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Introduction

This work is devoted to the construction of three types of L functions. In particular we consider an algebraic group G (over a number field) whose L-group L_G has the form of a semi-direct product $(SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \rtimes W_K$ where the Weil group W_K acts on the connected component $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ via permuting the factors. ${}^LG$ has a natural 8 degree representation σ' on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. We then consider an automorphic cuspidal representation Π of $G(\mathbb{A})$; Langlands has associated to Π and σ' a L function $L(\Pi, \sigma', s)$. The purpose of this work is to give a Rankin type integral representation of $L(\Pi, \sigma', s)$ and thereby deduce the functional equation and meromorphic continuation (with exact location of the possible finite number of poles) of this L function.

In April, 1985 Professor Paul Garrett communicated to us that he succeeded in constructing an integral representing the Dirichlet series of 3 holomorphic modular forms. His work was very important to us. We started to analyze his work by determining the structure on the space $P_3\backslash GSp_3$ under the action of $(GL_2 \times GL_2 \times GL_2)^0$. We found there exists a finite number of orbits and all the orbits except the unique open orbit are negligible (in the sense of [P-R-(I)]). This leads to the natural generalization where we consider the action of $GL_2(\mathbb{K})^0$ ($\mathbb{K}$ a semi-simple Abelian algebra of degree 3 over K) on $P_3\backslash GSp_3$. There are three types of semi-simple Abelian algebras $\mathbb{K}$ over K (i.e. $\mathbb{K} = K \oplus K \oplus K$, $K \oplus K_1$ with K_1 a quadratic extension of K, and K_2, a cubic extension of K).

In each of these cases we have an embedding of $GL_2(\mathbb{K})^0 = \{g \in GL_2(\mathbb{K}) | \det g \in K^\times\}$ into GSp_3. Under the action of $GL_2(\mathbb{K})^0$ on $P_3\backslash GSp_3$ there are a finite number of orbits and all the orbits except the unique open orbit are negligible. This allows us, following the methods of [P-R-(I)], to construct a Rankin type integral representation of the L function $L(\Pi, \sigma', s)$. As a consequence we deduce that $L(\Pi, \sigma', s)$ has at most 4 poles. We are able to define local γ_v and L_v factors (associated to the local component Π_v) for all finite places v in K. This is based on a very detailed study of the analytic properties of the Siegel type Eisenstein series on GSp_n (general n). We emphasize here that these methods are similar to those in [P-R-(I)] and [P-R-(II)].

It is possible to analyze $L(\Pi, \sigma', s)$ also by using the work of Shahidi and Langlands. This is an instance where Π has a standard Whittaker model. This work is based on a study of a cuspidal Eisenstein series on a group of type "twisted" D_4.

We describe the contents of the manuscript.

In §0 we review the Langlands construction of the L function $L(\Pi, \sigma', s)$ mentioned above. Also we consider the functional equation satisfied by $L_S(\Pi, \sigma', s)$ given in [Sh].

In §1 we determine in *Lemma 1.1* the exact orbit structure of $GL_2(\mathbb{K})^0$ in the space $P_3\backslash GSp_3$ = the variety of maximal isotropic subspaces of k^6 (relative to a skew symmetric form). We find in *Corollary 1* to *Lemma 1.1* the isotropy group of the various orbits. We note that the isotropy group of the unique open $GL_2(\mathbb{K})^0$ orbit in $P_3\backslash GSp_3$ is an Euler subgroup in the sense of [P]. This guarantees that there is a nice local theory associated to the local zeta integrals of §3 (i.e. functional equation, etc.).

In §2 we prove the basic identity relating to the Rankin integral of a Eisenstein series as GSp_3 and a cusp form (in Π) on $GL_2(\mathbb{K})^0$ (integrated over $GL_2(\mathbb{K})^0\backslash GL_2(\mathbb{A}_{\mathbb{K}})^0$) to a zeta integral of a partial Whittaker transform $\Phi_{\omega,\omega',s}$ (of an element in the induced representation $I_s = \mathrm{ind}_{P_3}^{GSp_3}(\ldots)$) and a Whittaker function based in Π. This allows us to show that this Rankin integral is then Eulerian.

In §3 we develop the local theory of the zeta integral constructed in §2. Namely, we prove in *Proposition 3.1* the generic uniqueness principle about the zeta integral (3-1). As a consequence we have the functional equation of the zeta integral in *Corollary 1* to *Proposition 3.1*. Then we begin the calculation of (3-1) when all the data are "unramified". We show in *Theorem 3.1* that in such an instance the zeta integral equals the local factor $L_v(\Pi_v, \sigma', (s+1)/2)$ times the product of two Abelian zeta functions in the numerator. The importance of this numerator is that globally it precisely gives the correct normalizing factor for the Eisenstein series used in the Rankin integral. The proof of *Theorem 3.1* requires two parts. We need first to determine an explicit generating series representing the zeta integral in terms of the characters of $SL_2 \times SL_2 \times SL_2(\mathbb{C})$ finite dimensional modules. Secondly, we must relate this generating series to the factor $L_v(\Pi_v, \sigma', s)$. The key idea here is that $L_v(\Pi_v, \sigma', s)$ can be interpreted as a "Poincare polynomial" determining the decomposition of $SL_2 \times SL_2 \times SL_2(\mathbb{C})$ acting on the space of polynomials on the vector space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. In this context we consider the isogeny between O(4) and $SL_2 \times SL_2$. Then $L_v(\Pi_v, \sigma', s)$ represents the Rankin product of O(4) and SL_2; in particular the representation of $O(4) \times SL_2$ in $\mathscr{P}$ is viewed as the restriction of the oscillator representation of the dual reductive pair $O(4)_{\text{compact}} \times Sp_2$ to its maximal compact subgroup $O(4)_{\text{compact}} \times U(2)$. From this fact we finish the proof of *Theorem 3.1*. We note that such a use of an oscillator representation plays a critical role in the calculation of more general Rankin integral representations of the type $G \times GL_{\mathrm{n}}$ where G is any classical group.

In Appendix 3 of §3 we consider the question of rationality of the local zeta integrals; here we use techniques adapted from [P-R-(III)].

In §4 we study the analytic properties of a special family of Eisenstein series constructed from an Abelian character ($g \to |\det g|^s$) on the Siegel parabolic in GSp_n. *We note here that this discussion is valid for general n.* This analysis is a continuation of the work in [P-R-(I)]. Here we make a detailed analysis of the intertwining operators that arise in the constant term of the given family of Eisenstein series. In particular we require a complete analysis of the local intertwining operators. In fact we find the "good" normalizing factor for such operators (*Theorem 4.2*) that cancels off the possible poles. The main idea in this section is to decompose such local intertwining operators into a product of two intertwining operators. The first operator corresponds basically to the intertwining operator coming from the "open cell" in $P_r \backslash GSp_r$ (where $r \leqslant n$). The second operator is an intertwining operator arising from an induced representation of GL_n, coming from a maximal parabolic having a Levi component of the form $GL_{n-r} \times GL_r$. The analytic analysis of the first operator is accomplished in the Appendix to §4 using basic invariant theory of the linear action of GL_n on the space Sym_n of $n \times n$ symmetric matrices. The analytic properties of the second family of intertwining operators is determined in a similar but somewhat easier manner.

One critical point here is that once the correct normalizing factor is found for each family of operators, it does not suffice to multiply the two factors together to obtain the correct normalizing factor for the given fixed intertwining operator. We may introduce extraneous poles this way (this is precisely the problem of why we cannot use the usual Harish Chandra theory to analyze our given intertwining operators). However, by a very subtle analysis of the irreducibility properties of the representations I_s (for general s) we can get rid of the extraneous poles. We note that this works for general n if K is a local nonarchimedean field. In the archimedean case we can do this only for the cases $n \leqslant 3$. The main difficulty is the determination of the values of s where I_s is reducible (for K archimedean).

Then in §5 we collect together the results of §2 to §4 and prove (*Theorem 5.1*) that the restricted $L_S(\Pi, \sigma', s)$ has a meromorphic continuation with possible poles at $s = 0, \frac{1}{4}, \frac{3}{4}$ and 1. At this point we introduce a new family of sections in $\mathrm{Ind}_{P_3}^{GSp_3}(\ldots)$ as input information for the construction of Eisenstein series. This is done for each finite place v. From this we are able to define a local factor $L_v(\Pi_v, \sigma', s)$ (associated to Π_v) which has the form $1/P(q_v^{-s})$ with P a polynomial in $\mathbb{C}[X]$ so that $P(0) = 1$. Moreover, we also get an $\varepsilon(\Pi_v, \sigma_v', s)$ factor which has no zeroes or poles. Then as a consequence we can define $L_{\text{fin}}(\Pi, \sigma', s)$ and show (*Theorem 5.2*) that L_{fin} has a finite number of possible poles (at $s = 0, \frac{1}{4}, \frac{3}{4}$ and 1). Moreover, we have a functional equation relating $L_{\text{fin}}(\Pi, \sigma', s)$ to $L_{\text{fin}}(\Pi, \sigma', 1 - s)$ with an ε_{fin} factor built from the $\varepsilon(\Pi_v, \sigma_v', s)$ factors defined above.

Notation

(1) If K is a local nonarchimedean field, let $\mathcal{O}$ be the ring of integers and π, the prime element in $\mathcal{O}$. Let $q = \text{card}\,[\mathcal{O}/\pi\mathcal{O}]$.

(2) Let K be a local nonarchimedean field and ψ a nontrivial additive character in K. We say ψ has order 0 if ψ is trivial on $\mathcal{O}$, but not on $\pi^{-1}\mathcal{O}$.

(3) Let ζ_v be the usual local zeta function of a field K_v. If S is any finite set of primes in a number field K, then $\zeta_S = \Pi_{v \notin S}\zeta_v$. We use a similar notation for any general L_S function with local factor L_v.

(4) Let $S(X)$ denote the Schwartz Bruhat space of functions on X. If X is an Archimedean object, then $S(X) = C_c^\infty(X)$ the C^∞ functions of compact support in X.

(5) Let G be a group and L a subgroup with $\chi\colon L \to K^\times$ a homomorphism. Then we consider the quasicharacter in L given by $l \rightsquigarrow |\chi(l)|_K^s$. We consider the map of $S(G)$ to $\mathrm{ind}_L^G(l \rightsquigarrow |\chi(l)|_K^s)$ given by the map

$$\phi \rightsquigarrow \int_L \phi(\ell g)|\chi(\ell)|^s \, d\ell.$$

We say an element of $\mathrm{ind}_L^G(\ell \rightsquigarrow |\chi(\ell)|^s)$ is an *entire* section (in s) if it can be obtained by the above construction.

(6) We consider the formation of Eisenstein series in the following special case. Namely we let G be a reductive group, P a parabolic subgroup with a decomposition MN with M the Levi component and N the unipotent radical. Let δ_P be the usual Jacobian associated to P. Then the map $m \rightsquigarrow \delta_P(m)$ is a character on M and we form the family of induced representations $(\mathrm{ind}_P^G(|\delta_P|^s)$ as given above. We note that we can do this both locally at primes v or globally relative to $\mathbb{A}$. In particular a global "entire" section f_s is of the type above where the object $\phi \in S(G(\mathbb{A})) = \otimes\, S_v(G_v)$ is a finite linear combination of functions of the form $\otimes \phi_v$ with $\phi_v =$ the characteristic function of the special maximal compact subgroup K_v of G_v for almost all primes v. This implies that f_s is a finite linear combination of $\otimes_v\, (\tilde{f}_s)_v$ where for almost all v, $(\tilde{f}_s)_v$ is the unique K_v invariant function in $\mathrm{Ind}_{P_v}^{G_v}(|\delta_P|^s)$. Then the family of Eisenstein series we consider has the form

$$\sum_{\gamma \in P_K \backslash G_K} f_s(\gamma g).$$

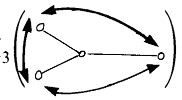
Here the series converges for Re (s) sufficiently large. Moreover we know from the Langlands theory that the Eisenstein series above has a meromorphic continuation to $\mathbb{C}$ provided that $(\tilde{f}_s)_\infty$ is a K_∞ finite function for all ∞ primes.

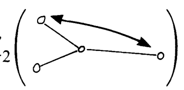
§0. *L* functions associated to "twisted" D_4

We recall here a particular case of the construction of L functions given in [L] and [Sh].

Let G be a connected group over a number field K isomorphic to a "*twisted*" *adjoint group of type* D_4. Then we know that the associated L group ${}^LG$ has the form $\mathrm{Spin}_8 \rtimes W_K$ (semi-direct) with Spin_8, the usual Spin group, and W_K the Weil group of K. We know that W_K acts on the Dynkin diagram of D_4 via the following method. Namely there exists a homomorphism of $W_K \to S_3$ = symmetries of D_4. The kernel of such a map factors through an appropriate normal extension L of K, i.e., $W_K/W_L \cong$ Gal $(L/K) \rightsquigarrow S_3$. Then the action of W_K on D_4 is defined through Gal (L/K), and the appropriate ${}^LG$ is thus constructed. We note that the possibilities of this image of W_K in S_3 are

(i) S_3 itself,

(ii) $\mathbb{Z}_3$ ,

(iii) $\mathbb{Z}_2$ , or

(iv) $\{1\}$.

We consider the relevant parabolic ${}^LP$ of ${}^LG$ which is obtained from the Dynkin diagram of D_4 by removing the center point of the diagram. In such an instance ${}^LP = {}^LM \cdot {}^LN$ where ${}^LM$, the Levi component has the form $(SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times \mathbb{C}^\times) \rtimes W_K$. Here W_K acts by permuting the three SL_2 factors (induced from the homomorphism of $W_K \to$ Gal $(L/K) \rightsquigarrow S_3$) and fixing the $\mathbb{C}^\times$ factor. ${}^LN$ is the unipotent radical of ${}^LP$.

The group M in G (*twisted adjoint type* D_4) then has the following structure. Modulo the centre Z_M, $Z_M\backslash M$ can be *isogenous* to one of the four possibilities

(i) $R_{K_1/K}(PGL_2)$, K_1 a non Galois cubic extension of K,
(ii) $R_{K_1/K}(PGL_2)$, K_1 a Galois cubic extension of K,
(iii) $R_{K_1/K}(PGL_2) \times PGL_2$, K_1 a quadratic extension of K and
(iv) $PGL_2 \times PGL_2 \times PGL_2$.

Here $R_{\tilde{K}/K}$ represents restriction of scalars. We note that it is important to distinguish cases (i) and (ii) since the corresponding ${}^LM$ will be *distinct*.

We consider the *adjoint representation* σ of ${}^LM$ on the space $LA({}^LN)$. In particular this representation has the following form. It is the direct sum of two representations. The first piece is the one dimensional representation which is trivial on $(SL_2 \times SL_2 \times SL_2 \times \{1\}) \rtimes W_K$ and maps $(\{1, 1, 1, r\}, 1)$ to r^2. The second subspace is an eight dimensional representation given by the following data. If $g = (g_1, g_2, g_3, \lambda) \in SL_2 \times SL_2 \times SL_2 \times \mathbb{C}^\times$, then $\sigma'(g) = (\sigma_2(g_1) \otimes \sigma_2(g_2) \otimes \sigma_2(g_3))\lambda$ where σ_2 is the standard two dimensional representation of SL_2. The element $(1, \xi)$ ($\xi \in W_K$) maps via σ' to the permutation $\xi'(v_1 \otimes v_2 \otimes v_3) = v_{\xi'(1)} \otimes v_{\xi'(2)} \otimes v_{\xi'(3)}$ where ξ' is the permutation defined through the map $W_K \to \mathrm{Gal}\,(L/K) \rightsquigarrow S_3$.

We now let Π be an automorphic cuspidal representation of $M(\mathbb{A})$. We are going to associate to Π and σ' above the Rankin triple L function $L(\Pi, \sigma', s)$.

The explicit procedure of such a construction must be modified slightly in order to accomodate the lack of suitable local factors at bad primes.

Indeed we look at the associated local groups G_v (of G given above) which are *quasi split*. As above, we can define the L groups and the relevant parabolics ${}^LP_v$ in ${}^LG_v$. In such an instance $Z_v\backslash M_v$ is *isogeneous* to cases (ii), (iii) or (iv) (given above). Moreover the representation $\sigma_v\colon {}^LM_v \to \mathrm{Aut}\,({}^LW_v)$ has exactly the same description as in the global case.

