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Higher asymptotics of the complex Monge–Ampère equation

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Introduction

The complex Monge–Ampère equation

$$J(u) = (-1)^n \det \begin{pmatrix} u & u_{\bar{j}} \\ u_i & u_{i\bar{j}} \end{pmatrix} = 1, \quad u|_{b\Omega} = 0, \quad u > 0 \text{ in } \Omega \qquad (0.1)$$

on a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ was introduced by Charles Fefferman [4] in his study of the Bergman kernel and the geometry of such domains. The importance of this equation stems from the fact that it possesses an invariance property under biholomorphic mappings which insures that its solution is a canonical, biholomorphically invariant defining function for Ω, as well from its interpretation that the Kähler metric $(\log 1/u)_{j\bar{k}}\, dz^j\, d\bar{z}^k$ has constant negative Ricci curvature. Fefferman showed that it is possible to construct locally a smooth defining function ϱ for $M = b\Omega$ which solves $J(\varrho) = 1$ to order $n + 1$. Such a solution is uniquely determined to order $n + 2$, but is arbitrary beyond that point. In [5] Fefferman showed how his approximate solution ϱ could be used to construct scalar invariants of strictly pseudoconvex hypersurfaces and applied these results to the asymptotic expansion of the Bergman kernel. His construction of invariants is limited, however, by the indeterminacy of ϱ at order $n + 2$. In this paper we show that it is possible to construct more invariants of arbitrarily high weight and involving arbitrarily high order derivatives of a defining function for M, by studying the higher asymptotics of solutions to (0.1).

The study of (0.1) was renewed by Cheng–Yau [2], who proved the existence of a unique solution $u \in C^\infty(\Omega) \cap C^{n+3/2-\varepsilon}(\bar{\Omega})$ for $\varepsilon > 0$ on a smooth bounded strictly pseudoconvex domain Ω. Lee–Melrose [8] established that the Cheng–Yau solution has an asymptotic expansion at $b\Omega$ of the form

$$u \sim \varrho \sum_{k=0}^{\infty} \eta_k (\varrho^{n+1} \log \varrho)^k, \quad \eta_k \in C^\infty(\bar{\Omega}). \qquad (0.2)$$

We study the local determination of the coefficient functions η_k by $b\Omega$. Although u, hence also η_k to infinite order at $b\Omega$, is globally uniquely determined, locally an asymptotic solution (0.2) is determined only up to the choice of one arbitrary function on $b\Omega$. The main result is Theorem 2.11, which asserts that for $k \geqslant 1$ each η_k mod $0(\varrho^{n+1})$ is independent of this choice, so is locally uniquely determined by $b\Omega$.

In order to prove Theorem 2.11 we follow the approach of Lee–Melrose and explicitly rewrite the equation $J(u) = 1$ as a totally characteristic differential equation. This greatly facilitates the analysis for u of the form (0.2). It is also possible to prove Theorem 2.11 by carrying out the analysis directly in terms of the J operator, using identities similar to those used by Fefferman in his construction of the approximate solution. Such a proof of Theorem 2.11(i) is given in [1]. The higher asymptotics of solutions of the Monge–Ampère equation have also been studied by Lee [7]. He showed that the coefficient of the first log term on the boundary is a local invariant and proved versions of Proposition 2.16, Remark 4.13(b) and a result similar to Proposition 4.1.

At the expense of some overlap with [8], our treatment is self-contained. §1 shows how the Monge–Ampère equation can be rewritten in terms of totally characteristic operators. §2 uses the result of §1 to study the asymptotics of formal solutions. §3 shows how to construct scalar invariants from the higher asymptotics. In §4, a formula for the linear part of the coefficient of the first log term on the boundary is given and it is shown that there are nonspherical strictly pseudoconvex hypersurfaces whose Monge–Ampère solutions have no log terms.

The results of this paper were announced in [6], which also contains applications of these results to the asymptotic expansion of the Bergman kernel and derives further information about the invariants from invariant-theoretic considerations.

1. The Monge–Ampère equation as a totally characteristic operator

We begin by showing how the Monge–Ampère equation $J(u) = 1$ may be written as a nonlinear totally characteristic differential equation. This is a slightly different treatment of results of [8]. Recall [9] that a linear totally characteristic differential operator P on a manifold with boundary is an operator which can be written as a polynomial in smooth vector fields tangent to the boundary. Thus in a coordinate system (x, ϱ), where $\varrho \geqslant 0$ is a defining function for the boundary and x restricts to a coordinate system on the boundary, P is of the form $Pu = \Sigma_{|\alpha|+j \leqslant m} p_{\alpha,j} \partial_x^\alpha (\varrho \partial_\varrho)^j u$ for some C^∞

coefficients $p_{\alpha,j}(x, \varrho)$. For our purposes a nonlinear totally characteristic operator will be a polynomial with no constant term in linear totally characteristic operators; thus a finite sum of terms of the form $P_1 u \cdot \cdots \cdot P_r u$ for linear totally characteristic P_i, $r \geqslant 1$. The nonlinear totally characteristic operators form a commutative ring under multiplication, and also a module over C^∞.

Let M be a strictly pseudoconvex hypersurface in $\mathbb{C}^n$ and V be a neighborhood of M which is split into two pieces by M. If Ω is the strictly pseudoconvex side of V, then $\Omega \cup M$ is a manifold with boundary M. We are interested in the equation $J(u) = 1$ in Ω, with $u > 0$ in Ω, $u = 0$ on M. Elementary operations with determinants show that

$$J(u) = (-1)^n u \det\left(u_{j\bar{k}} - \frac{u_j u_{\bar{k}}}{u}\right) = u^{n+1} \det\left(\log \frac{1}{u}\right)_{j\bar{k}}, \tag{1.1}$$

which can be interpreted in terms of the volume form of the Kähler metric $g_{j\bar{k}} = (\log 1/u)_{j\bar{k}}$. We thus study briefly the geometry of such a metric.

Fix a C^∞ defining function ϱ for M, with $\varrho > 0$ in Ω. We assume that Ω is chosen small enough that at all points of Ω $d\varrho \neq 0$ and $\varrho_{j\bar{k}}$ is negative definite on $T_b^{1,0} = \ker \partial\varrho \subset T^{1,0}\Omega$. As in [8], since $\varrho_{j\bar{k}}$ is nondegenerate on $T_b^{1,0}$, $T_b^{1,0}$ has a well defined one dimensional orthogonal complement in $T^{1,0}\Omega$ with respect to $\varrho_{j\bar{k}}$. Consequently there is a smooth $(1, 0)$ vector field ξ on $\bar{\Omega}$ uniquely determined by the conditions

$$\xi \lrcorner \partial\bar{\partial}\varrho = 0 \bmod \bar{\partial}\varrho, \quad \partial\varrho(\xi) = 1. \tag{1.2}$$

Thus there is a smooth function r so that $\varrho_{j\bar{k}}\xi^j = r\varrho_{\bar{k}}$; evaluating on $\xi^{\bar{k}}$ gives $r = \varrho_{j\bar{k}}\xi^j\xi^{\bar{k}}$. Set $N = \operatorname{Re} \xi$, $T = \operatorname{Im} \xi$. Then one has $N\varrho = 1$, $T\varrho = 0$.

Working locally if necessary, choose smooth $(1, 0)$ vector fields $Z_1, \ldots, Z_{n-1}$ on $\bar{\Omega}$ satisfying $\partial\varrho(Z_i) = 0$ so that the Z_i form an orthonormal basis for $T_b^{1,0}$ with respect to $-\varrho_{j\bar{k}}|_{T_b^{1,0}}$. Also set $Z_n = \xi$. Assuming that Ω has been chosen small enough that $g_{j\bar{k}} = (\log 1/\varrho)_{j\bar{k}}$ is positive definite everywhere on Ω, one finds from $g_{j\bar{k}} = -(\varrho_{j\bar{k}})/\varrho + (\varrho_j\varrho_{\bar{k}})/\varrho^2$ and the definitions of Z_i, ξ, that $\sqrt{\varrho}Z_i$, $1 \leqslant i \leqslant n - 1$, and $(\varrho Z_n)/\sqrt{1 - r\varrho}$ form an orthonormal basis for $T^{1,0}\Omega$ with respect to $g_{j\bar{k}}$. So if $\omega^1, \ldots, \omega^n$ is the basis for $T^{1,0*}\Omega$ dual to $Z_1, \ldots, Z_n$, then each ω^i is smooth in $\bar{\Omega}$, $(\omega^i)/\sqrt{\varrho}$, $1 \leqslant i \leqslant n - 1$, and $(\omega^n\sqrt{1 - r\varrho})/\varrho$ are orthonormal with respect to $g^{j\bar{k}}$, and $\omega^n = \partial\varrho$. Consequently the Kähler form $\omega = \partial\bar{\partial} \log 1/\varrho$ is given by

$$\omega = \sum_{i=1}^{n-1} \frac{\omega^i \wedge \overline{\omega^i}}{\varrho} + (1 - r\varrho)\frac{\omega^n \wedge \overline{\omega^n}}{\varrho^2}. \tag{1.3}$$

Comparing with $\omega = -(\partial\bar{\partial}\varrho)/\varrho + (\partial\varrho \wedge \bar{\partial}\varrho)/\varrho^2$ and recalling $\omega^n = \partial\varrho$, it follows that

$$\partial\bar{\partial}\varrho = -\sum_{i=1}^{n-1} \omega^i \wedge \overline{\omega^i} + r\omega^n \wedge \overline{\omega^n}. \tag{1.4}$$

We will subsequently need the following formula for $\partial\bar{\partial}$.

