COMPOSITIO MATHEMATICA

NOLAN R. WALLACH

On the irreducibility and inequivalence of unitary representations of gauge groups

Compositio Mathematica, tome 64, nº 1 (1987), p. 3-29

<http://www.numdam.org/item?id=CM_1987__64_1_3_0>

Compositio Mathematica 64: 3–29 (1987)

On the irreducibility and inequivalence of unitary representations of gauge groups

NOLAN R. WALLACH
Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

Received 19 December 1986; accepted 2 March 1987

Introduction

Let M be a connected, smooth, orientable, paracompact, n dimensional manifold. Let $\langle\ ,\ \rangle$ be a Riemannian structure and ω a volume form on M. Let U be a compact, semi-simple Lie group and let G be the group of all smooth mappings of M into U that equal the identity outside of a compact set with pointwise multiplication. Let B denote an $Ad(U)$-invariant inner product on $\mathfrak{u}$, the Lie algebra of U. Then certain "non-local" unitary representations, $T = T_{\langle\ ,\ \rangle, B, \omega}$, of G were introduced in [I] for $U = SU(2)$ and in [GGV, I, II], [AKT] for general U (see §5).

Our main results on these representations are

(1) If $n \geqslant 3$ then T is irreducible.

(2) Let $n = 2$. Write $\omega = \varrho \mathrm{Vol}_{\langle\ ,\ \rangle}$ ($\mathrm{Vol}_{\langle\ ,\ \rangle}$ a volume element of M with respect to $\langle\ ,\ \rangle$). Fix $\mathfrak{h}$ a maximal abelian subalgebra of $\mathfrak{u}$ and let $\|\cdot\cdot\cdot\|_B$ denote the norm on $\mathfrak{h}^*$ corresponding to B. If $\|\alpha\|_B > (8\pi|\varrho(x)|)^{1/2}$ for $x \in M$ and all roots α of $\mathfrak{u}$ relative to $\mathfrak{h}$ then T is irreducible.

Whenever one has $\langle\ ,\ \rangle$, B, ω as above then one has an inner product $(\ ,\ )_{\langle\ ,\ \rangle, B, \omega}$ on, $\Omega^1_c(M; \mathfrak{u})$, the compactly supported smooth one forms on M with values in $\mathfrak{u}$ (see §6).

(3) If $\dim M = 2$ then assume that $\langle\ ,\ \rangle$, B, ω satisfy the condition in (2). If $\dim M \geqslant 2$ then $T_{\langle\ ,\ \rangle, B, \omega}$ and $T_{\langle\ ,\ \rangle_1, B_1, \omega_1}$ are either equivalent or disjoint. They are equivalent if and only if $(\ ,\ )_{\langle\ ,\ \rangle, B, \omega} = (\ ,\ )_{\langle\ ,\ \rangle_1, B_1, \omega_1}$.

Condition (2) can be made to hold by varying ϱ and or B. (1), (3) for $n \geqslant 5$ and $U = SU(2)$ and ω, ω_1 the Riemannian volume elements are due to [I] (as are the main "algebraic" ideas in the proof of the general case). (1), (2), (3) without any conditions are asserted for $n \geqslant 2$ and M non-compact in [GGV, I]. But that paper is severely flawed. In [GGV, II] there is a proof of (1), (3) for $n \geqslant 4$ if M is the interior of a compact manifold with boundary although it is not clear if this condition is necessary to their proof, since finite volume and non-compact seem to be enough) for the Riemannian volume elements. In [AKT], (1) is proved for $n \geqslant 3$ as is (2) and a slightly weaker form of (3) for $\|\alpha\|_B > (32\pi|\varrho(x)|)^{1/2}$, $x \in M$ for the same class of manifolds as in [GGV, II] (since they refer to this paper for the details of the proof of irreducibility). They also indicate the likelihood that the 32 can be replaced by 8. We note that slight modifications of the original argument are necessary in the case of compact M. Thus the new results in this paper involve establishing the validity of (1), (2), (3) for general manifolds, replacing a 32 by an 8 and a stronger criterion for disjointness.

In this paper, the first four sections contain technical results on Gaussian measures. The representation theory is in Sections 5 and 6. We suggest on first reading that the reader start with Section 6 and refer back to the necessary preliminaries.

As indicated above the main line of the proof of irreducibility is contained in [I] and [GGV, I]. The differences involve precise results on uniform mutual singularity of measures on spaces of distributions (our results can be found in §3 and §4). We give a complete proof of the "algebraic" aspects of the proof of irreducibility in §5, 6 for several reasons. One is that [GGV, I, II] and [AKT] make use of undocumented "well known" results on direct integrals (which are essentially proved in §5). Secondly, there is a rather subtle argument regarding singularity of convolutions in [AKT], Lemma 3.2, for the case when $n = 2$ that we don't understand (this of course, is not meant to imply that it is wrong). We avoid this argument (which also appears in [GGV, II], however there seems to be no problem with it if $n \geqslant 3$). Thirdly, the details of our argument are necessary in order to prove (3). Fourthly, we fix a minor error in [GGV, II]. Finally, our proof should be accessible to novices to quantum field theory and probability theory.

1. Gaussian measures

Let V be a locally convex, separable, topological vector space over $\mathbb{R}$. Let $(\ ,\ )$ be a continuous, positive definite, symmetric, bilinear form (inner product for short) on V. Let H be the Hilbert space completion of V with respect to $(\ ,\ )$. If W is a topological vector space then we use the notation W' for the space of all continuous linear functionals on W endowed with the weak topology.

If W is a finite dimensional subspace of V and if Ω is a Borel set in W' then we set $Z_{W,\Omega} = \{\lambda \in V' : \lambda_{|W} \in \Omega\}$. $Z_{W,\Omega}$ is called a *cylinder set*. Let A_W be the isomorphism of W onto W' given by $A_W(v)(u) = (u,v)$. Let $\mathrm{d}_W x$ denote the

Lebesgue measure on W corresponding to an orthonormal basis of W relative to $(\ ,\ )$. If $\dim W = n$ then we set (cf. [GV], IV, 3.1)

$$\mu(Z_{W,\Omega}) = (2\pi)^{-n/2} \int_{A_W^{-1}(\Omega)} \mathrm{e}^{-(x,x)/2}\, \mathrm{d}_W x.$$

Let $\mathscr{B} = \mathscr{B}(V')$ denote the σ-algebra of sets generated by the cylinder sets. We assume that $\mu = \mu_{(\ ,\ )}$ has a countably additive extension to $\mathscr{B}$. In this case μ is a probability measure which is called the Gaussian measure associated with $(\ ,\ )$.

The following simple lemma will be used often in this paper.

LEMMA 1.1. *Let $v \in V$ be such that $(v, v) = 1$. Then if $r \geqslant 1$*

$$\mu(\{\lambda \in V' : |\lambda(v)| \leqslant r\}) \geqslant 1 - \mathrm{e}^{-r^2/2}.$$

Proof. By definition, the measure of the indicated set is

$$(2\pi)^{-1/2} \int_{-r}^{r} \mathrm{e}^{-x^2/2}\, \mathrm{d}x = (1/\Gamma(1/2)) \int_0^{r^2/2} x^{-1/2}\, \mathrm{e}^{-x}\, \mathrm{d}x$$

$$= 1 - (1/\Gamma(1/2)) \int_{r^2/2}^{\infty} x^{-1/2}\, \mathrm{e}^{-x}\, \mathrm{d}x \geqslant 1 - (2^{1/2}/\Gamma(1/2)) \int_{r^2/2}^{\infty} \mathrm{e}^{-x}\, \mathrm{d}x$$

$$= 1 - (1/\pi)^{1/2}\, \mathrm{e}^{-r^2/2} \geqslant 1 - \mathrm{e}^{-r^2/2}.$$

We now record two results. The first will be used later. The second is standard but it gives a simple instance of the technique that we will use to prove singularity of measures.

LEMMA 1.2. *Let $S, T \subset V'$. Suppose that for each $\varepsilon > 0$ there exists $X_\varepsilon \in \mathscr{B}$ such that $\mu(X_\varepsilon) \geqslant 1 - \varepsilon$, $X_\varepsilon + \lambda = X_\varepsilon$ for $\lambda \in T$ and $\mu(X_\varepsilon + \lambda) \leqslant \varepsilon$ for $\lambda \in S$. Then there exists $Y \in \mathscr{B}$ such that $\mu(Y) = 1$, $Y + \lambda = Y$ for $\lambda \in T$ and $\mu(Y + \lambda) = 0$ for $\lambda \in S$.*

Proof. Set $Z_n = X_{2^{-n}}$. Put $Y = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} Z_n$.

If $h \in H$ then we define $\lambda_h \in V'$ by $\lambda_h(v) = (v, h)$. If $\lambda \in V'$ then we say $\lambda \in H$ if $\lambda = \lambda_h$ for some $h \in H$.

LEMMA 1.3. *If $\lambda \in V'$. $\lambda \notin H$ then there exists $X \in \mathscr{B}$ such that $\mu(X) = 1$ and $\mu(X + \lambda) = 0$.*

Proof. Since $\lambda \notin H$ there exists for each $n \geqslant 1$, $v_n \in V$ such that $\lambda(v_n) \geqslant n$ and $(v_n, v_n) = 1$. Set $Z_n = \{\xi \in V' : |\xi(v_n)| < n/2\}$. Then $Z_n + \lambda \subset V' - Z_n$. So Lemma 1.1 implies that

$$\mu(Z_n) \geqslant 1 - e^{-n^2/8}$$

and

$$\mu(Z_n + \lambda) \leqslant \mu(V' - Z_n) \leqslant e^{-n^2/8}.$$

Now apply the previous lemma with $S = \{\lambda\}$, T the empty set.

2. Some observations about the first Sobelev space

We first prove a simple lemma which will be adequate to prove our results on Gaussian measures for $n \geqslant 3$. If η is a 1-form on $\mathbb{R}^n$, $\eta = \Sigma\, \eta_i\, dx_i$ then set

$$\|\eta\|^2 = \int \left(\sum (\eta_i)^2\right) dx.$$

We use the notation $B(x; r)$ for the open r-ball with center x in $\mathbb{R}^n$. $\|\cdots\|$ will denote the usual Hilbert space norm on $\mathbb{R}^n$.

LEMMA 2.1. *Assume that $n \geqslant 3$. Then there exists $C(n) > 0$ depending only on n such that for each $0 < \varepsilon < 1/2$ there exists $f_\varepsilon \in C_c^\infty(B(0; 1))$ with*
(1) $\|df_\varepsilon\| = 1$,
(2) $f_\varepsilon(x) \geqslant \varepsilon^{-n/2+1} C(n)$ for $\|x\| \leqslant \varepsilon$.

Proof. Let $h \in C^\infty(\mathbb{R})$ be such that $h(x) = 1$ for $x \leqslant 1$ and $h(x) = 0$ for $x \geqslant 2$. Set (as usual) $r(x) = \|x\|$. Put $\varphi_\varepsilon(x) = h(r(x)/\varepsilon)$. Then $\varphi_\varepsilon \in C_c^\infty(B(0; 1))$ if $0 < \varepsilon < 1/2$ and $d\varphi_\varepsilon = h'(r/\varepsilon)dr/\varepsilon$. Let Ω_n denote the volume of the $n - 1$ dimensional Euclidean sphere. Then

$$\|d\varphi_\varepsilon\|^2 = (\Omega_n/\varepsilon^2) \int_0^\infty r^{n-1} h'(r/\varepsilon)^2\, dr = \varepsilon^{n-2}\Omega_n \int_0^\infty r^{n-1} h'(r)^2\, dr.$$

Set

$$C(n)^{-2} = \Omega_n \int_0^\infty r^{n-1} h'(r)^2\, dr$$

and put $f_\varepsilon = \varepsilon^{-n/2+1} C(n) \varphi_\varepsilon$.

We now prove an analogous result for $n = 2$. In this case one can show (as was pointed out to us by Roger Nussbaum) the estimates are best possible.