Then consider the decomposition of $\Pi = \otimes_v \Pi_v$ into irreducible local factors. We consider a finite prime v where Π_v is a unramified spherical principal series. Then if G_v is quasisplit, we note that M_v is quasisplit. It is then possible by the Satake isomorphism Theorem to associate to Π_v a certain conjugacy class in ${}^LM_v$ of the form (g_v, Fr) where Fr corresponds to a *Frobenius generator* of the cyclic group $\mathrm{Gal}\,(L_v/K_v)$ and g_v is a certain semisimple element in ${}^LM_v^0$. In any case, we define the local factor associated to Π_v and σ'_v as

$$L_v(\Pi_v, \sigma'_v, s) = [\det\,(I_8 - \sigma'_v(g_v, \mathrm{Fr})q_v^{-s})]^{-1}. \tag{0-1}$$

Then let S be the set of primes which include the Archimedean ones and those finite v where either G_v is not quasisplit or Π_v is not a spherical principal series. We then define the restricted L_S function associated to Π and σ':

$$L_S(\Pi, \sigma', s) \equiv \prod_{v \notin S} L_v(\Pi_v, \sigma'_v, s). \tag{0-2}$$

Then the general theory of [L] implies that $L_S(\Pi, \sigma', s)$ has a *meromorphic continuation* in the s variable. Moreover, we also know from [Sh] that there exists an associated functional equation of the following form:

$$L_S(\Pi, \sigma', s)\zeta_S(2s) \equiv \varepsilon_S(\Pi, \sigma, s) \qquad\qquad (0\text{-}3)$$
$$L_S(\Pi, \sigma', 1 - s)\zeta_S(1 - 2s)$$

where ε is a finite product of local factors given in terms of local intertwining operators in [Sh].

At this point we can describe the local factors in L_S in more detail.

For instance assume that the local factor Π_v determines a spherical principal series representation of $R_{K_1/K}(PGL_2)$ (K_1 a cubic Galois extension of K). Then the corresponding conjugacy class in ${}^LM_v$ has the form

$$((I_2, I_2, A), Fr)$$

where A is some diagonal matrix and Fr acts via the permutation $v_1 \otimes v_2 \otimes v_3 \to v_2 \otimes v_3 \otimes v_1$. If Π_v determines a spherical principal series representation of $R_{K_1/K}(PGL_2) \times PGL_2$ (K_1 quadratic extension of K), then the corresponding conjugacy class in ${}^LM_v$ has the form

$$((A, I_2, B), \mathrm{Fr})$$

where A and B are diagonal matrices and Fr acts via the permutation $v_1 \otimes v_2 \otimes v_3 \to v_2 \otimes v_1 \otimes v_3$.

Finally if Π_v determines a spherical principal series of $PGL_2 \times PGL_2 \times PGL_2$, the corresponding conjugacy class in ${}^LM_v$ has the form

$$((A, B, C), \mathrm{Fr})$$

where A, B and C are diagonal matrices and Fr acts trivially on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$!

§1. Orbit structure

We let $\mathbb{K}$ be a semisimple Abelian algebra of degree 3 over a global field K. In such an instance we know that $\mathbb{K}$ is (i) a field extension of degree 3 over K, or (ii) the direct sum (as algebras) $K_1 \oplus K$ where K_1 is a field extension of degree 2 over K, or (iii) $K \oplus K \oplus K$, 3 copies of the field K.

We let $GL_2(\mathbb{K})$ = the *general linear group* of $\mathbb{K}$ of degree 2. In the cases above (i) $GL_2 = Gl_2(\mathbb{K})$, $\mathbb{K}$ cubic extension of K, or (ii) $GL_2 = GL_2(K_1) \times GL_2(K)$ or (iii) $GL_2 = GL_2(K) \times GL_2(K) \times GL_2(K)$.

On the space $\mathbb{K} \oplus \mathbb{K}$ we consider the skew symmetric form given by: $A[(x, y), (x', y')] = xy' - x'y$. Then taking the canonical trace form on $\mathbb{K}$ we form a K-valued skew symmetric form $\mathrm{tr}_{\mathbb{K}/K} \circ A = A'$.

We let $GSp(A')$ be the group of similitudes relative to A'. In particular $GSp(A') \cong GSp_3(K)$.

Moreover let $GL_2(\mathbb{K})^0 = \{g \in GL_2(\mathbb{K}) | g \in GSp_3(K)\}$. Thus in particular we note that in case (i) $GL_2(\mathbb{K})^0 = \{g \in GL_2(\mathbb{K}) | (\det g) \in K^{\times}\}$, in case (ii) $GL_2(\mathbb{K})^0 = \{g = (g_1, g_2) | \det g_1 = \det g_2\}$ and case (iii) $GL_2(\mathbb{K})^0 = \{(g_1, g_2, g_3) | \det g_1 = \det g_2 = \det g_3\}$.

We let P be the parabolic subgroup of $GSp_3(K)$ which stabilizes a fixed maximal isotropic subspace of A'. Then the space $P\backslash GSp_3$ consists of all maximal isotropic subspaces.

The first point is to determine the $GL_2(\mathbb{K})^0$ orbits in the space $P\backslash GSp_3$. This calculation is similar to that given in [P-R-(I)] where we define the $GSp_n \times GSp_n$ orbits in $P\backslash GSp_{2n}$.

LEMMA 1.1

(1) *Let* $\mathbb{K} = K \oplus K \oplus K$. *Consider the decomposition of* A' *as a direct sum* $A'_1 \oplus A'_2 \oplus A'_3$ *where* A'_i *is the standard 2 dimensional alternating form. Let* $V_i = \{(z_1, z_2, z_3) \in K^2 \times K^2 \times K^2 | z_\ell = z_k = 0 \text{ if } k \neq i, \ell \neq i\}$. *Then define* $X_{(a_1,a_2,a_3)} = \{X \in P\backslash GSp_3 | X \cap V_i \text{ has dimension } a_i\}$. *Here* $0 \leqslant a_i \leqslant 1$. *Then the* $GL_2(\mathbb{K})^0$ *orbits consist of the sets* $X_{(0,0,0)}$, $X_{(1,0,0)}$, $X_{(0,1,0)}$, $X_{(0,0,1)}$ *and* $X_{(1,1,1)}$.

(2) *Let* $\mathbb{K}$ *be a cubic extension of* K. *Let* $Z = \{(t, 0) | t \in \mathbb{K}\}$ *and* $Y = \{(\beta_1, \beta_2) | \beta_1 \in K \cdot 1, \beta_2 \in \mathbb{K}$ *so that* $\mathrm{tr}_{\mathbb{K}/K}(\beta_2) = 0\}$. *Then* Z *and* Y *are maximal isotropic subspaces and the* $GL_2(\mathbb{K})^0$ *orbits in* $P\backslash GSp_3$ *consist of* $(Z)GL_2(\mathbb{K})^0$ *and* $(Y) \cdot GL_2(\mathbb{K})^0$.

(3) *Let* $\mathbb{K} = K_1 \oplus K$, K_1 *a quadratic extension of* K. *Consider the decomposition of* A' *as a direct sum* $A'_1 \oplus A'_2$ *where* A_1 (A_2 resp.) *is the standard alternating form on* K_1 (K resp). *Let* $V_1 = \{(z, 0) | z \in K_1 \oplus K_1\}$ *and* $V_2 = \{(0, w) | w \in K \oplus K\}$. *Then define*

$$X_1 = \{X \in P\backslash GSp_3 | \dim (X \cap V_1) = 1\}$$

$$X_2 = \{X \in P\backslash GSp_3 | X \cap V_1$$

is the $GL_2(K_1)^0$ *orbit of the isotropic subspace* $Y' = \{(\beta_1, \beta_2) \in K_1 \oplus K_1 | \beta_1 \in K \cdot 1$ *and* $\beta_2 \in K_1$ *with* $\mathrm{tr}_{K_1/K}(\beta_2) = 0\}\}$ *and*

$$X_3 = \{X \in P\backslash GSp_3 | X \cap V_1$$

is the $GL_2(K_1)^0$ *orbit of the isotropic subspace* $Z' = \{(t, 0) \in K_1 \oplus K_1 | t_1 \in K_1\}\}$

$$(\text{here } GL_2(K_1)^0 = \{g \in GL_2(K_1) | \det g \in K^\times\}).$$

Then X_1, X_2 *and* X_3 *are the* $GL_2(\mathbb{K})^0$ *orbits in* $P\backslash GSp_3$.

Proof. We first consider case (2).

Assume that we are given a $GL_2(\mathbb{K})^0$ orbit X in $P\backslash GSp_3$. Then we can assume that X contains a subspace W which contains the element (1, 0) (recall $SL_2(\mathbb{K})$ operates transitively on the set $\mathbb{K} \oplus \mathbb{K} - (0, 0)$). Thus W has a K-basis of the form $\{(1, 0), (x_1, y_1), (x_2, y_2)\}$ where $\mathrm{tr}_{\mathbb{K}/K}(y_i) = 0$. If y_1 and y_2 are not K-linearly independent, then W is $GL_2(\mathbb{K})^0$ conjugate to W', a subspace which contains a basis of the form $\{(1, 0), (x', 0), (0, y')\}$ (with $\mathrm{tr}_{\mathbb{K}/K}(y') = 0$ and $\mathrm{tr}_{\mathbb{K}/K}(x'y') = 0\}$) or $\{(1, 0), (x', 0), (z', 0)\}$. In the first case if we apply $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} y' & 0 \\ 0 & (y')^{-1} \end{smallmatrix}\right)$ to W', we obtain Y given above.

Now we can assume that y_1 and y_2 are K-linearly independent. Then we can apply an element of the form $\left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$ ($\alpha \in \mathbb{K}$) to W to get a subspace W' which has a basis of the form

$$\{(1, 0), (0, y_1), (x, y_2)\} \quad \text{where} \quad \mathrm{tr}_{\mathbb{K}/K}(y_1) = 0$$

and $\mathrm{tr}_{\mathbb{K}/K}(xy_1) = 0$. If $x \in K \cdot 1$, then we are finished proving our statement.

However, if $x \notin K \cdot 1$ then x can be written as $A \cdot 1 + By_2/y_1$ with A, $B \in K$ and $B \neq 0$ (we note that 1 and y_2/y_1 span the K subspace $\{\xi | \mathrm{tr}_{\mathbb{K}/K}(\xi y_1) = 0\}$. Then we can apply the element

$$\begin{bmatrix} 1 & 0 \\ -By_1^{-1} & 1 \end{bmatrix} \quad \text{to the subspace } Y \text{ to achieve } W'.$$

We now consider case (3). Given a subspace $X_0 \in P \backslash GSp_3$, we see that dim $(X \cap V_1)$ and dim $(X \cap V_2)$ are invariants of the $GL_2(\mathbb{K})^0$ orbit of X.

In particular we assume that X_0 has the property that dim $(X_0 \cap V_1) = 2$. This implies that $X_0 \cap V_2$ has dimension 1; thus the $GL_2(\mathbb{K})^0$ orbit of X_0 intersected by V_1 consists of at most two $GL_2(K_1)$ orbits given in (3) above (i.e. $(Y')GL_2(K_1)^0$ and $(Z')GL_2(K_1)^0$). Thus it follows that the space X_0 belongs to either X_1 or X_2 given in (3)!

Then we assume X_0 satisfies dim $(X_0 \cap V_1) = 1$.

Moreoever, we can assume (up to $GL_2(\mathbb{K})^0$ conjugacy) that X_0 contains an element of the form $((1, 0), (1, 0))$ $(\in (K_1 \oplus K_1) \oplus (K \oplus K))$. On the other hand, $X_0 \cap V_1$ must be spanned by $((x, \mu), (0, 0))$ with $x \in K$ and $\mu \in K_1$ satisfying $\mathrm{tr}_{K_1/K}(\mu) = 0$. However, we can conjugate by an element of the form $(\left(\begin{smallmatrix} 1 & 0 \\ \xi & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$ in $GL_2(\mathbb{K})^0$ to deduce that $X_0 \cap V_1 =$ span of $((0, \mu), (0, 0))$. Finally we see easily that X_0 has a basis of the form $\{((1, 0), (1, 0)), ((0, \mu), (0, 0)), ((1, x), (y_1, y_2))\}$ with $\mathrm{tr}_{K_1/K}(x) + y_2 = 0$. But again we conjugate this span by an element of the form $(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ \xi' & 1 \end{smallmatrix}\right))$ to achieve a new subspace X_0' with basis $\{((1, 0), (1, 0)), ((0, \mu), (0, 0)), ((1, x), (0, y_2))\}$. However, we note that

$$\begin{aligned} &\text{Span } \{((1, 0), (1, 0)), ((0, \mu), (0, 0)), ((0, x), (0, y_2))\} \\ &\quad = \text{Span } \{((1, 0), (1, 0)), ((0, \mu), (0, 0)), ((1, x), (1, y_2))\}. \end{aligned}$$

Then the latter space is conjugate to X_0' via an element of the form $((\begin{smallmatrix}1&0\\0&1\end{smallmatrix}),(\begin{smallmatrix}1&0\\\xi''&1\end{smallmatrix}))$.

Finally we consider case (1).

Suppose X is a given $GL_2(\mathbb{K})^0$ orbit with $T \in X$ having the property that $T \cap V_1$ is a one dimensional subspace of V_1; in particular this implies that $T \cap (V_2 \oplus V_3)$ is a two dimensional space. Thus $X \cap (V_2 \oplus V_3)$ can be decomposed into $GL_2(K \oplus K)^0$ orbits (here $GL_2(K \oplus K)^0 = \{(g_1, g_2) \in GL_2(K) \times GL_2(K) | \det g_1 = \det g_2\})$ of a given type. In particular X has a representative T_0 so that either (1) $\dim T_0 \cap V_2 = \dim T_0 \cap V_3 = 1$ or (2) $\dim T_0 \cap V_2 = \dim T_0 \cap V_3 = 0$ (this is just the $GL_2(K \oplus K)^0$ version of (1) above; this is a special case of Theorems proved in [P-R-(I)] for the action of $GSp_1 \times GSp_1$ on $P \backslash GSp_2$!).

Thus we must show that $X_{(0,0,0)}$ is a single $GL_2(\mathbb{K})^0$ orbit. First we observe that $T \in X_{(0,0,0)}$ has the property that $T \cap (V_1 \oplus V_2)$ is a *one dimensional subspace* of $V_1 \oplus V_2$. Indeed if $T \cap (V_1 \oplus V_2)$ is two dimensional, then any vector $z \in T$ with $z \notin T \cap (V_1 \oplus V_2)$ $(z = (z_1, z_2, z_3) \in V_1 \oplus V_2 \oplus V_3)$ has the property that $(z_1, z_2, 0) \in T$; hence $(0, 0, z_3) \in T \cap V_3$ and thus $T \cap V_3 \neq \{0\}$!

Thus for $T \in X_{(0,0,0)}$ we have that both $T \cap (V_1 \oplus V_2)$ and $T \cap (V_2 \oplus V_3)$ are one dimensional subspaces! In particular if $e_1 = (0, 1)$ and $e_0 = (1, 0)$ belong to $K \oplus K$, then we find a $T' \in X_{(0,0,0)}$ which is $GL_2(\mathbb{K})^0$ equivalent to T and which contains the vectors $(e_0, -e_0, 0)$ and $(0, e_0, -e_0)$ (using the decomposition $V_1 \oplus V_2 \oplus V_3$ and identifying each V_i with $K \oplus K$). Thus a third basis element of T' has the form $(e_1, e_1, e_1) + (\lambda e_0, \mu e_0, \varrho e_0)$ with λ, μ and ϱ scalars in K.

However, we observe that if we take the space T_0 spanned by $\{(e_1, e_1, e_1), (e_0, -e_0, 0), (0, e_0, -e_0)\}$, then the element $((\begin{smallmatrix}1&0\\\lambda&1\end{smallmatrix}), (\begin{smallmatrix}1&0\\\mu&1\end{smallmatrix}), (\begin{smallmatrix}1&0\\\varrho&1\end{smallmatrix})) \in GL_2(\mathbb{K})^0$ applied to T_0 yields T' above! Q.E.D.

From *Lemma* 1.1 we can determine each orbit as a homogeneous space.

COROLLARY 1 TO LEMMA 1.1

(1) (a) $X_{(0,0,0)} = R_{(0,0,0)} \backslash GL_2(\mathbb{K})^0$ with

$$R_{(0,0,0)} \cong \left\{\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}, \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}, \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right) | \alpha, \beta \in K^x\right\}$$
$$\times \left\{\left(\begin{bmatrix}1 & t_1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & t_2\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & t_3\\ 0 & 1\end{bmatrix}\right) | t_1 + t_2 + t_3 = 0\right\}.$$

(b) $X_{(1,0,0)} = R_{(1,0,0)} \backslash GL_2(\mathbb{K})^0$ with

$$R_{(1,0,0)} \cong \left\{\left(\begin{pmatrix}\alpha & t\\ 0 & \beta\end{pmatrix}, g, g\right) | g \in GL_2(K),\ t \in K,\ \alpha\beta = \det g\right\}.$$ We have similar structure for $X_{(0,1,0)}$ and $X_{(0,0,1)}$.