LEMMA 1.5: *If f is a function on Ω, then*

$$\partial\bar{\partial}f = \sum_{i,j=1}^{n} [\tfrac{1}{2}(Z_i\bar{Z}_j + \bar{Z}_jZ_i) + a_i\delta_{i\bar{j}}N + E_{i\bar{j}}]f \quad \omega^i \wedge \overline{\omega^j},$$

where $\delta_{i\bar{j}}$ is the Kronecker δ, $a_i = -1$ *if* $1 \leqslant i \leqslant n-1$, $a_n = r$, and $E_{i\bar{j}}$, $1 \leqslant i, j \leqslant n$, are smooth vector fields on $\bar{\Omega}$ satisfying $\partial\varrho(E_{i\bar{j}}) = \bar{\partial}\varrho(E_{i\bar{j}}) = 0$.

Proof. Using the summation convention, $\bar{\partial}f = \bar{Z}_jf\overline{\omega^j}$, so

$$\begin{aligned}\partial\bar{\partial}f &= Z_i\bar{Z}_jf\omega^i \wedge \overline{\omega^j} + \bar{Z}_jf\partial\overline{\omega^j}\\ &= Z_i\bar{Z}_jf\omega^i \wedge \overline{\omega^j} + \bar{Z}_nf\partial\bar{\partial}\varrho + \sum_{j=1}^{n-1} c_{k\bar{l}}\bar{Z}_jf\omega^k \wedge \overline{\omega^l}\end{aligned}$$

for some smooth coefficients $c_{k\bar{l}}$. However for real f, $\partial\bar{\partial}f$ is pure imaginary, so upon averaging this formula with its negative conjugate and using $\operatorname{Re} Z_n = N$ and (1.4), one obtains Lemma 1.5.

Following [2] and [8] we rewrite the Monge-Ampère equation $J(u) = 1$. With ϱ a smooth defining function for M as above, introduce a new unknown function f by writing $u = \varrho e^{-f}$, and set $\mathscr{M}(f) = [J(u)]/[J(\varrho)] = [J(\varrho e^{-f})]/[J(\varrho)]$. The equation then becomes $\mathscr{M}(f) = J(\varrho)^{-1}$, and $J(\varrho)^{-1}$ is a smooth positive function in $\bar{\Omega}$. The next result shows how $\mathscr{M}(f)$ may be written as a totally characteristic operator.

PROPOSITION 1.6:

$$\mathscr{M}(f) = [1 + (\varrho N)^2f - n(\varrho N)f + Q(\varrho Nf, (\varrho N)^2f) + \varrho\mathscr{T}(f)]e^{-(n+1)f},$$

where $Q(x, y)$ is a polynomial in two variables with real coefficients having no constant or linear terms, and $\mathscr{T}(f)$ is a nonlinear totally characteristic differential operator.

Proof. Setting $dv = n!dz^1 \wedge d\bar{z}^{\bar 1} \wedge \ldots \wedge dz^n \wedge d\bar{z}^{\bar n}$, (1.1) can be rewritten $J(u)\, dv = u^{n+1} (\partial\bar\partial \log 1/u)^n$. Let $u = \varrho e^{-f}$ to obtain

$$J(\varrho e^{-f})e^{(n+1)f}\, dv = \varrho^{n+1}(\omega + \partial\bar\partial f)^n, \tag{1.7}$$

where $\omega = \partial\bar\partial \log 1/\varrho$ is given by (1.3). Using Lemma 1.5, we can write $\omega + \partial\bar\partial f = A_{i\bar j}\omega^i \wedge \overline{\omega^j}$, where $A_{i\bar j}$ are the components of the matrix

$$A = \begin{array}{c} \\ n-1\left\{\begin{array}{c} \\ \\ \end{array}\right. \\ \\ \\ \end{array}\begin{pmatrix} \overbrace{\begin{array}{c} 1/\varrho\delta_{i\bar j} + \tfrac12(Z_i\bar Z_j + \bar Z_j Z_i)f \\ -\ \delta_{i\bar j}Nf + E_{i\bar j}f \end{array}}^{n-1} & \tfrac12(Z_i\bar Z_n + \bar Z_n Z_i)f + E_{i\bar n}f \\ \tfrac12(Z_n\bar Z_j + \bar Z_j Z_n)f + E_{n\bar j}f & \begin{array}{c} (1 - r\varrho)/\varrho^2 + \tfrac12(Z_n\bar Z_n + \bar Z_n Z_n)f \\ +\ rNf + E_{n\bar n}f \end{array} \end{pmatrix}.$$

Hence

$$(\omega + \partial\bar\partial f)^n = (\det A)n!\omega^1 \wedge \overline{\omega^1} \wedge \ldots \wedge \omega^n \wedge \overline{\omega^n}. \tag{1.8}$$

When $f = 0$, $\det A = (1 - r\varrho)/(\varrho^{n+1})$, so substituting (1.8) into (1.7) with $f = 0$ gives $n!\omega^1 \wedge \overline{\omega^1} \wedge \ldots \wedge \omega^n \wedge \overline{\omega^n} = [J(\varrho)]/(1 - r\varrho)\, dv$. Using this in (1.8) and substituting again into (1.7), this time with $f \neq 0$, shows that (1.7) may be rewritten

$$J(\varrho e^{-f})e^{(n+1)f} = J(\varrho)\frac{\varrho^{n+1}}{1 - r\varrho}\det A,$$

or,

$$\mathcal{M}(f)e^{(n+1)f} = \frac{\varrho^{n+1}}{1 - r\varrho}\det A. \tag{1.9}$$

Now $(\varrho^{n+1})/(1 - r\varrho)\det A = \det B$, where

$$B = \begin{pmatrix} \begin{array}{c} \delta_{i\bar j}(1 - \varrho Nf) + \dfrac{\varrho}{2}(Z_i\bar Z_j + \bar Z_j Z_i)f \\ +\ \varrho E_{i\bar j}f \end{array} & \dfrac{\varrho}{2}(Z_i\bar Z_n + \bar Z_n Z_i)f + \varrho E_{i\bar n}f \\ \dfrac{\varrho^2}{1 - r\varrho}[\tfrac12(Z_n\bar Z_j + \bar Z_j Z_n)f + E_{n\bar j}f] & \begin{array}{c} 1 + \dfrac{\varrho^2}{1 - r\varrho}[N^2f + T^2f + rNf \\ +\ E_{n\bar n}f] \end{array} \end{pmatrix}.$$

But all of Z_i, $\bar Z_i$, $1 \leqslant i \leqslant n-1$, T, $E_{i\bar j}$, ϱN, ϱZ_n, $\varrho\bar Z_n$ are vector fields tangent to the boundary. It follows that B is of the form

$$B = \begin{pmatrix} \delta_{i\bar j}(1 - \varrho Nf) + \varrho P_{i\bar j}f & P_{i\bar n}f \\ \varrho P_{n\bar j}f & 1 + \varrho^2N^2f + \varrho P_{n\bar n}f \end{pmatrix},$$

where each $P_{i\bar{j}}$, $1 \leqslant i, j \leqslant n$, is a linear totally characteristic operator. Expanding, one finds that $\det B = (1 - \varrho Nf)^{n-1}(1 + \varrho^2 N^2 f) + \varrho\mathscr{T}(f)$ for some nonlinear totally characteristic operator $\mathscr{T}$. But

$$(1 - \varrho Nf)^{n-1}(1 + \varrho^2 N^2 f) \;=\; 1 + (\varrho N)^2 f - n\varrho Nf + Q(\varrho Nf, (\varrho N)^2 f)$$

for a polynomial Q consisting only of quadratic and higher terms, so substitution into (1.9) concludes the proof of Proposition 1.6.

2. Asymptotics of solutions

In this section we analyze the local determination of formal solutions of $J(u) = 1, u|_M = 0$. We are interested in solutions of the form (0.2). Thus let $\mathscr{A}$ denote the space of formal expansions

$$\sum_{k=0}^{\infty} \eta_k(\varrho^{n+1} \log \varrho)^k, \tag{2.1}$$

where $\eta_k \in C^\infty(\bar{\Omega})$ and ϱ is a C^∞ defining function for M as in §1. Two such expansions are identified if the corresponding coefficient functions η_k agree to infinite order along M for all $k \geqslant 0$. Thus the space $\mathscr{A}$ may alternately be interpreted as equivalence classes of functions on $\bar{\Omega}$ modulo smooth functions vanishing to infinite order along M. An expansion (2.1) is C^∞ in $\bar{\Omega}$ iff $\eta_k = 0$ (to infinite order) for $k \geqslant 1$. If $\varrho = \lambda\tilde{\varrho}$ for some $0 < \lambda \in C^\infty(\bar{\Omega})$ and other defining function $\tilde{\varrho}$, then it is easily seen that (2.1) may be rewritten as $\Sigma_{k=0}^{\infty} \tilde{\eta}_k(\tilde{\varrho}^{n+1} \log \tilde{\varrho})^k$, where $\tilde{\eta}_k \in C^\infty(\bar{\Omega})$ satisfy

$$\tilde{\eta}_k \;=\; \lambda^{(n+1)k}\eta_k + 0(\varrho^{n+1}). \tag{2.2}$$

In particular $\mathscr{A}$ is independent of the choice of a defining function. $\mathscr{A}$ has a natural structure as a ring and a C^∞ module.

