beta = \beta'$  &  $\gamma = \gamma'$ . By (3.3) we get  $r_\beta = q$  for every  $\beta$ . Therefore  $j_p^*\Theta = q(\sum_{\beta=1}^{m'} \Xi'_\beta)$ . Part a) of (3.1) follows now from [3] (p. 86). In particular  $P_0 = P$ . Part b) of (3.1) follows from a) and (3.3). Q.E.D.

#### 4. Proof of Theorem 3.1 c)

First we should like to include a comment concerning the restrictions made on the ground field  $k$ . Theorem 3.1 c) itself, as well as Propositions (4.2) and (4.5), are stated for  $k$  of characteristic zero. However, the only real use of this assumption occurs in the proof of Lemma 4.5.4, while all the remaining arguments are valid under the weaker condition that  $\text{char}(k)$  does not divide  $q$ .

(4.1) In view of (3.1 a) we can identify  $P_0$  with  $P$  and  $j_p$  with  $j$ . Let  $\Xi'$  be an irreducible component of  $\Xi$ , let  $e \in \Xi'$  and let  $j(e) = \sum_{i=1}^n (L_i - \kappa_i)$ .

Throughout this section, for every  $s = 1, \dots, n$  we denote by  $L_s$  a divisor  $\xi_s + D_{ss}(a_s) - a_s$ , where  $\xi_s \in \phi_s^* \text{Cl}(\Xi_e)$ . If  $\xi_s$  is not fixed by the context we choose it arbitrarily. According to (2.4)  $\mathcal{L}_s = \text{Cl}(L_s)$ . Assume  $p_s^* \Theta_s \supset \Xi'$  Equivalently for every  $e \in \Xi'$ ,  $h^0(C_s, L_s) = d_s \geq 1$ . Let  $e$  be a sufficiently general nonsingular point of  $\Xi'$ , let  $u_1, \dots, u_{d_s}$  be a basis of  $H^0(C_s, L_s)$  and let  $w_1, \dots, w_{d_s}$  be a basis of  $H^0(C_s, \omega_{C_s}(-L_s))$ . As usual when we say that a sufficiently general point of some irreducible variety satisfies some property this means that every point of certain dense open subset satisfies the property. In particular we assume that  $d_s$  is the lowest possible value of  $h^0(C_s, L_s)$  for  $e \in \Xi'$ . The translation of  $T_e \Xi'$  to  $0 \in P$  defines a hyperplane in  $T_0 P$ . Let  $h \in (T_0 P)^*$  vanish on it and  $h \neq 0$ . The regular differential forms  $u_\mu \otimes w_\nu$  correspond to linear forms on  $T_0 J_s$  via the natural isomorphism  $\alpha_s^*: (T_0 J_s)^* \rightarrow H^0(C_s, \omega_{C_s})$ .

**PROPOSITION 4.2.** *Let the assumptions be as in (3.1) and let  $\text{char}(k) = 0$ . Suppose  $\Xi'$  is an irreducible component of  $\Xi$  and let  $p_s^* \Theta_s \supset \Xi'$  for some  $s$ . Then for every point  $e$  of a certain dense open subset of  $\Xi'$  the following properties hold:*

- a) *for every  $\mu, \nu = 1, \dots, d_s$  the equality  $p_s^*(u_\mu \otimes w_\nu)|_{T_0 P} = \alpha_{\mu\nu} h$  holds, where  $\alpha_{\mu\nu} \in k$ ;*
- b)  $\det(\alpha_{\mu\nu})_{\mu, \nu=1}^{d_s} \neq 0$ .

Before giving the proof of (4.2) we shall deduce (3.1 c) from it. Let  $\Xi'$  be an arbitrary irreducible component of  $\Xi$ , let  $e$  be a sufficiently general point of  $\Xi'$  and let  $Q_1, \dots, Q_n$  be nonzero polynomials such that  $Q_i = 0$  is the equation of the tangent cone of  $p_i^* \Theta_i$  at  $e$ . If  $p_i^* \Theta_i \not\supset \Xi'$  for some  $t$  we put  $Q_t = 1$ . If  $p_s^* \Theta_s \supset \Xi'$  then according to the theorem of Riemann-Kempf [8]  $Q_s = c \cdot \det(p_s^*(u_\mu \otimes w_\nu))_{\mu, \nu=1}^{d_s}$ , where  $c \in k$  and  $c \neq 0$ . From (4.2) follows that  $Q_s|_{T_0 P} = c_s \cdot h^{d_s}$ , where  $c_s \neq 0$ . Hence  $Q_1 \dots Q_n|_{T_0 P} = c' \cdot h^{d_1 + \dots + d_n}$ , where  $c' \neq 0$ . According to (3.1 b)  $\Theta \cdot P = q\Xi$  therefore  $d_1 + \dots + d_n = q$ . The inequality  $\sum_{s=1}^n h^0(C_s, L_s) \geq q$  holds for every  $e \in \Xi'$ , hence for every  $e \in \Xi$ , because of the semicontinuity theorem.

(4.3) *Proof of (4.2 a).* Let  $e$  be a sufficiently general point of  $\Xi'$ . The assumption  $h^0(C_s, L_s) \geq 1$  and the equality  $\text{deg}(L_s) = g_s - 1$  (cf. 3.1 b) imply  $C_s \not\cong \mathbb{P}^1$ . By the semicontinuity theorem  $p_s(\Xi') + \kappa_s \subset W_{g_s-1}^{d_s-1}(C_s)$ . Hence  $p_s^*(T_e \Xi')$  is contained in  $T_{p_s(e)}(W_{g_s-1}^{d_s-1} - \kappa_s)$ . The translation of the last vector space to 0 is defined by the simultaneous vanishing of  $u_\mu \otimes w_\nu$ ,  $\mu, \nu = 1, \dots, d_s$  [1]. Hence  $p_s^*(u_\mu \otimes w_\nu)|_{T_0 P} = \alpha_{\mu\nu} h$  since  $e$  is a nonsingular point of  $\Xi'$ . Proposition 4.2 a) is proved.