(c) $X_{(1,1,1)} = R_{(1,1,1)}\backslash GL_2(\mathbb{K})^0$ with

$$R_{(1,1,1)} \cong \left\{\left(\begin{pmatrix}\alpha_1 & t_1\\ 0 & \beta_1\end{pmatrix}, \begin{pmatrix}\alpha_2 & t_2\\ 0 & \beta_2\end{pmatrix}, \begin{pmatrix}\alpha_3 & t_3\\ 0 & \beta_3\end{pmatrix}\right) | \alpha_i, \beta_j \in K^\times,\ t_i \in K', \text{ and } \alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_3\right\}.$$

(2) (a) $(Y)GL_2(\mathbb{K})^0 = R\backslash GL_2(\mathbb{K})^0$ with

$$R = \left\{\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} | \alpha, \beta \in K^\times\right\} \times \left\{\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} | \mathrm{Tr}_{\mathbb{K}/K}(t) = 0\right\}.$$

(b) $(Z)GL_2(\mathbb{K})^0 = S\backslash GL_2(\mathbb{K})^0$ with

$$S = \left\{\begin{pmatrix}\alpha & t\\ 0 & \beta\end{pmatrix} | \alpha, \beta \in \mathbb{K}^\times,\ t \in \mathbb{K} \text{ with } a\beta \in K^\times\right\}.$$

(3) (a) $X_1 = R\backslash GL_2(\mathbb{K})^0$ with

$$R \cong \left\{\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}, \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right) | \alpha, \beta, \in K^\times\right\}$$
$$\times \left\{\left(\begin{pmatrix}1 & T\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\right) | \mathrm{Tr}_{K_1/K}(T) + t = 0\right\}.$$

(b) $X_2 = S\backslash GL_2(\mathbb{K})^0$ with

$$S = \left\{\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}, \begin{pmatrix}\alpha' & 0\\ 0 & \beta'\end{pmatrix}\right) | \alpha, \beta, \alpha', \beta' \in K^\times, \alpha\beta = \alpha'\beta'\right\}$$
$$\times \left\{\left(\begin{pmatrix}1 & T\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}\right) | \mathrm{Tr}_{K_1/K}(T) = 0, \gamma \in K\right\}.$$

(c) $X_3 = T\backslash GL_2(\mathbb{K})^0$ with

$$T = \left\{\left(\begin{pmatrix}\alpha & T\\ 0 & \beta\end{pmatrix}, \begin{pmatrix}\alpha' & t\\ 0 & \beta'\end{pmatrix}\right) | \alpha\beta = \alpha'\beta', T \in K_1 \text{ and } t \in K\right\}.$$

REMARK 1.1. We note that the subgroups $R_{(0,0,0)}$ in (1) and R in (2) and (3) above are *Euler subgroups* of $GL_2(\mathbb{K})^0$ in the sense of [P]. This comment is expanded upon in §3. Moreover, we note that the isotropy groups $\neq R_{(0,0,0)}$ or R have the property that each contains as a normal subgroup the unipotent radical of some parabolic subgroup of $GL_2(\mathbb{K})^0$. This is the defining condition of being "negligible" given in [P-R-(II)]. We use this property in an essential way in §2.

§2. Basic identity

We let $GSp_3(\mathbb{A})$, $P(\mathbb{A})$ and $GL_2(\mathbb{K})^0(\mathbb{A})$ be the adelized groups given in §1.
We start with a cusp form F on $GL_2(\mathbb{K})(\mathbb{A})$. That is F belongs to

$$L^2_{\text{cusp}}(GL_2(\mathbb{K})(K)\backslash GL_2(\mathbb{K})(\mathbb{A})).$$

Moreover, we assume that F transforms according to a fixed central character $\tilde{\omega}$ of $Z(\mathbb{A})$ (where Z = the center of $GL_2(K)$).

If $F \in \Pi$ an irreducible cuspidal representation of $GL_2(\mathbb{K})(\mathbb{A})$, we let ω_Π be the central character of Π restricted to $Z(\mathbb{A})$. If $\mathbb{K} = K \oplus K \oplus K$ then $\tilde{\omega} = \omega_\Pi = \omega_1 \otimes \omega_2 \otimes \omega_3$ with $\Pi = \Pi_1 \otimes \Pi_2 \otimes \Pi_3$ and ω_i = central character of Π_i (a representation of $GL_2(\mathbb{A})$), and if $\mathbb{K} = K_1 \oplus K$ then $\tilde{\omega} = \omega_\Pi = \omega_1 \otimes \omega_2$ with $\Pi = \Pi_1 \otimes \Pi_2$ and ω_1 = central character of Π_1 (representation of $GL_2(\mathbb{A}_{K_1})$) and ω_2 = central character of Π_2 (representation of $GL_2(\mathbb{A})$).

Then we note P is the semidirect product

$$GL_3(K) \cdot K^\times \cdot \mathrm{Sym}_3(K)$$

given by

$$(A, \lambda, Z) \rightsquigarrow \left[\begin{array}{c|c} A & 0 \\ \hline 0 & \lambda \cdot {}^tA^{-1} \end{array}\right] \left[\begin{array}{c|c} I & Z \\ \hline 0 & I \end{array}\right].$$

Then we consider the character on the group $GL_3(\mathbb{A}) \cdot \mathbb{A}^\times \cdot \mathrm{Sym}_3(\mathbb{A})$ given by

$$(A, \lambda, Z) \xrightarrow{\lambda_{\omega,\omega',s}} \omega(\det A)\omega'(\lambda)|\det A|^{s+2}|\lambda|^{-(3/2s+3)}.$$

Then we form the induced representation

$$\mathrm{Ind}_{P(\mathbb{A})}^{GSp_3(\mathbb{A})}(\lambda_{\omega,\omega',s}) = I_{\omega,\omega',s}.$$

At this point we require a certain compatibility between ω, ω', and $\tilde{\omega}$.

Namely, we want to insure that the characters $\tilde{\omega}$ and $\lambda_{\omega,\omega',s}$ when restricted to the Center Z' of GSp_3 are equal. We note here that Z' has *finite index* in Center $(GL_2(\mathbb{K})^0)$. In concrete terms this means that in (i) ($\mathbb{K}$ cubic extension of K) $\tilde{\omega}^{-1}(\lambda) = \omega(\lambda^3)\omega'(\lambda^2)$ where $\tilde{\omega}$ is a character on $GL_1(\mathbb{A}) = \mathbb{A}_{\mathbb{K}}^\times$, in (ii) ($\mathbb{K} = K_1 \oplus K$, K_1 quadratic extension of K) $\tilde{\omega}^{-1}(\lambda) = \omega_1^{-1}\omega_2^{-1}(\lambda) = \omega(\lambda^3)\omega'(\lambda^2)$ where ω_1 and ω_2 are characters on $\mathbb{A}_{K_1}^\times$ and $\mathbb{A}_K^\times$ (with $\tilde{\omega} = \omega \otimes \omega_2$), and in (iii) ($\mathbb{K} = K \oplus K \oplus K$) $\tilde{\omega}^{-1}(\lambda) = \omega_1^{-1} \cdot \omega_2^{-1} \cdot \omega_3^{-1}(\lambda) = \omega(\lambda^3)\omega'(\lambda^2)$ where ω_i are characters on $\mathbb{A}_K^\times$ (with $\tilde{\omega} = \omega_1 \otimes \omega_2 \otimes \omega_3$).

Then we let $\Phi_{\omega,\omega',s} \in I_{\omega,\omega',s}$.

We form the associated Eisenstein series

$$E(\Phi_{\omega,\omega',s}, g) = \sum_{P(K)\backslash GSp_3(K) \ni \gamma} \Phi_{\omega,\omega',s}(\gamma g). \tag{2-1}$$

The series converges for s such that Re (s) is large. Moreover, $E(\Phi_{\omega,\omega',s})$ has a meromorphic continuation in s to all of $\mathbb{C}$ provided the data in the formation of $\Phi_{\omega,\omega',s}$ is K-finite for the ∞ primes (see (6) in the Notation Section).

Then we form the *Rankin integral* of F and $E(\Phi_{\omega,\omega',s})$,

$$\int_{Z'(\mathbb{A})GL_2(\mathbb{K})^0(K)\backslash GL_2(\mathbb{K})^0(\mathbb{A})} E(\Phi_{\omega,\omega',s}, x)F(x)\, dx. \tag{2-2}$$

We note that since Z' is of finite index in Center $(GL_2(\mathbb{K})^0)$ we see that $Z'(\mathbb{A})\backslash$Center $(GL_2(\mathbb{K})^0)(\mathbb{A})$ is *compact*. Hence the above integral (2-2) is absolutely convergent (since $E(\Phi_{\omega,\omega',s})$ is slowly increasing and $F(\ )$ is rapidly decreasing on $Z'(\mathbb{A}) \cdot GL_2(\mathbb{K})^0(K)\backslash GL_2(\mathbb{K})^0(\mathbb{A})$).

The basic identity involves computing (2-2) in another fashion using Lemma 1.1. Namely we decompose

$$E(\Phi_{\omega,\omega',s}, g) = \sum_{P_3\backslash GSp_3} = \sum_{P_3\backslash GSp_3/GL_2(\mathbb{K})^0(K)}.$$

In particular this means that (2-2) equals a finite sum

$$\sum_{\gamma_l} \int_{Z'(\mathbb{A})L_{\gamma_l}(K)\backslash GL_2(\mathbb{K})^0(\mathbb{A})} \Phi_{\omega,\omega',s}(\gamma_l g)F(g)\, dg \tag{2-3}$$

where γ_l runs over a set of representatives of $GL_2(\mathbb{K})^0$ orbits in $P\backslash GSp_3$ and $L_{\gamma_l} = \{g \in GL_2(\mathbb{K})^0 | g^{-1}\gamma_l g \in P\}$.

In *Corollary* 1 to *Lemma* 1.1 we determine each L_{γ_l} explicitly. Then we decompose the integral in (2-3) as

$$\int_{Z'(\mathbb{A})L_{\gamma_l}(\mathbb{A})\backslash GL_2(\mathbb{K})^0(A)} \Phi_{\omega,\omega',s}(\gamma_l g) \left(\int_{L_{\gamma_l}(K)\backslash L_{\gamma_l}(\mathbb{A})} F(\ell g)\, d\ell \right) dg. \tag{2-4}$$

However, we can determine the inner integral. If L_{γ_l} is negligible in the sense of *Remark* 1.1, then there is a normal subgroup N of L_{γ_l} where N_l is the unipotent radical of a parabolic in $GL_2(\mathbb{K})^0$. Thus we can write

$$\int_{L_{\gamma_l}(K)\backslash L_{\gamma_l}(\mathbb{A})} F(\ell g)\, d\ell = \int_{L_{\gamma_l}(K)N_l(\mathbb{A})\backslash L_{\gamma_l}(\mathbb{A})} \left(\int_{N_l(K)\backslash N_l(\mathbb{A})} F(n\ell g)\, dn \right) d\ell.$$

But by the above hypothesis (F is cusp form) we see that the inner integral vanishes identically for *all* ℓg!

Thus the remaining isotropy subgroup L_{γ_1} is a Euler subgroup (see *Remark* 1.1).

We recall here that if Π is an irreducible automorphic cuspidal representation of $GL_2(\mathbb{A}_L)$ (L = some global field which is a finite extension of K), then Π admits a nondegenerate Whittaker model. That is, given a non-zero additive character ψ on $\mathbb{A}_K K$ and $f \in \Pi$, then there exists $W_f^\psi \in \mathrm{Ind}_{N(\mathbb{A}_L)}^{GL_2(\mathbb{A}_L)}(\psi)$ so that

$$f = \sum_{\gamma \in N \backslash P'} W_f^\psi(\gamma g)$$

where $N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$, $P = \{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}\}$ and ψ is the character $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \to \psi(\mathrm{tr}_{L/K}(x)) = \Pi_{v \in K}(\Pi_{w/v} \psi_v(\mathrm{tr}_{L_w/K_v}(x_w)))$.

We choose certain subgroups now in $GL_2(\mathbb{K})^0$. In case (i) ($\mathbb{K}$ cubic extension), let $W = \{\begin{pmatrix} 1 & z \cdot 1_K \\ 0 & 1 \end{pmatrix} | z \in K\}$. In case (ii) ($\mathbb{K} = K_1 \oplus K$, K_1 quadratic extension), $W = \{\begin{pmatrix} 1 & z \cdot 1_K \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}) | z \in K\}$. And in case (iii) ($\mathbb{K} = K \oplus K \oplus K$) let $W = \{(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}) | z \in K\}$. In each case we consider the additive character on W given by $z \rightsquigarrow \psi(z)$. Then we let

$$\Phi_{\omega,\omega',s}^{\bar{\psi}}(g) = \int_{\mathbb{A}} \Phi_{\omega,\omega',s}(z \cdot g)\psi(3z)\, \mathrm{d}z.$$

Here $z \in W$ is embedded into $GL_2(\mathbb{K})^0$ by the above prescription.

In case (i) ($\mathbb{K}$ cubic extension), let $N = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} | x \in \mathbb{K}\}$. In case (ii) ($\mathbb{K} = K_1 \oplus K$, K_1 quadratic extension), let $N = \{(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}) | x \in K_1, y \in K\}$. And in case (iii) ($\mathbb{K} = K \oplus K \oplus K$), let $N = \{(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}) | x, y, z \in K\}$.

Thus we can compute (2-2) explicitly.

THEOREM 2.1

(1) *Let $\mathbb{K}$ be a cubic extension of K; let Π be an irreducible automorphic cuspidal representation of $GL_2(\mathbb{A}_\mathbb{K})$. Then* (2-2) *equals*

$$\int_{Z'(\mathbb{A}) \cdot N(\mathbb{A}) \backslash GL_2(\mathbb{K})^0(\mathbb{A})} \Phi_{\omega,\omega',s}^{\bar{\psi}}(g) W_F^\psi(g)\, \mathrm{d}g \tag{2-5}$$

with $F \in \Pi$.

(2) *Let $\mathbb{K} = K_1 \oplus K$ with K_1 a quadratic extension of K. Let Π_1 and Π_2 be irreducible automorphic cuspidal representations of $GL_2(\mathbb{A}_{K_1})$ and $GL_2(\mathbb{A}_K)$. Then* (2-2) *equals*

$$\int_{Z'(\mathbb{A}) N(\mathbb{A}) \backslash GL_2(\mathbb{K})^0(\mathbb{A})} \Phi_{\omega,\omega',s}^{\bar{\psi}}(g) W_{F_1}^\psi(g) W_{F_2}^\psi(g)\, \mathrm{d}g \tag{2-6}$$

with $F = F_1 \otimes F_2 \in \Pi_1 \otimes \Pi_2$.

(3) *Let* $\mathbb{K} = K \oplus K \oplus K$. *Let* Π_1, Π_2 *and* Π_3 *be irreducible automorphic cuspidal representations of* $GL_2(\mathbb{A}_K)$. *Then* (2-2) *equals*

$$\int_{Z'(\mathbb{A})N(\mathbb{A})\backslash GL_2(\mathbb{K})^0(\mathbb{A})} \Phi^{\bar{\psi}}_{\omega,\omega',s}(g) W^{\psi}_{F_1}(g) W^{\psi}_{F_2}(g) W^{\psi}_{F_3}(g)\, dg \tag{2-7}$$

with $F = F_1 \otimes F_2 \otimes F_3 \in \Pi_1 \otimes \Pi_2 \otimes \Pi_3$.

Remark. Before we start the proof of *Theorem* 2.1 we note that all of the integrals (2-5), (2-6) and (2-7) are Eulerian integrals. Indeed we note that we can choose $\Phi_{\omega,\omega',s} = \otimes_v \Phi_{\omega_v,\omega'_v,s}$ and $W^{\psi}_F = \otimes_v W^{\psi_v}_{F_v}$. In such a case (2-5) equals

$$\prod_v \left(\int_{Z'_v(K_v)\cdot N_v\backslash GL_2(\mathbb{K}_v)^0} \Phi^{\bar{\psi}}_{\omega_v,\omega'_v,s}(g_v) W^{\psi_v}_{F_v}(g_v)\, dg_v \right). \tag{2-8}$$

A similar formula holds for (2-6) and (2-7). See the beginning of §3 for the definition of $W^{\psi_v}_{F_v}$.

Proof of Theorem 2.1. We calculate (2-4) by substituting $F(g) \equiv \sum_{a\in\mathbb{K}^\times} W^{\psi}_F ((\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix})g)$. Then we integrate F over the space $M(K)\backslash M(\mathbb{A})$ where $M = \{(\begin{smallmatrix} 1 & \xi \\ 0 & 1\end{smallmatrix}) | \mathrm{tr}_{\mathbb{K}/K}(\xi) = 0\}$. Hence we get

$$\int_{M(K)\backslash M(\mathbb{A})} F\left(\begin{pmatrix} 1 & \xi \\ 0 & 1\end{pmatrix} g\right) d\xi \equiv \sum_{a\in\mathbb{K}^\times} \left(\int_{M(K)\backslash M(\mathbb{A})} \psi(\mathrm{tr}_{\mathbb{K}/K}(a\xi))\, d\xi \right)$$

$$\times\; W^{\psi}_F\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g\right) \equiv \sum_{a\in K^\times} W^{\psi}_F\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g\right).$$

Here we use the fact that $\mathbb{K} = K\cdot 1_{\mathbb{K}} \oplus M$.