LEMMA 2.2. *Let* $n = 2$. *Given* $0 < C < 1$ *there exists for each* $0 < \varepsilon < 1$, $f_{\varepsilon,C} \in C_c^\infty(B(0;\, 1))$ *such that*
(1) $\|df_{\varepsilon,C}\| = 1$,
(2) $f_{\varepsilon,C}(x) \geqslant (C/(2\pi)^{1/2})|\log \varepsilon|^{1/2}$ for $\|x\| \leqslant \varepsilon$.

Before we give the proof we recall some well known (or easily proved) sophomore calculus results. If f is a continuous function on $\mathbb{R}^2$ then we say that $\partial f/\partial x = u$ and $\partial f/\partial y = v$ in L^2 if u and v are square integrable and whenever $g \in C_c^\infty(\mathbb{R}^2)$

$$\int f(x, y) \frac{\partial g}{\partial x}(x, y)\, dx\, dy = -\int u(x, y) g(x, y)\, dx\, dy,$$

$$\int f(x, y) \frac{\partial g}{\partial y}(x, y)\, dx\, dy = -\int v(x, y) g(x, y)\, dx\, dy.$$

Under this condition we write $df = u dx + v dy$ and we say that df exists in L^2.

The following assertion is an easy calculation using Stoke's theorem.

(1) Let h be a piecewise smooth function on $\mathbb{R}$ with supp $h \subset (-\infty, a]$, $a < \infty$, such that h is constant in a neighborhood of 0. Set $f(x) = h(r(x))$. Then df exists in L^2 and $df = h'(r)dr$.

If $f \in L^1(\mathbb{R}^2)$ and if $g \in L^2(\mathbb{R}^2)$ then we set (as usual)

$$f * g(z) = \int f(u) g(z - u) du = \int f(z - u) g(u) du.$$

The following result is also standard.

(2) Let f be continuous on $\mathbb{R}^2$ with supp $f \subset B(0;\, 1 - \eta)$ for some $0 < \eta < 1$ and suppose that df exists in L^2. Then given $\varepsilon > 0$ there exists $0 < \delta < \eta$ and $\varphi \in C_c^\infty(B(0;\, \delta))$ such that

$$\|d(\varphi * f)\| \leqslant \|df\|,$$

$$\sup_{x \in B(0;1)} |\varphi * f(x) - f(x)| < \varepsilon.$$

Indeed, the first inequality is true for any non-negative φ with L^1-norm one without any assumption on δ, and the second is an easy consequence of uniform continuity.

We now give the proof of Lemma 2.2. For each $0 < \delta < 1 - \varepsilon$ define $h_{\varepsilon,\delta}$ by

$$h_{\varepsilon,\delta}(x) = \begin{cases} 1 & x \leqslant \varepsilon \\ \dfrac{\log x - \log(1-\delta)}{\log \varepsilon - \log(1-\delta)} & \varepsilon \leqslant x \leqslant 1-\delta \\ 0 & 1-\delta \leqslant x. \end{cases}$$

Set $\varphi_{\varepsilon,\delta}(z) = h_{\varepsilon,\delta}(r(z))$ for $z \in \mathbb{R}^2$. If we apply (1) above we have

$$\|d\varphi_{\varepsilon,\delta}\|^2 = \frac{2\pi}{(\log \varepsilon - \log(1-\delta))^2} \int_\varepsilon^{1-\delta} dr/r$$

$$= \frac{2\pi}{|\log \varepsilon| \left|1 - \dfrac{\log(1-\delta)}{\log \varepsilon}\right|}.$$

Set $C(\varepsilon, \delta)^2 = 2\pi/|1 - (\log(1-\delta))/\log \varepsilon|$. Let $\mu > 0$ be given. Then (2) above implies that there exists $u \in C_c^\infty(B(0;\eta))$, $0 < \eta < \delta$, with $u \geqslant 0$ and $\int u = 1$ such that

$$\|d(u * \varphi_{\varepsilon,\delta})\| \leqslant C(\varepsilon, \delta)/|\log \varepsilon|^{1/2}$$

and

$$u * \varphi_{\varepsilon,\delta}(x) \geqslant 1 - \mu \quad \text{for} \quad \|x\| \leqslant \varepsilon.$$

Put $g_{\varepsilon,\delta,\mu} = u * \varphi_{\varepsilon,\delta}/\|d(u * \varphi_{\varepsilon,\delta})\|$. Then $\|dg_{\varepsilon,\delta,\mu}\| = 1$ and

$$g_{\varepsilon,\delta,\mu}(x) \geqslant (1-\mu)C(\varepsilon,\delta)^{-1}|\log \varepsilon|^{1/2} \quad \text{for} \quad \|x\| \leqslant \varepsilon.$$

Now $\lim_{\delta,\mu\to 0}(1-\mu)C(\varepsilon,\delta)^{-1} = (2\pi)^{-1/2}$. So we can take $f_{\varepsilon,C} = g_{\varepsilon,\delta,\mu}$ for δ and μ sufficiently small.

The following simple covering Lemma is sufficient for our purposes.

LEMMA 2.3. *Let $n \geqslant 2$. If $0 < \varepsilon < 1/4$ then there exist $z_1, z_2, \ldots, z_N \in B(0; 5/4)$ with $N \leqslant (2\sqrt{n} + 1)^n/\varepsilon^n$ such that $B(0; 1) \subset U_{j=1}^N B(z_j; \varepsilon)$.*

Note. $B(z_j; 1) \subset B(0; 5/2)$.

Proof. This is standard. For each $x \in B(0; 1)$ let $m_i \in Z$ be the unique element that satisfies $m_i \leqslant x_i\sqrt{n}/\varepsilon < m_i + 1$. Put $m(x) = (m_1, \ldots, m_n)$. Then $\|x - \varepsilon m(x)/\sqrt{n}\| < \varepsilon$. Also $|m_i| < \sqrt{n}/\varepsilon + 1$. Take the z_j to be an enumeration of the set $\{\varepsilon m(x)/\sqrt{n}: x \in B(0; 1)\}$. Clearly, there are at most $2(\sqrt{n}/\varepsilon + 3)^n$ such points.

3. Singularity of translates of Gaussian measures

Let M be a smooth, orientable, paracompact, connected manifold of dimension n. Let W be a finite dimensional vector space over $\mathbb{R}$. If M is non-compact set $V = C_c^\infty(M; W)$, the smooth compactly supported functions on M with values in W. If M is compact then we fix once and for all a base point $x_0 \in M$ and set $V = \{f \in C^\infty(M; W): f(x_0) = 0\}$. If K is a compact set in M set $C_K^\infty(M; W)$ equal to the smooth functions from M to W that equal 0 outside of K. We endow $C_K^\infty(M; W)$ with the topology of uniform convergence with all derivatives. If M is non-compact then we endow V with the "union topology" (cf. [GV], p. 330). If M is compact then we use the topology of uniform convergence with all derivatives. Then V is either a nuclear space or a "union of nuclear spaces" (cf. [GV], p. 330). Let $\Omega^1(M; W)$ denote the space of all 1-forms on M with values in W. Let $\langle\ ,\ \rangle$ be a Riemannian structure on M, B an inner product on W and ω a volume form on M. If $\alpha, \beta \in \Omega^1(M; W)$ then $\alpha_x, \beta_x \in \mathrm{Hom}_{\mathbb{R}}(TM_x, W)$. We write $(\alpha, \beta)_x$ for the Hilbert-Schmidt inner product of α_x with β_x. That is, $(\alpha, \beta)_x = Tr(\beta_x^* \alpha_x)$. If $f, g \in V$ then we set

$$(f, g) = \int_M (\mathrm{d}f, \mathrm{d}g)_x \omega.$$

Let μ denote the corresponding Gaussian measure on V' (§1). Then μ is countably additive ([GV], Theorem 6, p. 333).

If $v \in W'$ and if $x \in M$ ($x \neq x_0$ if M is compact) then we set $v_x(f) = v(f(x))$ for $f \in V$. Then $v_x \in V'$ and $v_x = 0$ if and only if $v = 0$. We set $\|v\|_B = \sup_{B(w,w)=1} |v(w)|$. We also write $\omega = \varrho\, \mathrm{Vol}_{\langle,\rangle}$ as in the introduction. If it is necessary to indicate the dependence of $(\ ,\ )$ and μ on $\langle\ ,\ \rangle$, B, ω then we write $(\ ,\ )_{\langle,\rangle,B,\omega}$ and $\mu_{\langle,\rangle,B,\omega}$.

PROPOSITION 3.1. *Let* $v_1, \ldots, v_d \in W' - (0)$. *Let* $\langle\ ,\ \rangle_i$, B_i, ω_i *be as above for* $i = 1, 2$. *Set* $\mu_i = \mu_{\langle,\rangle_i,B_i,\omega_i}$. *If* $n \geqslant 3$ *there is no additional condition. If* $n = 2$ *we assume*

$$\|v_i\|_{B_1} > \sqrt{8\pi|\varrho_1(x)|} \quad \text{for} \quad x \in M.$$

Then if $n \geqslant 2$ *there exists* $X \in \mathscr{B}$ *such that* $\mu_i(X) = 1$ *and* $\mu_2(X + \Sigma(v_i)_{x_i}) = 0$ *for* $x_1, \ldots, x_d \in M$ ($M - x_0$ *if* M *is compact*) *with* $x_i \neq x_j$ *if* $i \neq j$.

We will derive this result from a lemma which will also be used in §5.

LEMMA 3.2. *Let* μ_i, $i = 1.2$ *be as in Proposition 3.1. Assume that* $v \in W^* - \{0\}$ *and* if $\dim M = 2$ *that* $\|v\|_{B_1} > (8\pi|\varrho_1(x)|)^{1/2}$ for $x \in M$. *If* U *is an open subset of* M (*not containing* x_0 *if* M *is compact*) *then there exists* $Y \in \mathscr{B}$ such that $\mu_l(Y) = 1$, $Y + \varphi_y = Y$ for $y \in M - U$, $\varphi \in W^*$, *and* $\mu_2(Y + v_x) = 0$ *for* $x \in U$.

We first show that Lemma 3.2 implies Proposition 3.1. So assume it. Set $M^d = \{(x_1, \ldots, x_d) \in \times^d M : x_i \neq x_j$ if $i \neq j (x_i \neq x_0$ if M is compact$)\}$.

(1) If $x \in M^d$ then there exists an open neighborhood, U_x, of x in M^d and $Y_x \in \mathscr{B}$ such that $\mu_1(Y_x) = 1$ and $\mu_2(Y_x + \Sigma(v_j)_{y_j}) = 0$ for $y \in U_x$.

Let us show how (1) implies the proposition. Then we will use Lemma 3.2 to prove (1). Clearly, M^d is separable. There is therefore a countable sub-covering $\{U_{x_j}\}$ of $\{U_x\}_{x \in M^d}$. Set $U_j = U_{x_j}$ and $Y_j = Y_{x_j}$. Then

$$\mu_1(Y_i) = 1 \quad \text{and} \quad \mu_2\Big(Y_i + \sum_J (v_j)_{y_j}\Big) = 0 \quad \text{for} \quad y \in U_i.$$

Take $X = \bigcap Y_i$.

We now derive (1). Let $x = (x_1, \ldots, x_d)$. Let W_i, $i = 1, \ldots, d$ be an open neighborhood of x_i such that $W_i \cap W_j = \varnothing$ if $i \neq j$ and $x_0 \notin W_i$ if M is compact. Let $U_x = W_1 \times \ldots \times W_d$. Let Y_x be the "Y" of Lemma 3.2 for $U = W_1$ and $v = v_1$. Then $\mu_1(Y_x) = 1$. If $y = (y_1, \ldots, y_d) \in U_x$ then $y_j \in M - W_1$ for $j \geqslant 2$. Thus $Y_x + \Sigma_{j \geqslant 2}(v_j)_{y_j} = Y_x$. Thus $\mu_2(Y_x + \Sigma_{j \geqslant 1}(v_j)_{y_j}) = 0$ for $y \in U_x$.