(4.4) Assume  $\det(\alpha_{\mu\nu}) = 0$ . This is equivalent to the existence of a nonzero section  $u = \sum_{\mu=1}^{d_s} \beta_{\mu} u_{\mu} \in H^0(C_s, L_s)$  such that for every  $w_v, p_s^*(u \otimes w_v)|_{T_0P} = 0$ . Let us recall that  $1 - i = 0$  on  $B$  and  $1 - i = q_P$  on  $P$  (cf. (1.3)). So  $(1 - i_*)(T_0P) = T_0P$  since  $\text{char}(k) \nmid q$ . Hence  $p_s^*(u \otimes w_v)|_{T_0P} = 0$  if and only if  $i^*p_s^*(u \otimes w_v) = p_s^*(u \otimes w_v)$ . Let  $M \in |L_s|$  be the divisor of the section  $u$ . The condition  $i^*p_s^*(u \otimes w_v) = p_s^*(u \otimes w_v)$  for any  $v = 1, \dots, d_s$  is equivalent to the following: any  $\omega \in H^0(C_s, \omega_{C_s})$  such that  $(\omega) \geq M$  satisfies  $i^*p_s^*\omega = p_s^*\omega$  (here  $(\omega) = (\omega)_0 - (\omega)_{\infty}$  is the divisor of  $\omega$ ). The last equality is equivalent to  $i_s^*\omega = 0$  if  $t \neq s$  and  $i_{ss}^*\omega = \omega$ . So (4.2 b) is a consequence of the following proposition.

**PROPOSITION 4.5.** *Let the assumptions be as in (4.2). If  $e$  is a sufficiently general point of  $\Xi'$ , then for every  $M \in |L_s|$  there exists  $\omega \in H^0(C_s, \omega_{C_s})$  such that  $(\omega) \geq M$  and  $i_{ss}^*\omega \neq \omega$ .*

*Proof.* The proof of this proposition occupies the rest of the section. From (2.5) and the condition  $\text{Supp}(D_{ss}) \cap \Delta_s = \emptyset$  follows that  $\kappa_s + {}^t\kappa_s = K_{C_s}$ . Therefore

$$j_s(-e) = -L_s + \kappa_s = (K_{C_s} - L_s) - {}^t\kappa_s.$$

By (2.4) we obtain

$$\phi_s^* \text{Cl}(\Xi_{-e}) + {}^tD_{ss}(a_s) - a_s = K_{C_s} - L_s$$

for every point  $a_s \in C_s$ . The idea of the proof of (4.5) is for every  $M \in |L_s|$  to find  $\omega \in H^0(C_s, \omega_{C_s})$  such that  $(\omega) = M + \phi_s^*\Xi_{-e} + {}^tD_{ss}(a_s) - a_s$  and then prove that for sufficiently general  $e \in \Xi'$  and  $a_s \in C_s$  this differential form satisfies  $i_{ss}^*\omega \neq \omega$ .

(4.5.1) **LEMMA.** *Let  $\Xi = \Xi_1 + \dots + \Xi_m$  where  $\Xi_{\mu}$  are the irreducible components of  $\Xi$ . Suppose  $\Xi' = \Xi_{\alpha}$ . Let  $P_{\alpha}$  be the corresponding direct summand of  $P$  and let  $\pi_{\alpha}: P \rightarrow P_{\alpha}$  be the corresponding projection. Then the image of the composition  $\pi_{\alpha} \circ \phi_s: C_s \rightarrow P \rightarrow P_{\alpha}$  is 1-dimensional.*

*Proof.* Let  $\{1, \dots, m\} = B \cup \Gamma$ , where  $\beta \in B$  iff  $\dim \text{Im}(\pi_{\beta} \circ \phi_s) = 1$  and  $\gamma \in \Gamma$  iff  $\pi_{\gamma} \circ \phi_s(C_s) = 0$ . Suppose  $\alpha \in \Gamma$ . Let  $e \in \Xi_{\alpha} \setminus \bigcup_{\mu \neq \alpha} \Xi_{\mu}$ . If  $\gamma \in \Gamma$  then  $\phi_s^* \text{Cl}(\Xi_{\gamma} + e) = 0$ . If  $\beta \in B$  then  $\phi_s(C_s) \not\subset \Xi_{\beta} + e$  since  $\Xi_{\beta}$  is symmetric and  $e \notin \Xi_{\beta}$  by assumption. Hence  $\zeta_{s\beta} = \phi_s^*(\Xi_{\beta} + e)$  is a well-defined effective



divisor and  $a_s \notin \text{Supp } \xi_{s\beta}$ . The divisor  $\xi_s = \sum_{\beta \in B} \xi_{s\beta}$  belongs to  $\phi_s^* \text{Cl}(\Xi_e)$  hence by (2.4)

$$j_s(e) = \xi_s + D_{ss}(a_s) - a_s - \kappa_s = L_s - \kappa_s$$

Now,  $h^0(C_s, L_s) = h^0(C_s, L_s + a_s) - 1$  since  $\xi_s$  is effective and  $a_s \notin \text{Supp } \xi_s$ . The proof of (3.2) implies  $h^0(C_s, L_s) = 0$ . Thus  $j_s(e) = L_s - \kappa_s \notin \Theta_s$ . This contradicts the assumption  $p_s^* \Theta_s \supset \Xi'$ .

(4.5.2) LEMMA. *There exists a closed subset  $W \subsetneq \Xi'$  such that if  $e \in \Xi' \setminus W$ , then  $\phi_s(C_s) \not\subset \text{Supp}(\Xi_{-e})$ .*

*Proof.* We keep the notation of (4.5.1). First let  $W' = \Xi_\alpha \cap \bigcup_{\mu \neq \alpha} \Xi_\mu$ . If  $e \notin W'$ , then  $\phi_s(C_s) \not\subset \Xi_\mu - e$  for any  $\mu \neq \alpha$ . Suppose  $\phi_s(C_s) \subset \Xi'_{-e}$ . Then  $e \in \bigcap_{x \in C_s} (\Xi' - \phi_s(x)) = W''$ . According to (4.5.1)  $\text{codim}_p W'' \geq 2$ . Put  $W = W' \cup W''$ . Q.E.D.

Let  $e \in \Xi' \setminus W$ . Then  $\xi_s = \phi_s^* \Xi_{-e}$  is a well-defined effective divisor and  $a_s \in \text{Supp}(\xi_s)$  since  $\phi_s(a_s) = 0 \in \Xi_{-e}$ . We have already seen that there exists a differential form  $\omega$  such that  $(\omega) = M + (\xi_s - a_s) + {}'D_{ss}(a_s)$ . In particular  $\omega \in H^0(C_s, \omega_C)$ .