Then we have that (2-4) equals

$$\int_{Z'(\mathbb{A})\left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta\end{smallmatrix}\right) | \alpha,\beta\in K^\times\right\} M(\mathbb{A})\backslash GL_2(\mathbb{K})^0(\mathbb{A})} \Phi_{\omega,\omega',s}(g) \sum_{a\in K^\times} W^{\psi}_F\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g\right) dg. \tag{2-9}$$

But we note that $(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta\end{smallmatrix}) = (\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda\end{smallmatrix})(\begin{smallmatrix} \alpha' & 0 \\ 0 & 1\end{smallmatrix})$ with $\lambda = \beta$ and $\alpha' = \alpha/\beta$. Hence (2-9) equals

$$\int_{Z'(\mathbb{A})M(\mathbb{A})\backslash GL_2(\mathbb{K})^0(\mathbb{A})} \Phi_{\omega,\omega',s}(g) W^{\psi}_F(g)\, dg. \tag{2-10}$$

But then combining $M \oplus W = N$ we deduce that (2-10) equals

$$\int_{Z'(\mathbb{A})N(\mathbb{A})\backslash GL_2(\mathbb{K})^0(\mathbb{A})} \Phi^{\bar{\psi}}_{\omega,\omega',s}(g) W^{\psi}_{F}(g)\, dg. \tag{2-11}$$

This completes the proof of case (1).
We note that proofs of (2-6) and (2-7) are similar and hence omitted.
Q.E.D.

Thus the problem becomes one of evaluating the local integrals

$$\int_{Z'_v \cdot N_v\backslash GL_2(\mathbb{K}_v)^0} \Phi^{\bar{\psi}_v}_{\omega_v,\omega'_v,s}(g_v) W^{\psi_v}_{F_v}(g_v)\, dg_v = \int_{Z'_v \cdot M_v\backslash GL_2(\mathbb{K}_v)^0} \Phi_{\omega_v,\omega'_v,s}(g_v) W^{\psi_v}_{F_v}(g_v)\, dg_v.$$

§3. Local theory

In this section we develop the local theory of the zetal integral defined at the end of §2. In particular we prove the generic uniqueness principle (*Proposition 3.1*) of this zeta integral; this fact coupled with rationality of the zeta integral allows us to establish a functional equation for the zeta integral (*Corollary 1* to *Proposition 3.1*). Then we compute the local zeta integral when all the data are *unramified* (*Theorem 3.1*). Next we establish in *Proposition 3.2* that the general zeta integral is absolutely convergent for Re (s) large. Then in *Proposition 3.3* we show that the data can be chosen in such a way that the zeta integral becomes independent of s. In Appendix 1 to §3 we discuss the validity of the above Propositions in the Archimedean case. In Appendix 2 of §3 we compute the local zeta integral in the unramified case using the representation theory of a certain oscillator representation. In Appendix 3 to §3 we consider the question of rationality of the zeta integral in the case $\mathbb{K} = K \oplus K \oplus K$.

We let K be a local field. We let ψ be a character of order zero on K.

We let $\mathrm{ind}_P^{GSp_3}(\lambda_{\omega,\omega',s}) = I_{\omega,\omega',s}$ be defined as in §2.

We let $\Phi_{\omega,\omega',s} \in I_{\omega.\omega',s}$.

We assume in case (i) ($\mathbb{K}$, a cubic extension of K) that Π is an arbitrary, admissible, irreducible, generic representation of $GL_2(\mathbb{K})$, in case (ii) ($\mathbb{K} = K_1 \oplus K$, K_1 a quadratic extension of K) that $\Pi = \Pi_1 \otimes \Pi_2$ is an arbitrary, irreducible, admissible, generic representation of $GL_2(K_1) \times GL_2(K)$, and in case (iii) ($\mathbb{K} = K \oplus K \oplus K$) that $\Pi = \Pi_1 \otimes \Pi_2 \otimes \Pi_3$ is an admissible, irreducible, generic representation of $GL_2(K) \times GL_2(K) \times GL_2(K)$.

Then in any case when we write W_F^ψ for $F \in \Pi$ we mean: W_F^ψ in case (i), $W_F^\psi = W_{F_1}^\psi \otimes W_{F_2}^\psi$ in case (ii), and $W_F^\psi = W_{F_1}^\psi \otimes W_{F_2}^\psi \otimes W_{F_3}^\psi$ in case (iii).

Then we consider the local zeta integral

$$Z(\Phi_{\omega,\omega',s}F) = \int_{Z' \cdot M \backslash GL_2(\mathbb{K})^0} \Phi_{\omega,\omega',s}(g) W_F^\psi(g)\, dg. \tag{3-1}$$

We assume at this point that the local integral above converges for s such that Re (s) large (proved below in *Proposition* 3.2).

Moreover, for such s (3-1) defines a $GL_2(\mathbb{K})^0$ invariant bilinear form on

$$I_{\omega,\omega',s} \otimes \Pi.$$

Or in other words, (3-1) defines an element in $\mathrm{Bil}_{GL_2(\mathbb{K})^0}(I_{\omega,\omega',s}, \Pi)$. We then prove the following uniqueness statement (similar to that in [J-P-S] and [P-R-(III)]). If $v = +\infty$, we require in Bil . . . (,) separate continuity in each component (relative to C^∞ topology).

PROPOSITION 3.1. *The space* $\mathrm{Bil}_{GL_2(\mathbb{K})^0}(I_{\omega,\omega',s}, \Pi)$ *is at most a one dimensional space except for a finite number* (*countable and discrete set*) *of values of s if* $v < \infty$ (*if* $v = +\infty$).

Proof. The proof depends on the enumeration of the $GL_2(\mathbb{K})^0$ orbits in $P \backslash GSp_3$ given in *Lemma* 1.1.

We assume that $v < \infty$.

We begin with case (iii). The strategy of the proof is to analyze each $GL_2(\mathbb{K})^0$ orbit separately and determine

$$\mathrm{Hom}_{GL_2(\mathbb{K})^0}(\mathrm{ind}_{L_{\gamma_i}(K)}^{GL_2(\mathbb{K})^0}(\lambda_{\omega,\omega',s}(\gamma_i)), \Pi_1 \otimes \Pi_2 \otimes \Pi_3)$$

where $\lambda_{\omega,\omega',s}(\gamma)$ is the restriction of $\lambda_{\omega,\omega's}$ to $L_{\gamma_i}(K)$. However, by use of Frobenius reciprocity the dimension of the latter space is majorized by the dimension of

$$\mathrm{Hom}_{L_{\gamma_i}(K)}(\check{\Pi}_1 \otimes \check{\Pi}_2 \otimes \check{\Pi}_3, \lambda_{\omega,\omega',s}(\gamma_i)^{-1} \otimes \chi_i)$$

with χ_i a certain character on L_{γ_i}.

At this point we need to enumerate the L_{γ_i} in *Lemma* 1.1. Indeed starting with the X_5 orbit, then $\mathrm{Hom}_{L_{\gamma(5)}}(\check{\Pi}_1 \otimes \check{\Pi}_2 \otimes \check{\Pi}_3, \lambda_{\omega,\omega',s}(\gamma_5)^{-1} \otimes \chi_5) \equiv \mathrm{Hom}_{L'_{\gamma(5)}}((\check{\Pi}_1)_N \otimes (\check{\Pi}_2)_N \otimes (\check{\Pi}_3)_N, \lambda_{\omega,\omega',s}(\gamma_5)^{-1} \otimes \chi_5)$. Here $L'_{\gamma_5} = \{((\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}), (\begin{smallmatrix}\alpha' & 0\\ 0 & \beta'\end{smallmatrix}), (\begin{smallmatrix}\alpha'' & 0\\ 0 & \beta''\end{smallmatrix})) | \alpha\beta = \alpha'\beta' = \alpha''\beta''\}$. Moreover, $\lambda_{\omega,\omega',s}(\gamma_5)[(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}), (\begin{smallmatrix}\alpha' & 0\\ 0 & \beta'\end{smallmatrix}), (\begin{smallmatrix}\alpha'' & 0\\ 0 & \beta''\end{smallmatrix})] = \omega(\alpha\alpha'\alpha'')|\alpha\alpha'\alpha''|^{s+2}|\alpha\beta|^{-((3/2)s+3)}\omega'(\alpha\beta)$. But we know that $(\check{\Pi}_i)_N$ is at most a two dimensional space and a module of $\{(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix})|\alpha, \beta \in K^\times\}$ of finite type. This implies that $\mathrm{Hom}_{L'_{\gamma(5)}}(\ \ldots\ ,\ \ ) = 0$ for all but a finite number of s.

Then we consider in a similar way the orbit X_2. In particular for $\mathrm{Hom}_{L_{\gamma(2)}}(\ldots\ ,\ \ ) \neq 0$ we require that $\check{\Pi}_2$ is dual to $\check{\Pi}_3$ and that $(\check{\Pi}_1)_N$ carries the character $\tilde{\chi}(r)|r|^{s+2}$ on the subgroup $\{(\begin{smallmatrix}r & 0\\ 0 & r^{-1}\end{smallmatrix})|r \in K^\times\}$ with $\tilde{\chi}$ a fixed character. Hence again we deduce that $\mathrm{Hom}_{L_{\gamma(2)}}(\ \ldots\ ,\ \ ) = 0$ for all but a finite number of s. The same arguments work for the orbits X_3 and X_4.

Finally we are left with the orbit X_1. Here L_{γ_1} is an Euler subgroup of $GL_2(\mathbb{K})^0$. We recall the construction of the Kirillov model $K(\Pi)$ of an irreducible admissible representation of Π of $GL_2(K)$. Then we note that $\mathrm{Hom}_{L_{\gamma(1)}}(\ldots\ ,\ \ ) \neq 0$ is equivalent to having a nonzero form on the space $K(\Pi_1) \otimes K(\Pi_2) \otimes K(\Pi_3)$ which transforms according to the character $\lambda_{\omega,\omega',s}(\gamma_1)$ on L_{γ_1}. But $L_{\gamma_1} \subseteq M$ (defined in §2) and $\lambda_{\omega,\omega',s}(\gamma_1)$ is trivial on M. Thus we first look for M invariant forms on $K(\Pi_1) \otimes K(\Pi_2) \otimes K(\Pi_3)$.

We note that each $K(\Pi_i)$ has a finite composition series relative to the group $\{(\begin{smallmatrix}\alpha & x\\ 0 & \beta\end{smallmatrix})|\alpha, \beta \in K^\times, x \in K\}$. The series has two terms. The unique submodule consists of the space $S(K^\times)$; the quotient $K(\Pi_i)/S(K^\times)$ transforms according to at most a two-dimensional representation of $\{(\begin{smallmatrix}\alpha & x\\ 0 & \beta\end{smallmatrix})|\alpha, \beta \in K^\times, x \in K\}$ given by (i) a direct sum of two characters (if two-dimensional) or a character (if one-dimensional) or (ii) a two-dimensional indecomposable representation. Thus $\{(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix})|x \in K\}$ acts trivially in such two-dimensional spaces.

Then the space $K(\Pi_1) \otimes K(\Pi_2) \otimes K(\Pi_3)$ admits a composition series of the following form: $S(K^\times) \otimes S(K^\times) \otimes S(K^\times) \subseteq K(\Pi_1) \otimes S(K^\times) \otimes S(K^\times) \subseteq K(\Pi_1) \otimes K(\Pi_2) \otimes S(K^\times) \subseteq K(\Pi_1) \otimes K(\Pi_2) \otimes K(\Pi_3)$. Hence each successive quotient (except the first submodule) is, at most, a two-dimensional module of $\{[\begin{smallmatrix}\alpha_1 & x_1\\ 0 & \beta_1\end{smallmatrix}] \times [\begin{smallmatrix}\alpha_2 & x_2\\ 0 & \beta_2\end{smallmatrix}] \times [\begin{smallmatrix}\alpha_3 & x_3\\ 0 & \beta_3\end{smallmatrix}]|\alpha_i, \beta_i \in K^\times, x_i \in K\}$. (Moreover, it is trivial on two of these factors and a two-dimensional module for the third).

Then any M invariant distribution on $S(K^\times) \otimes S(K^\times) \otimes S(K^\times)$ is supported on the set $\{(x, x, x)|\ x \in K^\times\}$. A simple argument shows that there is at most one distribution that is supported on this set and transforms relative to $L_{\gamma_1}/M = \{((\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}), (\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}), (\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}))|\alpha, \beta \in K^\times\}$ according to the character $|\alpha/\beta|^{(s+2)/2}\tilde{\chi}_1(\alpha)\tilde{\chi}_2(\beta)$ with $\tilde{\chi}_1$, $\tilde{\chi}_2$ fixed characters (*for all but a finite number of values of* s).

On the other hand M acts trivially on the other composition factors above. And again it is easy to see that for all but a finite number of values of s, there are no functionals on these composition factors which transform according to the character on L_{γ_1}/M given in the preceding paragraph. The point here is that each such composition factor transforms according to a finite number of fixed characters on L_{γ_1}/M.

Again we emphasize here that L_{γ_1} is an Euler subgroup of $GL_2(\mathbb{K})^0$. This makes the proof of the uniqueness proceed in a more direct fashion.

We note that the proof in cases (i) and (ii) is similar to case (iii) and is omitted!

We note also that the proof of the Archimedean cases works in a similar fashion. We remark here that the uniqueness statement for the Archimedean case is valid in the differentiable category of representations. Q.E.D.

Thus from *Proposition* 3.1 we can deduce a local functional equation for the zeta integrals (3-1).

Indeed we asume that the integrals (3-1) *are rational functions in* q^{-s} *if* $v < \infty$ (*meromorphic functions in* $s \in \mathbb{C}$ *if* $v = \infty$). *In the case* $v = +\infty$ *we must assume in addition that the integrals* (3-1) *define separately continuous maps in the space* $\mathrm{Bil}_{GL_2(\mathbb{K})^0}(I_{\omega,\omega',s}, \Pi)$ (*relative to the* C^∞ *topology on the spaces* $I_{\omega,\omega',s}$ *and* Π). We call this hypothesis $(*)$. We note that in Appendix 1 (see Remark 2) and Appendix 3 to §3, we prove the validity of $(*)$.

CORROLARY 1 TO PROPOSITION 3.1. *Assume* $(*)$ *is valid. Let* M_{w_n} *be the canonically* GSp_3 *intertwining operator defined in* §4 *mapping the space* $I_{\omega,\omega',s}$ *to* $I_{\omega^{-1},\omega^3(\omega')^{-1},-s}$.

Then there exists $\Gamma(\Pi, \omega, \omega', s)$ *a meromorphic function in s so that*

$$Z(M_{w_n}(\Phi_{\omega,\omega',s}), F) = \Gamma(\Pi, \omega, \omega', s) Z(\Phi_{\omega,\omega',s}, F)$$

where $F \in \Pi$ (*if* $v < \infty$, *then* $\Gamma(\Pi, \omega, \omega', s)$ *is a rational function in* q^{-s}).

Proof. The proof follows directly from the uniqueness principle of *Proposition* (3-1) and the meromorphic properties of the continuation of (3-1). Q.E.D.

At this point we must compute the local zeta integral (3-1) in a more explicit way. The first case of interest is when Π is an unramified principal series.

In particular we assume that in (3-1) F is a spherical vector of Π, i.e., the unique fixed vector under the maximal compact. We assume in the ensuing calculation that the residual characteristic of $K \neq 3$.

However, first we note that the explicit determination of the term $\Phi_{\omega,\omega',s}$ restricted to the Borel subgroup of $GL_2(\mathbb{K})^0$ is of interest by itself (see [P-R-(II)]).

We first consider the case when $\mathbb{K} = K \oplus K \oplus K$.

From *Lemma* 1.1, the $GL_2(\mathbb{K})^0$ principal orbit in $P\backslash GSp_3$ has a representative of the form $Z = K(e_0, e_0, e_0) \oplus \{(\alpha e_1, \beta e_1, \gamma e_1) | \alpha + \beta + \gamma = 0\}$.

Let T be an element in GSp_3 such that $\{K(e_1, 0, 0) \oplus K(0, e_1, 0) \oplus K(0, 0, e_1)\}T = Z$. Then we consider the elements of GSp_3 given by

$$T \cdot \left\{ \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \middle| x \in K \right\}$$

$$\times \left\{ \left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix}, \begin{pmatrix} \alpha_3 & 0 \\ 0 & \beta_3 \end{pmatrix} \right) \middle| \alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_3 \right\}.$$

We decompose such elements via the Iwasawa decomposition in GSp_3

$$\left[\begin{array}{c|c} A & * \\ \hline 0 & \mu \cdot {}^t A^{-1} \end{array} \right] K$$

with $K \in GSp_3(\mathcal{O}_v)$.

Then we note that $|\mu| = |\alpha_1\beta_1| = |\alpha_2\beta_2| = |\alpha_3\beta_3|$, and

$$|\det A| = |\mu|^3 \max \{|\beta_1\beta_2\beta_3 x|, |\alpha_i\beta_j\beta_k|\}^{-1}.$$

We note that x varies in K and $\alpha_i \in K^\times$ and $\beta_i \in K^\times$ so that $\alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_3$.

Next we consider the case when $\mathbb{K}$ is a cubic extension of K. In such a case there are two possibilities for $\mathbb{K}$. Namely either $\mathbb{K}$ is the unique unramified extension of K or $\mathbb{K}$ is an extension of K where $\pi^3 = \pi_K$.