We are left with the proof of Lemma 3.2. As above the following "local" assertion implies the lemma.

(2) If $x \in U$ then there exist a neightborhood, U_x, of x in U and $Y_x \in \mathscr{B}$ such that $\mu_1(Y_x) = 1$, $Y_x + \varphi_y = Y_x$ for $\varphi \in W^*$, $y \in M - U$ and $\mu_2(Y_x + v_y) = 0$ if $y \in U_x$.

We are left with the proof of (2). Since the proof in the case of $n \geqslant 3$ fairly simple and contains most of the essential ideas for the more delicate case of $n = 2$, we will now give the complete proof for the case $n \geqslant 3$. It would be worthwhile to read this even if there is only interest in the case $n = 2$ which we will prove in the next section.

Let $x \in U$. Let (U_1, Ψ) be a chart for U such that $x \in U_1$ and

(i) $\Psi(U_1) = B(0; 3)$,

(ii) $\Psi(x) = 0$.

Set $V_1 = (\Psi)^{-1}(B(0;\ 5/2))$. On $\mathbb{R}^n$ we use the usual Riemannian structure and Lebesgue measure for the volume element. We will write $\|\cdots\|_i$ for the pointwise norm on Hom (TM^*, W) corresponding to $\langle\ ,\ \rangle_i$ and B_i. We fix an arbitrary inner product, B, on W^* and write $\|\ \|$ for the norm on $(\mathbb{R}^n)^* \otimes W^*$ corresponding to the usual inner product tensored with B.
(A) There exist constants D_1, $D_2 > 0$ such that if $f \in C_c^\infty(V_1;\ W)$ then

$$D_1 \int_{B(0;3)} \|\mathrm{d}(f \cdot \Psi^{-1})(x)\|^2\ \mathrm{d}x \leqslant \int_M \|\mathrm{d}f\|_{i,x}^2 \omega_i$$

$$\leqslant D_2 \int_{B(0;3)} \|\mathrm{d}(f \cdot \Psi)^{-1}\ (x)\|^2\ \mathrm{d}x.$$

Let $\{y_1, \ldots, y_n\}$ be the local coordinates on U_1 corresponding to Ψ. Set

$$g_{p,q,i}(z) = \langle \mathrm{d}y_p, \mathrm{d}y_q \rangle_{i,x}, \quad z \in U_1.$$

Then there exist α_i, $\beta_i > 0$ such that if $z \in$ Closure (V_1) then

$$\alpha_i \langle y, y \rangle \leqslant \sum g_{p,q,i}(z) y_p y_q \leqslant \beta_i \langle y, y \rangle, \quad y \in \mathbb{R}^n.$$

Also, $\omega_{i|U_1} = u_i\ \mathrm{d}y_1 \Lambda \mathrm{d}y_2 \Lambda \ldots \Lambda \mathrm{d}y_n$. There exist γ_i, $\delta_i > 0$ such that

$$\gamma_i \leqslant |u_i(z)| \leqslant \delta_i, \quad z \in \text{Closure}\ (V_1).$$

There exist M_i, $m_i > 0$ such that

$$m_i B(v, v) \leqslant B_i(v, v) \leqslant M_i B(v, v), \quad v \in W^*.$$

Take $D_1 = \min\{\gamma_i \alpha_i m_i\}$, $D_2 = \max\{\delta_i \beta_i M_i\}$. (A) now follows.

Note. We will also use this result in our proof in the case $n = 2$.

Set $U_x = \Psi^{-1}(B(0;\ 1))$.

(B) There exist E, $F > 0$ such that if $1/4 > \varepsilon > 0$ there exists an open covering W_i, $i = 1, \ldots, N \leqslant E\varepsilon^{-n}$, of U_x with $W_i \subset V_1$ and there exist $f_{i,\varepsilon} \in C_c^\infty(V_1)$ such that
  (a) $\|\mathrm{d}f_{i,\varepsilon}\| = 1$ and
  (b) $f_{i,\varepsilon}(z) \geqslant F\varepsilon^{-n/2+1}$ for $z \in W_i$.

Indeed, let f_ε be as in Lemma 2.1 and $z_1, \ldots, z_N$ be as in Lemma 2.3. Set $W_i = \Psi^{-1}(B(z_i;\ \varepsilon))$. Put $g_{i,\varepsilon}(y) = f_\varepsilon(\Psi(y) - z_i)$ if $y \in V_1$ and 0 otherwise. Then $\|\mathrm{d}g_{i,\varepsilon}\|_1 \leqslant (D_2)^{1/2}$ by (A) and $g_{i,\varepsilon}(y) \geqslant C(n)\varepsilon^{-n/2+1}$. Set $f_{i,\varepsilon} = g_{i,\varepsilon}/\|\mathrm{d}g_{i,\varepsilon}\|_1$, $F = C(n)/(D_2)^{1/2}$ and $E = (2n^{1/2} + 1)$. This proves (B).

Let $w \in W$ be such that $B_1(w, w) = 1$ and $v(w) = \|v\|_{B_1}$. Set $\alpha_{i,\varepsilon} = f_{i,\varepsilon} \otimes w$. Then $\|d\alpha_{i,\varepsilon}\| = 1$. If $u \in W_j$ then

$$(v)_u(\alpha_{j,\varepsilon}) = f_{j,\varepsilon}(u)\|v\|_{B_1} \geqslant C\varepsilon^{-n/2+1}$$

with $C = F\|v\|_{B_1}$.

Set $Z_{j,\varepsilon} = \{\lambda \in V' : |\lambda(\alpha_{j,\varepsilon})| < C\varepsilon^{-n/2+1}/2\}$. Then Lemma 1.1 implies that

$$\mu_1(Z_{j,\varepsilon}) \geqslant 1 - e^{-C^2\varepsilon^{-n+2}/8}.$$

If $y \notin U$ then $\alpha_{j,\varepsilon}(y) = 0$ so $Z_{j,\varepsilon} + \varphi_y = Z_{j,\varepsilon}$ for $\varphi \in W^*$. Also, if $u \in U_x$ then

$$Z_{j,\varepsilon} + v_u \subset \{\lambda \in V' : \lambda(\alpha_{j,\varepsilon}/\|\alpha_{j,\varepsilon}\|_2) \geqslant C\varepsilon^{-n/2+1}/2\|\alpha_{j,\varepsilon}\|_2\}.$$

Hence Lemma 1.1 implies that

$$\mu_2(Z_{j,\varepsilon} + v_u) \leqslant \exp(-C^2\varepsilon^{-n+2}/8\|d\alpha_{j,\varepsilon}\|_2^2).$$

(A) implies that $\|\alpha_{j,\varepsilon}\|_2 \leqslant (D_2/D_1)^{1/2}$. Set $\xi = C^2 D_1/8D_2$. Take $Z_\varepsilon = \bigcap_j Z_{j,\varepsilon}$. Then

$$\mu_1(Z_\varepsilon) \geqslant 1 - E\varepsilon^{-n}e^{-C^2\varepsilon^{-n+2}/8},$$

$$\mu_2(Z_\varepsilon + v_u) \leqslant e^{-\xi\varepsilon^{-n+2}}$$

if $u \in U_x$ and $Z_\varepsilon + \varphi_y = Z_\varepsilon$ if $\varphi \in W^*$ and $y \in M - U$. Since $\lim_{\varepsilon\to 0}\varepsilon^{-n}e^{-C^2\varepsilon^{-n+2}/8} = 0$ if $n \geqslant 3$, this implies that given $\varepsilon > 0$ there exists $Y_\varepsilon \in \mathscr{B}$ such that $\mu_1(Y_\varepsilon) = 1$, $Y_\varepsilon + \varphi_y = Y_\varepsilon$ for $\varphi \in W^*$, $y \in M - U$ and $\mu_2(Y_\varepsilon + v_u) = 0$ for $u \in U_x$. This combined with the argument in the proof of Lemma 1.2 completes the proof of (2) in the case $n \geqslant 3$.

4. The proof of Lemma 3.2 for $n = 2$

In this section we assume that $n = 2$, otherwise the assumptions and notation are as in §3. To simplify notation, we denote $\langle\ ,\ \rangle_1$, B_1, ω_1 by the same symbols without the subscripts. We will be using some elementary Riemannian geometry. For this we refer to [H], Chapter 1. Let $d(x, y)$ be the Riemannian distance on M. If $m \in M$ then we denote by $\|v\|_m$ the norm of

$v \in TM_m$ relative to $\langle\ ,\ \rangle_m$. Put $B(m; x; r) = \{v \in TM_m: \|v - x\|_m < r\}$ for $x \in TM_m$. If $x \in M$ we set $B_x(r) = \{y \in M: \mathrm{d}(x, y) < m\}$. For $m \in M$, let $\exp_m$ be the (geodesic) exponential map of M at m. For each $m \in M$ let $s_m > 0$ be such that $\exp_m: B(m; 0; s_m) \to B_m(s_m)$ is a surjective diffeomorphism.

If we identify $T(TM_m)_v$ for $v \in TM_m$ with TM_m in the canonical way then $\mathrm{d}(\exp_m)_0 = I$ for $m \in M$. This implies that for each $\delta > 0$ there exists $0 < \eta_m(\delta) < s_m$ such that if $f \in C_c^\infty(B_m(\eta_m(\delta))$ then

$$\int_M \|\mathrm{d}f\|^2\omega \leqslant (1 + \delta)|\varrho(m)| \int_{B(m;0;\eta_m(\delta))} \|\mathrm{d}(f \circ \exp_m)\|_m^2 \mathrm{d}_m x. \qquad (*)$$

Here $\mathrm{d}_m x$ is the Lebesgue measure on TM_m corresponding to an orthonormal basis relative to $\langle\ ,\ \rangle_m$.

Fix $x \in U$. Choose $s > 0$ such that $B_x(3s) \subset U$. Put $r(\delta) = \min\{s/3, \eta_x(\delta)/3\}$. Set $U_1 = B_x(3r(\delta))$, $V_1 = B_x(5r(\delta)/2)$, $W_1 = B_x(r(\delta))$. We now prove the assertion of (2), §3, for $U_x = W_1$ if δ is chosen to be sufficiently small. Let $0 < \zeta < 1$, and let $f_{\varepsilon,1-\zeta}$ be as in lemma 2.2. Set $u_{\varepsilon,\zeta}(x) = f_{\varepsilon,1-\zeta}(x/r(\delta))$. Then $\|u_{\varepsilon,\zeta}\| = 1$ (here we are using the norms as in §2) and

$$u_{\varepsilon,\zeta}(x) \geqslant (1 - \zeta)\, |(\log \varepsilon)/2\pi|^{1/2}, \quad \|x\| \leqslant \varepsilon r(\delta).$$

Let $z_1, \ldots, z_N$ ($N \leqslant (1 + 2\sqrt{2})^2\varepsilon^{-2} = E\varepsilon^{-2}$) be as in Lemma 2.3 for $\varepsilon > 0$. Put $Z_i(\delta) = \exp_x(B(x; r(\delta)z_i; r(\delta)\varepsilon))$. Define $\xi_{i,\varepsilon,\delta,\zeta} \in C_c^\infty(V_1)$ by $\xi_{i,\varepsilon,\delta,\zeta}(y) = 0$ if $y \notin V_1$ and $\xi_{i,\varepsilon,\delta,\zeta}(\exp_x(y)) = u_{\varepsilon,\zeta}(y - r(\delta)z_i)$ for $y \in B(x; r(\delta)z_i; r(\delta)\varepsilon)$. Put $f_{i,\varepsilon,\delta,\zeta} = \xi_{i,\varepsilon,\delta,\zeta}/\|\mathrm{d}\xi_{j,\varepsilon,\delta,\zeta}\|$. Then $(*)$ implies that

$$f_{i,\varepsilon,\delta,\zeta}(z) \geqslant (1 - \zeta)|(\log \varepsilon)/2\pi(1 + \delta)\varrho(x)|^{1/2} \text{ for } z \in Z_i(\delta). \qquad (**)$$