There is a filtration of $\mathscr{A}$ that is relevant for the inductive calculations used to solve the Monge–Ampère equation. For s a nonnegative integer, let $\mathscr{A}_s$ denote the elements of $\mathscr{A}$ which "vanish to order s". Precisely, an expansion (2.1) is in $\mathscr{A}_s$ if for all $k \geqslant 0$, $\eta_k\varrho^{(n+1)k} = 0(\varrho^s)$. Then $\mathscr{A} = \mathscr{A}_0 \supset \mathscr{A}_1 \supset \ldots$. One checks easily that $\mathscr{A}_s \cdot \mathscr{A}_t \subset \mathscr{A}_{s+t}$; in particular each $\mathscr{A}_s$ is preserved under multiplication by functions in $C^\infty(\bar{\Omega})$, and $\varrho^t\mathscr{A}_s \subset \mathscr{A}_{s+t}$. Also we have:

$$\begin{aligned} &\text{If } f_j \in \mathscr{A}_{s_j}, \quad \text{where} \quad s_j \geqslant 0 \quad \text{and} \quad s_j \to \infty \text{ as } j \to \infty, \\ &\text{then } \sum_{j=0}^{\infty} f_j \text{ exists and defines an element of } \mathscr{A}. \end{aligned} \tag{2.3}$$

In fact, for all N sufficiently large the partial sums $\Sigma_{j=0}^{N} f_j$ agree to any fixed finite order, so coefficient functions η_k for $\Sigma_{j=0}^{\infty} f_j$ can be computed to any finite order by adding a partial sum of the series. Then smooth functions η_k can be chosen with the prescribed Taylor expansions, uniquely up to functions vanishing to infinite order, thus expressing $\Sigma_{j=0}^{\infty} f_j$ as an element of $\mathscr{A}$. (2.3) can be applied to show that if $f \in \mathscr{A}$ then $e^f \in \mathscr{A}$. In fact, if f is the expansion (2.1), write $f = \eta_0 + \tilde{f}$ where $\tilde{f} = \Sigma_{k=1}^{\infty} \eta_k(\varrho^{n+1} \log \varrho)^k$, so that $e^f = e^{\eta_0} e^{\tilde{f}}$. Then $e^{\eta_0} \in C^{\infty}(\bar{\Omega})$, and $e^{\tilde{f}} = \Sigma_{j=0}^{\infty} f_j, f_j = 1/(j!)(\tilde{f})^j \in \mathscr{A}_{(n+1)j}$, is a series as in (2.3).

LEMMA 2.4: *If $\mathscr{T}$ is a (linear or) nonlinear totally characteristic operator and s is a nonnegative integer, then $\mathscr{T}\colon \mathscr{A}_s \to \mathscr{A}_s$ and $\mathscr{T}\colon \varrho^s\mathscr{A} \to \varrho^s\mathscr{A}$.*

Proof. Since a nonlinear totally characteristic operator is a polynomial in linear totally characteristic operators and both $\mathscr{A}_s$ and $\varrho^s\mathscr{A}$ are rings, it suffices to consider the linear case. As a linear totally characteristic operator is a polynomial in vector fields tangent to the boundary, it suffices to consider such a vector field. In local coordinates (x, ϱ) as in the beginning of §1, a basis for these consists of $\partial/(\partial x^i)$, $\varrho(\partial/\partial\varrho)$. The result now follows easily by direct calculation.

A consequence of Proposition 1.6, Lemma 2.4 and the preceding observations is that $\mathscr{M}\colon \mathscr{A} \to \mathscr{A}$. In order to construct formal solutions of the Monge–Ampère equation an analysis of the behavior of $\mathscr{M}$ under perturbations is necessary, and is given in the next two lemmas. Let I be the indicial operator $I = (\varrho N)^2 - n\varrho N - (n + 1)$.

LEMMA 2.5: *Let $f \in \mathscr{A}_1$, $g \in \mathscr{A}_s$, $s \geqslant 1$. Then*

$$\mathscr{M}(f + g) = \mathscr{M}(f) + I(g) \bmod \mathscr{A}_{s+1}.$$

Proof. Use Proposition 1.6. Since ϱN, $(\varrho N)^2 f \in \mathscr{A}_1$, $\varrho N g$, $(\varrho N)^2 g \in \mathscr{A}_s$, and Q consists only of quadratic and higher terms, it follows upon expanding Q and multiplying out the terms that $Q(\varrho Nf + \varrho Ng, (\varrho N)^2 f + (\varrho N)^2 g) = Q(\varrho Nf, (\varrho N)^2 f) \bmod \mathscr{A}_{s+1}$. A general nonlinear totally characteristic operator may have linear terms, so an argument similar to that for Q shows that $\mathscr{T}(f + g) = \mathscr{T}(f) \bmod \mathscr{A}_s$. Hence $\varrho\mathscr{T}(f + g) = \varrho\mathscr{T}(f) \bmod \mathscr{A}_{s+1}$. Also

$$\begin{aligned} e^{-(n+1)(f+g)} &= e^{-(n+1)f} e^{-(n+1)g} = e^{-(n+1)f}(1 - (n + 1)g + \mathscr{A}_{s+1}) \\ &= e^{-(n+1)f} - (n + 1)g + \mathscr{A}_{s+1} \end{aligned}$$

since $e^{-(n+1)f} = 1 + \mathscr{A}_1$. Thus from Proposition 1.6 one obtains

$$\begin{aligned}\mathscr{M}(f+g) &= [1 + (\varrho N)^2 f - n(\varrho N)f + Q(\varrho Nf, (\varrho N)^2 f) + \varrho\mathscr{T}(f) \\ &\qquad + (\varrho N)^2 g - n(\varrho N)g + \mathscr{A}_{s+1}] \\ &\qquad \times [e^{-(n-1)f} - (n+1)g + \mathscr{A}_{s+1}] \\ &= \mathscr{M}(f) + [(\varrho N)^2 g - n(\varrho N)g]\, e^{-(n+1)f} - (n+1)\, g + \mathscr{A}_{s+1} \\ &= \mathscr{M}(f) + I(g) + \mathscr{A}_{s+1},\end{aligned}$$

again using $e^{-(n+1)f} = 1 + \mathscr{A}_1$.

LEMMA 2.6: *Let $f, g \in \mathscr{A}$, $0 \leqslant s \in \mathbb{Z}$. Then*

$$\mathscr{M}(f + \varrho^s g) - \mathscr{M}(f) \in \varrho^s \mathscr{A}.$$

Proof. First, by Lemma 2.4, if P is a linear totally characteristic operator then $P(f + \varrho^s g) = Pf + \varrho^s \mathscr{A}$. Thus it follows upon multiplying out the terms that if $\mathscr{S}$ is a nonlinear totally characteristic operator, then $\mathscr{S}(f + \varrho^s g) = \mathscr{S}(f) + \varrho^s \mathscr{A}$. But by Proposition 1.6, $\mathscr{M}(f) = [1 + \mathscr{S}(f)]e^{-(n+1)f}$ for a nonlinear totally characteristic operator $\mathscr{S}$. Consequently

$$\begin{aligned}\mathscr{M}(f + \varrho^s g) &= [1 + \mathscr{S}(f) + \varrho^s \mathscr{A}]e^{-(n+1)f}e^{-(n+1)\varrho^s g} \\ &= [\mathscr{M}(f) + \varrho^s \mathscr{A}](1 + \varrho^s \mathscr{A}) = \mathscr{M}(f) + \varrho^s \mathscr{A},\end{aligned}$$

and Lemma 2.6 is proved.

Lemma 2.5 shows that on $\mathscr{A}_s/\mathscr{A}_{s+1}$, $\mathscr{M}(f + \cdot)$ reduces to the indicial operator I. The next lemma analyzes I on this space. Since I preserves both $\mathscr{A}_s$ and $\mathscr{A}_{s+1}$ it induces an operator on $\mathscr{A}_s/\mathscr{A}_{s+1}$. If $g \in \mathscr{A}$, let $[g]_s \in \mathscr{A}/\mathscr{A}_{s+1}$ be the equivalence class of g.

LEMMA 2.7: *Let $0 \leqslant s \in \mathbb{Z}$, $h \in \mathscr{A}_s$.*

(i) *If $s \neq n + 1$, there is a unique $[g]_s \in \mathscr{A}_s/\mathscr{A}_{s+1}$ solving $I[g]_s = [h]_s$. If $h \in C^\infty(\bar{\Omega})$* mod *$\mathscr{A}_{s+1}$, then g can be chosen in $C^\infty(\bar{\Omega})$.*
(ii) *if $s = n + 1$, it may happen that there is no solution in $\mathscr{A}_s$ to $I[g]_s = [h]_s$. But if $h \in C^\infty(\bar{\Omega})$* mod *$\mathscr{A}_{n+2}$, then there is a solution $[g]_s$. In this case $[g]_s$ is uniquely determined up to addition of $[\psi\varrho^{n+1}]_s$, $\psi \in C^\infty(\bar{\Omega})$.*

Proof. Any $g \in \mathscr{A}_s$ is of the form

$$g = \varrho^s \sum_{j=0}^{l} \alpha_j (\log \varrho)^j \mod \mathscr{A}_{s+1}, \tag{2.8}$$

where $l = [s/(n+1)]$ and $\alpha_j \in C^\infty(\bar{\Omega})$. Thus we try to find a solution of this form. In computing $I(g)$, if any of the differentiations hit α_j then the resulting term is easily seen to be in $\mathscr{A}_{s+1}$. Consequently

$$I(g) = \sum_{j=0}^{l} \alpha_j I(\varrho^s (\log \varrho)^j) \mod \mathscr{A}_{s+1}.$$