We note that the $GL_2(\mathbb{K})^0$ principal orbit in $P\backslash GSp_3$ has a representative of the form $Z = K(1, 0) \oplus \{(0, \xi) | \mathrm{tr}_{\mathbb{K}/K}(\xi) = 0\}$. Let T be the element in GSp_3 such that $\{(0, z) | z \in \mathbb{K}\}T = Z$. Then we consider the element of GSp_3 given by

$$T \cdot \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in K \right\} \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \middle| \alpha, \beta \in K^\times \right\}.$$

When we decompose such an element in GSp_3 via the Iwasawa decomposition (as above) we have ($\mathbb{K}$ unramified extension of K)

$$|\mu| = |\alpha\beta|$$

and

$$|\det A| = |\mu|^3 \max \{|\alpha\beta^2|, |x\beta^3|\}^{-1}$$

when we assume α and β belong to $K^\times$ and $x \in K$ (recall that $\mathbb{K}^\times = K^\times \cdot W$ where W is a compact subgroup of $\mathbb{K}^\times$ in the case where $\mathbb{K}$ is an unramified extension of K).

Finally we consider the case where $\mathbb{K} = K_1 \oplus K$ with K_1 a quadratic extension of K.

We note that the $GL_2(\mathbb{K})^0$ principal orbit in $P\backslash GSp_3$ has a representative of the form $Z = K\{((1, 0), (1, 0))\} \oplus \{((0, T), (0, t)) | \mathrm{tr}_{K_1/K}(T) + t = 0\}$. Let T be an element in GSp_3 such that

$$\{((0, w), (0, x)) | w \in K_1, x \in K\} T = Z.$$

Then we consider the element in GSp_3 given by

$$T \cdot \left\{\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) | x \in K\right\} \left\{\left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix}\right) | \alpha_1 \beta_1 = \alpha_2 \beta_2 \right\}$$

and consider its Iwasawa decomposition (as above). Again we deduce (if K_1 is an unramified extension of K)

$$|\mu| = |\alpha_1 \beta_1| = |\alpha_2 \beta_2|$$

and

$$|\det A| = |\mu|^3 \max \{|\beta_1^2 \beta_2 t|, |\alpha_1 \beta_1 \beta_2|, |\alpha_1 \beta_1^2|, |\alpha_2 \beta_1 \beta_2|, |\alpha_2 \beta_1^2|\}.$$

Here we assume $t \in K$ and α_1, β_1, α_2 and β_2 belong to $K^\times$.

Then we consider the space $X = ZM\backslash GL_2(\mathbb{K})^0$ and observe that we can choose a coordinate system on X such that

(i) if $\mathbb{K} = K \oplus K \oplus K$, then

$$X = \left\{\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) | t \in K\right\}$$
$$\cdot \left\{\left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} \lambda\beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}\right) | \lambda, \alpha, \beta \in K^\times \right\}$$
$$\cdot \{\text{a compact subgroup of } GL_2(\mathbb{K})^0\},$$

(ii) if $\mathbb{K}$ = a cubic extension which is unramified over K, then

$$X = \left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} | t \in K\right\} \cdot \left\{\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} | \alpha \in K^\times \right\}$$
$$\cdot \{\text{a compact subgroup of } GL_2(\mathbb{K})^0\},$$

(iii) if $\mathbb{K} = K_1 \oplus K$ where K_1 is unramified over K, then

$$X = \left\{\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) \Big| t \in K\right\}$$

$$\times \left\{\left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right) \Big| \lambda \text{ and } \alpha \text{ in } K^{\times}\right\}$$

$$\cdot \{\text{a compact subgroup of } GL_2(\mathbb{K})^0\}.$$

At this point we recall the formula for the special values of Whittaker functions.

Indeed let $\sigma_{(s_1,s_2)} = \mathrm{ind}_{B_2(L)}^{GL_2(L)} \left(\begin{smallmatrix} \alpha & * \\ 0 & \beta \end{smallmatrix}\right) \to |\alpha|_L^{s_1} |\beta|_L^{s_2} |\frac{\alpha}{\beta}|_L^{1/2})$ (here L is some local field). Let $F_{(s_1,s_2)}$ be the spherical vector of the representation and $W_{(s_1,s_2)}$ the corresponding Whittaker vector. We know that

$$W_{(s_1,s_2)} \begin{bmatrix} \Pi_L^{\ell} & 0 \\ 0 & \Pi_L^m \end{bmatrix} = 0 \quad \text{if } m > \ell$$

and

$$W_{(s_1,s_2)} \begin{bmatrix} \Pi_L^{\ell} & 0 \\ 0 & \Pi_L^m \end{bmatrix} = q_L^{(m-\ell)/2} \operatorname{tr} \left[\varrho_{(\ell,m)} \left(\begin{bmatrix} q_L^{-s_1} & 0 \\ 0 & q_L^{-s_2} \end{bmatrix} \right) \right]$$

where $\varrho_{(\ell,m)}$ is the finite dimensional irreducible representation of $GL_2(\mathbb{C})$ with highest weight parameterized by the pair (ℓ, m) with $\ell \geqslant m$.

Then if $\mathbb{K} = K \oplus K \oplus K$ we have that (3-1) equals

$$\int \Phi^{\bar{\psi}}_{\omega,\omega',s} \left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} \lambda\beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \right)$$

$$W_{(s_1,s_2)} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} W_{(s_1',s_2')} \begin{pmatrix} \lambda\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} W_{(s_1'',s_2'')} \begin{pmatrix} \lambda\beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \tag{3-3}$$

$$\{|\lambda|^{-3}|\alpha|^{-2}|\beta|^{-2}\} d^{\times}(\lambda)\, d^{\times}(\alpha)\, d^{\times}(\beta).$$

In order to insure the compatibility conditions on centers (see §2) we assume that $\omega(x) = |x|^{\mu}$ and $\omega'(x) = |x|^{-\mu}$ where $\mu = -(s_1 + s_2 + s_1' + s_2' + s_1'' + s_2'')$. Then we use the fact $W_{(\varrho_1,\varrho_2)}\left(\begin{smallmatrix} \lambda t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) = \omega_{(\varrho_1,\varrho_2)}(t^{-1}) W_{(\varrho_1,\varrho_2)}\left(\begin{smallmatrix} \lambda t^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $\omega_{(\varrho_1,\varrho_2)}$ is the central character of $\sigma_{(\varrho_1,\varrho_2)}$ and $\omega_{\varrho_1,\varrho_2}(t) = |t|^{(\varrho_1+\varrho_2)}$. Also in the integral above we make the change of variables $\alpha' = \lambda\alpha^2$ and $\beta' = \lambda\beta^2$ to deduce that (3-3) equals

$$\int_{(I)} f_\psi(s, \lambda, \alpha', \beta') W_{(s_1,s_2)}\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix} W_{(s_1',s_2')}\begin{pmatrix}\alpha' & 0\\ 0 & 1\end{pmatrix}$$

$$W_{(s_1'',s_2'')}\begin{pmatrix}\beta' & 0\\ 0 & 1\end{pmatrix} |\lambda|^{-(s_1+s_2)/2}|\alpha'|^{-(s_1'+s_2')/2} \tag{3-4}$$

$$|\beta'|^{-(s_1''+s_2'')/2}\{|\lambda\alpha'\beta'|^{-1}\}\, d^\times(\alpha')^1\, d^\times(\beta')\, d^\times(\lambda) + \int_{(II)} \cdots .$$

Here . . . represents the same integrand as in the first integral above. Moreover, (I) $= \{(\alpha, \beta, \lambda)|\alpha, \beta$ and λ have even order$\}$ and (II) $= \{(\alpha, \beta, \lambda)|\alpha, \beta$ and λ have odd order$\}$.

In addition

$$f_\psi(s, \lambda, \alpha', \beta') \equiv |\lambda\alpha'\beta'|^{(1/2)(s+\mu)+1} \int \psi(x)$$

$$\max\{|x|, |\alpha'\beta'|^{1/2}, |\lambda\alpha|^{1/2}, |\lambda\beta|^{1/2}, |\alpha'|, |\beta'|, |\lambda|\}^{-(s+\mu+2)}\, dx. \tag{3-5}$$

However, to evaluate (3-5) we use the easily established identity (with $\Lambda \geqslant 0$):

$$\int_K \psi(x) \max\{|x|, q^{-\Lambda}\}^{-\lambda}\, dx \equiv \left(\frac{1 - q^{-\lambda}}{(qq^{-\lambda})^\Lambda}\right)\cdot\left(\frac{1 - (qq^{-\lambda})^{\Lambda+1}}{1 - (qq^{-\lambda})}\right).$$

Then using the values of Whittaker functions given above we can express (3-4) as a series of the form

$$\sum_{\left\{(\ell,m,t)\in \mathbf{Z}_+\times\mathbf{Z}_+\times\mathbf{Z}_+ \middle| {\ell\equiv m\equiv t\equiv 0 \atop \ell\equiv m\equiv t\equiv 1}\bmod 2\right\}} \operatorname{tr}(\varrho_{(\ell,0)}(A))\operatorname{tr}(\varrho_{(m,0)}(B)) \tag{3-6}$$

$$\operatorname{tr}(\varrho_{(t,0)}(C))(\det A)^{-\ell/2}(\det B)^{-m/2}(\det C)^{-t/2}\left(\frac{(1 - q^{-(2+\mu)}X)}{(q^{-(1+\mu)}X)^{\Lambda(\ell,m,t)}}\right)$$

$$\times\left(\frac{1 - (q^{-(1+\mu)}X)^{\Lambda(\ell,m,t)+1}}{1 - (q^{-(1+\mu)}X)}\right)(q^{-(1+\mu)}X)^{(\ell+m+t)/2}$$

where

$$A = \begin{bmatrix} q^{-s_1} & 0 \\ 0 & q^{-s_2} \end{bmatrix}, \quad B = \begin{bmatrix} q^{-s'_1} & 0 \\ 0 & q^{-s'_2} \end{bmatrix}, \quad C = \begin{bmatrix} q^{-s''_1} & 0 \\ 0 & q^{-s''_2} \end{bmatrix},$$

$X = q^{-s}$ and $\Lambda(\ell, m, t) = \min\{\ell, m, t\}$.

We note at this point that the function $A \rightsquigarrow \operatorname{tr}(\varrho_{(\ell,0)}(A))(\det A)^{-\ell/2}$ depends only on the SL_2 part of A; that is, if $A = A_1 \cdot \left(\begin{smallmatrix} \xi & 0 \\ 0 & \xi \end{smallmatrix}\right)$ with $\xi \in \mathbb{C}^\times$ and $A_1 \in SL_2$, then the above function is independent of ξ.

We now consider in the case of $\mathbb{K}$, a unramified cubic extension of K, the evaluation of (3-1) for spherical data. In such an instance (3-1) equals

$$\int_{K^\times} \Phi^{\bar{\psi}}_{\omega,\omega',s}\left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\right) W_{(s_1,s_2)}\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} |\lambda|^{-1}\, d^\times(\lambda) \tag{3-8}$$

(we note here the integration is over $K^\times$). Again we choose $\omega(x) = |x|^\mu$ and $\omega'(x) = |x|^{-\mu}$ where $\mu = -3(s_1 + s_2)$ (see §2). Then we see that (3-8) equals

$$\int_{K^\times} f_\psi(s, \lambda) W_{(s_1,s_2)}\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} |\lambda|_K^{-3}\, d^\times(\lambda) \tag{3-9}$$

with

$$f_\psi(s, \lambda) = |\lambda|^{(3/2)(s+\mu)+3} \int_K \psi(x) \max(|x|, |\lambda|)^{-(s+\mu+2)}\, dx.$$

Then when we evaluate f_ψ (as above) we obtain that (3-8) equals the series

$$\sum_{m\in\mathbb{Z}_+} \operatorname{tr}(\varrho_{(m,0)}(A))(\det A)^{-m/2}\left(\frac{(1 - q^{-(2+\mu)}X)}{(q^{-(1+\mu)}X)^m}\right)\left(\frac{1 - (q^{-(1+\mu)}X)^{m+1}}{1 - (q^{-(1+\mu)}X)}\right)$$
$$\times (q^{-(1+\mu)}X)^{3m/2} \tag{3-10}$$

where

$$A = \begin{bmatrix} q^{-3s_1} & 0 \\ 0 & q^{-3s_2} \end{bmatrix} \quad \text{and} \quad X = q^{-s}.$$

Next we consider the case when $\mathbb{K} = K_1 \oplus K$ with K_1, quadratic unramified over K. Then when we evaluate (3-1) we get

$$\int \Phi^{\bar{\psi}}_{\omega,\omega',s}\left(\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}\lambda\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\right) W_{(s_1,s_2)}\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix} W_{(s'_1,s'_2)}\begin{pmatrix}\lambda\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}$$

$$\times\ |\lambda|_K^{-3}|\alpha|_K^{-2}\, d^{\times}(\lambda)\, d^{\times}(\alpha). \tag{3-11}$$

Again we choose $\omega(x) = |x|^{\mu}$ and $\omega'(x) = |x|^{-\mu}$ where $\mu = -(2(s_1 + s_2) + (s'_1 + s'_2))$ (see §2).

Then following the above examples we make the changes of variable $\alpha' = \lambda\alpha^2$ and deduce that (3-11) equals

$$\int_{(\mathrm{I})} f_{\psi}(s, \lambda, \alpha') W_{(s_1,s_2)}\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix} W_{(s'_1,s'_2)}\begin{pmatrix}\alpha' & 0\\ 0 & 1\end{pmatrix} |\lambda|^{-(s_1+s_2)}|\alpha'|^{-(s'_1+s'_2)/2}$$

$$\times\ \{|\lambda|^{-2}|\alpha'|^{-1}\}\, d^{\times}(\lambda)\, d^{\times}(\alpha') + \int_{(\mathrm{II})} \cdots \tag{3-12}$$

where (I) $= \{(\lambda, \alpha)|\lambda$ and α have even order$\}$ and (II) $= \{(\lambda, \alpha)|\lambda$ and α have odd order$\}$. In addition

$$f_{\psi}(s, \lambda, \alpha') \equiv |\lambda|^{(s+\mu)+2}|\alpha'|^{(1/2)(s+\mu)+1}\int \psi(t) \max\{|t|, |\lambda|, |\alpha\lambda|^{1/2}, |\alpha|\}^{-(s+\mu+2)}\, dt.$$

Then when we evaluate f_{ψ} we deduce that (3-12) equals the series

$$\sum_{\{(\ell,m)\in \mathbf{Z}_+\times\mathbf{Z}_+|\ell\equiv m\equiv 0 \text{ or } \ell\equiv m\equiv 1 \bmod 2\}} \mathrm{tr}(\varrho_{(m,0)}(A))\ \mathrm{tr}\,(\varrho_{(\ell,0)}(B))$$

$$\times\ (\det A)^{-m/2}\,(\det B)^{-\ell/2}\left(\frac{(1 - q^{-(2+\mu)}X)}{(q^{-(1+\mu)}X)^{\Lambda(m,\ell)}}\right)$$

$$\times\left(\frac{1 - (q^{-(1+\mu)}X)^{\Lambda(m,\ell)+1}}{1 - (q^{-(1+\mu)}X)}\right)(q^{-(1+\mu)}X)^{m+\ell/2} \tag{3-13}$$

with

$$A = \begin{bmatrix} q^{-2s_1} & 0\\ 0 & q^{-2s_2}\end{bmatrix},\quad B = \begin{bmatrix} q^{-s'_1} & 0\\ 0 & q^{-s'_2}\end{bmatrix}\quad \text{and}\quad X = q^{-s}.$$

Also $\Lambda(\ell, m) = \min(\ell, m)$.

Thus we are at the point of explicitly evaluating (3-1) for spherical data. We assume in Theorem 3.1 that residual characteristic of $K \neq 3$.

THEOREM 3.1. *Let $\Phi^0_{\omega,\omega',s} \in I_{\omega,\omega',s}$ be the unique vector which is $GSp_3(\mathcal{O}_v)$ fixed and normalized so that $\Phi^0_{\omega,\omega',s}(e) = 1$. Moreover assume that $\omega(x) = |x|^{\mu}$ and $\omega'(x) = |x|^{-\mu}$ where* (i) $\mu = -(s_1 + s_2 + s_1' + s_2' + s_1'' + s_2'')$ ($\mathbb{K} = K \oplus K \oplus K$), (ii) $\mu = -(2(s_1 + s_2) + s_1' + s_2')$ ($\mathbb{K} = K_1 \oplus K$, *$K_1$ quadratic unramified*), *and* (iii) $\mu = -3(s_1 + s_2)$ ($\mathbb{K}$, *cubic unramified*). *Then*

(1) *for $\mathbb{K} = K \oplus K \oplus K$ and $F = F_1 \otimes F_2 \otimes F_3$ with F_i a spherical vector in $\Pi_i \subseteq \sigma_{(s_1^i, s_2^i)}$, we have*

$$Z(\Phi^0_{\omega,\omega',s}, F) \equiv \frac{1}{\zeta(s + \mu + 2)\zeta(2s + 2\mu + 2)}$$

$$\times\ [\det (I_8 - \sigma(A) \otimes \sigma(B) \otimes \sigma(C) q^{-(s+1+2\mu)/2})]^{-1}$$

with

$$A = \begin{bmatrix} q^{-s_1} & 0 \\ 0 & q^{-s_2} \end{bmatrix}, \quad B = \begin{bmatrix} q^{-s_1'} & 0 \\ 0 & q^{-s_2'} \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} q^{-s_1''} & 0 \\ 0 & q^{-s_2''} \end{bmatrix}.$$

(2) *For $\mathbb{K} = K_1 \oplus K$ (K_1, an unramified quadratic extension) and $F = F_1 \otimes F_2$ with F_i a spherical vector in $\Pi_i \subseteq \sigma_{(s_1^i, s_2^i)}$, we have*

$$Z(\Phi^0_{\omega,\omega',s}, F) \equiv \frac{1}{\zeta(s + \mu + 2)\zeta(2s + 2\mu + 2)}$$

$$\times\ [\det (I_8 - \delta(A, B, t') q^{-(s+1+2\mu)/2})]^{-1}$$

with

$$A = \begin{bmatrix} q^{-2s_1} & 0 \\ 0 & q^{-2s_2} \end{bmatrix}, \quad B = \begin{bmatrix} q^{-s_1'} & 0 \\ 0 & q^{-s_2'} \end{bmatrix},$$

and $\delta(A, B, t)$ is the operator $(I_2 \otimes \sigma(A) \otimes \sigma(B)) \circ t'$ where $t'(v_1 \otimes v_2 \otimes v_3) = v_2 \otimes v_1 \otimes v_3$.