Let $w \in W$ be such that $v(w) = \|v\|_B$ and $B(w, w) = 1$. Put

$$\alpha_{i,\varepsilon,\delta.\zeta} = f_{i,\varepsilon,\delta,\zeta} \otimes w.$$

Then $\|\mathrm{d}\alpha_{i,\varepsilon,\delta.\zeta}\| = 1$ and if $u \in Z_i(\delta)$ then

$$\begin{aligned} v_u(\alpha_{i,\varepsilon,\delta.\xi}) &= |v\|_B f_{i,\varepsilon,\delta,\zeta}(u) \\ &\geqslant \|v\|_B(1 - \zeta)|(\log \varepsilon)/2\pi(1 + \delta)\varrho(x)|^{1/2}. \end{aligned}$$

Our assumption on v implies that $\|v\|_B > (8\pi|\varrho(x)|)^{1/2}$. Thus we can choose ζ and δ so small that

$$\|v\|_B(1 - \zeta)|(\log \varepsilon)2\pi(1 + \delta)\varrho(x)|^{1/2} \geqslant (1 + \gamma)(2 + \gamma)|\log \varepsilon|^{1/2}$$

for some $\gamma > 0$. Fix these values of δ and ζ. Set $\alpha_{j,\varepsilon} = \alpha_{j,\varepsilon,\delta,\zeta}$ and $U_x = W_1$.
Set $Z_{j,\varepsilon} = \{\lambda \in V' \colon |\lambda(\alpha_{j,\varepsilon})| < (2 + \gamma)|\log \varepsilon|^{1/2}\}$. Then lemma 1.1 implies that

$$\mu(Z_{j,\varepsilon}) \geqslant 1 - \exp(((2 + \gamma)^2 \log \varepsilon)/2).$$

$$\geqslant 1 - \varepsilon^{2+\gamma}.$$

If $u \in M - U$ and $\varphi \in W$ then $\varphi_u(\alpha_{j,\varepsilon}) = 0$ so $Z_{j,\varepsilon} + \varphi_u = Z_{j,\varepsilon}$. Also if $u \in Z_j$ then $Z_{j,\varepsilon} + (v)_u \subset \{\lambda \in V' \colon \lambda(\alpha_{j,\varepsilon}/\|d\alpha_{j,\varepsilon}\|_2) \geqslant \gamma(2 + \gamma)|\log \varepsilon|^{1/2}/\|d\alpha_{j,\varepsilon}\|_2\}$. Thus, if we set $\xi_j = (\gamma(2 + \gamma)/\|d\alpha_{j,\varepsilon}\|_2)^2/2$ then Lemma 1.1 implies that

$$\mu_2(Z_{j,\varepsilon} + v_u) \leqslant e^{-\xi_j|\log \varepsilon|} = \varepsilon^{\xi_j}.$$

Now §3 (A) implies that there exists a constant $D > 0$ independent of j, ε such that $\|d\alpha_{j,\varepsilon}\| < D$. Thus if we set $\xi = \gamma/2D^2$ then $\xi_j > \xi$. Hence if $u \in Z_j$ then

$$\mu_2(Z_{j,\varepsilon} + v_u) \leqslant \varepsilon^{\xi}$$

Put $Z_\varepsilon = \bigcap_j Z_{j,\varepsilon}$. Then

$$\mu(Z_\varepsilon) \geqslant 1 - E\varepsilon^{-2}\varepsilon^{2+\gamma},$$

$Z_\varepsilon + \varphi_u = Z_\varepsilon$ for $\varphi \in W^*$, $u \in M - U$ and

$$\mu_2(Z_\varepsilon + v_y) \leqslant \varepsilon^{\xi} \quad \text{and} \quad u \in U_x.$$

Thus given $\varepsilon > 0$ there exists $Y_\varepsilon \in \mathscr{B}$ such that $\mu(Y_\varepsilon) \geqslant 1 - \varepsilon$, $Y_\varepsilon + \varphi_y = Y_\varepsilon$ for $y \in M - U$, $\varphi \in W^*$ and $\mu_2(Y_\varepsilon + v_u) \leqslant \varepsilon$ for $u \in U_x$. The result now follows from the argument in the proof of Lemma 1.2.

5. Some representation theory

As in §3, let M be a smooth, paracompact, connected, orientable manifold. Let $(W, (\ ,\ ))$ be a finite dimensional real inner product space. Fix a Riemannian structure, $\langle\ ,\ \rangle$ and a volume element, ω, on M. If M is compact then fix a base point, x_0, set $V = C_c^\infty(M; W)$ if M is non-compact and $V = \{f \in C^\infty(M; W) \colon f(x_0) = 0\}$ if M is compact. We set $Q(f, g) = (df, dg)$ as in §3. Let μ be the Gaussian measure on V' corresponding to Q. We define a unitary representation, S, of V on $L^2(V', \mu)$ by $S(v)f(\lambda) = e^{i\lambda(v)}f(\lambda)$.

Let $v_1, v_2, \ldots$ be a fixed sequence of non-zero (not necessarily distinct) elements of W'. If $\dim M = 2$ then we assume that $\|v_i\| > (8\pi|\varrho(x)|)^{1/2}$ for $x \in M$ and all i. If $I = (i_1, \ldots, i_d)$ define for $x = (x_1, \ldots, x_d) \in M^d$ (see §3), $\Psi_I(x)(f) = \Sigma_j v_{i_j}(f(x_j))$, for $f \in V$. Then Ψ_I defines a continuous mapping of M^d into V' (with the weak topology). It is easily seen that $\Psi(M^d) \in \mathscr{B}$. If E is a Hilbert vector bundle over M^{d+k} for $d \geqslant 0, k \geqslant 0$ and if ω_1 is a volume form on M then we set $L^2(E, \omega_1)$ equal to the space of square integrable cross sections of E (here we use the product measure ω_1^{d+k} on M^{d+k}). If $f \in L^2(E, \omega_1)$, $v \in V$ and $I = (i_1, \ldots, i_d)$ then we set

$$\sigma_{I,E}(v)f(x) = e^{i\Psi_I(x_1,\ldots,x_d)(v)}f(x).$$

Let $\langle\ ,\ \rangle_1$, B_1, ω_1 be respectively a Riemannian structure on M, an inner product on W and a volume form on M. Let Q_1 be the inner product defined as above on V using $\langle\ ,\ \rangle_1$, B_1 and ω_1 in place of $\langle\ ,\ \rangle$, B, ω. Let μ_1 be the corresponding Gaussian measure on V'.

LEMMA 5.1. *Let $d > 0$. If C is a bounded linear operator from $L^2(V', \mu)$ to $L^2(V', \mu_1) \hat{\otimes} L^2(E, \omega_1)$ such that $CS(v) = S(v) \otimes \sigma_{I,E}(v)C$ for all $v \in V$ then $C = 0$.*

Proof. We write $\Psi_I(x) = \Psi_I(x_1, \ldots, x_d)$. Set $\Omega(\lambda) = 1$ for all $\lambda \in V'$. We note that $\text{Closure}(\text{span}(S(v)\Omega)) = L^2(V', \mu)$. Indeed, span $\{e^{i\cdot(v)}: v \in V\}$ is dense in $L^2(V', \mu)$ (cf. [Gu, §7.2]). This implies that $C = 0$ if and only if $C\Omega = 0$. Let $f = C\Omega$. We assume that $f \neq 0$ and derive a contradiction. Then we can look upon f as a function on V' with values in $L^2(E)$. Thus we can write $f(\lambda, x) \in E_x$. Let

$$D = \text{Closure}(\text{span}\{(S(v) \otimes \sigma_{I,E}(v))f: v \in V\}).$$

Then C is a continuous linear map of $L^2(V', \mu)$ onto D. On $V' \times M^{d+k}$ we put the product measure, $\mu \times \omega_1^{d+k}$. Let $\Gamma(\lambda, x) = \lambda + \Psi_I(x) \in V'$ for $\lambda \in V'$, $x \in M^{d+k}$. Then Γ is continous. We define a measure, γ, on V' as follows:

$$\gamma(X) = \int_{\Gamma^{-1}(X)} \|f(\lambda, x)\|_x^2 d\mu_1(\lambda)\omega_1^{d+k}.$$

Thus

$$\gamma(X) = \int_{X \times M^{d+k}} \|f(\lambda - \Psi_I(x), x)\|_x^2 d\mu_1(\lambda - \Psi_I(x))\omega_1^{d+k}. \quad (*)$$

Then the representation of V on D given by the restriction of $S \otimes \sigma_{I,E}$ to D is equivalent with the representation, β, of V on $L^2(V', \gamma)$ with $\beta(v)\varphi(\lambda) = e^{i\lambda(v)}\varphi(\lambda)$. Thus C induces a continuous linear map, C_1, of $L^2(V', \mu)$ into $L^2(V', \gamma)$ such that $C_1 S(v) = \beta(v)C_1$, $v \in V$ and $C_1\Omega = \Omega_1$ (Ω_1 is the constant function 1 on V' looked upon as an element of $L^2(V', \gamma)$). Now

$$(C_1 S(v)\Omega, \Omega_1) = (\beta(v)\Omega_1, \Omega_1) = \int_{V'} e^{i\lambda(v)}\, d\gamma(\lambda).$$

On the other hand

$$(C_1 S(v)\Omega, \Omega_1) = (S(v)\Omega, C_1^*\Omega_1).$$

Set $C_1^*\Omega_1 = h$. Then

$$(C_1 S(v)\Omega, \Omega_1) = \int_{V'} e^{i\lambda(v)}\, \overline{h(\lambda)}\, d\mu(\lambda).$$

Since both γ and $\overline{h(\lambda)}\, d\mu(\lambda)$ are cylinder set measures (finite valued σ-additive measures on $\mathscr{B}$), we see that this implies that $d\gamma = \overline{h(\lambda)}\, d\mu(\lambda)$.

Proposition 3.1 implies that there exists $X \in \mathscr{B}$ such that $\mu(X) = 1$ and $\mu_1(X + \Psi_I(x)) = 0$ for all $x \in M^{d+k}$. Thus $\gamma(X) = \int_X \overline{h(\lambda)}\, d\mu(\lambda) = \int_{V'} \overline{h(\lambda)}\, d\mu(\lambda) = (\Omega_1, \Omega_1) = \|f\|^2$. On the other hand $\gamma(X) = 0$ by (*). This is a contradiction, so the lemma follows.

We now assume that $v_1, \ldots, v_r$ are distinct and satisfy the hypothesis above. Let $E_1, E_2, \ldots, E_r$ be Hermitian vector bundles over M. We define an action of V on each E_i by $\sigma_j(v)_{x|(E_j)_x} = e^{iv_j(v(x_j))} I$. Let $E = E_1 \oplus E_2 \oplus \ldots \oplus E_r$ with action $\oplus\, \sigma_i = \sigma$. Let $\otimes^d E$ be the d-fold tensor product bundle over M^d with the corresponding tensor product action of V, $\otimes^d\sigma$. Let S_d, the symmetric group on d letters, act on M^d by permuting the coordinates. We also let $s \in S_d$ act on $\otimes^d E$ by $e_1 \otimes \ldots \otimes e_d$ over $(x_1, \ldots, x_d)$ goes to $s(e_1 \otimes \ldots \otimes e_d) = e_{s1} \otimes \ldots \otimes e_{sd}$ over $(x_{s1}, \ldots, x_{sd})$. Then $\otimes^d\sigma(v)$ commutes with the action of S_d.

By our definition of M^d the action of S_d is free. We therefore have a manifold $N^d = S_d \backslash M^d$. Let π be the canonical projection of M^d onto N^d. We note that ω^d "pushes down" to a measure on N^d. We write $L^2(\otimes^d E)$ for $L^2(\otimes^d E, \omega^d)$. Let H^d be the space of all $f \in L^2(\otimes^d E)$ such that $sf(x_1, \ldots, x_d) = f(x_{s1}, \ldots, x_{sd})$. We define a representation, τ, of V on by $\tau(v)f(x) = \otimes^d\sigma(v)_x f(x)$.