(4.5.3) LEMMA. *Let  $X$  be an irreducible, nonsingular nonrational curve. Let  $Y \in \text{Div}(X \times X)$  be an effective divisor, let  $\sigma: J(X) \rightarrow J(X)$  be the corresponding endomorphism and let  $\sigma^*: H^0(X, \Omega_X^1) \rightarrow H^0(X, \Omega_X^1)$  be the linear map induced by  $\sigma$  via the natural identification  $H^0(JX, \Omega_{JX}^1) \simeq H^0(X, \Omega_X^1)$ . There is a finite set  $S \subset X$  such that  $z \times X \not\subset \text{Supp}(Y)$  for every  $z \in X \setminus S$  and the following holds: if  $\omega \in H^0(X, \Omega_X^1)$  and  $(\omega) \geq Y(z)$ , then  $(\sigma^* \omega) \geq z$ .*

*Proof.* The cases  $g(X) = 1$  and  $\sigma = 0$  are trivial so we can assume that  $g(X) \geq 2$  and that the divisors  $Y(z)$  do not all belong to one linear system. We identify  $\mathbb{P}^{g-1}$  with  $\mathbb{P}(H^0(X, \Omega_X^1)^*)$ , denote by  $\lambda: X \rightarrow \mathbb{P}^{g-1}$  the canonical map and by  $\sigma_*$  the dual of  $\sigma^*$ . Let  $x_0$  be a point of  $X$ , let  $\alpha: X \rightarrow J(X)$  be the map  $\alpha(x) = \text{Cl}(x - x_0)$  and let  $r = \text{deg } Y(x_0)$ . If the map  $\sigma \circ \alpha: X \rightarrow (\sigma \circ \alpha)(X) \subset J(X)$  is inseparable, then  $\sigma^* = 0$  and the lemma is trivial. So we can assume that  $\sigma \circ \alpha$  is a separable map. Consider the composition

$$\gamma = \alpha_r \circ \beta: X \rightarrow X^{(r)} \rightarrow J(X)$$

where  $\beta(z) = Y(z)$ ,  $\alpha_r(x_1 + \dots + x_r) = \text{Cl}(x_1 + \dots + x_r - rx_0)$ . Since  $\gamma = \sigma \circ \alpha$  and  $\sigma \circ \alpha$  is separable there exists an open dense subset  $U \subset X$  such that for any  $z \in U$  we have  $z \times X \not\subset \text{Supp}(Y)$  and  $\gamma$  has nonvanishing derivative at every point  $z \in U$ . Let  $S = X \setminus U$ . Translating the line  $\gamma_* T_z X$  to 0 we obtain a regular map  $\gamma_*: U \rightarrow \mathbb{P}^{g-1}$ . Obviously  $\gamma_* = \sigma_* \circ \lambda$ . Let  $\omega \in H^0(X, \Omega_X^1)$  and let  $H$  be the hyperplane in  $\mathbb{P}^{g-1}$  corresponding to  $\omega$ . The well-known calculation of the derivative of  $\alpha$ , shows that if  $(\omega) \geq Y(z)$  then the translation to 0 of  $\alpha_* T_{Y(z)} X^{(r)}$  is contained in  $H$ . Therefore  $\gamma_*(z) \in H$ , or equivalently  $\lambda(z) \in \sigma^* H$ . Q.E.D.

We prove below that if  $e \in \Xi'$  and  $a_s \in C_s$  are sufficiently general then the divisor  $\xi_s = \phi_s^* \Xi_{-e}$  is without multiple points. Assuming this let us finish the proof of (4.5). Consider  $\omega \in H^0(C_s, \omega_{C_s})$  such that  $(\omega) = M + (\xi_s - a_s) + 'D_{ss}(a_s)$ , where  $e \in \Xi'$ ,  $a_s \in C_s$  are sufficiently general and  $\xi_s = \phi_s^*(\Xi_{-e})$ . We will apply (4.5.3) for  $X = C_s$ ,  $Y = 'D_{ss}$ ,  $\sigma = i_{ss}$ . We can assume that  $a_s \notin (\text{Supp } M) \cup S$  where  $S \subset C_s$  is the finite set of points defined in (4.5.3). By (4.5.4)  $a_s \notin \text{Supp}(\xi_s - a_s)$ . Furthermore  $a_s \notin \text{Supp } 'D_{ss}(a_s)$  since  $D_{ss}$  is without fixed points by assumption. So  $a_s \notin \text{Supp}(\omega)$ . From (4.5.3) we obtain  $(i_{ss}^* \omega) \geq a_s$ , or equivalently  $a_s \in \text{Supp}(i_{ss}^* \omega)$ . Consequently  $i_{ss}^* \omega \neq \omega$ . Proposition 4.5 is proved.

(4.5.4) LEMMA. Assume  $\text{char}(k) = 0$ . Let  $e \in \Xi'$  and  $a_s \in C_s$  be sufficiently general points of  $\Xi'$  (resp.  $C_s$ ). Then the effective divisor  $\xi_s = \phi_s^* \Xi_{-e}$  is without multiple points.

*Proof.* Let us choose and fix a point  $b \in C_s$  and let  $\psi: C_s \rightarrow P$  be the map  $\psi(x) = x - b - i(x - b)$ . We have

$$\phi_s(x) = x - a_s - i(x - a_s) = \psi(x) - \psi(a_s).$$

Let  $f = e - \psi(a_s)$ . Then  $\xi_s = \phi_s^* \Xi_{-e} = \psi^* \Xi_{-f}$ . We shall use the notation of (4.5.1) and its proof. Let  $\mu_\beta: \Xi_\beta \times C_s \rightarrow P$  be the map  $\mu_\beta(e, x) = e - \psi(x)$ . From the definition of  $\mathbf{B}$  obviously follows

*Claim I.* For any  $\beta \in \mathbf{B}$ ,  $\mu_\beta$  is an epimorphism.

From (4.5.1) and Claim I it follows that the map  $\mu': \Xi' \times C_s \rightarrow P$  defined by  $\mu'(e, a_s) = e - \psi(a_s)$  is an epimorphism. By (4.5.2) we can choose an open subset  $U \subset P$  such that for any  $f \in U$

1.  $\psi(C_s) \cap (\Xi_\gamma - f) = \emptyset$  if  $\gamma \in \Gamma$  and  $\psi(C_s) \not\subset (\Xi_\beta - f)$  if  $\beta \in \mathbf{B}$ ;

2. For any  $\beta \in \mathbf{B}$  and any  $(e, x) \in \mu_\beta^{-1}(f)$ , the point  $e$  is nonsingular in  $\Xi_\beta$  and the map  $\mu_\beta$  is étale at  $(e, x)$  (here we use the assumption  $\text{char}(k) = 0$ ).

These properties imply  $\psi(C_s) \not\subset \Xi_{-f}$  and  $\xi_s = \psi^*\Xi_{-f} = \sum_{\beta \in \mathbf{B}} \xi_{s\beta}$ , where  $\xi_{s\beta} = \psi^*(\Xi_\beta - f)$ .