(3) *For $\mathbb{K}$, a cubic unramified extension and F a spherical vector in $\Pi \subseteq \sigma_{(s_1, s_2)}$, we have*

$$Z(\Phi^0_{\omega,\omega',s}, F) \equiv \frac{1}{\zeta(s + \mu + 2)\zeta(2s + 2\mu + 2)}$$

$$\times\ [\det (I_8 - \delta(A, t) q^{-(s+1+2\mu)/2})]^{-1}$$

with

$$A = \begin{bmatrix} q^{-3s_1} & 0 \\ 0 & q^{-3s_2} \end{bmatrix}, \text{ and } \delta(A, t) \text{ is the operator } (I_2 \otimes I_2 \otimes \sigma(A)) \circ t$$

$$\text{where } t(v_1 \otimes v_2 \otimes v_3) = v_2 \otimes v_3 \otimes v_1.$$

Proof. We start with (3-6) and compare that calculation to *Lemma* i in Appendix 2. In particular we write $A = A'(\begin{smallmatrix} \xi & 0 \\ 0 & \xi \end{smallmatrix})$ with $A' \in SL_2$; we do the same for B and C. Thus we have that tr $(\varrho_{(\ell,0)}(A))(\det A)^{-\ell/2} = $ tr $(\varrho_{(\ell,0)}(A'))$. And we consider (3-6) with A', B' and C' replacing A, B and C. Then we can apply *Lemma* i in Appendix 2. In particular we deduce that the coefficient of the term $(\ell_1, 0) \otimes (\ell_2, 0) \otimes (j, 0)$ (in the $SL_2 \times SL_2 \times SL_2$ decomposition of $S^*(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$) is given by

$$\left(\frac{1}{1 - Y^2}\right)\left(\frac{1 - (Y^2)^{\Lambda(\ell_1,\ell_2,j)}}{1 - Y^2}\right) Y^{\ell_1 + \ell_2 + j - 2\Lambda(\ell_1,\ell_2,j)}$$

where $\Lambda(\ell_1, \ell_2, j) = \min(\ell_1, \ell_2, j)$.

Thus if we replace $q^{-(1+\mu)}X = Y^2$ in (3-6), we then deduce that (3-6) equals

$$(1 - Y^4)(1 - q^{-1}Y^2)\det[I_8 - \sigma(A') \otimes \sigma(B') \otimes \sigma(C')Y]^{-1}.$$

But we know that if $A'[\begin{smallmatrix} \xi_A & 0 \\ 0 & \xi_A \end{smallmatrix}] = A$, then $\sigma(A) = \sigma(A') \cdot \xi_A$. Then $\sigma(A) \otimes \sigma(B) \otimes \sigma(C) = \sigma(A') \otimes \sigma(B') \otimes \sigma(C') \cdot (\xi_A \cdot \xi_B \cdot \xi_C)$. But it is possible to choose ξ_A, ξ_B, and ξ_C up to sign so that $\xi_A \cdot \xi_B \cdot \xi_C = q^{+\mu/2}$. Hence $\det(I_8 - \sigma(A') \otimes \sigma(B') \otimes \sigma(C')Y) \equiv \det(I_8 - \sigma(A) \otimes \sigma(B) \otimes \sigma(C)q^{-\mu-1/2}X^{1/2})$.

Thus we have that (3-6) equals

$$(1 - q^{-(2+2\mu)}X^2)(1 - q^{-2-\mu}X)$$

$$[\det(I_8 - \sigma(A) \otimes \sigma(B) \otimes \sigma(C)q^{-\mu-1/2}X^{1/2})]^{-1}.$$

Next we consider the series (3-10). Again we let $A = A'[\begin{smallmatrix} \xi_A & 0 \\ 0 & \xi_A \end{smallmatrix}]$ and $B = B'[\begin{smallmatrix} \xi_B & 0 \\ 0 & \xi_B \end{smallmatrix}]$ with A' and B' belonging to SL_2. Thus (3-10) has the form (with $Y^2 = q^{-(1+\mu)}X$)

$$(1 - q^{-1}Y^2) \sum_{\{(m,\ell) \in \mathbf{Z}_+ \times \mathbf{Z}_+ | m \equiv \ell \equiv 0 \text{ or } m \equiv \ell \equiv 1 \bmod 2\}}$$

$$\frac{1 - (Y^2)^{\Lambda(m,\ell)+1}}{1 - Y^2} Y^{2\ell + m - 2\Lambda(\ell,m)} \operatorname{tr}(\varrho_{m,0)}(A')) \operatorname{tr}(\varrho_{(\ell,0)}(B')).$$

But we compare this series to (ii) in Appendix 2. Thus we have that (3-10) equals

$$(1 - Y^4)(1 - q^{-1}Y^2)[\det (I_8 - \delta(A, B, t')Y)]^{-1}.$$

Again we note that $\sigma(A) \otimes \sigma(B) = \xi_A \xi_B \sigma(A') \otimes \sigma(B')$. Then it is possible to choose ξ_A and ξ_B up to sign so that $\xi_A \xi_B = q^{+\mu/2}$. Hence

$$\det (I_8 - \delta(A', B', t')Y) = \det (I_8 - \delta(A, B, t')q^{-\mu-1/2}X^{1/2}).$$

Thus (3-10) equals

$$(1 - q^{-(2+2\mu)}X^2)\,(1 - q^{-(2+\mu)}X)$$

$$[\det (I_8 - \delta(A, B, t')q^{-\mu-1/2}X^{1/2})]^{-1}.$$

Finally we consider the series (3-13). Again we let $A = A'[\begin{smallmatrix}\xi_A & 0\\ 0 & \xi_A\end{smallmatrix}]$ with $A' \in SL_2$. Then such a series has the form (with $Y^2 = q^{-(1+\mu)}X$)

$$(1 - q^{-1}Y^2) \sum_{m=0}^{m=\infty} \frac{1 - (Y^2)^{m+1}}{1 - Y^2}\, Y^m \operatorname{tr} (\varrho_{(m,0)}(A')).$$

But then a direct calculation with $A' = (\begin{smallmatrix}\varrho & 0\\ 0 & \varrho^{-1}\end{smallmatrix})$ shows the above series equals (see Appendix 2).

$$\frac{(1 - q^{-1}Y^2)(1 - Y^4)}{(1 - \varrho Y)(1 - \varrho^{-1}Y)(1 - \varrho Y^3)(1 - \varrho^{-1}Y^3)}.$$

But we note that the above denominator equals

$$\det [I_8 - \delta(A', \sigma)Y]$$

where $\delta(A', \sigma)$ is the operator given by the composition $\sigma(I_2) \otimes \sigma(I_2) \otimes \sigma(A') \circ t$ with $t(v_1 \otimes v_2 \otimes v_3) = v_2 \otimes v_3 \otimes v_1$.

Then we note that $\delta(A, t) = \xi_A \delta(A', t)$. But again we can choose ξ_A up to sign so that $\xi_A = q^{+\mu/2}$. Thus $\det (I_8 - \delta(A', t)Y) = \det (I_8 - \delta(A, t)q^{-\mu-1/2}X^{1/2})$.

Thus (3-13) equals

$$(1 - q^{-(2+2\mu)}X^2)(1 - q^{-2-\mu}X)$$

$$[\det (I_8 - \delta(A, t)q^{-\mu-1/2}X^{1/2})]^{-1}.$$

Q.E.D.

At this point we must determine when the integrals $Z(\Phi_{\omega,\omega',s}, F)$ defined by (3-1) are absolutely convergent (*with no assumption on residual characteristic*).

We note first that if we vary $\Phi_{\omega,\omega',s_0} \in I_{\omega,\omega',s_0}$ (for a *fixed* s_0), then any element of $I_{\omega,\omega',s}$ has the form $\Phi_{\omega,\omega',s_0} \cdot \Phi^0_{s-s_0}$ (where $\Phi^0_{s-s_0} = \Phi^0_{1,1,s-s_0}$ is the unique $GSp_3(\mathcal{O}_v)$ invariant vector in $I_{1,1,s-s_0}$ normalized so that $\Phi^0_{s-s_0}(e) = 1$). We know that $\Phi_{\omega,\omega',s_0}$ is a bounded function on $Z'M \backslash GL_2(\mathbb{K})^0$ (since $Z'M \backslash GL_2(\mathbb{K})^0$ embeds into a compact space $P \backslash GSp_3$ and $\Phi_{\omega,\omega',s_0}$ is a smooth section in I_{ω,ω',s_0}). Thus an arbitrary function in $I_{\omega,\omega',s}$ is majorized by $|\Phi_{\omega,\omega',s_0}| \cdot |\Phi_{s-s_0}| \leqslant |\Phi_{s-s_0}| \cdot K$ (with $K = \sup |\Phi_{\omega,\omega',s_0}(*)|$). Thus $Z(\Phi_{\omega,\omega',s}, F)$ is majorized by an integral of the form

$$\int_{Z'M \backslash GL_2(\mathbb{K})^0} |\Phi^0_{s-s_0}(g)| |W_F^{\psi}(g)| \, dg. \tag{3-14}$$

Then we use the coordinates on $Z'M \backslash GL_2(\mathbb{K})^0$ given above. In particular from the above calculations we deduce that (3-14) is majorized by integrals of the form (in the case $\mathbb{K} = K \oplus K \oplus K$ and $v < \infty$)

$$\int \phi_1(\lambda)\phi_2(\alpha')\phi_3(\beta')\delta_1(\lambda)\delta_2(\alpha')\delta_3(\beta')|\lambda\alpha'\beta'|^{\{\mathrm{Re}(s-s_0)\}/2}$$

$$\left\{\int \max\{|x|, |\alpha'\beta'|^{1/2}, |\lambda\alpha'|^{1/2}, |\lambda\beta'|^{1/2}, |\alpha'|, |\beta'|, |\lambda|\}^{-\{\mathrm{Re}(s-s_0)+2\}} \, dx\right\}$$

$$d^{\times}(\lambda)d^{\times}(\alpha')d^{\times}(\beta').$$

Here we have used the fact that any $W \in W(\Pi, \psi)$ has the form $W\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) =$ a linear combination $\phi(a)X(a)$ with $\phi \in S(K)$ and X some character on $K^{\times}$. Moreover, we assume in the above integral that the $\phi_i > 0$ and that the characters $\delta_i > 0$.

First we note that the inner integral above is bounded by (provided Re (s) is large enough)

$$\int_K \max\{|x|, |\alpha'|, |\beta'|, |\lambda|\}^{-\{\mathrm{Re}(s-s_0)+2\}} \, dx.$$

But by explicitly determining this integral we deduce that it is bounded by a rational function in q^{-s} times

$$\max\{|\alpha'|, |\beta'|, |\lambda|\}^{-\{\mathrm{Re}(s-s_0)+1\}}.$$

Thus the above integral is bounded by

$$\int \phi_1(\lambda)\phi_2(\alpha')\phi_3(\beta')\delta_1(\lambda)\delta_2(\alpha')\delta_3(\beta')|\lambda\alpha'\beta'|^{\{\mathrm{Re}(s-s_0)\}/2}$$

$$\times\ \max\{|\lambda|, |\alpha'|, |\beta'|\}^{-\{\mathrm{Re}(s-s_0)+1\}}\ \mathrm{d}^{\times}(\lambda)\ \mathrm{d}^{\times}(\alpha')\ \mathrm{d}^{\times}(\beta').$$

Thus an easy exercise shows that the above integral is absolutely convergent for Re (s) large!

We note a similar argument works for proving the absolute convergence of (3-1) in the cases $\mathbb{K}$ cubic, and $\mathbb{K} = K_1 \oplus K$ with K_1 quadratic.

PROPOSITION 3.2. *If* $\mathbb{K}$ *has one of the following forms* ($\mathbb{K}$ *cubic*, $\mathbb{K} = K_1 \oplus K$ *and* $\mathbb{K} = K \oplus K \oplus K$), *then the family of zeta integrals* (3-1) *is absolutely convergent for* Re (s) *large.*

Indeed we first consider the GSp_3 homogeneous quotient space $P''\backslash GSp_3$ where

$$P'' = \left\{\left(\begin{array}{c|c} A & Z \\ \hline 0 & {}^tA^{-1}\end{array}\right) \in P|\det A = 1\right\}.$$

Then there exists a linear map of $S(P''\backslash GSp_3)$ *onto* $I_{\omega,\omega',s}$ given by

$$\phi \rightsquigarrow \int \phi\left(\left[\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & & \\ & & t & & 0 & \\ \hline & & & \lambda & & \\ & & & & \lambda & \\ & & & & & \lambda t^{-1}\end{array}\right] g\right)\omega^{-1}(t)|t|^{-(s+2)}$$

$$(\omega')^{-1}(\lambda)|\lambda|^{(3/2)s+3}\,\mathrm{d}^{\times}(t)\ \mathrm{d}^{\times}(\lambda).$$

We know that $GL_2(\mathbb{K})^0$ has a unique open orbit in $P''\backslash GSp_3$. This open orbit determines the following induced representation of $GL_2(\mathbb{K})^0$, i.e.

$$S(M\cdot\mathscr{D}\backslash GL_2(\ )^0)$$

where $\mathscr{D} = \{((\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}), (\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}), (\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}))|\,|\alpha|_K = |\beta|_K = 1\}$ if $\mathbb{K} = K \oplus K \oplus K$, $\{((\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}), (\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}))\,|\alpha, \beta \in K^*, |\alpha| = |\beta| = 1\}$ if $\mathbb{K} = K_1 \oplus K$ with K_1 quadratic, and $\{(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix})|\alpha, \beta \in K^{\times}, |\alpha|_K = |\beta|_K = 1\}$ with $\mathbb{K}$ a cubic extension.

Then the image of $S(M\mathscr{D}\backslash GL_2(\mathbb{K})^0)$ under the above intertwining map becomes the following *compactly* induced representation of $GL_2(\mathbb{K})^0$, i.e.

$$H_{\omega,\omega',s} = c - \text{ind}\,[M \cdot R\backslash GL_2(\mathbb{K})^0,\ z \cdot r \rightsquigarrow \lambda_{\omega,\omega',s}(\gamma_1)(r)]$$

where $z \in M$ and $r \in R$.

We recall here that if $\mathbb{K} = K \oplus K \oplus K$, $\lambda_{\omega,\omega',s}(((\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}), (\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}), (\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}))) = |\alpha/\beta|^{(s/2)+1}\omega(\alpha^2\beta)\omega'(\alpha\beta)$ with similar formulae in the remaining cases.

Our problem here is to study the zeta integral (3-1) when the input information $\Phi_{\omega,\omega',s} \in H_{\omega,\omega',s}$.

Proposition 3.3. *Let $v < \infty$. Then there exist a function $h_{\omega,\omega',s}$ in $H_{\omega,\omega',s}$ and $F \in \Pi$ so that $Z_v(h_{\omega,\omega',s}, F) = 1$.*

If $v = \infty$ and $s_0 \in \mathbb{C}$, then there exist a function $h_{\omega,\omega',s}$ (which is dependent on s_0 and finite under the maximal compact subgroup of $GL_2(\mathbb{K})^0$) and $F \in \Pi$ so that $Z(h_{\omega,\omega',s}, F)$ is analytic at $s = s_0$ and is nonvanishing at s_0!

Proof. We assume first that $\mathbb{K} = K \oplus K \oplus K$.