A direct computation then gives

$$\begin{aligned} I(g) = \varrho^s \Bigg\{ & I(s)\alpha_l (\log \varrho)^l + [I(s)\alpha_{l-1} + l(2s-n)\alpha_l](\log \varrho)^{l-1} \\ & + \sum_{j=0}^{l-2} [I(s)\alpha_j + (j+1)(2s-n)\alpha_{j+1} + (j+1)(j+2)\alpha_{j+2}] \\ & \times (\log \varrho)^j \Bigg\} \mod \mathscr{A}_{s+1}, \end{aligned}$$

where $I(s) = s^2 - ns - (n+1)$. Thus if h is written $h = \varrho^s \Sigma_{j=0}^{l} \beta_j (\log \varrho)^j \mod \mathscr{A}_{s+1}$, then the equation $I[g]_s = [h]_s$ holds iff the following equations hold on M.

$$\begin{aligned} & I(s)\alpha_l = \beta_l \\ & I(s)\alpha_{l-1} + l(2s-n)\alpha_l = \beta_{l-1} \\ & I(s)\alpha_j + (j+1)(2s-n)\alpha_{j+1} + (j+1)(j+2)\alpha_{j+2} = \beta_j, \\ & \quad 0 \leqslant j \leqslant l-2. \end{aligned} \tag{2.9}$$

If $s \neq n+1$ then $I(s) \neq 0$, so these equations can be successively solved to yield a solution g. Each α_j is uniquely determined on M so g is uniquely determined mod $\mathscr{A}_{s+1}$. In order that $h \in C^\infty(\bar{\Omega})$ mod $\mathscr{A}_{s+1}$ it is necessary and sufficient that $\beta_j = 0$ on M for $j \geqslant 1$. In this situation it is clear that the solution α_j to (2.9) satisfies $\alpha_j = 0$ on M for $j \geqslant 1$, so $g \in C^\infty(\bar{\Omega})$. Thus (i) is proved.

Since $I(n+1) = 0$, when $s = n+1$ the equations (2.9) become

$$0 = \beta_1, \quad (n+2)\alpha_1 = \beta_0. \tag{2.10}$$

Hence a necessary condition that there will be a solution is $\beta_1 = 0$ on M, which means $h \in C^\infty(\bar{\Omega}) \bmod \mathscr{A}_{n+2}$. In this case α_1 is determined but α_0 on M remains arbitrary, so $[g]_s$ is determined only up to $[\psi\varrho^{n+1}]_s$, $\psi \in C^\infty(\bar{\Omega})$. This concludes the proof of Lemma 2.7.

REMARK: It is also possible to prove a version of Lemma 2.7 describing the kernel and cokernel of I on all of $\mathscr{A}_s$ rather than just on $\mathscr{A}_s/\mathscr{A}_{s+1}$. This can be carried out either by inductively using Lemma 2.7 or by a direct analysis upon introducing a special coordinate system.

Finally we are in a position to construct the asymptotic solutions u to $J(u) = 1$, $u|_M = 0$. Fefferman [4] has shown that there is a smooth defining function ϱ for M solving $J(\varrho) = 1 + 0(\varrho^{n+1})$, with ϱ uniquely determined modulo $0(\varrho^{n+2})$. (An alternate proof of this fact from the point of view of totally characteristic operators is given in [8], Theorem 8.13.) Fix such a defining function ϱ. If u is a formal solution of $J(u) = 1$ of the form (0.2), then the smooth function $\varrho\eta_0$ solves $J(\varrho\eta_0) = 1 + 0(\varrho^{n+1})$, so necessarily $\eta_0 = 1 + 0(\varrho^{n+1})$.

THEOREM 2.11:

(i) *Let $a \in C^\infty(M)$. Then there is a unique asymptotic expansion of the form (0.2) solving $J(u) = 1$ to infinite order, for which $(\eta_0 - 1)/(\varrho^{n+1}) = a$ on M.*

(ii) *For $k \geqslant 1$, each η_k* mod $0(\varrho^{n+1})$ *is independent of the choice of a.*

(iii) *For $k \geqslant 1$, each η_k* mod $0(\varrho^{n+1})$ *is also independent of the smooth defining function ϱ solving $J(\varrho) = 1 + 0(\varrho^{n+1})$.*

Proof. As before set $u = \varrho e^{-f}$; then $J(u) = 1$ becomes $\mathscr{M}(f) = J(\varrho)^{-1}$. If f is written $f = \Sigma_{k=0}^{\infty} \gamma_k(\varrho^{n+1}\log\varrho)^k$, then $\eta_0 = e^{-\gamma_0}$, so the condition $\eta_0 = 1 + 0(\varrho^{n+1})$ becomes $\gamma_0 = 0(\varrho^{n+1})$, or equivalently $f \in \mathscr{A}_{n+1}$. In order to prove (i) it suffices to prove its analogue for f; namely that there is a unique $f \in \mathscr{A}_{n+1}$ solving $\mathscr{M}(f) = J(\varrho)^{-1}$ to infinite order, for which $\gamma_0/(\varrho^{n+1}) = -a$ on M.

To begin, by Lemma 2.5, for $f \in \mathscr{A}_{n+1}$, $\mathscr{M}(f) = \mathscr{M}(0+f) = \mathscr{M}(0) + I(f) \bmod \mathscr{A}_{n+2} = 1 + I(f) \bmod \mathscr{A}_{n+2}$. Thus $\mathscr{M}(f) = J(\varrho)^{-1} \bmod \mathscr{A}_{n+2}$ becomes

$$I[f]_{n+1} = [J(\varrho)^{-1} - 1]_{n+1}. \tag{2.12}$$

Since $J(\varrho)^{-1} - 1 \in C^\infty(\bar{\Omega}) \cap \mathscr{A}_{n+1}$, by Lemma 2.7(ii) there is a solution $f \in \mathscr{A}_{n+1}$ to (2.12) with $[f]_{n+1}$ uniquely determined up to addition of $[\psi\varrho^{n+1}]_{n+1}$, $\psi \in C^\infty(\bar{\Omega})$. The prescription of $(\gamma_0)/(\varrho^{n+1})$ on M fixes this ambiguity, thus determining $[f]_{n+1}$. Pick some representative f in the determined class and set $f_{n+1} = f$ and $h_{n+2} = \mathscr{M}(f) - J(\varrho)^{-1} \in \mathscr{A}_{n+2}$.

We now proceed inductively to show that for $s \geqslant n + 1$ there is $f_s \in \mathscr{A}_{n+1}$ for which $h_{s+1} = \mathscr{M}(f_s) - J(\varrho)^{-1} \in \mathscr{A}_{s+1}$, $(\gamma_0)/(\varrho^{n+1}) = -a$ on M, and that $[f_s]_s$ is uniquely determined by these requirements. Given f_{s-1}, $s \geqslant n + 2$, set $f_s = f_{s-1} + g_s$ for $g_s \in \mathscr{A}_s$. Then by Lemma 2.5,

$$\mathscr{M}(f_s) = \mathscr{M}(f_{s-1}) + I(g_s) \bmod \mathscr{A}_{s+1}, \text{ so}$$

$$h_{s+1} = \mathscr{M}(f_s) - J(\varrho)^{-1} = h_s + I(g_s) \bmod \mathscr{A}_{s+1}$$

and it follows that the requirement $h_{s+1} \in \mathscr{A}_{s+1}$ is equivalent to $I[g_s]_s = -[h_s]_s$. Lemma 2.7(i) implies that there is a solution g_s, with $[g_s]_s$ uniquely determined. Thus $[f_s]_s$ is uniquely determined as well and the induction is complete. By (2.3) the series $f_{n+1} + \Sigma_{s=n+2}^{\infty} g_s = \lim_{s\to\infty} f_s$ defines an element of $\mathscr{A}_{n+1}$ which is the unique solution to our problem, and (i) is proved.

We next establish the version of (ii) for f: for $k \geqslant 1$, each $\gamma_k \bmod 0(\varrho^{n+1})$ is independent of the choice of a, which is equivalent to the statement that $f \bmod \varrho^{n+1}\mathscr{A}$ is independent of a. It suffices to show that if $a, a' \in C^\infty(M)$ then the corresponding sequences f_s, f_s' of approximate solutions constructed above may be chosen so that $f_s - f_s' \in \varrho^{n+1}\mathscr{A}$ for all s. In order to recognize when an equivalence class mod $\mathscr{A}_{s+1}$ has a representative in $\varrho^{n+1}\mathscr{A}$, we have

LEMMA 2.13: *Let $s \geqslant 0$ and let $g \in \mathscr{A}_s$ be written in the form (2.8). Then there is $\tilde{g} \in \mathscr{A}_s \cap \varrho^{n+1}\mathscr{A}$ with $[g]_s = [\tilde{g}]_s$ if and only if $\alpha_l|_M = 0$.*

Proof. First if $\tilde{g} \in \varrho^{n+1}\mathscr{A}$ then $\tilde{g} = \varrho^{n+1} \Sigma_{k=0}^{\infty} \eta_k(\varrho^{n+1} \log \varrho)^k$ for some $\eta_k \in C^\infty(\bar{\Omega})$. Thus for all k the coefficient of $(\log \varrho)^k$ in $\tilde{g}$ vanishes to order $(n + 1)(k + 1)$. Now if also $\tilde{g} \in \mathscr{A}_s$ and we write $\tilde{g}$ as in (2.8) with coefficients $\tilde{\alpha}_j$, then the coefficient of $(\log \varrho)^l$ is $\varrho^s\tilde{\alpha}_l \bmod 0(\varrho^{s+1})$. As $s < (n + 1) \times (l + 1)$ it follows immediately that $\tilde{\alpha}_l|_M = 0$. If $[g]_s = [\tilde{g}]_s$, then $\alpha_j|_M = \tilde{\alpha}_j|_M$ for all j, so $\alpha_l|_M = 0$ too.