The following result is undoubtedly a very special case of a well known result that is true for totally discontinuous actions of discrete groups. We include a proof since it is short.

LEMMA 5.2. *There exists an open subset, F^d, of M^d such that π is injective on F^d and $N^d - \pi F^d$ has measure 0.*

Proof. Whitney has shown [W] that we may assume that M is a closed analytic submanifold of $\mathbb{R}^N$ for some large N. Choose a non-constant real analytic function, f, on M. Set $F^d = \{(x_1, \ldots, x_d): f(x_1) > f(x_2) > \ldots > f(x_d)\}$. Then clearly $sF^d \cap F^d$ is empty if $s \neq 1$. Set $f_{ij}(x_1, \ldots, x_d) = f(x_i) - f(x_j)$ for $i \neq j$. Then f_{ij} is real analytic and non-constant on M^d for $i \neq j$. Now, the complement to $\bigcup_{s \in S_d} sF^d$ is $\bigcup_{i \neq j}\{x \in M^d: f_{ij}(x) = 0\}$. Since the zero set of a non-constant real analytic function has measure 0 relative to any volume form, the result follows.

We will "abuse notation" and think of π as projecting onto F^d. Also H^d is, under these identifications, just $L^2(\otimes^d E|_{F^d})$ with the same action of V. Set $\otimes^d E|_{F^d} = E^d$. Then E^d splits into a direct sum $\oplus E_I^d$ over $I = (i_1, \ldots, i_d)$, $1 \leqslant i_j \leqslant r$ and

$$\otimes^d \sigma(v)|_{(E_I^d)_x} = e^{i\Psi_I(x)(v)} I.$$

Let $\tau_{I,E}$ be the representation of V on $L^2(E_I^d)$ given by $\tau_{I,E}(v)f(x) = \tau(v)f(x)$.

LEMMA 5.3. *Let $U \subset N^d$ be open. Let C be a continous linear operator on $L^2(V', \mu) \hat{\otimes} L^2(E_I^d)$ such that $C(S(v) \otimes \tau_{I,E}(v)) = (S(v) \otimes \tau_{I,E}(v))C$ for all $v \in V$. Then $C(L^2(V', \mu) \hat{\otimes} L^2(E_{I|U}^d)) \subset L^2(V', \mu) \hat{\otimes} L^2(E_{I|U}^d)$.*

Proof. Let $Z = N^d - U$. Then

$$L^2(V', \mu) \hat{\otimes} L^2(E_I^d) = L^2(V', \mu) \hat{\otimes} L^2(E_{I|U}^d) \oplus L^2(V', \mu) \hat{\otimes} L^2(E_{I|Z}^d)$$

a direct sum of invariant subspaces under $S \otimes \tau_{I,E}$. Thus we must show that if C is a continuous linear operator from $L^2(V', \mu) \hat{\otimes} L^2(E_{I|U}^d)$ to $L^2(V', \mu) \hat{\otimes} L^2(E_{I|Z}^d)$ such that $C(S(v) \otimes \tau_{I,E}(v)) = (S(v) \otimes \tau_{I,E}(v))C$ for all $v \in V$ then $C = 0$. We first reduce this result to a special case. Let $x \in U$ and let $y \in F^d$ be such that $\pi(y) = x$. Then there exist open neighborhoods $W_1, \ldots, W_d$ of $y_1, \ldots, y_d$ such that $W_i \cap W_j = \varnothing$ if $i \neq j$ and $W_x = W_1 \times W_2 \times \ldots \times W_d \subset F^d$. Now $\bigcup_x \pi W_x = U$. A countable number of the πW_x cover. Let P_x be the projection of $L^2(E_{I|U}^d)$ onto $L^2(E_{I|\pi W_x}^d)$ given by multiplication by the characteristic function of πW_x. If $C(I \otimes P_x) = 0$ for all

$x \in \pi^{-1}U$ then $C = 0$. We may thus assume that $U = W_1 \times W_2 \times \ldots \times W_d$ as above.

For this special case we need the following simple observation:
(1) $N^d - U = \pi((\bigcup_i \times^d(M - W_i)) \bigcap M^d)$.
Set $Z_i = \pi((\times^d(M - W_i)) \bigcap M^d)$. Let Q_i be the projection of $L^2(E|_Z)$ onto $L^2(E^d_{I|Z_i})$ given by multiplication by the characteristic function of Z_i. Then it is enough to prove that $C_i = (I \otimes Q_i)C = 0$ for all $i = 1, \ldots, d$. So assume that $C_i \neq 0$. We now follow precisely the same line of argument as in Lemma 5.1. Let $f \in L^2(E^d_{I|U})$ be such that $g = C_i f \neq 0$. As above define

$$\sigma_1(X) = \int_{X \times U} \|f(\lambda - \Psi_I(x), x)\|_x^2 d\mu(\lambda)\omega^d$$

and

$$\sigma_2(X) = \int_{X \times Z_i} \|g(\lambda - \Psi_I(x), x)\|_x^2 d\mu(\lambda)\omega^d. \quad (**)$$

Let ϱ_i be the representation of V on $L^2(V', \sigma_i)$ given by $(\varrho_i(v)u)(\lambda) = e^{i\lambda(v)} u(\lambda)$. Then C_i induces a continuous linear map, L, of $L^2(V', \sigma_1)$ to $L^2(V', \sigma_2)$ such that if Ω_i is the function identically equal to 1 on V' as an element of $L^2(V', \sigma_i)$ then $L\Omega_1 = \Omega_2$ and $L\varrho_1(v) = \varrho_2(v)L$ f or $v \in V$. As above, this implies that there exists $h \in L^2(V', \sigma_1)$ such that $d\sigma_2 = hd\sigma_1$.

Lemma 3.2 implies that there exists $X \in \mathscr{B}$ such that $\mu(X + \Psi_I(x)) = 0$ for $x \in U$ and $\mu(X + \Psi_I(x)) = 1$ for $x \in Z_i$ (take X to be the X in Lemma 3.2 corresponding to W_i and v_i). Thus $\sigma_1(X) = 0$ and $\sigma_2(X) = \|g\|^2$. On the other hand, (*) implies that $\sigma_2(X) = 0$. We have derived our desired contradiction. The lemma now follows.

If H is a Hilbert space then we denote by End H the space of all bounded linear operators of H to H with the strong topology (the topology defined by the semi-norms $\|Tv\|$, $v \in H$). If $S \subset$ End H then we set Comm $(S) = \{A \in \text{End } H: AT = TA \text{ for all } T \in S\}$. The Von Neumann density Theorem asserts that if S is a subalgebra of End H, containing I, such that if $T \in S$ then $T^* \in S$ then Comm(Comm(S)) = Closure(S) (cf. [D], p. 42).

LEMMA 5.4. *Let the notation be as in the previous lemma. If* $v \in V$ *then* $S(v) \otimes I \subset$ Closure(Span$\{S(v) \otimes \tau_{I,E}(v): v \in V\}$).

Proof. If X is a Borel set in N^d let χ_X denote the characteristic function of X. If $f \in L^\infty(N^d)$ let M_f be the operator of multiplication by f on $L^2(E)$. Then

$$\text{Closure}(\text{span}\{M_{\chi_U}: U \text{ open in } N^d\}) \supset \{M_f: f \in L^\infty(N^d)\}.$$

Now the previous Lemma implies that

$$\mathrm{Comm}(\mathrm{Comm}(\mathrm{span}\{S(v) \otimes \tau_{I,E}(v)\colon v \in V\})) \supset \{M_{\chi_U}\colon U \text{ open in } N^d\}.$$

Thus the density theorem implies that

$$\mathrm{Closure}(\mathrm{span}\{S(v) \otimes \tau_{I,E}(v)\colon v \in V\}) = A$$

contains the operators $I \otimes M_f$, $f \in L^\infty(N^d)$. Since $I \otimes \tau_{I,E}(v)$ is such an operator, we see that A contains the operators

$$(I \otimes \tau_{I,E}(-v))(S(v) \otimes \tau_{I,E}(v)) = S(v) \otimes I \quad \text{for} \quad v \in V.$$

For the next lemma we assume in addition that if $v_i \neq v_j$ and if $(v_i, v_j) > 0$ then $v_i - v_j \in \{v_1, \ldots, v_r\}$. We note that this implies that if $v_i - v_j \neq 0$ then $\|v_i - v_j\| \geqslant \min\{\|v_k\|\colon k = 1, \ldots, r\}$. Indeed, if $(v_i, v_j) \leqslant 0$ then $\|v_i - v_j\|^2 \geqslant |v_i\|^2 + \|v_j\|^2$. If $(v_i, v_j) > 0$ and $v_i \neq v_j$ then $v_i - v_j = v_k$ for some k.

LEMMA 5.5. *Let $d, d' \geqslant 0$ and suppose that there exists a continuous non-zero linear map of $L^2(V', \mu) \hat{\otimes} L^2(E_I^d)$ into $L^2(V', \mu) \hat{\otimes} L^2(E_J^{d'})$ such that $C(S(v) \otimes \tau_{I,E}(v)) = (S(v) \otimes \tau_{J,E}(v))C$ for all $v \in V$. Then $d = d'$ and $I = J$.*

Proof. Suppose that $d \neq d'$. Then by replacing C by C^*, if necessary, we may assume that $d > d'$. If we argue as above it is enough to prove that C is 0 on

$$L^2(V', \mu) \hat{\otimes} L^2(E^d_{I|W_1 \times \cdots \times W_d})$$

for $W_1 \times \cdots \times W_d \subset F^d$ and $W_i \cap W_j = \Phi$ if $i \neq j$. We thus replace C by its restriction to this space. If Q_i is the usual projection of $L^2(E_J^{d'})$ onto $L^2(E_J^{d'}|_{(\times^{d'}(M - W_i)) \cap F^{d'}})$ then Lemma 3.2 combined with the argument in Lemma 5.3 implies that $(I \otimes Q_i)C = 0$.

Since $d > d'$ it is easily seen that $\bigcup_{i=1}^d \times^{d'}(M - W_i) = \times^d M$. This proves the result for $d \neq d'$. So assume that $d = d'$. In this case we may argue as in the proof of Lemmas 5.2 and 5.3 to see that if $\varphi \in L^\infty(N^d)$ then $C(I \otimes M_\varphi) = (I \otimes M_\varphi)C$. This implies that if we define a new action, γ, on $L^2(E_J^d)$ given by

$$\gamma(v)f(x) = e^{\iota(\Psi_I(x) - \Psi_J(x))(v)} f(x)$$

then $C(S(v) \otimes I) = (S(v) \otimes \gamma(v))C$.

Our new hypothesis implies that if $v_{i_k} - v_{j_k} \neq 0$ and if dim $M = 2$ then $\|v_{i_k} - v_{j_k}\| > (8\pi|\varrho(x)|)^{1/2}$ for $x \in M$. So if $I \neq J$ then Lemma 5.1 implies that $C = 0$.

6. Unitary representations of groups of smooth mappings

Let U be a compact Lie group. We write 1 for the identity element of U. Let M be as in §3. Let $\langle\ ,\ \rangle$ be a Riemannian structure on M and let ω be a volume element for M. Let $G = \{g \in C^\infty(M; U)$: $g(m) = 1$ outside of a compact set$\}$. If K is a compact subset of M then we set $G_K = \{g \in G$: $g(m) = 1$ if $m \notin K\}$. We endow G_K with the topology of uniform convergence with all derivatives and look upon G as $\bigcup_K G_K$. Then G is a topological group under pointwise multiplication.