*Claim II.* If  $f \in U$ , then for any  $\beta \in \mathbf{B}$ ,  $\xi_{s\beta}$  is without multiple points.

Let  $x \in \text{Supp } \xi_{s\beta}$  and  $e = \psi(x) + f$ . From 2 it follows that  $T_{e-f}(\Xi_\beta - f)$  and  $\psi_* T_x C_s$  span  $T_{e-f}P$ . Hence the multiplicity of  $x$  is equal to 1.

Let  $\Pi_\beta$  be the closure in  $\Xi_\beta \times C_s \times P$  of the graph of the map  $\mu_\beta: \mu_\beta^{-1}(U) \rightarrow U$  and  $\Gamma_\beta$  be its projection in  $C_s \times P$ . Let  $q_1, q_2$  be the projections of  $C_s \times P$  onto the corresponding factors. If  $f \in U$  the divisors  $\xi_{s\beta}$  and  $q_{1*}(q_2^*(f) \cdot \Gamma_\beta)$  are without multiple points and their supports coincide. Thus  $q_{1*}(q_2^*(f) \cdot \Gamma_\beta) = \xi_{s\beta}$ .

*Claim III.* If  $\beta, \beta' \in \mathbf{B}$  and  $\beta \neq \beta'$  then  $\Gamma_\beta \neq \Gamma_{\beta'}$ . Assume  $\Gamma_\beta = \Gamma_{\beta'}$ . Let  $e' \in \Xi_\beta$  and  $x \in C_s$  be sufficiently general points such that  $f' = e' - \psi(x) \in U$ . By definition  $(x, f') \in \Gamma_\beta$ . Since  $\Gamma_\beta = \Gamma_{\beta'}$ ,  $f' + \psi(x) = e'$  is contained in  $\Xi_{\beta'}$ . Hence  $\Xi_\beta \subset \Xi_{\beta'}$  – a contradiction.

From Claim III and the equality  $q_{1*}(q_2^*(f) \cdot \Gamma_\beta) = \xi_{s\beta}$  it follows that if  $f$  is a sufficiently general point of  $P$  then  $\text{Supp } \xi_{s\beta} \cap \text{Supp } \xi_{s\beta'} = \emptyset$  if  $\beta \neq \beta'$  and  $\beta, \beta' \in \mathbf{B}$ . This and Claim II show that  $\xi_s$  is without multiple points. Q.E.D.

Theorem 3.1 is proved.

### 5. Corollaries and applications of Theorem 3.1

In the text theorem we shall identify the effective divisors  $\Theta, \Xi$  with the corresponding subschemes of  $J$  (resp.  $P$ ). These subschemes are reduced (see [4]).

**THEOREM 5.1.** *Assume the conditions of (3.1) are satisfied and  $\text{char}(k) = 0$ . then  $\Xi \subset \text{Sing}_q \Theta$ . If  $x$  is a nonsingular point of  $\Xi$ ,  $h_x = 0$  is the equation of  $T_x \Xi$  in  $T_x P$  and  $Q_x = 0$  is the equation in  $T_x J$  of the tangent cone of  $\Theta$  at  $x$ , then  $h_x = c \cdot (Q_x|_{T_x P})^{1/q}$ , where  $c \in k$  and  $c \neq 0$ . Let  $e \in \Xi$  and  $e = \sum_{i=1}^n (L_i - \kappa_i)$  (cf. (4.1)). Then one of the following holds:*

- i)  $\sum_{i=1}^n h^0(C_i, L_i) = q$  and for any  $s$  such that  $h^0(C_s, L_s) \geq 1$  and any  $M \in |L_s|$  there is a differential form  $\omega \in H^0(C_s, \Omega_{C_s}^1)$  such that  $(\omega) \geq M$  and  $i^*p_s^*\omega \neq p_s^*\omega$ ;

- ii)  $\sum_{t=1}^n h^0(C_t, L_t) \geq q + 1$ ;
- iii)  $\sum_{t=1}^n h^0(C_t, L_t) = q$  and for any  $s$ , such that  $h^0(C_s, L_s) \geq 1$ , and any  $M \in |L_s|$  the differential forms  $\omega \in H^0(C_s, \Omega_{C_s}^1)$ , such that  $(\omega) \geq M$ , satisfy the equality  $i^*p_s^*\omega = p_s^*\omega$ .

A point  $e \in \Xi$  is nonsingular if and only if i) holds.

*Proof.* Most of (5.1) has been already proved in Sections 3 and 4. We must only prove that if  $e \in \text{Sing } \Xi$  and  $\sum_{t=1}^n h^0(C_t, L_t) = q$  then iii) holds. Let  $h^0(C_s, L_s) \geq 1$  (equivalently  $p_s^*\Theta_s \ni e$ ). We claim that  $p_s^*\Theta_s$  contains any irreducible component of  $\Xi$  which contains  $e$ . To prove this assume that  $\Xi'$  is such a component. Let  $e' \in \Xi'$  be a generic point of  $\Xi'$  and  $j(e') = \sum_{t=1}^n (L'_t - \kappa_t)$ . The point  $e$  is a specialization of  $e'$ , thus  $h^0(C_t, L_t) \leq h^0(C_t, L'_t)$  for any  $t$ . From (3.2)  $\sum_{t=1}^n h^0(C_t, L'_t) \geq q$  hence  $h^0(C_t, L'_t) = h^0(C_t, L_t)$  for any  $t$ , hence  $h^0(C_s, L'_s) \geq 1$ , equivalently  $\Xi' \subset p_s^*\Theta_s$ . Let  $d_s$  be the greatest integer such that  $p_s(\Xi') \subset W_{g_s-1}^{d_s-1} - \kappa_s$  for any irreducible component  $\Xi'$  of  $\Xi$  which contains  $e$ . The argument above shows that  $h^0(C_s, L_s) = d_s$  and  $p_s(e) \notin W_{g_s-1}^{d_s} - \kappa_s$ . So there is a neighborhood  $U$  of  $e$  in  $P$  such that  $p_s(U \cap \Xi) + \kappa_s \subset W_{g_s-1}^{d_s-1} \setminus W_{g_s-1}^{d_s}$ . The argument of (4.3) shows that  $p_s^*(u \otimes w)|_{T_0P} = 0$  for any sections  $u \in H^0(C_s, L_s)$ ,  $w \in H^0(C_s, \omega_{C_s}(-L_s))$  since  $e$  is a singular point of  $\Xi$ . This is equivalent to iii).  $\square$