Conversely suppose that $\alpha_l|_M = 0$, and set $\tilde{g} = \varrho^s \Sigma_{j=0}^{l-1} \alpha_j(\log \varrho)^j$. Then certainly $\tilde{g} \in \mathscr{A}_s$ and $[g]_s = [\tilde{g}]_s$. Additionally we have $\tilde{g} = \varrho^{n+1} \Sigma_{j=0}^{l-1} \varrho^{s-(j+1)(n+1)}\alpha_j(\varrho^{n+1} \log \varrho)^j$. But $s \geqslant l(n + 1)$, so for $0 \leqslant j \leqslant l - 1$ it is the case that $s - (j + 1)(n + 1) \geqslant 0$, and we have explicitly exhibited $\tilde{g}$ as an element of $\varrho^{n+1}\mathscr{A}$. Lemma 2.13 is proved.

Now let $a, a' \in C^\infty(M)$ and consider the sequences f_s, f_s' constructed in the proof of Theorem 2.11(i). For each s, f_s and f_s' are determined modulo $\mathscr{A}_{s+1}$. We show by induction on s that f_s and f_s' may be taken so that $f_s - f_s' \in \varrho^{n+1}\mathscr{A}$. First consider $s = n + 1$. f_{n+1} was chosen to be a solution of (2.12) with $(\gamma_0)/(\varrho^{n+1}) = -a$ on M. Choose some smooth extension of a to $\bar{\Omega}$; then we have

$$f_{n+1} = -a\varrho^{n+1} + \gamma_1\varrho^{n+1}\log\varrho \mod \mathscr{A}_{n+2}, \tag{2.14}$$

and by (2.10) $\gamma_1|_M$ is determined by

$$(n+2)\gamma_1 = \frac{J(\varrho)^{-1} - 1}{\varrho^{n+1}} \quad \text{on} \quad M. \tag{2.15}$$

As the right-hand side of (2.15) is independent of a, it follows that $\gamma_1 = \gamma_1'$. Thus $f_{n+1} - f_{n+1}' = (a' - a)\varrho^{n+1} \mod \mathscr{A}_{n+2}$. Clearly $(a' - a)\varrho^{n+1} \in \varrho^{n+1}\mathscr{A}$, so f_{n+1}, f_{n+1}' may be chosen so that $f_{n+1} - f_{n+1}' \in \varrho^{n+1}\mathscr{A}$.

Assume next that $s \geqslant n + 2$ and f_{s-1}, f_{s-1}' have been constructed with $f_{s-1} - f_{s-1}' \in \varrho^{n+1}\mathscr{A}$. Recall that $f_s = f_{s-1} + g_s$, where $g_s \in \mathscr{A}_s$ is the solution of $I[g_s]_s = -[h_s]_s$, and $h_s = \mathscr{M}(f_{s-1}) - J(\varrho)^{-1} \in \mathscr{A}_s$. Write g_s, g_s' in the form (2.8) with coefficients α_j, α_j', and similarly for $-h_s, -h_s'$ with coefficients β_j, β_j'. Since $f_{s-1} - f_{s-1}' \in \varrho^{n+1}\mathscr{A}$ it follows from Lemma 2.6 that $h_s - h_s' \in \varrho^{n+1}\mathscr{A}$. Hence by Lemma 2.13 it must be that $\beta_l = \beta_l'$ on M. g_s is obtained by solving equations (2.9) and from the first of these equations one obtains $\alpha_l = \alpha_l'$ on M. Thus by Lemma 2.13 again it follows that g_s, g_s' may be chosen so that $g_s - g_s' \in \varrho^{n+1}\mathscr{A}$, so that the same is true for f_s, f_s' and the induction is complete.

The proof of Theorem 2.11(ii) is concluded by passing back from f to u. It has been shown that if f, f' are the solutions to $\mathscr{M}(f) = J(\varrho)^{-1}$ with $(\gamma_0)/(\varrho^{n+1}) = -a, -a'$, resp., then $f - f' \in \varrho^{n+1}\mathscr{A}$. But $u = \varrho e^{-f}$, so $u - u' = \varrho e^{-f}[1 - e^{(f-f')}]$. As $1 - e^{f-f'} \in \varrho^{n+1}\mathscr{A}$ and $e^{-f} \in \mathscr{A}$ we obtain $u/\varrho - u'/\varrho \in \varrho^{n+1}\mathscr{A}$. So upon writing u and u' in the form (0.2) one concludes that $\eta_k - \eta_k' = 0(\varrho^{n+1})$ and (ii) is proved.

As for (iii), let $\tilde{\varrho}$ be another smooth solution to $J(\varrho) = 1 + 0(\varrho^{n+1})$. Then $r = \lambda\tilde{\varrho}$, where $0 < \lambda \in C^\infty(\bar{\Omega})$ and $\lambda = 1 + 0(\varrho^{n+1})$. If u is an asymptotic expansion (0.2) solving $J(u) = 1$ to infinite order, then as in (2.2) u can be expanded in terms of $\tilde{\varrho}$ and the new coefficients $\tilde{\eta}_k$ satisfy $\tilde{\eta}_k = \lambda^{(n+1)k+1}\eta_k$ mod $0(\varrho^{n+1})$. Clearly $\tilde{\eta}_k = \eta_k$ mod $0(\varrho^{n+1})$, thus establishing (iii), and the proof of Theorem 2.11 is complete.

According to Theorem 2.11, each of the functions η_k mod $0(\varrho^{n+1})$, $k \geqslant 1$, is canonically associated to the strictly pseudoconvex hypersurface M. In particular $b\eta_k = \eta_k|_M$ is a canonically determined function on M. We conclude §2 by showing that $b\eta_1$ is of particular importance in that it determines whether or not any log terms can occur in any solution to the Monge–Ampère equation.

Proposition 2.16: *Suppose that* $\eta_1|_M = 0$. *If* u *is any solution of the form (0.2) to* $J(u) = 1$, *then* $\eta_k = 0$ *to infinite order for all* $k \geqslant 1$. *In particular, all such formal solutions are smooth.*

Proof. As in Theorem 2.11 we carry out the analysis in terms of f. So let $u = \varrho e^{-f}$ where $f = \Sigma_{k=0}^{\infty} \gamma_k(\varrho^{n+1} \log \varrho)^k$; since $\gamma_1 = -\eta_1$ on M it must be shown that if $\gamma_1|_M = 0$ and f is any solution to $\mathscr{M}(f) = J(\varrho)^{-1}$ then $\gamma_k = 0$ to infinite order for all $k \geqslant 1$. This is established by analyzing the inductive construction of solutions in Theorem 2.11. We show that each approximate solution f_s may be chosen to be $C^\infty(\bar{\Omega})$, i.e., to have no log terms. For $s = n + 1$, (2.14) shows that this is exactly our hypothesis $\gamma_1|_M = 0$. Suppose the result is true for f_{s-1}; then $f_s = f_{s-1} + g_s$ where $g_s \in \mathscr{A}_s$ is a solution of $I[g_s]_s = -[h_s]_s$, and $h_s = \mathscr{M}(f_{s-1}) - J(\varrho)^{-1} \in \mathscr{A}_s$. But since $f_{s-1} \in C^\infty(\bar{\Omega})$ it follows that $\mathscr{M}(f_{s-1}) \in C^\infty(\bar{\Omega})$ so also $h_s \in C^\infty(\bar{\Omega})$. By Lemma 2.7(i) one can choose $g_s \in C^\infty(\bar{\Omega})$ and we are done.

3. Scalar invariants of strictly pseudoconvex hypersurfaces

In [5], C. Fefferman posed the problem of constructing all scalar invariants of strictly pseudoconvex hypersurfaces and showed how some such invariants arise from the smooth defining function ϱ satisfying $J(\varrho) = 1 + 0(\varrho^{n+1})$. We show here that further scalar invariants can be constructed from the higher asymptotics of a solution to $J(u) = 1$; more particularly from the functions η_k mod $0(\varrho^{n+1})$, $k \geqslant 1$, of Theorem 2.11. We begin by briefly reviewing the definition and Fefferman's construction of scalar invariants. For more details on these matters see [5, 6].

First recall Moser's normal form [3]. A real analytic strictly pseudoconvex hypersurface $N \subset \mathbb{C}^n$ containing 0 is said to be in normal form if it is defined by an equation of the form

$$2u = |z|^2 + \sum_{\substack{|\alpha|, |\beta| \geqslant 2 \\ l \geqslant 0}} A^l_{\alpha\bar{\beta}} z^\alpha \bar{z}^\beta v^l, \tag{3.1}$$

where $z \in \mathbb{C}^{n-1}$, $w = u + iv \in \mathbb{C}$ so that $(z, w) \in \mathbb{C}^n$, α, β are lists of indices between 1 and $n - 1$, and the coefficients $A^l_{\alpha\bar{\beta}} \in \mathbb{C}$ satisfy:

(i) $A^l_{\alpha\bar\beta}$ is symmetric with respect to permutations of the indices in α and β, resp.

(ii) $\overline{A^l_{\alpha\bar\beta}} = A^l_{\beta\bar\alpha}$

(iii) $\operatorname{tr} A^l_{2\bar2} = 0$, $\operatorname{tr}^2 A^l_{2\bar3} = 0$, $\operatorname{tr}^3 A^l_{3\bar3} = 0$.

Here, for $p, q \geqslant 2$, $A^l_{p\bar q}$ is the bisymmetric tensor $(A^l_{\alpha\bar\beta})_{\substack{|\alpha|=p\\|\beta|=q}}$ on $\mathbb{C}^{n-1}$ and the traces are the usual tensorial traces with respect to $\delta_{i\bar j}$. We sometimes identify the hypersurface N with the collection of coefficients $(A^l_{\alpha\bar\beta})$. It can happen that two different normal forms are biholomorphically equivalent, i.e., there are biholomorphic maps Φ from one neighborhood of the origin to another with $\Phi(0) = 0$ and $\Phi(N) = \tilde N$, where N and $\tilde N$ are both hypersurfaces in normal form.