Let $\mathfrak{u}$ be the Lie algebra of U which we identify with TU_1 (the tangent space at 1) as usual. Fix B, an $Ad(U)$-invariant inner product on $\mathfrak{u}$. If x, $y \in U$ then set $R(y)x = xy^{-1}$. If W is a finite dimensional vector space over $\mathbb{R}$ then let $\Omega^1(M, W)$ denote the space of all smooth 1-forms on M with values in W. If K is a compact subset of M then we set $\Omega^1_K(M; W)$ equal to the space of all $\eta \in \Omega^1$ $(M; W)$ such that $\eta_x = 0$ for $x \notin K$. We endow $\Omega^1_K(M; W)$ with the topology of uniform convergence with all derivatives. We set $\Omega^1_c(M; W) = \bigcup_K \Omega^1_K(M; W)$ with the corresponding union topology.

Before we introduce the main results of this paper let us record a result which we feel is necessary in the course of their proof. The argument below is based on a suggestion of A. Borel.

LEMMA 6.1. *$d(C^\infty_c(M; W))$ is closed in $\Omega^1_c(M; W)$.*

Proof. It is enough to prove this result for $W = \mathbb{R}$. Let N be a connected, paracompact, orientable, smooth n-dimensional manifold. Set $Z^k_c(N) = \{\eta \in \Omega^k_c(N)$: $d\eta = 0\}$ and put $B^k_c(N) = d\Omega^{k-1}_c(N)$. Then $H^k_c(N) = Z^k_c(N)/B^k_c(N)$ is called the k-th (de Rham) cohomology of M with compact support. If $\eta \in \Omega^k_c(N)$ and if $\nu \in \Omega^{n-k}(N)$ then set

$$(\eta|\nu) = \int_N \eta \Lambda \nu.$$

Then $(d\eta|\nu) = (-1)^{k-1}(\eta|d\nu)$ for $\eta \in \Omega^{k-1}_c(N)$, $\nu \in \Omega^{n-k}(N)$ and if we set $Z^k(N) = \{\eta \in \Omega^k(N)$: $d\eta = 0\}$, $B^k(N) = d\Omega^{k-1}(N)$, $H^k(N) = Z^k(N)/B^k(N)$ then ([deR; §22, §23]) $(\ |\ )$ induces a nondegenerate pairing of $H^k_c(N)$ with

$H^{n-k}(N)$ (i.e. $(\eta|H^{n-k}(N)) = 0$ implies $\eta = 0$). Thus if $\dim H^{n-1}(N) < \infty$ then $\dim H_c^1(N) < \infty$.

(1) If $\dim H^{n-1}(N) < \infty$ then $\mathrm{d}C_c^\infty(N)$ is closed in $\Omega_c^1(N)$.

Indeed, $Z_c^1(N)$ is clearly closed in $\Omega_c^1(N)$. Let Z be a finite dimensional subspace of $Z_c^1(N)$ such that $Z_c^1(N) = \mathrm{d}C_c^\infty(N) \oplus Z$. If N is compact, then choose $x_0 \in N$ and let $V = \{f \in C^\infty(N): f(x_0) = 0\}$. If N is non-compact then set $V = C_c^\infty(N)$. We set $A(v, z) = \mathrm{d}v + z$ for $v \in V$, $z \in Z$. Then

$$A: V \times Z \to Z_c^1(N)$$

is continuous and bijective. Since $V \times Z$ is an LF space this implies that A^{-1} is continuous. Thus $A(V \times \{0\}) = \mathrm{d}C_c^\infty(N)$ is closed.

We now return to M. Let $\{U_i\}$ be a covering of M such that all non-empty finite intersections of the U_i are contractible (e.g., take a covering by convex neighborhoods relative to $\langle\ ,\ \rangle$). For $m = 1, 2, \ldots,$ define $N_m = \bigcup_{i \leqslant m} U_i$. Then $\dim H^{n-1}(N_m) < \infty$ for all m. If K is a compact subset of M then $K \subset N_m$ for some m. Thus (1) implies that $\mathrm{d}C_K^\infty(M)$ is closed. This completes the proof.

We now return to the situation at the beginning of this section. Let β: $G \to \Omega_c^1(M; \mathfrak{u})$ be defined by

$$\beta(g)_x = \mathrm{d}R(g(x))_{g(x)}\mathrm{d}g_x.$$

(i.e., $\beta(g) = (\mathrm{d}g)g^{-1}$). We set $V = \Omega_c^1(M; \mathfrak{u})$ and define a representation π of G on V by $(\sigma(g)\eta)_x = Ad(g(x))(\eta_x(v))$, $g \in G$, $x \in M$, $v \in TM_x$. Then $\beta(xy) = \beta(x) + \sigma(x)\beta(y)$. We will sometimes write $Ad(g)\eta$ for $\sigma(g)\eta$.

For η, $v \in V$, let

$$(\eta, v) = \int_M (\eta_x, v_x)_x \omega.$$

Here we use the inner product on $TM_x^* \otimes \mathfrak{u}$ corresponding to $\langle\ ,\ \rangle$ and B. It will sometimes be necessary to write

$$(\ ,\ ) = (\ ,\ )_{\langle , \rangle, B, \omega}.$$

Let H denote the Hilbert space completion of V with respect to $(\ ,\ )$. Let μ be the Gaussian measure corresponding to this inner product (§1). Then μ is countably additive (cf. [GV], Theorem 6, p. 332). We note that if $g \in G$ then $\sigma(g)$ extends to a unitary operator on H and (σ, H) is a unitary representation of G.

If $f \in L^2(V', \mu)$ and if $g \in G$ then we set $T(g)f(\lambda) = e^{i\lambda(\beta(g))} f(\lambda \cdot \sigma(g))$. Then $(T, L^2(V', \mu))$ is a (strongly continuous) unitary representation of G. We will also write $T = T_{\langle , \rangle, B, \omega}$.

Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{u}$. Let Δ be the root system of $\mathfrak{u}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$ (here sub $\mathbb{C}$ indicates complexification). If $\alpha \in \Delta$ then $\alpha|_{\mathfrak{h}} = i\tilde{\alpha}$ with $\tilde{\alpha} \in \mathfrak{h}'$. Let $\|\alpha\|_B$ be the norm of $\tilde{\alpha}$ relative to $B|_{\mathfrak{h}}$. The main theorem is

THEOREM 6.2. *Assume that U is semi-simple. If* dim $M = 2$ *then we assume that* $\|\alpha\|_B > (8\pi|\varrho(x)|)^{1/2}$, *for* $x \in M$ ($\omega = \varrho \mathrm{Vol}_{\langle , \rangle}$) *and* $\alpha \in \Delta$. *If* dim $M \geqslant 2$ *then* $T_{\langle , \rangle, B, \omega}$ *is irreducible. Let* $\langle \ , \ \rangle_1, B_1, \omega_1$ *be an arbitrary triple as above and let* μ_1 *be the corresponding Gaussian measure on* V'. *If* C *is a non-zero bounded operator from* $L^2(V', \mu)$ *to* $L^2(V', \mu_1)$ *such that* $CT_{\langle , \rangle, B, \omega}(g) = T_{\langle , \rangle_1, B_1, \omega_1}(g)C$ *for* $g \in G$ *then*

$$(\ , \ )_{\langle , \rangle, B, \omega} = (\ , \ )_{\langle , \rangle_1, B_1, \omega_1}.$$

The proof of this result will involve more notation and concepts. For the moment we assume that V is a locally convex, separable, topological vector space over $\mathbb{R}$. Let $(\ , \ )$ be an inner product on V and let μ be the corresponding Gaussian measure on V'. We assume that V_1, V_2 are closed subspaces of V such that $V = V_1 \oplus V_2$ and that $(V_1, V_2) = 0$. Let μ_i be the corresponding Gaussian measure on V_i' for $i = 1, 2$. We assume that μ_i is countably additive for $i = 1, 2$. We identify V_1' with $\{\lambda \in V' : \lambda(V_2) = 0\}$. Then $V' = V_1' \oplus V_2'$. It is easily seen that $\mu = \mu_1 \times \mu_2$ (product measure). Thus Fubini's theorem implies that the map S from $L^2(V_1', \mu_1) \hat{\otimes} L^2(V_2', \mu_2)$ to $L^2(V, \mu)$ given by $S(f \otimes g)(\lambda + v) = f(\lambda)g(v)$ is a unitary isomorphism. Here $\hat{\otimes}$ denotes completed tensor product.

We will also make use of the Fock space. Let $\mathscr{F}(H) = \oplus_{n \geqslant 0} \hat{S}^n(H)_{\mathbb{C}}$. Here $\hat{S}^n(H)$ is the completed n-fold symmetric power of H (the inner product is defined by $(v^n, v^n) = \|v\|^{2n}$). The subscript $\mathbb{C}$ will always indicate complexification with the Hermitian extension if there is an inner product. If $h \in H$ then set

$$\mathrm{EXP}\, h = \sum_{n \geqslant 0} h^n/\sqrt{n!}.$$

Here $h^0 = 0$ and $h^n = h \otimes h \otimes \ldots \otimes h$, n-times. Then $(\mathrm{EXP}\, h_1, \mathrm{EXP}\, h_2) = e^{(h_1, h_2)}$. The next results that we will be describing can be found in [Gu], §2.1, §7.2. Span$\{\mathrm{EXP}\, v : v \in V\}$ is dense in $\mathscr{F}(H)$. We set for $v \in V$,

$e_v(\lambda) = \exp(i\lambda(v) + (v, v)/2)$. Then span$\{e_v: v \in V\}$ is dense in $L^2(V', \mu)$. Furthermore $\{\text{EXP}(v): v \in V\}$ and $\{e_v: v \in V\}$ are linearly independent sets in their respective spaces. Since $(e_v, e_w) = e^{(v,w)}$ for $v, w \in V$ we can define a natural isometry $F_V: L^2(V', \mu) \to \mathscr{F}(H)$ by $F_V(e_v) = \text{EXP } v$.

If $V = V_1 \oplus V_2$ as above then $\mathscr{F}(H) = \mathscr{F}(H_1) \hat{\otimes} \mathscr{F}(H_2)$ and $F_V(L^2(V_1', \mu_1)1_{V_2'}) = \text{Closure}(\text{span}\{\text{EXP } v: v \in V_1\})$. At this point we will explain and fix an error in [GGV, II]. In that paper they look at a situation such as this and consider the space $Q = \{f \in L^2(V', \mu): f(\cdot + \lambda) = f(\cdot)$ for $\lambda \in V_2'\}$. We assert that if $V_2 \neq H_2$ then Q is not defined. Indeed, let $\lambda \in V_2' - H_2$. Then there exists $X \in \mathscr{B}$ such that $\mu(X) = 1$, $\mu(X - \lambda) = 0$ (Lemma 1.2). Let χ_X be the characteristic function of X. If $f \in L^2(V', \mu)$ then $\chi_X f = f$. But $\chi_X(\cdot + \lambda) = \chi_{X-\lambda} = 0$ as an element of $L^2(V', \mu)$. Thus if "$f \in Q$" then $\chi_{X+\lambda} f = f$ so $f = 0$ in L^2. We replace this nonsense with the following result.