We note that $GL_2(\mathbb{K})^0 = P^0 \cdot K^0$ where $P^0 = \{((\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}1 & y\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}1 & z\\ 0 & 1\end{smallmatrix}) | x, y, z \in K\} \cdot \{((\begin{smallmatrix}\alpha & 0\\ 0 & \alpha'\end{smallmatrix}), (\begin{smallmatrix}\beta & 0\\ 0 & \beta'\end{smallmatrix}), (\begin{smallmatrix}\gamma & 0\\ 0 & \gamma'\end{smallmatrix})) | \alpha\alpha' = \beta\beta' = \gamma\gamma'\}$ and $K^0 = SL_2(\mathcal{O}_v) \times SL_2(\mathcal{O}_v) \times SL_2(\mathcal{O}_v)$ if $v < \infty$ or $K^0 = 0(2, \mathbb{R}) \times 0(2, \mathbb{R}) \times 0(2, \mathbb{R})$ or $U(2) \times U(2) \times U(2)$ if $v = \infty$ (depending on whether ∞ is real or complex).

Then we define an element in $H_{\omega,\omega',s}$ as follows. Indeed we let $\phi_{\omega,\omega',s}$ belong to $\text{ind}_{MR}^{P^0}(z \cdot r \rightsquigarrow \lambda_{\omega,\omega',s}(\gamma_1)(r))$ and satisfy the condition of transforming on the right according to a fixed character on $K^0 \cap P^0$ which is trivial on the subgroup $\{((\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}1 & y\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}1 & z\\ 0 & 1\end{smallmatrix})) | x, y, z \in K\}$. Let $\tilde{\phi}$ be a function on K^0 which transforms on the left by $K^0 \cap P^0$ by the same character. Then let

$$h_{\omega,\omega',s}(g) = h_{\omega,\omega',s}(p \cdot k) = \phi_{\omega,\omega',s}(p)\tilde{\phi}(k).$$

Such a function is clearly well defined and determines an element in $H_{\omega,\omega',s}$.

We now determine that actually such $\phi_{\omega,\omega',s}$ exist. At this point we note that P^0 equals the direct product of MR and $\{((\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}), I, I) | x \in K\} \cdot \{(I, (\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}), (\begin{smallmatrix}\beta & 0\\ 0 & \beta^{-1}\end{smallmatrix})) | \alpha, \beta \in K^x\}$. Then we let $\phi_{\omega,\omega',s}(m \cdot r \cdot ((\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}), (\begin{smallmatrix}\beta & 0\\ 0 & \beta^{-1}\end{smallmatrix}))) \equiv \lambda_{\omega,\omega',s}(\gamma_1)(r)\chi_{\mathcal{O}}(x)\omega_{\Pi_2}^{\mathcal{O}}(\alpha)\omega_{\Pi_3}^{\mathcal{O}}(\beta)$ (if $v < \infty$) where $\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$, and $\omega_{\Pi_2}^{\mathcal{O}}$ and $\omega_{\Pi_3}^{\mathcal{O}}$ represent the characters ω_{Π_2} and ω_{Π_3} supported on $\mathcal{O}^x$. On the other hand if $v = +\infty$, then $\phi_{\omega,\omega',s}(\ldots) \equiv \lambda_{\omega,\omega',s}(\gamma_1)(r)h_1(x)h_2(\alpha)h_3(\beta)$ where $h_1 \in C_c^\infty(\mathbb{R})$, h_2, $h_3 \in C_c^\infty(\mathbb{R}^x)$.

It is straightforward to verify that if $v < \infty$ then $\phi_{\omega,\omega',s}(((\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}), (\begin{smallmatrix}\beta & 0\\ 0 & \beta^{-1}\end{smallmatrix}))((\begin{smallmatrix}1 & z\\ 0 & 1\end{smallmatrix}))((\begin{smallmatrix}1 & w\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}))((\begin{smallmatrix}\varrho_1 & 0\\ 0 & \varrho_1^{-1}\end{smallmatrix}), (\begin{smallmatrix}\varrho_2 & 0\\ 0 & \varrho_2^{-1}\end{smallmatrix}), (\begin{smallmatrix}\varrho_3 & 0\\ 0 & \varrho_3^{-1}\end{smallmatrix}))) \equiv \omega^4(\omega')^2\omega_{\Pi_1}(\varrho_1) \times \omega_{\Pi_2}(\varrho_2)\omega_{\Pi_3}(\varrho_3)\phi_{\omega,\omega',s}(((\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}), (\begin{smallmatrix}\beta & 0\\ 0 & \beta^{-1}\end{smallmatrix})))$. (Here $\varrho_i \in \mathcal{O}^x$ and $z, w, t \in \mathcal{O}$).

Then we consider the zeta integral (3-1) for such $h_{\omega,\omega',s}$. Here we use as coordinates on $Z \cdot M \backslash GL_2(\mathbb{K})^0$ the group $\{((\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix})(\begin{smallmatrix}\lambda & 0\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}\lambda\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}), (\begin{smallmatrix}\lambda\beta & 0\\ 0 & \beta^{-1}\end{smallmatrix}))| x \in K, \lambda, \alpha, \beta \in K^\times\}$. Then (3-1) is equal to

$$\int \chi_{\mathcal{O}}\left(\frac{x}{\lambda}\right) \omega_{\Pi_2}^{\mathcal{O}}(\alpha)\omega_{\Pi_3}^{\mathcal{O}}(\beta)\omega^2\omega'(\lambda)|\lambda|^{(1/2)s-2}\tilde{\Phi}(k_1, k_2, k_3) \tag{3-15}$$

$$\times\ W_{F_1}^{\psi}\left(\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}k_1\right) W_{F_1}^{\psi}\left(\begin{pmatrix}\lambda\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}k_2\right) W_{F_3}^{\psi}\left(\begin{pmatrix}\lambda\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}k_3\right)$$

$$\times \psi(x)\, \mathrm{d}x\, \mathrm{d}^\times(\lambda)\, \mathrm{d}^\times(\alpha)\, \mathrm{d}^\times(\beta).$$

We observe that the inner integral

$$\int_K \chi_{\mathcal{O}}\left(\frac{x}{\lambda}\right)\psi(x)\, \mathrm{d}x \;=\; |\lambda|\chi_{\mathcal{O}}(\lambda).$$

Then we asume that $\tilde{\Phi} = \tilde{\Phi}_1 \otimes \tilde{\Phi}_2 \otimes \tilde{\Phi}_3$ where $\tilde{\Phi}_1$ is defined by

$$\tilde{\Phi}_1(k) = \begin{cases} (\omega^2\omega')^2\omega_{\Pi_1}(\alpha) \text{ on } \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\cdot SL_2^{(m)} | x \in \mathcal{O}, \alpha \in \mathcal{O}^\times\right\} \\ 0 \qquad\qquad\qquad \textit{otherwise.}\end{cases}$$

We similarly define ($i = 2, 3$)

$$\tilde{\Phi}_i(k) = \begin{cases} \omega_{\Pi_i}(\alpha) \text{ on } \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\cdot SL_2^{(m)} | x \in \mathcal{O}, \alpha \in \mathcal{O}^\times\right\} \\ 0 \qquad\qquad \textit{otherwise.}\end{cases}$$

Here $SL_2^{(m)} = \{\gamma \in SL_2(\mathcal{O})_v | \gamma \equiv I \bmod \pi^m\}$ (m chosen so that $(\omega^2 \cdot \omega')^2\omega_{\Pi_1}$, ω_{Π_2}, and ω_{Π_3} are trivial on the subgroup $\{\alpha \in \mathcal{O}^\times | \alpha \equiv 1 \bmod \pi^m\}$.

Then we consider an integral of the form

$$\int_{SL_2(\mathcal{O}_v)} W_{F_1}^{\psi}\left(\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}k_1\right)\tilde{\Phi}_1(k_1)\, \mathrm{d}k_1.$$

We also choose m so that W_{F_1} is invariant by $SL_2^{(m)}$. Then the above integral is a nonzero multiple of

$$\int_{\mathcal{O}\times\mathcal{O}^\times} W_{F_1}^{\psi}\left(\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\delta & 0\\ 0 & \delta^{-1}\end{pmatrix}\right)(\omega^2\omega')^2\omega_{\Pi_1}(\delta)\,\mathrm{d}x\,\mathrm{d}^\times(\delta)$$

$$\equiv\left(\int_{\mathcal{O}}\psi(x\lambda)\,\mathrm{d}x\right)\left(\int_{\mathcal{O}^\times} W_{F_1}\left(\begin{pmatrix}\lambda\delta^2 & 0\\ 0 & 1\end{pmatrix}\right)(\omega^2\omega')^2(\delta)\,\mathrm{d}^\times(\delta)\right).$$

But since $\chi_{\mathcal{O}}(\lambda)$ appears in the integral above then the first term is nonzero and independent of $\lambda\in\mathcal{O}$.

We repeat the same calculation for $W_{F_i}^{\psi}$ i.e.

$$\int_{\mathcal{O}\times\mathcal{O}^\times} W_{F_i}^{\psi}\left(\begin{pmatrix}\lambda\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\delta & 0\\ 0 & \delta^{-1}\end{pmatrix}\right)\omega_{\Pi_i}(\delta)\,\mathrm{d}x\,\mathrm{d}^\times(\delta)$$

$$\equiv\left(\int_{\mathcal{O}}\psi\left(\frac{x\lambda}{\alpha^2}\right)\mathrm{d}x\right)\left(\int_{\mathcal{O}^\times} W_{F_i}^{\psi}\left(\begin{pmatrix}\lambda\alpha\delta & 0\\ 0 & (\alpha\delta)^{-1}\end{pmatrix}\right)\omega_{\Pi_i}(\delta)\,\mathrm{d}^\times(\delta)\right).$$

Again, $\alpha\in\mathcal{O}^\times$ and $\lambda\in\mathcal{O}$; thus the first integral is nonvanishing and independent of α and λ. Finally by change of variables $\delta\to\alpha\delta$ in the second integral we obtain

$$\omega_{\Pi_i}^{-1}(\alpha)\left(\int_{\mathcal{O}^\times} W_{F_i}^{\psi}\left(\begin{pmatrix}\lambda\delta^2 & 0\\ 0 & 1\end{pmatrix}\right)\mathrm{d}^\times(\delta)\right).$$

Thus the integral (3-15) equals

$$\begin{gathered}\int\chi_{\mathcal{O}}(\lambda)\omega^2\omega'(\lambda)|\lambda|^{(1/2)s-2}\\ \left\{\int_{\mathcal{O}^\times} W_{F_1}^{\psi}\left(\begin{pmatrix}\lambda\delta^2 & 0\\ 0 & 1\end{pmatrix}\right)(\omega^2\omega')^2(\delta)\,\mathrm{d}^\times(\delta)\right\} \qquad (3\text{-}16)\\ \left\{\prod_{i=2}^{i=3}\left(\int_{\mathcal{O}^\times} W_{F_i}^{\psi}\left(\begin{pmatrix}\lambda\delta^2 & 0\\ 0 & 1\end{pmatrix}\right)\mathrm{d}^\times(\delta)\right)\right\}\mathrm{d}^\times(\lambda).\end{gathered}$$

Finally we note that $W_{F_i}\left(\begin{smallmatrix} v & 0 \\ 0 & 1 \end{smallmatrix}\right)$ represents any $\xi_i \in S(K^\times)$ (by suitable choice of F_i). We let support $(\xi_i) \subset \mathcal{O}^\times$. Then we deduce that (3-16) equals

$$\int_{\mathcal{O}^\times} \omega^2\omega'(\lambda) \left\{ \int_{\mathcal{O}^\times} \xi_1(\lambda\delta^2)(\omega^2\omega')^2(\delta)\, \mathrm{d}^\times(\delta) \right\}$$

$$\times \left\{ \int_{\mathcal{O}^\times} \xi_2(\lambda\delta^2)\, \mathrm{d}^\times(\delta) \right\} \left\{ \int_{\mathcal{O}^\times} \xi_3(\lambda\delta^2)\, \mathrm{d}^\times(\delta) \right\} \mathrm{d}^\times(\lambda).$$

Finally we let $\xi_1 = (\omega^2\omega')^{-1}$, $\xi_2 = \xi_3 = 1$. Thus the above integral is nonvanishing. Hence (3-15) is *nonvanishing and independent of s*!

We let $v = +\infty$. We know that $\tilde{\Phi} = \tilde{\Phi}_1 \otimes \tilde{\Phi}_2 \otimes \tilde{\Phi}_3$ can be chosen so that

$$\left(\prod_{i=1}^{i=3} \int W_{F_i}^{\psi}(g_i k_i)\phi_i(k_i)\, \mathrm{d}k_i \right) = \prod_{i=1}^{i=3} W_{F_i}^{\psi}(g_i)$$

(the $W_{F_i}^{\psi}$ are finite functions relative to the maximal compact subgroup of $SL_2(K_v)$ and the ϕ_i can be chosen to be suitable matrix coefficients of such a subgroup.)

Thus we have that (3-15) equals

$$\int h_1\left(\frac{x}{\lambda}\right) h_2(\alpha)h_3(\beta)\omega^2\omega'(\lambda)|\lambda|^{s/2-2}\, W_{F_1}^{\psi}\left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$\times\ W_{F_2}^{\psi}\left(\begin{pmatrix} \lambda\alpha^2 & 0 \\ 0 & 1 \end{pmatrix}\right) W_{F_3}^{\psi}\left(\begin{pmatrix} \lambda\beta^2 & 0 \\ 0 & 1 \end{pmatrix}\right)\left(\omega_{\Pi_2}^{-1}(\alpha)\,\frac{1}{|\alpha|^2}\right)$$

$$\times \left(\omega_{\Pi_3}^{-1}(\beta)\,\frac{1}{|\beta|^2}\right) \psi(x)\, \mathrm{d}x\, \mathrm{d}^\times(\lambda)\, \mathrm{d}^\times(\alpha)\, \mathrm{d}^\times(\beta).$$

Then following the same idea as above

$$\int_K h_1\left(\frac{x}{\lambda}\right) \psi(x)\, \mathrm{d}x = |\lambda|\check{h}_1(\lambda).$$

Thus our integral reduces to one of the form

$$\int X_1(\lambda)X_2(\alpha)X_3(\beta)F_s(\lambda, \alpha, \beta)\, \mathrm{d}^\times(\lambda)\, \mathrm{d}^\times(\alpha)\, \mathrm{d}^\times(\beta)$$

where $F_s(\lambda, \alpha, \beta)$ is a C^∞ function on compact sets $\{s| \, |s - s_0| < \varepsilon\} \times \{\lambda| \, |\lambda - \lambda_0| < \varepsilon\} \times \{\alpha| \, |\alpha - \alpha_0| < \varepsilon\} \times \{\beta| \, |\beta - \beta_0| < \varepsilon\}$ with $\lambda_0 \neq 0$, $\alpha_0 \neq 0$, and $\beta_0 \neq 0$. In such an instance we can choose the X_1, X_2, and X_3 with compact support in $\{\lambda| \, |\lambda - \lambda_0| < \varepsilon\}$, $\{\alpha| \, |\alpha - \alpha_0| < \varepsilon\}$ and $\{\beta| \, |\beta - \beta_0| < \varepsilon\}$ respectively, so that the above integral is analytic and nonvanishing at $s = s_0$. We note here that the choice of $h_{\omega,\omega',s}$ is also necessarily K_∞ finite (by the choice of the ϕ_i given above)!

We note similar proofs work for the remaining cases! Q.E.D.

Appendix 1 to §3

The proofs of §3 are given for the case when the local field K is non Archimedean. However some statements remain valid for the Archimedean case also. We point out which of the statements in §3 are true in such an instance.

Indeed we consider the cases when $K = \mathbb{R}$ or $K = \mathbb{C}$. Then when we calculate (3-3) we find that

$$f_\psi(s, \lambda, \alpha, \beta) \equiv |\lambda\alpha\beta|^{(s+\mu)/2+1} \int_K \psi(x)\{|x|^2 + A|\alpha + \beta + \lambda|^2\}^{-(s+\mu+2)/2} \, dx$$

where

$$|Z| = \begin{cases} |Z|_\infty & \text{if } K = \mathbb{R} \\ |Z \cdot \bar{Z}|_\infty^2 & \text{if } K = \mathbb{C} \end{cases}$$

and

$$\psi(Z) = \begin{cases} e^{2\pi\sqrt{-1}Z} & \text{if } K = \mathbb{R} \\ e^{2\pi\sqrt{-1}(Z+\bar{Z})} & \text{if } K = \mathbb{C} \end{cases}$$

Thus we see using the above formulae combined with (3-3) that *Propositions* 3.2 and 3.3 remain valid for $K = \mathbb{R}$ or $\mathbb{C}$.

REMARK. The calculation of (3-3) for unramified data (for $K = \mathbb{R}$ or $\mathbb{C}$) cannot be done at the present moment but probably with some care it is a reasonable classical calculation involving Whittaker type functions (see Appendix 3 to §3).

We consider the problem of showing the analyticity of the zeta integral (3-1) for the case when K is an Archimedean field.