DEFINITION 3.2: An invariant of weight $w \geqslant 0$ is a polynomial P in the normal form coefficients $(A^l_{\alpha\bar\beta})$ satisfying

$$P(\tilde A^l_{\alpha\bar\beta}) = |\det \Phi'(0)|^{-(2w)/(n+1)} P(A^l_{\alpha\bar\beta})$$

whenever Φ is a biholomorphism as above from one normal form $N = (A^l_{\alpha\bar\beta})$ to another $\tilde N = (\tilde A^l_{\alpha\bar\beta})$.

A particularly important example of equivalent normal forms is obtained from the biholomorphism $\Phi = \Phi_\delta$, where $\Phi_\delta(z, w) = (\delta z, \delta^2 w)$, $\delta > 0$. If $\mathrm{N} = (A^l_{\alpha\bar\beta})$ is any normal form then $\tilde N = \Phi_{\delta^{-1}}(N)$ is the normal form $\tilde N = (\delta^{|\alpha|+|\beta|+2l-2} A^l_{\alpha\bar\beta})$, which is hereafter denoted by $N\delta$. As $\det \Phi'_{\delta^{-1}} = \delta^{-(n+1)}$ it follows that if P is an invariant of weight w then

$$P(N\delta) = \delta^{2w} P(N), \quad \delta > 0. \tag{3.3}$$

Thus w measures the homogeneity of P with respect to the dilations δ: $N \to N\delta$ on the space of normal forms.

Fefferman's construction of invariants from a smooth approximate solution to $J(u) = 1$, $u|_M = 0$ is based on the observation that this equation is biholomorphically invariant. If M, $\tilde M$ are strictly pseudoconvex hypersurfaces and Φ is a biholomorphism from a neighborhood V of M to a neighborhood $\tilde V$ of $\tilde M$ with $\Phi(M) = \tilde M$, then for any solution u to $J(u) = 1$ on V, the function $\tilde u$ given by

$$\tilde u \circ \Phi = |\det \Phi'|^{2/(n+1)} u \tag{3.4}$$

is a solution to $J(\tilde u) = 1$ on $\tilde V$. This transformation law can alternately be formulated by introducing a new variable $z_0 \in \mathbb{C}^* = \mathbb{C} - \{0\}$ which transforms under a biholomorphism Φ as above by the rule $\tilde z_0 = z_0(\det \Phi')^{-1}$;

then $|z_0|^{2/(n+1)}u$ is invariantly defined on $\mathbb{C}^* \times V$, which itself can be invariantly interpreted as the canonical bundle of V with its zero section removed. Thus if ϱ is a smooth approximate solution to the Monge–Ampère equation, uniquely determined modulo $0(\varrho^{n+2})$, then $|z_0|^{2/(n+1)}\varrho$ is invariantly defined, also modulo $0(\varrho^{n+2})$.

From the invariance of $|z_0|^{2/(n+1)}\varrho$ it follows that the Kähler–Lorentz metric g on $\mathbb{C}^* \times V$ with Kähler form $i\partial\bar{\partial}(|z_0|^{2/(n+1)}\varrho)$ is biholomorphically invariant. Fefferman's "Weyl invariants" are obtained from the scalar invariants of g. Explicitly, let $R = R_{i\bar{j}k\bar{l}}$ be the curvature tensor of g and $\nabla^m R$ be one of its covariant derivatives of order m. So $\nabla^m R$ is a covariant tensor of rank $m + 4$ on $\mathbb{C}^* \times V$, holomorphic in some indices and antiholomorphic in others. A scalar invariant of g is a function of the form $W = \operatorname{contr}(\nabla^{m_1}R \otimes \cdots \otimes \nabla^{m_s}R) \in C^\infty(\mathbb{C}^* \times V)$, where the contraction is taken with respect to $g_{i\bar{j}}$ for some pairing of holomorphic with antiholomorphic indices, and all indices are assumed to be contracted. The weight of W is defined to be $w = s + \frac{1}{2}\sum_{i=1}^{s} m_i$. Of course not all Weyl invariants of g are well defined on $\mathbb{C}^* \times M$ because of the $0(\varrho^{n+2})$ ambiguity in ϱ. However Fefferman showed that if $w \leqslant n$ then the restriction of W to $\mathbb{C}^* \times M$ is well defined independent of which approximate solution ϱ one uses. Additionally, if this construction is applied to the normal form N given by (3.1) then for $w \leqslant n$ the value of W at $z_0 = 1$, $z = w = 0$, is a polynomial in the coefficients $(A^l_{\alpha\bar{\beta}})$ and is an invariant in the sense of Definition 3.2. Fefferman also proved that any invariant of weight $\leqslant n - 20$ is a linear combination of these "Weyl invariants".

Now according to Theorem 2.11, associated to a strictly pseudoconvex hypersurface M are the functions η_k mod $0(\varrho^{n+1})$, $k \geqslant 1$, as well as the defining function ϱ mod $0(\varrho^{n+2})$. If Φ is a biholomorphism from M to $\tilde{M}$ and u is an infinite order solution to $J(u) = 1$ near M, then $\tilde{u}$ given by (3.4) is an infinite order solution to $J(\tilde{u}) = 1$ near $\tilde{M}$. Modulo $0(\varrho^{n+2})$, ϱ and $\tilde{\varrho}$ also transform by (3.4), so upon transforming the expansion (0.2) one finds that the η_k satisfy

$$\tilde{\eta}_k \circ \Phi = |\det \Phi'|^{-2k}\eta_k \mod 0(\varrho^{n+1}),\ k \geqslant 1. \tag{3.5}$$

Restricting to M and comparing with Definition 3.2, it follows that the function $b\eta_k = \eta_k|_M$ transforms like an invariant of weight $k(n + 1)$. In order to show that it defines an invariant in that sense, we have

PROPOSITION 3.6: *Let N be the normal form (3.1), and $k \geqslant 1$. Then any partial derivative of order $\leqslant n$ evaluated at the origin of the function η_k is a universal polynomial in the $(A^l_{\alpha\bar{\beta}})$.*

Proof. This amounts to going back through §§1,2 and verifying the polynomial dependence of the derivatives of the various functions occurring there. One must check that the inductive construction used to prove Theorem 2.11 can be carried out in such a way that at each step of the induction the Taylor coefficients of the coefficient functions γ_k are all polynomials in the Moser normal form coefficients $(A^l_{\alpha\bar\beta})$. The details are left to the interested reader.

As a consequence of Proposition 3.6, if N is a normal form then $b\eta_k = \eta_k(0)$ is a polynomial in the $(A^l_{\alpha\bar\beta})$, which by (3.5) is an invariant of weight $k(n+1)$. In order to construct invariants involving derivatives of the η_k we proceed as follows. First observe that (3.5) may be reformulated as stating that $|z_0|^{-2k}\eta_k$ is invariantly defined on $\mathbb{C}^* \times V$ modulo $0(\varrho^{n+1})$. Thus the tensors $\nabla^l(|z_0|^{-2k}\eta_k)$ are also invariantly defined, where ∇^l is a covariant derivative of order l with respect to the metric g, holomorphic in some indices and antiholomorphic in others. These tensors can now be included in the list of ingredients to be used in the process of taking tensor products and contracting which was used to form scalar invariants from the covariant derivatives of curvature of g in Fefferman's construction. This results in

THEOREM 3.7: *Let $s, t \geqslant 0$; for $1 \leqslant i \leqslant s$ let $0 \leqslant m_i \leqslant n-3$ and for $1 \leqslant j \leqslant t$ let $0 \leqslant l_j \leqslant n$ and $1 \leqslant k_j$. Then the expression*

$$\operatorname{contr}\,(\nabla^{m_1}R \otimes \cdots \otimes \nabla^{m_s}R \otimes \nabla^{l_1}(|z_0|^{-2k_1}\eta_{k_1}) \otimes \cdots \otimes \nabla^{l_t}(|z_0|^{-2k_t}\eta_{k_t})), \tag{3.8}$$

when computed for a normal form N and evaluated at $z_0 = 1, z = w = 0$, is a polynomial in the $(A^l_{\alpha\bar\beta})$. This polynomial is an invariant of weight

$$w = s + \frac{1}{2}\sum_{i=1}^{s} m_i + \sum_{j=1}^{t}\,[(n+1)k_j + \tfrac{1}{2}l_j].$$

As before contr *indicates complete contraction with respect to $g_{i\bar j}$ for some pairing of the indices.*

Note. In case $s = 0$, $t = 1$, $l = 0$, the above invariant is $b\eta_k$.

Proof. First any component of the tensor $\nabla^{m_i}R$ involves at most $m_i + 2$ derivatives of g, so $m_i + 4$ derivatives of ϱ. As $m_i + 4 \leqslant n + 1$ and ϱ is determined mod $0(\varrho^{n+2})$, it follows that all components of $\nabla^{m_i}R$

are independent of the ambiguity in ϱ. Similarly the components of $\nabla^{l_j}(|z_0|^{-2k_j}\eta_{k_j})$ involve at most l_j derivatives of η_{k_j} and $l_j + 1$ derivatives of ϱ so are also independent of the ambiguity of ϱ and η_{k_j}. Moreover the components are all expressible as universal polynomials in derivatives of $g_{i\bar{j}}$, $g^{i\bar{j}}$, $|z_0|^{-2k_j}\eta_{k_j}$. As in [5], when evaluated at $z_0 = 1$, $z = w = 0$ for a normal form, the derivatives of $g_{i\bar{j}}$, $g^{i\bar{j}}$ which occur are universal polynomials in the $(A^l_{\alpha\bar{\beta}})$, and the same is true for $|z_0|^{-2k_j}\eta_{k_j}$ in view of Proposition 3.6. Thus when evaluated at $z_0 = 1$, $z = w = 0$, the expression (3.8) is a polynomial in the Moser normal form coefficients.