LEMMA 6.3. *Let $V = V_1 \oplus V_2 = V_3 \oplus V_4$ be two decompositions of V as above with closures in H, H_i, $i = 1, 2, 3, 4$. Suppose that $H_1 \cap H_3 = (0)$. Then $L^2(V_1', \mu_1)1_{V_2'} \cap L^2(V_3', \mu_3)1_{V_4'} = \mathbb{C}1_{V'}$. Here the μ_i are the Gaussian measures corresponding to $(\ ,\ )|_{V_i}$ respectively and we assume that the properties above of (V_1, V_2) are satisfied by (V_3, V_4).*

Proof. By the above we must show that $(\oplus \hat{S}^n(H_1)) \cap (\oplus \hat{S}^n(H_3)) = \mathbb{C}1$. So suppose that $a = \Sigma\, a_n \in (\oplus \hat{S}^n(H_1)) \cap (\oplus \hat{S}^n(H_3))$. Then comparing homogeneity, we see that $a_n \in \hat{S}^n(H_1) \cap \hat{S}^n(H_3)$ for all n. Thus we need only show that $\hat{S}^n(H_1) \cap \hat{S}^n(H_3) = (0)$ for $n > 0$. If $n = 1$ this just says that $H_1 \cap H_3 = (0)$ as assumed. So assume, inductively, the desired result for n ($\geqslant 1$). If $a \in H$ then define $\partial_a: \hat{S}^{n+1}(H) \to \hat{S}^n(H)$ by $\partial_a x^{n+1} = (n + 1)(x, a)x^n$. Then ∂_a defines a bounded operator. Furthermore, if $x \in \hat{S}^{n+1}(H_1)$ and $\partial_a x = 0$ for all $a \in H_1$ then $x = 0$. Now $\partial_a(\hat{S}^{n+1}(H_i)) \subset \hat{S}^n(H_i)$ for $i = 1, 2$ and $a \in H_1$. Thus the inductive hypothesis implies that $\partial_a(\hat{S}^{n+1}(H_1) \cap \hat{S}^{n+1}(H_3)) = (0)$ hence $\hat{S}^{n+1}(H_1) \cap \hat{S}^{n+1}(H_3) = (0)$.

We now begin the proof of Theorem 6.2. Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{u}$. Let $\mathfrak{h}^\perp = \{X \in \mathfrak{u}: B(\mathfrak{h}, X) = 0\}$. Put $V_1 = \Omega_c^1(M; \mathfrak{h})$, and $V_2 = \Omega_c^1(M; \mathfrak{h}^\perp)$. Then $V = V_1 \oplus V_2$ as above. We fix a base point x_0 (as usual) if M is compact, and set $A = \{f \in C^\infty(M; \mathfrak{h}): f(x_0) = 0\}$ if M is compact and $A = C_c^\infty(M; \mathfrak{h})$ otherwise. Endow A with the topology given as in §3, 4. We look upon A as an abelian topological group. Define $W(a) = T(\exp a)$ for $a \in A$. Then W defines a (strongly continuous) unitary representation of A of $L^2(V', \mu)$.

As above $L^2(V', \mu) = L^2(V_1', \mu_1) \otimes L^2(V_2', \mu_2)$. Under this identification the action W is given as follows

$$W(a)f(\lambda, \xi) = e^{i\lambda(da)}f(\lambda, \xi \cdot \sigma(\exp a)), \quad \lambda \in V_1', \quad \xi \in V_2', \quad a \in A.$$

Thus if we define $W_1(a)f(\lambda) = e^{i\lambda(da)}f(\lambda)$ for $f \in L^2(V_1', \mu_1)$ and $W_2(a)f(\xi) = f(\xi \cdot \sigma(\exp a))$ for $f \in L^2(V_2', \mu_2)$ then under the above identification $W = W_1 \otimes W_2$. We now analyze each of the representations W_i, $i = 1, 2$.

Let H_i denote the Hilbert space completion of V_i, $i = 1, 2$. Let $F_i = F_{V_i}$: $L^2(V_i', \mu_i) \to \mathscr{F}(H_i)$ (defined as above). We note that $W_2(a)e_h = e_{\sigma(\exp a)h}$ for $h \in V_2$. Thus

$$F_2 \cdot W_2(a) = (\oplus_{m \geqslant 0} S^m(\sigma)(\exp a)) \cdot F_2, \quad a \in A.$$

Here $S^m(\sigma)$ is the representation on $\hat{S}^m(H_2)$ corresponding to $\sigma(a)|_{H_2}$.

We have $(\mathfrak{h}^\perp)_{\mathbb{C}} = \oplus_{\alpha \in \Delta}(\mathfrak{u}_{\mathbb{C}})_\alpha$, an orthogonal direct sum relative to the Hermitian extension of B to $\mathfrak{u}_{\mathbb{C}}$ (here, as usual, $(\mathfrak{u}_{\mathbb{C}})_\alpha$, is the α-root space). Let $\alpha_1, \ldots, \alpha_r$ be an enumeration of Δ. Thus the complexification of the vector bundle $\mathrm{Hom}(TM, \mathfrak{h}^\perp)$ splits into a direct sum $E_1 \oplus E_2 \oplus \ldots \oplus E_r$ where E_i is the complexification of TM^* and A acts on E_j via $\beta_j(v)_x u = e^{i\tilde{\alpha}_j(a(x))}u$ for $u \in (E_j)_x$. Let β denote the action $\beta_1 \oplus \ldots \oplus \beta_r$ on $E_1 \oplus \ldots \oplus E_r = E$. Then $H_2 = L^2(E)$. Here we are using the notation in §5. Thus $\hat{\otimes}^d H_2 = L^2(\otimes^d E)$ where $\otimes^d E$ is a vector bundle over M^d. The action $\otimes^d \sigma$ goes over to $\otimes^d \beta$ as in §5. Under this identification the action of S_d on $\hat{\otimes}^d H_2$ given by $s(v_1 \otimes \ldots \otimes v_d) = v_{s1} \otimes \ldots \otimes v_{sd}$ corresponds to $(s^{-1}f)(x_1, \ldots, x_d) = s^{-1}f(x_{s1}, \ldots x_{sd})$. Let F^d be as in Lemma 5.2. Thus, as in §5, $\hat{S}^d(H_2) = L^2(\otimes^d E|_{F^d})$. We write (as in §5) $\otimes^d E|_{F^d} = E^d$. Then $E^d = \oplus E_I^d$ an orthogonal direct sum over $I = (i_1, \ldots, i_d)$, $1 \leqslant i_j \leqslant r$ (here we have replaced the v_j with $\tilde{\alpha}_j$). The conditions on the roots in Theorem 6.2 imply that the $\tilde{\alpha}_j$ satisfy all of the conditions on the v_j. Indeed, it is standard that if α, τ are roots and if $\alpha \neq \tau$, and $B(\tilde{\alpha}, \tilde{\tau}) > 0$ then $\alpha - \tau$ is a root. We therefore see that $\mathscr{F}(H_2) \simeq \mathbb{C}1 \oplus \oplus_{d>0} \oplus_I L^2(E_I^d)$ where the action on each $L^2(E_I^d)$ is given by $\tau_{I,E}$ as in §5.

We now analyze W_1. Lemma 6.1 implies that dA is closed in V_1. Let $Z = \{h \in H_1: (dA, h) = 0\}$. Then $H_1 = \mathrm{Closure}(dA) \oplus Z$. Define $Q(a, b) = (da, db)$, $a, b \in A$. Then Q defines a continuous inner product on A. Let μ_Q be the corresponding Gaussian measure on A'. We define a representation of A on $L^2(A', \mu_Q)$, S, by $S(a)f(\lambda) = e^{i\lambda(a)}f(\lambda)$. We note that as a representation of A, $L^2(V_1', \mu_1) \simeq L^2((dA)', \mu') \otimes \mathscr{F}(Z)$ with μ', the Gaussian measure on $(dA)'$ induced by (,) restricted to dA and the action of A on

$L^2((\mathrm{d}A)', \mu')$ is given by $\xi(a)f(\lambda) = \mathrm{e}^{i\lambda(\mathrm{d}a)}f(\lambda)$. Since d: $A \to \mathrm{d}A$ is continuous, linear and bijective the closed graph theorem implies that it is a topological isomorphism. The pullback of μ' is μ_Q. Thus as a representation of A, $(W_1, L^2(V_1', \mu_1))$ is equivalent with $(S \otimes I, L^2(A', \mu_Q) \otimes \mathscr{F}(Z))$. Let us recapitulate our analysis in the following result.

Lemma 6.4. *$(W, L^2(V', \mu))$ is unitarily equivalent with the direct sum of $(S \otimes I, L^2(A', \mu_Q) \otimes \mathscr{F}(Z))$ and*

$$\oplus_{d>0} \oplus_I (S \otimes I \otimes \tau_{I,E}, L^2(A', \mu_Q) \otimes \mathscr{F}(Z) \otimes L^2(E_I^d)).$$

Furthermore, the unitary equivalence, F, can be chosen so that $F(L^2(V_1', \mu_1)1_{V_2'}) = L^2(A', \mu_Q) \otimes \mathscr{F}(Z)$.

We use this decomposition to prove the theorem.
(1) Closure(span$\{T(g)1_{V'}: g \in G\}$) $= L^2(V', \mu)$.
Indeed, let C: $L^2(V', \mu) \to L^2(V', \mu)$ be such that $C \cdot W(a) = W(a) \cdot C$ for $a \in A$. Set $C_1 = FCF^{-1}$. Let P be the orthogonal projection of

$$L^2(A', \mu_Q) \otimes \mathscr{F}(Z) \oplus (\oplus_{d>0} \oplus_I L^2(A', \mu_Q) \otimes \mathscr{F}(Z) \otimes L^2(E_I^d))$$

onto $L^2(A', \mu_Q) \otimes \mathscr{F}(Z)$ and let $P_{d,I}$ be the orthogonal projection onto

$$L^2(A', \mu_Q) \otimes \mathscr{F}(Z) \otimes L^2(E_I^d).$$

Then Lemma 5.5 implies that $PC_1P_{d,I} = P_{d,I}C_1P = 0$ and $P_{d,I}C_1P_{d',J} = 0$ if $d \neq d'$ or if $d = d'$ and $I \neq J$. Thus Lemma 5.4 implies that

$$\text{Closure}(\text{span}\{FW(a)F^{-1}: a \in A\}) \supset \{\oplus S(a) \otimes I: a \in A\}.$$

Now,

$$\text{Closure}(\text{span}\{W(a): a \in A\}) = F^{-1}\,\text{Closure}(\text{span}\{FW(a)F^{-1}: a \in A\})F.$$

Thus, if we set $v(\eta)f(\lambda) = \mathrm{e}^{i\lambda(\eta)}f(\lambda)$ for $f \in L^2(V', \mu)$ and $\eta \in V$ then

$$F^{-1}\{\oplus S(a) \otimes I: a \in A\}F = \{v(\mathrm{d}a): a \in A\}.$$

If $x \in \mathfrak{u}$ then x is contained in a maximal abelian subalgebra of $\mathfrak{u}$. Thus

$$\text{Closure}(\text{span}\{T(g): g \in G\}) \supset \{v(\mathrm{d}f): f \in C_c^\infty(M; \mathfrak{u})\}.$$

Now $T(g)v(X)T(g)^{-1} = v(Ad(g)X)$ for $g \in G$, $X \in V$.

We assert that $L = \text{span}\{Ad(g)\mathrm{d}f: g \in G, f \in C_c^\infty(M; \mathfrak{u})\}$ is dense in H. In fact, the map $X \to Ad(\exp X)\mathrm{d}f$ is real analytic from $C_c^\infty(M; \mathfrak{u})$ to H. We may thus differentiate to find that

$$[X, \mathrm{d}Y] \in \text{Closure}(\text{span}\{Ad(g)\mathrm{d}f\colon g \in G, f \in C_c^\infty(M; \mathfrak{u})\})$$

for $X, Y \in C_c^\infty(M; \mathfrak{u})$. If $x, y \in \mathfrak{u}$ and if $f, g \in C_c^\infty(M)$ then consider $X = fx$, $Y = gy$. Then $[X, \mathrm{d}Y] = f\mathrm{d}g \otimes [x, y]$. Since $\mathfrak{u}$ is assumed to be semi-simple $[\mathfrak{u}, \mathfrak{u}] = \mathfrak{u}$. Thus if $X_1, \ldots, X_d$ is an orthonormal basis of $\mathfrak{u}$ then

$$\text{Closure}(\text{span}\{Ad(g)\mathrm{d}f\colon g \in G, f \in C_c^\infty(M; \mathfrak{u})\})$$

$$\supset \{\textstyle\sum \eta_i \otimes X_i\colon \eta_i \in \text{span}\{f\mathrm{d}g\colon f, g \in C_c^\infty(M)\}\}.$$

This latter set is obviously dense in H. We conclude that Closure(span$\{T(g)$: $g \in G\}$) contains $\{v(X)\colon X \in L\}$ with $L \subset V$ a dense subspace of H. The above described isomorphism of $L^2(V', \mu)$ with the Fock space on H now implies that span$\{v(X)1_{V'}\colon X \in L\}$ is dense in $L^2(V', \mu)$. This proves (1).