(5.2) Next we shall study some universal properties of Prym-Tjurin varieties. First we recall some definitions and results from [15]. Let  $(A, \theta)$  be a principally polarized abelian variety of dimension  $d$  and let  $Z$  be a one-dimensional cycle on  $A$ . We denote by  $\alpha(Z, \theta)$  the endomorphism of  $A$  defined by

$$\alpha(Z, \theta)(a) = S(Z \cdot (\theta_{a+b} - \theta_b))$$

where  $S$  denotes summation in  $A$  and  $b$  is any point of  $A$  such that  $\theta_{a+b}, \theta_b$  intersect properly  $Z$ . This endomorphism satisfies the following properties:

- i)  $r\alpha(Z, \theta) = \alpha(rZ, \theta)$  for any  $r \in \mathbb{Z}$ ;
- ii)  $\alpha(\theta^{d-1}, \theta) = (1/d) \text{deg } \theta^d \cdot 1_A = (d - 1)! \cdot 1_A$ ;
- iii) if  $Z_1, Z_2$  are two one-dimensional cycles on  $A$ , then  $Z_1 \stackrel{\text{num}}{=} Z_2$  if and only if  $\alpha(Z_1, \theta) = \alpha(Z_2, \theta)$ .

(5.3) Let  $m: A \times A \rightarrow A$  be the sum map and let  $p_1, p_2$  be the projections of  $A \times A$  onto the corresponding factors. The Poincaré divisor  $\Sigma = m^*\theta - p_1^*\theta - p_2^*\theta$  defines the polarization  $\lambda_\theta: A \rightarrow A^*$  by

$$\lambda_\theta(x) = p_{2*}((x \times A) \cdot \Sigma) = \text{Cl } (\theta_{-x} - \theta)$$

Let  $C$  be as in (1.1). Let  $a_s \in C_s, s = 1, \dots, n$  be fixed points and let  $\alpha: C \rightarrow J$  be the Abel map (cf. (2.1)). Suppose a morphism  $\psi: C \rightarrow A$  is given. It induces a homomorphism  $\sigma: J \rightarrow A$  such that for every  $x \in C_s, (\sigma \circ \alpha)(x) = \psi(x) + v_s$ . Here  $1 \leq s \leq n$  and  $v_s \in A$  are constants. Let  $T = (\psi \times \psi)^*\Sigma$  (more precisely we choose a divisor  $T \in (\psi \times \psi)^* \text{Cl}(\Sigma)$ ). The correspondence  $T$  defines an endomorphism  $\tau: J \rightarrow J$ .

Theorem 5.4 below is due to G. Welters [16]. We give a formulation which might be useful when one deals with correspondences. We give here a proof in order to verify that the theorem is true for arbitrary characteristics of the base field as well as for reducible curves.

**THEOREM 5.4** (G. Welters). *Let  $\psi: C \rightarrow A$  be a morphism and let  $\sigma: J \rightarrow A, T \in \text{Div}(C \times C)$  and  $\tau: J \rightarrow J$  be as above. The following conditions are equivalent:*

- a)  $\psi_*(C) \equiv_{\text{num}} (q/(d-1)!) \theta^{d-1}$  for some integer  $q \geq 1$
- b)  $\sigma \circ \tau = -q\sigma$  and  $\sigma$  is epimorphic.

*If these conditions hold then*

$$\tau \circ (\tau + q_j) = 0 \tag{9}$$

*Assume a) holds for  $q \geq 2$  and let  $\text{char}(k) \nmid q$ . Let  $D = T + \Delta$ , where  $\Delta$  is the diagonal of  $C \times C$ . Then  $D$  satisfies conditions (1.2) and (1.4), the restriction  $\sigma: P(C, i) \rightarrow A$  is an isogeny and  $\sigma(B) = 0$  (see (1.3)). Furthermore if we let  $j = \lambda_0^{-1} \circ \sigma \circ \lambda_0: A \rightarrow J$ , then  $j(A) = P(C, i)$  and  $(A, \theta), \sigma, j$  satisfy the properties required in Proposition 1.6.*

*Proof.* First note that it is sufficient to consider only morphisms  $\psi$  satisfying the additional condition  $\psi(a_s) = 0$  for every  $s = 1, \dots, n$ . Indeed, if  $\psi' = \sigma \circ \alpha$  and  $T' = (\psi' \times \psi')^*\Sigma$ , then the correspondence  $T'$  induces the same endomorphism of  $J$  as  $T$  does. So we can assume that  $\psi = \sigma \circ \alpha$ . The divisor  $E = (\sigma \times \sigma)^*\Sigma$  induces the homomorphism

$$\lambda_{\sigma \circ \theta} = {}^t \sigma \circ \lambda_0 \circ \sigma: J \rightarrow J^*$$

The map  $\alpha$  induces  $\alpha^*: J^* = \text{Pic}^0 J \rightarrow \text{Pic}^0 C = J$ . The endomorphism  $\tau: J \rightarrow J$  obtained from  $T = (\alpha \times \alpha)^*E$  is thus equal to  $\alpha^* \circ \lambda_{\sigma \circ \theta}$  (see [9] p. 155). Therefore

$$\tau = \alpha^* \circ {}^t \sigma \circ \lambda_0 \circ \sigma \tag{10}$$

So if  $x \in J$  then  $\tau(x) = \psi^*(\theta_{-\sigma(x)} - \theta)$ . Let  $Z = \sigma_* C$ . From the definition of  $\alpha(Z, \theta)$  (cf. (5.1)) it follows that

$$(\sigma \circ \tau)(x) = \sigma\psi^*(\theta_{-\sigma(x)} - \theta) = -\alpha(Z, \theta)(\sigma(x))$$

Thus  $\sigma \circ \tau = -\alpha(Z, \theta) \circ \sigma$ .

a)  $\Rightarrow$  b) *Proof:* We have  $Z \stackrel{\text{num}}{\cong} q/((d-1)!) \theta^{d-1}$ . Since  $\theta$  is ample the curve  $\psi(C) = \text{Supp}(Z)$  generates  $A$ . So  $\sigma$  is epimorphic. Using properties i), ii) and iii) of (5.2) we get:

$$\begin{aligned} (d-1)!(\sigma \circ \tau) &= -(d-1)!\alpha(Z, \theta) \circ \sigma \\ &= \alpha((d-1)!Z, \theta) \circ \sigma = -q\alpha(\theta^{d-1}, \theta) \\ &= -q(d-1)!\sigma \end{aligned} \tag{11}$$

Hence  $\sigma \circ \tau = -q\sigma$ .

b)  $\Rightarrow$  a) *Proof:* Using equalities (11) we get

$$\alpha((d-1)!Z, \theta) \circ \sigma = \alpha(q\theta^{d-1}, \theta) \circ \sigma$$

Since  $\sigma$  is epimorphic it follows that  $\alpha((d-1)!Z, \theta) = \alpha(q\theta^{d-1}, \theta)$ . By 5.2 iii) we obtain  $(d-1)!Z \stackrel{\text{num}}{\cong} q\theta^{d-1}$ .