To prove the invariance, for $\lambda \in \mathbb{C}^*$ consider the map M_λ: $\mathbb{C}^* \times V \to \mathbb{C}^* \times V$ defined by $M_\lambda(z_0, z) = (\lambda z_0, z)$. Then $M_\lambda^*(g) = |\lambda|^{2/(n+1)}g$, so also $M_\lambda^*(\nabla^{m_i}R) = |\lambda|^{2/(n+1)}\nabla^{m_i}R$. Additionally one has $M_\lambda^*(\nabla^{l_j}(|z_0|^{-2k_j}\eta_{k_j})) = |\lambda|^{-2k_j}\nabla^{l_j}(|z_0|^{-2k_j}\eta_{k_j})$. Now the covariant tensor $T = \nabla^{m_1}R \otimes \cdots \otimes \nabla^{m_s}R \otimes \nabla^{l_1}(|z_0|^{-2k_1}\eta_{k_1}) \otimes \cdots \otimes \nabla^{l_t}(|z_0|^{-2k_t}\eta_{k_t})$ has rank $4s + \Sigma\, m_i + \Sigma\, l_j$. Thus a complete contraction of T involves $2s + \frac{1}{2}\Sigma\, m_i + \frac{1}{2}\Sigma\, l_j$ contractions, each of which corresponds to one factor of g^{-1}, which satisfies $M_\lambda^* g^{-1} = |\lambda|^{-2/(n+1)}g^{-1}$. Putting all of this together, it follows that

$$M_\lambda^*(\text{contr } T) = |\lambda|^{(-2w)/(n+1)} \text{ contr } T, \text{ since}$$

$$-\frac{2w}{n+1} = \left(2s + \tfrac{1}{2}\sum m_i + \tfrac{1}{2}\sum l_i\right)\left(-\frac{2}{n+1}\right) + s\left(\frac{2}{n+1}\right) + \sum_{j=1}^{t}(-2k_j).$$

Otherwise put, contr T is of the form $|z_0|^{(-2w)/(n+1)}P(z)$ for some function P depending only on $z \in V$. Recalling the transformation law $\tilde{z}_0 = z_0(\det \Phi')^{-1}$ for z_0 under a biholomorphism Φ: $N \to \tilde{N}$ between two normal forms, it follows immediately that the value of (3.8) evaluated at $z_0 = 1$, $z = w = 0$, satisfies the transformation law of Definition 3.2 for an invariant of weight w. Thus Theorem 3.7 is proved.

Remark: There are other conditions that one can impose on an expression of the form (3.8) to insure that it is independent of the ambiguities of ϱ and the η_k and thus defines an invariant of weight w. For example, instead of simply counting the number of derivatives of each term one can count the total number of derivatives in a nonisotropic fashion, as in the Ambiguity Lemma of [5]. This leads to the conclusion that if $m_i \geqslant 0$ and $l_j \geqslant 0$ are such that $w \leqslant n + \Sigma_{j=1}^{t} k_j(n + 1)$, then the results of Theorem 3.7 still hold.

We also remark that not all invariants arise from the construction of Theorem 3.7. In fact, in [6] an example is given of a weight 5 invariant in $\mathbb{C}^2$ which is not of the form (3.8).

4. The coefficient of the first log term

Proposition 2.16 shows that the vanishing of the invariant $b\eta_1 = \eta_1|_M$ governs whether or not any log terms occur in solutions of the Monge–Ampère equation. In this section we derive an explicit formula for the linear part of $b\eta_1$ as a polynomial in the Moser normal form coefficients. We also show that there are hypersurfaces inequivalent to the sphere for which $b\eta_1 = 0$.

As an invariant of weight $w = n + 1$, $b\eta_1$ is a polynomial in the Moser normal form coefficients $(A^l_{\alpha\bar\beta})$ which is homogeneous in the sense of (3.3). Consequently when divided into homogeneous parts in the variables $(A^l_{\alpha\bar\beta})$, its linear part is a linear combination of $A^l_{\alpha\bar\beta}$ for which $|\alpha| + |\beta| + 2l = 2n + 4$. We would like to identify this linear part. To do so requires keeping track of the nonisotropic homogeneity of the terms in the Taylor expansion of a normal form. Recalling the coordinates (z, w) on $\mathbb{C}^n$ as in (3.1), define the strength of z_j, $\bar z_j$ to be 1, the strength of w, $\bar w$ to be 2, and the strength of a monomial in z, $\bar z$, w, $\bar w$ by extending this definition in the obvious way. The term in the normal form expansion of strength $2n + 4$ is then $\sum_{|\alpha|+|\beta|+2l=2n+4} A^l_{\alpha\bar\beta} z^\alpha \bar z^\beta v^l$.

For $\gamma \in \mathbb{C}$ define a second order differential operator L_γ on $\mathbb{C}^n$ by

$$L_\gamma \varphi = \sum_{j=1}^{n-1} (\varphi_{z_j\bar z_j} + z_j \varphi_{z_j \bar w} + \bar z_j \varphi_{\bar z_j w}) + 2u\varphi_{w\bar w} + (\gamma - 1)\varphi_u .$$

Note that L_γ maps polynomials of strength S to polynomials of strength $S - 2$.

PROPOSITION 4.1: *The linear part of $b\eta_1$ is*

$$[(n + 2)!(n + 1)!]^{-1} L_{n+1} L_n \ldots L_1 L_0 \left(\sum_{|\alpha|+|\beta|+2l=2n+4} A^l_{\alpha\bar\beta} z^\alpha \bar z^\beta v^l \right).$$

The proof uses the following lemma, which computes the linearization of the J operator at the origin. Let $r = 2u - |z|^2$.

LEMMA 4.2: *Let $S \geqslant 3$ and let $\psi = r + \varphi_1 + \varphi_2$, where φ_1 is a polynomial of strength S and φ_2 is a polynomial of strength $\geqslant 2S - 2$. Then $J(\psi) = 1 - L_0\varphi_1 +$ (terms of strength $\geqslant 2S - 4$).*

Proof. Set $\tilde{\varphi} = \varphi_1 + \varphi_2$. Then $J(r + \tilde{\varphi}) = (-1)^n \det(A + B)$, where

$$A = \begin{pmatrix} r & -z_j & 1 \\ -\bar{z}_i & -I & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \tilde{\varphi} & \tilde{\varphi}_{\bar{z}_j} & \tilde{\varphi}_{\bar{w}} \\ \tilde{\varphi}_{z_i} & \tilde{\varphi}_{z_i\bar{z}_j} & \tilde{\varphi}_{z_i\bar{w}} \\ \tilde{\varphi}_w & \tilde{\varphi}_{w\bar{z}_j} & \tilde{\varphi}_{w\bar{w}} \end{pmatrix}.$$

The cofactor matrix of A is $C = (-1)^{n-1}\begin{pmatrix} 0 & 0 & -1 \\ 0 & I & z_i \\ -1 & \bar{z}_j & 2u \end{pmatrix}$. Expanding $\det(A + B)$ and decomposing the terms into pieces homogeneous in the entries of A and B gives

$$\det(A + B) = \sum_{k=0}^{n+1} \sum_{I,J} (-1)^\sigma M_J^I(B) M_{\hat{J}}^{\hat{I}}(A), \tag{4.3}$$

where $\Sigma_{I,J}$ is the sum over all pairs I, J of subsets of $\{0, 1, \ldots, n\}$ of cardinality k (we label the rows and columns of A and B by indices running from 0 to n). $\hat{I}$, $\hat{J}$ are the corresponding complementary sets, M_J^I, $M_{\hat{J}}^{\hat{I}}$ are the respective minors of B and A, and for $I = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_k\}$ we have set $\sigma = \Sigma_{l=1}^k (i_l + j_l)$. The first term in (4.3), when $k = 0$, is $\det A = (-1)^n$. The second, when $k = 1$, is $\Sigma_{i,j=0}^n c_{ij} b_{ij} = (-1)^{n-1} L_0 \tilde{\varphi} = (-1)^{n-1} L_0 \varphi_1 +$ (terms of strength $\geqslant 2S - 4$). A straightforward analysis shows that the kth term in (4.3) is of strength $\geqslant k(S - 2)$, so for $k \geqslant 2$ these terms all have strength $\geqslant 2S - 4$. Thus $J(r + \tilde{\varphi}) = (-1)^n \det(A + B) = 1 - L_0\varphi_1 +$ (terms of strength $\geqslant 2S - 4$), and Lemma 4.2 is proved.

We next study the Taylor series of Fefferman's approximate solutions for perturbations of the hyperquadric. Recall that if ψ defines a strictly pseudoconvex hypersurface, then Fefferman's approximate solutions are defined by: $\psi_0 = \psi$,

$$\psi_1 = J(\psi)^{-1/(n+1)}\psi \tag{4.4}$$

and

$$\psi_p = \psi_{p-1}\left[1 + \frac{1 - J(\psi_{p-1})}{p(n + 2 - p)}\right], \quad 2 \leqslant p \leqslant n + 1, \tag{4.5}$$

and one has $J(\psi_p) = 1 + 0(\psi^p)$.