(2) If C is a continuous linear operator on $L^2(V', \mu)$ such that $C \cdot T(g) = T(g) \cdot C$ for all $g \in G$ then $C1_{V'} \in \mathbb{C}1_{V'}$.

Let us show how (2) now implies the first assertion of the Theorem. Let C be an operator as above that commutes with the action of $T(g)$ for $g \in G$. Then $C1_{V'} = c1_{V'}$. This implies that C acts by cI on span$\{T(g)1_{V'}\colon g \in G\}$. (1) implies that this space is dense so C acts by cI. Hence T is irreducible.

We now prove (2). The argument in the proof of (1) implies that $CF^{-1}(L^2(A', \mu_Q) \otimes \mathscr{F}(Z)) \subset F^{-1}(L^2(A', \mu_Q) \otimes \mathscr{F}(Z))$ since in particular $CW(a) = W(a)C$ for $a \in A$. But then $C(L^2(V_1', \mu_1)1_{V_2'}) \subset L^2(V_1', \mu_1)1_{V_2'}$. To complete the proof of (2), we need the following structural property of $\mathfrak{u}$ (this is where the semi-simplicity of U is used).

(3) There exists a maximal abelian subalgebra $\mathfrak{h}_1$ of $\mathfrak{u}$ such that $\mathfrak{h} \cap \mathfrak{h}_1 = (0)$.

We note that there exists $X \in \mathfrak{u}$ such that if $h \in \mathfrak{h}$ and if $[h, X] = 0$ then $h = 0$. Indeed, choose $X \in \mathfrak{h}^\perp$ such that its projection onto every root space is non-zero. Set $\mathfrak{u}_1 = \{Y \in \mathfrak{u}\colon [Y, X] = 0\}$. Choose $\mathfrak{h}_1$ to be a maximal abelian subalgebra of $\mathfrak{u}_1$. If $Y \in \mathfrak{u}$ and if $[Y, \mathfrak{h}_1] = (0)$ then, in particular, $[Y, X] = 0$. Thus $Y \in \mathfrak{h}_1$. So $\mathfrak{h}_1$ is maximal abelian in $\mathfrak{u}$. Since $\mathfrak{u}_1 \cap \mathfrak{h} = (0)$, $\mathfrak{h}_1 \cap \mathfrak{h} = (0)$. This proves (3).

Let $V_3 = \Omega_c^1(M; \mathfrak{h}_1)$ and $V_4 = \Omega_c^1(M; \mathfrak{h}_1^\perp)$. Let μ_3, μ_4 be the corresponding Gaussian measures on V_3' and V_4'. Then the above argument applied to $\mathfrak{h}_1$ instead of $\mathfrak{h}$ implies that $C(L^2(V_3', \mu_3)1_{V_4'}) \subset L^2(V_3', \mu_3)1_{V_4'}$. Now Lemma 6.3 implies that $L^2(V_1', \mu_1)1_{V_2'}, \mu_3)1_{V_4'} = \mathbb{C}1_{V'}$. This proves (2) and hence completes the proof of the first (irreducibility) part of Theorem 6.2.

We now prove the second assertion. Let C be as in the second part of Theorem 6.2. We use the notation in Lemma 5.1. We also write $V_\mathfrak{h}$ for

$\Omega_c^1(M; \mathfrak{h})$ and $V_{\mathfrak{h}^\perp}$ for $\Omega_c^1(M; \mathfrak{h}^\perp)$. We write $\mu_{\mathfrak{h}}$ (resp. $\mu_{1,\mathfrak{h}}$) for the Gaussian measures on $V'_{\mathfrak{h}}$ corresponding to the inner product $(\ ,\ )_{\langle,\rangle,B,\omega} = (\ ,\ )$ (respectively, $(\ ,\ )_{\langle,\rangle_1,B_1,\omega_1} = (\ ,\ )_1$).

Since $1_{V'}$ is a cyclic vector for $T_{\langle,\rangle,B,\omega} = T$, $C1_{V'} \neq 0$. If we argue as in the proof of (2) above using Lemma 5.1 we find that $C(L^2(V'_{\mathfrak{h}}, \mu_{\mathfrak{h}})1_{(V_{\mathfrak{h}^\perp})'}) \subset L^2(V'_{\mathfrak{h}}, \mu_{1,\mathfrak{h}})1_{(V_{\mathfrak{h}^\perp})}$ for all maximal abelian subalgebras $\mathfrak{h}$ of $\mathfrak{u}$. Thus we apply Lemma 6.3 and (3) we find that $C1_{V'} \subset \mathbb{C}1_{V'}$. Set Ω (respectively, Ω_1) equal to $1_{V'}$ as an element of $L^2(V', \mu)$ (respectively, $L^2(V', \mu_1)$). Then we assume that $C\Omega = \Omega_1$.

Since T is irreducible, C^*C is a multiple of the identity. So we may assume that $C^*\Omega_1 = \Omega$. This implies that

$$\langle T(g)\Omega, \Omega\rangle_{L^2(V',\mu)} = \langle T_{\langle,\rangle_1,B_1,\omega_1}(g)\Omega_1, \Omega_1\rangle_{L^2(V',\mu_1)}$$

for all $g \in G$. The obvious calculation of the left and right side of this equation implies that

$$e^{(\beta(g),\beta(g))/2} = e^{(\beta(g),\beta(g))_1/2} \quad \text{for all} \quad g \in G.$$

Now, if $X \in C_c^\infty(M; \mathfrak{u})$ then $\beta(\exp tX) = t\mathrm{d}X + O(t^2)$. We therefore conclude that

$$(\mathrm{d}X, \mathrm{d}X) = (\mathrm{d}X, \mathrm{d}X)_1 \quad \text{for} \quad X \in C_c^\infty(M, \mathfrak{u}). \tag{$*$}$$

We first show that $(*)$ implies that $B_1 = tB$ for some $t > 0$. Indeed, let $X_1, \ldots, X_d$ be an orthonormal basis of $\mathfrak{u}$ relative to B such that $B_1(X_i, X_j) = \lambda_i\delta_{i,j}$. Set $X = \Sigma f_i \otimes X_i$ with $f_i \in C_c^\infty(M)$. Then $(*)$ implies that

$$\sum \int_M \langle \mathrm{d}f_i, \mathrm{d}f_i\rangle\omega = \sum \lambda_i \int_M \langle \mathrm{d}f_i, \mathrm{d}f_i\rangle_1\omega_1.$$

Since this is true for all such f_i, it is clear that all the λ_i are equal to (say) t. If we change ω_1 to $t\omega_1$ we may thus assume that $B = B_1$.

The second part of Theorem 6.2 now follows from

Lemma 6.5. *If*

$$\int_M \langle \mathrm{d}f, \mathrm{d}f\rangle\omega = \int_M \langle \mathrm{d}f, \mathrm{d}f\rangle_1\omega_1$$

for all $f \in C_c^\infty(M)$ *then* $\langle\ ,\ \rangle_x\omega_x = \langle\ ,\ \rangle_{1,x}\omega_{1,x}$ *for all* $x \in M$.

Proof. Let $x \in M$. By taking local coordinates, we may assume that $M = \mathbb{R}^n$ and $x = 0$. We fix the usual inner product, $(\, , \,)$, on $(\mathbb{R}^n)^*$. Then $\langle v, v \rangle_y = (G(y)v, v)$ and $\langle v, v \rangle_{1,y} = (G_1(y)v, v)$. We may assume that $G(0) = I$. Also $\omega = u\mathrm{d}y$ and $\omega_1 = u_1 \mathrm{d}y$. We must prove that $u(0)I = u_1(0)G_1(0)$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be arbitrary. For $\varepsilon > 0$, let $\varphi_\varepsilon(y) = \varphi(y/\varepsilon)$. Then $\mathrm{d}\varphi_\varepsilon(y) = \varepsilon^{-1}\mathrm{d}\varphi(y/\varepsilon)$. A direct calculation yields

$$\int_{\mathbb{R}^n} (G(y)\mathrm{d}\varphi_\varepsilon(y), \mathrm{d}\varphi_\varepsilon(y))u(y)\mathrm{d}y$$

$$= \varepsilon^{n-2} \int (G(\varepsilon y)\mathrm{d}\varphi(y), \mathrm{d}\varphi(y))u(\varepsilon y)\mathrm{d}y.$$

If we divide this by ε^{n-2} and take the limit as $\varepsilon \to 0$ we find that

$$\int (G(0)\mathrm{d}\varphi(y), \mathrm{d}\varphi(y))u(0)\mathrm{d}y = \int (G_1(0)\mathrm{d}\varphi(y), \mathrm{d}\varphi(y))u_1(0)\mathrm{d}y$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. We may choose an orthonormal basis of $(\mathbb{R}^n)^*$ such that $G_1(0)$ is diagonal with entries $\xi_1, \ldots, \xi_n$. Set $v_i = u_1(0)\xi_i/u(0)$. Set

$$D = \sum \frac{\partial^2}{\partial x_i^2} \quad \text{and} \quad D_v = \sum v_i \frac{\partial^2}{\partial x_i^2}.$$

Then one has

$$\int \varphi D\varphi \mathrm{d}x = \int \varphi D_v \varphi \mathrm{d}x$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. This implies that $D = D_v$. Hence $v_i = 1$ for all i. This is the content of the lemma.

Acknowledgements

Research partially supported by a National Science Foundation grant.

This paper is an outgrowth of a course that was given during the Fall semester of 1986, at Rutgers, on representations of infinite dimensional groups. The intent of the course was to give an exposition of known results including [GGV, I]. However, there were so many misstatements and false assertions in that paper that the author began to work out a proof for himself. It was not until he had the results in this paper that he became aware of [GGV, II] and thereby [AKT]. He thanks the students in that course for their patience. He also thanks A. Borel and R. Goodman for helpful conversations, Steve Greenfield for his constructive criticism of the original draft of this paper, Hector Sussman for finding the missing π and Roger Nussbaum for helping us to realize that a logarithmic estimate is the best possible in dimension 2.

References

[AKT] S. Albeverio, R. Hoegh-Krohn and D. Testard: Irreducibility and reducibility for the energy representation of the group of mappings of a Riemmanian manifold into a compact semisimple Lie group. *J. Func. Anal.* 41 (1981) 378–396.

[deR] G. de Rham: *Variétés différentiables*, Publications de l'Institute de Mathématique de l'Université de Nancago, III, Actualités scientifiques et industrielles, 1222. Hermann, Paris (1960).

[D] J. Dixmier: *Les algèbres d'opérateurs dans l'espace hilbertien.* Gauthier-Villars, Paris (1969).

[GGV, I] I. M. Gelfand, M. I. Graev and A. Vershik: Representations of the group of smooth mappings of a manifold X into compact Lie group. *Compositio Math.* 35 (1977) 299–334.

[GGV, II] I.M. Gelfand, M.I. Graev and A. Vershik: Representations of the group of functions taking values in a compact Lie group. *Compositio Math.* 42 (1982) 217–243.

[Gu] A. Guichardet: Symmetric Hilbert spaces and related topics. *Lecture Notes in Mathematics*, Volum 261. Springer-Verlag, Berlin (1972).

[GV] I.M. Gelfand and N.Ya. Vilenkin: *Generalized functions*, Vol. 4 (Translated by A. Feinstein). Academic Press, New York (1964).

[H] S. Helagson: *Differential geometry, Lie groups, and symmetric spaces.* Academic Press, New York (1978).

[I] R. Ismagilov: On unitary representations of the group $C_0(X, G)$, $G = SU(2)$, Mat. Sb., 100 (1976), 117–131. *Math. U.S.S.R. Sb.* 29 (1976) 105–117.

[W] H. Whitney: *Geometric integration theory.* Princeton University Press, Princeton (1957).