*Equality (9).*  $\sigma \circ \tau = -q\sigma \Leftrightarrow \sigma \circ (\tau + q_J) = 0$ . Using (10) we obtain  $\tau(\tau + q_J) = \alpha^* \circ \sigma \circ \lambda_\theta \circ \sigma \circ (\tau + q_J) = 0$ .

If  $D = T + \Delta$  and  $i: J \rightarrow J$  is the endomorphism induced by  $D$  then  $\tau = i - 1$ . Thus the equality  $(i-1)(i+q-1) = 0$  follows from (9).

Using (2.2) one obtains the equality  $\lambda_\theta^{-1} = -\alpha^*$ . Hence  $j = -\alpha^* \circ \sigma \circ \lambda_\theta$ . From (10) it follows that

$$j \circ \sigma = -\tau = 1 - i \tag{12}$$

Let  $y \in A$  and let  $y = \sigma(x)$ . Then  $(\sigma \circ j)(y) = (\sigma \circ j \circ \sigma)(x) = -(\sigma \circ \tau)(x) = q\sigma(x) = qy$  (according to (12) and b)). Thus

$$\sigma \circ j = q_A \tag{13}$$

Now  $B = (i+q-1)J = (\tau+q)J$ . Thus  $\sigma(B) = 0$  in view of b). From (12) it follows that  $j: A \rightarrow P(C, i) = (1-i)J$  is epimorphic and from (13)

it follows that  $\text{Ker } \sigma = j(A_q)$  (points of order  $q$ ). Thus the restriction of  $\sigma$  on  $P(C, i)$  is an isogeny.

It remains to prove that the pull-back of  $\lambda_\Theta$  by  $j$  equals  $q\lambda_0$ . Indeed

$$\begin{aligned} \lambda_{j^*\Theta} &= {}'j \circ \lambda_\Theta \circ j = (-\lambda_0) \circ \sigma \circ (-\lambda_\Theta)^{-1} \circ \lambda_\Theta \circ j \\ &= \lambda_0 \circ \sigma \circ j = q\lambda_0 \quad (\text{according to (13)}). \end{aligned} \qquad \text{Q.E.D.}$$

**COROLLARY 5.5** (G. Welters). *Let  $C, D \in \text{Div}(C \times C)$  be as in (1.1). Assume (1.2) and (1.4) hold. Let  $(P_0, \Xi)$  be as in (1.6) and let  $\phi: C \rightarrow P_0$  be the morphism defined in (2.3). Then*

$$\phi_*(C) \stackrel{\text{num}}{\equiv} \frac{q}{(p-1)!} \Xi^{p-1}$$

where  $p = \dim P(C, i)$ .

*Proof.* We use the notation introduced in (2.5). An obvious prolongation of the arguments used in (2.5) shows that

$$T = (\phi \times \phi)^*\Sigma = D^* - \Delta^*$$

where

$$D^* = \sum_{r,s=1}^n D_{rs}^*, \quad \Delta^* = \sum_{s=1}^n \Delta_{ss}^*$$

and

$$D_{rs}^* = D_{rs} - {}'D_{rs}(a_s) \times C_s - C_r \times D_{rs}(a_r)$$

Therefore  $T$  induces the endomorphism  $\tau = i - 1$ . From (1.6 b) it follows that  $\sigma \circ \tau = -q\sigma$ , thus (5.4 b) holds. Q.E.D.

The following theorem is a generalization of a result of Masiewicki [14].

**THEOREM 5.6.** *Let  $C, D \in \text{Div}(C \times C)$  be as in (1.1). Let the assumption be as in (3.1). Suppose  $\text{char}(k) \nmid q$ . Let  $(A, \theta)$  be a principally polarized Abelian variety of dimension  $d$  and let  $\psi: C \rightarrow A$  be a morphism satisfying the*

following properties:

- i) there exist constants  $c_s \in A$  such that for every  $s = 1, \dots, n$  and every  $x \in C_s$

$$\psi(D(x)) = (1 - q)\psi(x) + c_s$$

(here  $\psi$  is extended to divisors by linearity);

- ii)  $\psi_*(C) \stackrel{\text{num}}{=} q/(d - 1)! \theta^{d-1}$ .

Then  $(A, \theta)$  is isomorphic to a direct summand of  $(P(C, i), \Xi)$  ( $\Xi$  is defined in (3.1)).

*Proof.* Let  $\sigma: J \rightarrow A$  and  $j: A \rightarrow J$  be the maps introduced in (5.3), (5.4). Condition i) is equivalent to the equality  $\sigma \circ i = (1 - q)\sigma$ .

*Claim.*  $j(A) \subset P(C, i)$

By definition  $j = \lambda_{\Theta}^{-1} \circ' \sigma \circ \lambda_{\theta}$ . We have to prove that  $(i + q - 1) \circ j = 0$ . This follows from the equalities

$$\begin{aligned} (i + q - 1) \circ \lambda_{\Theta}^{-1} \circ' \sigma \circ \lambda_{\theta} &= \lambda_{\Theta}^{-1} \circ' (i + q - 1) \circ' \sigma \circ \lambda_{\theta} \text{ (since } i' = i \text{ (1.5))} \\ &= \lambda_{\Theta}^{-1} \circ' (\sigma \circ (i + q - 1)) \circ \lambda_{\theta} = 0 \end{aligned}$$

So we can define a map  $\mu: A \rightarrow P(C, i)$  such that  $j = j_p \circ \mu$  where  $j_p: P(C, i) \hookrightarrow J$  denotes the inclusion. According to (3.1 a)  $j_p^* \Theta \equiv q\Xi$ . By (5.4)  $j^* \Theta \equiv q\theta$  therefore  $\mu^* \Xi \equiv \theta$ . This shows that  $\mu$  is a polarized isomorphism of  $A$  onto a direct summand of  $P(C, i)$ .

Next we give two examples of Prym–Tjurin varieties. For sake of simplicity we shall further assume that the ground field is  $\mathbb{C}$ . We denote by  $\approx$  homological equivalence.