LEMMA 4.6: *Let $\psi = r + \varphi$ where φ is a polynomial of strength $S \geqslant 3$, and let ψ_p be as above. Then for $0 \leqslant p \leqslant n + 1$,*

$$J(\psi_p) = 1 - \frac{(n+1-p)!}{p!(n+1)!} r^p L_p L_{p-1} \ldots L_0 \varphi$$

$$+ \text{(terms of strength } \geqslant 2S - 4).$$

Proof. We show by induction that there are differential operators P_p, Q_p with P_p preserving strength and Q_p lowering strength by 2, so that

$$\psi_p = r + P_p\varphi + (\text{strength} \geqslant 2S - 2) \tag{4.7}$$

and

$$J(\psi_p) = 1 - Q_p\varphi + (\text{strength} \geqslant 2S - 4). \tag{4.8}$$

For $p = 0$ this follows immediately from Lemma 4.2 with $P_0 = \text{Identity}$, $Q_0 = L_0$. (4.4) then gives (4.7) with $p = 1$ and $P_1 = 1/(n+1)rL_0 + 1$. In general, given (4.7), Lemma 4.2 yields (4.8) with $Q_p = L_0 P_p$. And given (4.7) and (4.8), (4.5) yields (4.7) for $p+1$, with $P_{p+1} = P_p + 1/[(n+1-p)(p+1)]rQ_p$.

In order to solve for Q_p, note that

$$Q_{p+1} = L_0 P_{p+1} = L_0 P_p + \frac{1}{(n+1-p)(p+1)} L_0 r Q_p$$

$$= \left(\frac{1}{(n+1-p)(p+1)} L_0 r + 1\right) Q_p.$$

So if we set $M_p = [p!(n+1)!]/[(n+1-p)!]Q_p$, then

$$M_0 = L_0, \quad M_p = [L_0 r + p(n+2-p)]M_{p-1}. \tag{4.9}$$

It is useful to know the commutator of L_0 and r^p. This is an easy direct calculation for $p = 1$; then the general case follows by a straightforward induction on p. The result is

$$[L_0, r^p] = pr^{p-1}\left[r\frac{\partial}{\partial u} - (n+2-p)\right], \quad p \geqslant 1. \tag{4.10}$$

Now the fact that $M_p = r^p L_p L_{p-1} \ldots L_0$ follows easily from (4.9) by induction using (4.10). This gives $Q_p = [(n+1-p)!]/[p!(n-1)!]r^p L_p L_{p-1} \ldots L_0$ and thus proves Lemma 4.6.

Now we can prove Proposition 4.1. As already noted, the linear part of $b\eta_1$ is a linear combination of $A^l_{\alpha\bar{\beta}}$ with $|\alpha| + |\beta| + 2l = 2n + 4$. Since the nonlinear terms in $b\eta_1$ involve only $A^l_{\alpha\bar{\beta}}$ with $|\alpha| + |\beta| + 2l < 2n + 4$, it follows that the linear part of $b\eta_1$ equals $\eta_1(0)$ for the hypersurface $\psi = r + \varphi = 0$, where $\varphi = -\Sigma_{|\alpha|+|\beta|+2l=2n+4} A^l_{\alpha\bar{\beta}} z^\alpha \bar{z}^\beta v^l$. But by §2 the higher asymptotic solutions u are of the form $u = \varrho e^{-f}$ where ϱ is a smooth solution of $J(\varrho) = 1 + 0(\varrho^{n+1})$ and $f = \Sigma_{k=0}^{\infty} \gamma_k(\varrho^{n+1} \log \varrho)^k$. The value of γ_1 on M is given by (2.15): $(n+2)\gamma_1 = [J(\varrho)^{-1} - 1]/(\varrho^{n+1}) = [1 - J(\varrho)]/(\varrho^{n+1})$ on M. So if u is written as (0.2), then

$$b\eta_1 = -b\gamma_1 = \frac{1}{n+2}\frac{J(\varrho) - 1}{\varrho^{n+1}} \text{ on } M. \tag{4.11}$$

In the present notation $\varrho = \psi_{n+1}$, so for our hypersurface $\psi = r + \varphi = 0$, $\eta_1(0)$ can be computed from (4.11) by applying Lemma 4.6 with $p = n + 1$, $S = 2n + 4$. The result is Proposition 4.1.

REMARK 4.13: We mention some results of [6] which are closely related to Proposition 4.1.

(a) It can be shown by combining Proposition 4.1 and invariant theory that the linear part of $b\eta_1$ can also be written

$$(n+2) \sum_{j=0}^{\left[\frac{n-2}{2}\right]} 2^{-2j} \binom{n}{j} \binom{n+2}{2j}^{-1} \operatorname{tr}^p A^{2j}_{p\bar{p}}, \quad \text{where} \quad p = n + 2 - 2j.$$

(b) When $n = 2$ one has $b\eta_1 = 4A^0_{4\bar{4}}$. (A straightforward calculation from Proposition 4.1 shows that when $n = 2$ the linear part of $b\eta_1$ is $4A^0_{4\bar{4}}$. In this case it turns out that $b\eta_1$ equals its linear part.)

(c) For $k \geqslant 2$ the linear part of $b\eta_k$ vanishes.

In conjunction with Proposition 2.16, we have

PROPOSITION 4.14: *There are real analytic strictly pseudoconvex hypersurfaces M which are not biholomorphically equivalent to the sphere, for which* $b\eta_1 = 0$.

Proof. Following a suggestion of D. Burns, we apply the Cauchy-Kowalewski theorem to the differential equation $K(\psi) = 0$, where $K(\psi) = [J(\psi_{n+1}) - 1]/(\psi^{n+1}_{n+1})$ and ψ_{n+1} is given by (4.4), (4.5). It is clear from (4.4), (4.5) that for $1 \leqslant p \leqslant n + 1$ the map $\psi \rightarrow \psi_p$ is a nonlinear differential operator of order $2p$ defined on functions ψ for which $J(\psi) > 0$, depending real analytically on ψ and its derivatives. Additionally, a close look at Fefferman's proof of the fact that $J(\psi_p) = 1 + 0(\psi_p^p)$ shows that $[J(\psi_p) - 1]/(\psi_p^p)$ is also a differential operator applied to ψ. In particular $K(\psi)$ is a nonlinear differential operator of order $2n + 4$, depending real analytically on ψ and its derivatives. We pose a Cauchy problem for K, taking as initial hypersurface $\{x_1 = \operatorname{Re} z_1 = 0\}$ and taking for Cauchy data that prescribed by the function $\psi_0 = 2u - |z|^2 + \operatorname{Re} z_1^2\bar{z}_1^{2n+2}$; i.e., require $\psi = \psi_0 \bmod x_1^{2n+4}$ near the origin. It must be shown that the data are consistent and that the problem is noncharacteristic. These follow, respectively, from the following:

$$\text{(i) } K(\psi_0)(0) = 0 \qquad \text{(ii) } \frac{d}{dt} K(\psi_0 + tx_1^{2n+4})(0)|_{t=0} \neq 0.$$

To prove (i) and (ii), apply Lemma 4.6 with $p = n + 1$. As for (i), write $\psi_0 = r + \varphi$ with $\varphi = \operatorname{Re} z_1^2\bar{z}_1^{2n+2}$. So Lemma 4.6 gives $J(\psi_{n+1}) = 1 - (n + 1)!^{-2}r^{n+1}L_{n+1}L_n \ldots L_0\varphi +$ (terms of strength $\geqslant 4n + 4$). Thus

$$K(\psi_0)(0) = -(n + 1)!^{-2}L_{n+1}L_n \ldots L_0\varphi = 0, \quad \text{proving (i).}$$

Similarly, for (ii) we have

$$K(\psi_0 + tx_1^{2n+4})(0) = -(n + 1)!^{-2}L_{n+1}L_n \ldots L_0(\operatorname{Re} z_1^2\bar{z}_1^{2n+4} + tx_1^{2n+4}) = ct$$

with $c \neq 0$.

Thus the Cauchy–Kowalewski theorem implies the existence of a real-analytic function χ near the origin so that $\psi = \psi_0 + x_1^{2n+4}\chi$ satisfies $K(\psi) = 0$. And since $K(\psi_0)(0) = 0$ it must be the case that $\chi(0) = 0$. Now the hypersurface $M = \{\psi = 0\}$ fulfills the requirements of Proposition 4.14. In fact by (4.11), $b\eta_1 = 1/(n + 2)K(\psi) = 0$ on M. That M is inequivalent to the sphere follows from the fact that ψ is in normal form through terms of strength $2n + 4$, so there is a normal form for M which agrees with ψ through terms of strength $2n + 4$. But the unique normal form for the sphere is $2u - |z|^2 = 0$.

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References

1. J. Bland, Local boundary behaviour of the canonical Einstein-Kähler metric on pseudo-convex domains, UCLA PhD. thesis, 1982.
2. S.-Y. Cheng and S.-T. Yau, On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation, *Comm. Pure Appl. Math* 33 (1980) 507–544.
3. S.S. Chern and J. Moser, Real hypersurfaces in complex manifolds, *Acta Math.* 133 (1974) 219–271.
4. C. Fefferman, Monge–Ampère equations, the Bergman kernel, and geometry of pseudo-convex domains, *Ann. Math.* 103 (1976) 395–416.
5. C. Fefferman, Parabolic invariant theory in complex analysis, *Adv. in Math.* 31 (1979) 131–262.
6. R. Graham, Scalar boundary invariants and the Bergman kernel, *Proceedings of the Special Year in Complex Analysis*, Univ. of Maryland, to appear.
7. J. Lee, Higher asymptotics of the complex Monge-Ampère equation and geometry of CR-manifolds, MIT PhD. thesis, 1982.
8. J. Lee and R. Melrose, Boundary behaviour of the complex Monge-Ampère equation, *Acta. Math.* 148 (1982) 159–192.
9. R. Melrose, Transformation of boundary problems, *Acta. Math.* 147 (1981) 149–236.