(5.7) Let  $X$  be an irreducible nonsingular curve,  $f: X \rightarrow \mathbb{P}^1$  be a map of degree  $d \geq 4$  and  $g_d^1$  be the linear system determined by  $f$ . Assume that any fiber of  $f$  contains at most one ramification point of multiplicity  $\leq 3$ . Consider the curve  $C \subset X^{(2)}$ ,  $C = \{(a + b): |g_d^1 - a - b| \neq \emptyset\}$ . Let  $D \in \text{Div}(C \times C)$  be the effective, reduced divisor for which  $\text{Supp } D = \{(a_1 + b_1, a_2 + b_2): |g_d^1 - a_1 - b_1 - a_2 - b_2| \neq \emptyset\}$ .

**PROPOSITION 5.8.** *The curve  $C$  is irreducible and non-singular;  $D$  satisfies the assumptions of (3.1), where  $q = d - 2$ ; and  $(P(C, i), \Xi) \cong (J(X), \Theta)$ .*



*Proof.* The nonsingularity of  $C$  is verified by easy local calculations. The irreducibility of  $X$  and the conditions imposed on  $f$  imply that the monodromy group  $H \subset S_d$  is transitive and is generated by transpositions and cycles of length 3. Consequently  $H = S_d$  or  $H = A_d$ . In both cases  $H$  acts transitively on pairs (i.e., is doubly transitive), hence  $C$  is irreducible. Let us check that  $D$  satisfies the assumptions of (3.1), where  $q = d - 2$ . By construction there exists a map  $F: C \rightarrow \mathbb{P}^1$  of degree  $d(d - 1)/2$ . Let  $G_{d(d-1)/2}^1$  be the corresponding linear system. Assume  $t \in \mathbb{P}^1$  is not a branch point of  $f$  and  $\{a_1, \dots, a_d\} = f^{-1}(t)$ . Then  $t$  is not a branch point of  $F$  as well and  $F^{-1}(t) = \{(a_j + a_k)\}_{1 \leq j < k \leq d}$ . The equality  $i^2 + (d - 4)i - (d - 3) = 0$  obviously follows from the equality

$$D(D(a_1 + a_2)) + (d - 4)D(a_1 + a_2) - (d - 3)(a_1 + a_2) = \binom{d - 3}{2} \sum_{1 \leq j < k \leq d} (a_j + a_k) \in \binom{d - 3}{2} G_{d(d-1)/2}^1$$

This holds since  $D(a_1 + a_2) = \sum_{3 \leq j < k \leq d} (a_j + a_k)$ ;

$$D(D(a_1 + a_2)) = \binom{d - 2}{2} (a_1 + a_2) + \binom{d - 3}{2} \times \sum_{j=3}^d ((a_1 + a_j) + (a_2 + a_j)) + \binom{d - 4}{2} \sum_{3 \leq j < k \leq d} (a_j + a_k).$$

If  $(x_1 + x_2, x_1 + x_2) \in D \cap \Delta$ , then  $|g_d^1 - 2x_1 - 2x_2| \neq \emptyset$  and if  $(2x, 2x) \in D \cap \Delta$ , then  $|g_d^1 - 4x| \neq \emptyset$ . Hence  $D \cap \Delta = \emptyset$  by the assumption on  $f$ .

We shall use (5.6) in order to prove the isomorphism  $(P(C, i), \Xi) \cong (J(X), \Theta)$ . Let  $x_0$  be a point on  $X$  and  $\alpha_m: X^{(m)} \rightarrow J(X)$  be the map  $\alpha_m(x_1 + \dots + x_m) = \text{Cl}(x_1 + \dots + x_m - mx_0)$ . Let  $\psi = \alpha_2|_C$ . Put  $c = (d - 3)\alpha_d f^*(t)$  ( $c$  does not depend on  $t \in \mathbb{P}^1$ ). Then

$$\psi(D(a_1 + a_2)) = (d - 3)\alpha_{d-2} \left( \sum_{j=3}^d a_j \right) = -(d - 3)\psi(a_1 + a_2) + c$$

Thus (5.6 i) is verified. The equality

$$\psi_* C \approx (d - 2)\alpha_1(X) \approx (q/(g(X) - 1)!) \Theta^{g(X)-1}$$

can be proved by the same arguments as those in [12]. By (5.6)  $(J(X), \Theta)$  is a direct summand of  $(P(C, i), \Xi)$ , so the proof will be finished if we prove that  $p = \dim P(C, i) = g(X)$ . Let  $n = \deg D(a_1 + a_2) = (d - 2)(d - 3)/2$ . According to [6] (p. 197)  $pq + n = g(C)$ . Let  $N_2$  (resp.  $N_3$ ) be the number of ramification points of  $f$  of multiplicity 2 (resp. 3). Then  $2g(X) - 2 = N_2 + 2N_3 - 2d$ . An easy calculation shows that the number of ramification points of  $F$  of multiplicity 2 (resp. 3) is  $(d - 2)N_2$  (resp.  $(d - 2)N_3$ ). Hence  $2g(C) - 2 = (d - 2)(N_2 + 2N_3) - d(d - 1)$ , hence  $g(X)q + n = g(C)$ , hence  $g(X) = p$ . Q.E.D.

(5.9) Let  $X, \tilde{X}$  be irreducible nonsingular curves,  $\pi: \tilde{X} \rightarrow X$  be a double étale covering,  $\sigma$  be the involution of  $\tilde{X}$  and let  $\Theta \subset P(\tilde{X}, \sigma)$  induce the canonical polarization. Assume there is a map  $f: X \rightarrow \mathbb{P}^1$  of degree 5 such that any fiber contains at most one ramification point of multiplicity  $\leq 3$ . Let  $g_5^1$  be the corresponding linear system. Let us consider the map  $\pi^{(5)}: \tilde{X}^{(5)} \rightarrow X^{(5)}$ . The subvariety of  $\tilde{X}^{(5)}$   $S = (\pi^{(5)})^{-1} g_5^1$  is a disjoint union of two subvarieties  $S = S_0 \cup S_1$ , such that  $S_1 = \sigma^{(5)}(S_0)$  [2]. Let  $C = S_0$ . For any point  $z \in \tilde{X}$  we shall denote  $\sigma(z)$  by  $z'$ . Let  $D \in \text{Div}(C \times C)$  be the reduced, effective divisor for which

$$\text{Supp } D = \{(x_1 + \dots + x_5, y_1 + \dots + y_5) : \pi_*(x_1 + \dots + x_5) \in g_5^1, \\ y_j = x_j \text{ for some } j, y_k = x'_k \text{ for } k \neq j\}.$$

**PROPOSITION 5.10.** *The curve  $C$  is irreducible and non-singular;  $D$  satisfies the assumptions of (3.1), where  $q = 4$ ;  $(P(C, i), \Xi) \cong (P(\tilde{X}, \sigma), \Theta)$ .*

*Proof.* The irreducibility and nonsingularity of  $C = S_0$  is proved in [2]. The map  $F = \pi^{(5)}|_C: C \rightarrow \mathbb{P}^1$  is of degree 16 and it defines a linear system  $G_{16}^1$  on  $C$ . Let  $z = (x_1 + \dots + x_5) \in C$ . Then  $D(z) = \sum_{j=1}^5 (\sigma^{(5)}(z) + x_j - x'_j)$  and  $D(D(z)) = 5z + \sum_{j \neq k} (z + x'_j + x'_k - x_j - x_k)$ . Hence  $D(D(z)) + 2D(z) - 3z = 2F^*(F(z)) \in 2G_{16}$ . Therefore  $i^2 + 2i - 3 = 0$ . If  $(z, z) \in (\text{Supp } D) \cap \Delta$ , then  $z = x_1 + x'_1 + x_3 + x'_3 + x_5$ . This is impossible since by assumption  $g_5^1$  cannot contain the divisor  $\pi_*(z) = 2\pi(x_1) + 2\pi(x_3) + \pi(x_5)$ . Hence  $(\text{Supp } D) \cap \Delta = \emptyset$ . Therefore  $D$  satisfies the assumptions of (3.1).

To prove the last statement of (5.10) we shall use (5.6). Let  $z_0 \in C$  and  $\psi: C \rightarrow P(\tilde{X}, \sigma)$  be the map  $\psi(z) = \text{Cl}(z - z_0)$ . If  $z = (x_1 + \dots + x_5)$ , then

the following equalities hold in  $J(\tilde{X})$ :

$$\psi(D(z)) = 4(x'_1 + \dots + x'_5) + (x_1 + \dots + x_5) + c_1;$$

$$\psi(D(z)) + 3\psi(z) = 4 \text{Cl}(\pi^*f^*F(z)) + c_2 = c;$$

where  $c_1, c_2$  and  $c$  are linear equivalence classes of divisors on  $\tilde{X}$  not depending on  $z$ . Obviously  $c \in P(\tilde{X}, \sigma)$  hence (5.6 i) holds. The equality

$$\psi_*C \approx (4/(g(X) - 2!))\Theta^{g(X)-2}$$

is a particular case of a general result proved in [2]. By (5.6)  $(P(\tilde{X}, \sigma), \Theta)$  is a direct summand of  $(P(C, i), \Xi)$ , so it remains to prove that  $p = \dim P(C, i) = g(X) - 1$ . This can be proved by the same argument as that at the end of the proof of (5.8). Q.E.D.

It is worthwhile to sketch the connection between these two examples and the intermediate Jacobians of threefolds with a pencil of Del Pezzo surfaces. Consider a threefold  $V$  with such a pencil  $\varphi: V \rightarrow \mathbb{P}^1$  satisfying the condition  $K_{V_t}^2 = 5$ . Denote by  $C$  the curve which parametrizes the lines of the fibers of  $\varphi$ . There exists a section of  $\varphi$   $s: \mathbb{P}^1 \rightarrow V$  and let  $\varepsilon: V' \rightarrow V$  be the blow-up of  $V$  along  $s(\mathbb{P}^1)$ . For a generic point  $t \in \mathbb{P}^1$  the fiber  $V'_t$  is an intersection of two quadrics in  $\mathbb{P}^4$  with a distinguished line  $l_0 = \sigma^{-1}s(t)$ . This line is intersected by five other lines  $l_1, \dots, l_5$  which do not intersect each other. Moving  $t \in \mathbb{P}^1$  we get a family of lines parametrized by a curve  $X$  which is a covering  $f: X \rightarrow \mathbb{P}^1$  of degree 5. Any line  $l$  of  $V'_t$  which does not intersect  $l_0$  intersects exactly two of the lines  $l_1, \dots, l_5$ . Conversely any pair  $l_j, l_k$  is intersected by exactly one line  $l$ , such that  $l \cdot l_0 = 0$ . Therefore  $C$  can be reconstructed from  $X$  by the construction of (5.7), where  $d = 5$ . Via this identification the correspondence  $D$  considered in (5.7) becomes the incidence correspondence. By [7] the intermediate Jacobian  $(J(V), \Theta_V)$  is isomorphic to  $(P(C, i), \Xi)$ . On the other hand blowing-down the surface  $W = \bigcup_{x \in X} l_x$  we get a threefold with a pencil of projective planes. Hence  $(J(V), \Theta_V) \cong (J(X), \Theta)$ . This gives another proof of (5.8) in this case.

Next, consider the case where  $V$  is a threefold with a pencil of Del Pezzo surfaces  $\varphi: V \rightarrow \mathbb{P}^1$  of degree 4 (i.e.,  $K_{V_t}^2 = 4$ ). We again denote by  $C$  the curve parametrizing the lines of the fibers of  $\varphi$ . Repeating the arguments above we get a threefold  $V'$  with a pencil of cubic surfaces  $V' \rightarrow \mathbb{P}^1$  with a distinguished line  $l_0$  in the generic fiber  $V'_t$ . There are ten other lines  $l_1, l'_1, \dots, l_5, l'_5$  which intersect  $l_0$  and which are divided into five pairs  $\{l_j, l'_j\}$ , such that  $l_j \cdot l'_j = 1$ . Moving  $t \in \mathbb{P}^1$  we get a curve  $\tilde{X}$  which parametrizes the

considered lines and which has an involution  $\sigma$  ( $\sigma(l_j) = l'_j$ ,  $\sigma(l'_j) = l_j$ ). Let  $X = \tilde{X}/\sigma$ . By construction  $X$  is a covering of  $\mathbb{P}^1$  of degree 5. Any line  $l \subset V'_i$  which does not intersect  $l_0$  intersects exactly one line of any of the pairs  $l_j$ ,  $l'_j$ ,  $j = 1, \dots, 5$ . Simple arguments, which we omit, show that  $C$  can be reconstructed from  $\tilde{X}$ ,  $X$ ,  $\sigma$  by the construction of (5.9). The correspondence  $D$  considered in (5.9) coincides with the incidence correspondence via this identification. The threefold  $V'$  has a structure of a conic bundle (project  $V'_i$  to  $\mathbb{P}^1_i$  from  $l_0$ ) and obviously  $X$  is isomorphic to the curve which parametrizes the singular fibers. The isomorphism of (5.10) reflects the fact that  $(P(C, i), \Xi) \cong (J(V), \Theta_\nu)$  (see [7]) and  $(J(V), \Theta_\nu) \cong (J(V'), \Theta_{\nu'}) \cong (P(\tilde{X}, \sigma), \Theta)$ .

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