We use the following coordinates in the space $Z \cdot M\backslash GL_2(\mathbb{K})^0$: $\{((\begin{smallmatrix}\lambda & 0\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}\lambda & 0\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}\lambda & 0\\ 0 & 1\end{smallmatrix}))|\lambda \in K^\times\} \cdot \{((\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}), (\begin{smallmatrix}\beta & 0\\ 0 & \beta^{-1}\end{smallmatrix}))|\alpha, \beta \in K^\times\} \cdot \{((\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}))|x \in K\}$. Then the integral (3-1) equals a finite linear combination of terms of the form

$$\int_{\mathbf{R}\times\mathbf{R}_+^\times\times\mathbf{R}_+^\times} f_s\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}, \begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}, \begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)$$

$$\left\{\int_{\mathbf{R}_+^\times} W_1\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix} W_2\begin{pmatrix}\lambda\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix} W_3\begin{pmatrix}\lambda\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}\right.$$

$$\left.\times\ e^{2\pi\sqrt{-1}(1+\alpha^2+\beta^2)\lambda x}|\lambda|^{s/2+1}\,d^\times(\lambda)\right\} dx\, d^\times(\alpha)\, d^\times(\beta). \tag{3-17}$$

Here $W_i \in \mathscr{W}_\psi(\Pi_i)$, W_i is K_∞ finite, and f_s is also K_∞ finite (K_∞ = the maximal compact subgroup of $GL_2(\mathbb{K})^0$).

The problem here is to analyze the analytic properties of the above integral. In the very procedure of this analysis, we show that the analytic continuation of (3-17) depends in a continuous way on the parameter f_s.

We study first

$$\int_{\mathbf{R}_+^\times} W_1\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix} W_2\begin{pmatrix}\lambda\alpha^2 & 0\\ 0 & 1\end{pmatrix} W_3\begin{pmatrix}\lambda\beta^2 & 0\\ 0 & 1\end{pmatrix} e^{2\pi\sqrt{-1}(1+\alpha^2+\beta^2)\lambda x}|\lambda|^{(1/2)s+1}\,d^\times(\lambda).$$

Then we assume that since Π_i is generic and unitary,

$$W\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix} = \sum \phi_i(x)\chi_i(x)|x|^{w_i}(\log|x|)^{m_i}$$

is a finite sum, $m_i \geqslant 0$, χ_i is a unitary character, $w_i \in \mathbb{R}$ and $\phi_i \in S(\mathbb{R})$.

Thus the first integral above equals terms of the type

$$\int_{\mathbf{R}_+^\times} \phi_1(\lambda)\phi_2(\lambda\alpha^2)\phi_3(\lambda\beta^2)\, e^{2\pi\sqrt{-1}(1+\alpha^2+\beta^2)\lambda x}|\lambda|^{(1/2)s+A_i}(\log|\lambda|)^{m_i}d\lambda, \tag{3-18}$$

where A_i is some specified complex number depending on the Π_i.

The first reduction is to employ integration by parts in the above integral.

We deduce that (3-18) equals

$$\sum_{i=0}^{i=m} Q_i(s) \int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^m$$

$$[\phi_1(\lambda)\phi_2(\lambda\alpha^2)\phi_3(\lambda\beta^2)\ \mathrm{e}^{2\pi\sqrt{-1}(1+\alpha^2+\beta^2)\lambda x}]\ |\lambda|^{(1/2)s+A_i+m}\ (\log|\lambda|)^i\ \mathrm{d}\lambda.$$

Here Q_i is a rational function in s.

We find that the term $(\mathrm{d}/\mathrm{d}\lambda)^m[\ldots]$ = a sum of terms of the form

$$P(\alpha, \beta, x)\tilde{\phi}_1(\lambda)\tilde{\phi}_2(\lambda\alpha^2)\phi_3(\lambda\beta^2)\ \mathrm{e}^{2\pi\sqrt{-1}(1+\alpha^2+\beta^2)\lambda x}$$

where P is a polynomial in α, β, and x (which has degree bounded by m).

We note in these calculations we are using the fact that for fixed α, β, and x, the function

$$\lambda \to \phi_1(\lambda)\phi_2(\lambda\alpha^2)\phi_3(\lambda\beta^2)\ \mathrm{e}^{2\pi\sqrt{-1}(1+\alpha^2+\beta^2)\lambda x}$$

is a Schwartz function!

Thus the next integral we analyze is

$$\int_{\mathbb{R}_+^\times} \tilde{\phi}_1(\lambda)\tilde{\phi}_2(\lambda\alpha^2)\phi_3(\lambda\beta^2)\ \mathrm{e}^{2\pi\sqrt{-1}(1+\alpha^2+\beta^2)\lambda x}|\lambda|^{(1/2)s+X}\ (\log|\lambda|)^m\ \mathrm{d}\lambda$$

(where X and m are given numbers) in terms of α, β and x. We must consider the behavior of such an integral when $\alpha \to \left\{\begin{smallmatrix}0\\+\infty\end{smallmatrix}\right.$, $\beta \to \left\{\begin{smallmatrix}0\\+\infty\end{smallmatrix}\right.$ and $x \to \infty$.

We consider $\alpha = r\cos\theta$, $\beta = r\sin\theta$ with $r \geqslant 0$ and $0 \leqslant \theta \leqslant \pi/2$. Then the above integral equals

$$\int_{\mathbf{R}_+^\times} \tilde{\phi}_1(\lambda)\tilde{\phi}_2(\lambda r^2\cos^2\theta)\tilde{\phi}_3(\lambda r^2\sin^2\theta)\ \mathrm{e}^{2\pi\sqrt{-1}(1+r^2)\lambda x}|\lambda|^{(1/2)s+X}\ (\log|\lambda|)^m\ \mathrm{d}\lambda.$$

The first analysis required is the determination of the function

$$(r, \theta, x) \rightsquigarrow f_s\left(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}, \begin{bmatrix}r\cos\theta & 0\\ 0 & \dfrac{1}{r\cos\theta}\end{bmatrix}\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix},\right.$$

$$\left.\begin{bmatrix}r\sin\theta & 0\\ 0 & \dfrac{1}{r\sin\theta}\end{bmatrix}\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\right)$$

as (r, θ, x) approach the various boundary data, i.e., $r \to \{{}^{0}_{\infty}$, $\theta \to \{{}^{0}_{\pi/2}$, and $x \to \pm\infty$.

We are required to study f_s in a more geometric way. Indeed we recall that $Z \cdot M\backslash Gl_2(\mathbb{K})^0$ is an open set in $P_3\backslash GSp_3$. The differentiable variety $P_3\backslash GSp_3$ can be realized as a closed submanifold of the projective space $P(\Lambda^3 \mathbb{R}^6)$ (Λ^3 = the third exterior power of $\mathbb{R}^6$). Indeed $P_3\backslash GSp_3$ is the set of all "subspaces" in $P(\Lambda^3 \mathbb{R}^6)$ which are "isotropic" relative to a fixed alternating form on $\mathbb{R}^6$. Moreover $P_3\backslash GSp_3$ is the closed orbit in $P(\Lambda^3 \mathbb{R}^6)$ of the unitary group U_3 acting on $P(\Lambda^3 \mathbb{R}^6)$ (i.e., $P_3\backslash GSp_3 \cong U_3 \cap P_3\backslash U_3 \cong O(3)\backslash U(3)$ by means of the standard identifications). Thus the differentiable structure on $P_3\backslash GSp_3$ is inherited from the projective space $P(\Lambda^3 \mathbb{R}^6)$. For instance, a typical coordinate patch in $P(\Lambda^3 \mathbb{R}^6)$ has the following structure. If we have fixed a global system of coordinates $\{z_{ijk}\}$ on $\Lambda^3 \mathbb{R}^6$, then a typical coordinate patch $M_{i_0 j_0 k_0} = \{Z \in P(\Lambda^3 \mathbb{R}^6) | z_{i_0 j_0 k_0}(Z) \neq 0\}$ has as coordinate $\{z_{ijk}/z_{i_0 j_0 k_0}\}$. In practical terms we identify the global coordinates on the space $\Lambda^3 \mathbb{R}^6$ as the linear space spanned by all 3×3 minors $\{z_{ijk}\}$ of a 3×6 matrix (i, j, k denote the columns of the given matrix in M_{36}). Then if we have a given map $\phi\colon M \to P_3\backslash GSp_3$ (where M is a given C^∞ manifold), then ϕ is differentiable at $p \in M$ if $\phi\colon M \to P(\Lambda^3 \mathbb{R}^6)$ is differentiable at p.

We consider the subspace Z in $P_3\backslash GSp_3$ given by $K(e_0, e_0, e_0) + \{(\alpha e_1, \beta e_1, \gamma e_1) | \alpha + \beta + \gamma = 0\}$. Then we choose $\{(e_0, e_0, e_0), (e_1, -e_1, 0), (0, e_1, -e_1)\}$ as a basis of Z. Relative to the standard basis $\{(e_0, 0, 0), (0, e_0, 0), (0, 0, e_0), (e_1, 0, 0), (0, e_1, 0), (0, 0, e_1)\}$ of $\mathbb{R}^6$, the above basis in matrix form is

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right].$$

When we apply

$$\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} r\cos\theta & 0 \\ 0 & (r\cos\theta)^{-1} \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} r\sin\theta & 0 \\ 0 & (r\sin\theta)^{-1} \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right)$$

to the above matrix, we get

$$\begin{bmatrix} 1 & r\cos\theta & r\sin\theta & x & rx\cos\theta & rx\sin\theta \\ 0 & 0 & 0 & 1 & \dfrac{-1}{r\cos\theta} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{r\cos\theta} & \dfrac{-1}{r\sin\theta} \end{bmatrix}.$$

Then, for instance, if r, x are large and θ is away from 0 and $\pi/2$, the above matrix equals

$$\begin{bmatrix} rx & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^{-1} \end{bmatrix} \begin{bmatrix} \frac{1}{rx} & \frac{\cos\theta}{x} & \frac{\sin\theta}{x} & \frac{1}{r} & \cos\theta & \sin\theta \\ 0 & 0 & 0 & 1 & \frac{-1}{r\cos\theta} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\cos\theta} & \frac{-1}{\sin\theta} \end{bmatrix}.$$

Hence

$$f_s\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} r\cos\theta & 0 \\ 0 & (r\cos\theta)^{-1} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix},\right.$$
$$\left.\begin{bmatrix} r\sin\theta & 0 \\ 0 & (r\sin\theta)^{-1} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) \equiv |x|^{-(s+2)} f_s(K(r, x, \theta)),$$

where $K(r, x, \theta) \in U(3)$ and has the property that $Z \cdot K(r, x, \theta)$ is the subspace spanned by the rows of the matrix (given as the second term in the product above).

On the other hand, it is possible to verify that the mapping $(w, t, \theta) \rightsquigarrow O(3)K(1/w, 1/t, \theta)$ is a C^∞ mapping from $\{w|\ |w| < \varepsilon\} \times \{t|\ |t| < \varepsilon\} \times \{\theta|\ |\theta - \theta_0| < \varepsilon\}$ into $O(3)\backslash U(3) = P_3\backslash GSp_3$. Indeed we show that the mapping $(w, t, \theta) \to Z \cdot K(1/w, 1/t, \theta)$ is a C^∞ mapping into $P(\Lambda^3\mathbb{R}^6)$ (here use the fact that the image of the above mapping for ε small enough lies in a fixed coordinate patch in $P(\Lambda^3\mathbb{R}^6)$ given above).

From these comments we deduce the mapping $(s, w, t, \theta) \rightsquigarrow f_s(K(1/w, 1/t, \theta))$ is a C^∞ mapping and in fact is analytic in the first variable.

The goal here is to determine the analyticity of the integral

$$\int_{A'} f_s\left(K\left(\frac{1}{w}, \frac{1}{x}, \theta\right)\right) w^{-r}(\sin\theta)^{r'}(\cos\theta)^{r''} x^{-r'''+s}\left(\frac{w^2}{1+w^2}\right)^{(s/2)+m+A'}$$
$$\left\{\int_0^\infty \phi_1\left(\lambda\frac{w^2}{1+w^2}\right)\phi_2\left(\lambda\frac{1}{1+w^2}\cos^2\theta\right)\right.$$
$$\times\ \phi_3\left(\lambda\frac{1}{1+w^2}\sin^2\theta\right)|\lambda|^{(s/2)+m+A}\, e^{2\pi\sqrt{-1}\lambda(1/x)}\, d\lambda\, dw\, dx\, d\theta.$$

Here $A' = \{w| \ |w| < \varepsilon\} \times \{x| \ |x| < \varepsilon\} \times \{\theta| \ |\theta - \theta_0| < \varepsilon\}$. Here r, r', r'', and r''' are nonnegative integers bounded by m.

The first point to observe is that the inner integral as a function of (s, w, θ) is C^∞ in the region $\{s \in \mathbb{C}| \ \mathrm{Re}\,(s) > -2(1 + m + A)\} \times \{w| \ |w| < \varepsilon\} \times \{\theta| \ |\theta - \theta_0| < \varepsilon\}$. The main point is to analyze the behavior in x at 0 of

$$\int_0^\infty \phi_1\left(\lambda \frac{w^2}{1 + w^2}\right) \phi_2\left(\lambda \frac{1}{1 + w^2} \cos^2\theta\right)$$

$$\times \ \phi_3\left(\lambda \frac{1}{1 + w^2} \sin^2\theta\right) |\lambda|^{(s/2)+m+A} \ \mathrm{e}^{2\pi\sqrt{-1}\lambda(1/x)} \ \mathrm{d}\lambda.$$

Here we let

$$A(\lambda, w, \theta) = \phi_1\left(\lambda \frac{w^2}{1 + w^2}\right) \phi_2\left(\lambda \frac{1}{1 + w^2} \cos^2\theta\right) \phi_3\left(\lambda \frac{1}{1 + w^2} \sin^2\theta\right).$$

Then the integral becomes

$$(*) = \int_0^\infty A(\lambda, w, \theta)|\lambda|^{(s/2)+m+A} \ \mathrm{e}^{2\pi\sqrt{-1}\lambda(1/x)} \ \mathrm{d}\lambda.$$

However the function $(\mathrm{d}/\mathrm{d}\lambda)^k A(\lambda, w, \theta)$ vanishes for λ large (i.e., ϕ_1, ϕ_2, and ϕ_3 are Schwartz functions). Thus we can apply integration by parts several times to get that

$$(*) = \sum_{k=0}^{k=N} c_k(s)|x|^{(s/2)+m+A+k} \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^k A(0, w, \theta)$$

$$+ \int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^N A(\lambda, w, \theta) h_{-N}(\lambda, x, s) \ \mathrm{d}\lambda.$$

Here $c_k(s)$ is a meromorphic function in s. The function

$$h_{-N}(\lambda, x, s) = \int_{u=0}^{u=\lambda} (u - \lambda)^N |u|^{(s/2)+m+A} \ \mathrm{e}^{2\pi\sqrt{-1}u(1/x)} \ \mathrm{d}u.$$

By a contour argument we also know that

$$h_{-N}(\lambda, x, s) = \oint_{\{u=\lambda+i\sigma, \sigma \geqslant 0\}} (u - \lambda)^N [u]^{(s/2)+m+A} \ \mathrm{e}^{2\pi\sqrt{-1}(u/x)} \ \mathrm{d}u$$

$$\equiv \mathrm{e}^{2\pi\sqrt{-1}(\lambda/x)} \int_{\sigma=0}^{\sigma=+\infty} \sigma^N [\lambda + \sqrt{-1}\sigma]^{(s/2)+m+A} \ \mathrm{e}^{-2\pi(\sigma/x)} \ \mathrm{d}\sigma.$$

We note here that Re $(\frac{1}{2}s + m + A) > -1$ is required for the convergence of the above integral. This is also the region in s where the asymptotic expansion of $(*)$ is valid as stated above!

Finally we note by change of variables that

$$\int_0^\infty \sigma^N |\lambda + \sqrt{-1}\sigma|^{(s/2)+m+A}\, \mathrm{e}^{-2\pi(\sigma/x)}\, \mathrm{d}\sigma$$

$$= x^{N+1} \int_0^\infty \sigma^N [\lambda + \sqrt{-1}\sigma x]^{(s/2)+m+A}\, \mathrm{e}^{-2\pi\sigma}\, \mathrm{d}\sigma.$$

Moreover we note that $(\mathrm{d}/\mathrm{d}\lambda)^k A(\lambda, w, \theta)$ has the form of a linear combination of terms

$$w^{\ell}(1 + w^2)^{-\ell'}(\sin\theta)^{\ell''}(\cos\theta)^{\ell'''}\tilde{\Phi}_1\left(\lambda \frac{w^2}{1 + w^2}\right)\tilde{\Phi}_2\left(\lambda \frac{1}{1 + w^2}\cos^2\theta\right)$$

$$\times\ \tilde{\Phi}_3\left(\lambda \frac{1}{1 + w^2}\sin^2\theta\right)$$

with $\ell, \ell', \ell'', \ell''' \geqslant 0$ and $\tilde{\Phi}_i \in S(\mathbb{R})$.

Then we use the asymptotic expansion of $(*)$ to analyze the analyticity of (3-17). The first term is of the form

$$c_k(s) \int_A f_s\left(K\left(\frac{1}{w}, \frac{1}{x}, \theta\right)\right)(\sin\theta)^{\varrho}(\cos\theta)^{\varrho'}|w|^{s+2m+2A+\varrho''}$$

$$\times\ (1 + w^2)^{-(s/2)-(m+A)-\varrho'''}|x|^{(3s/2)+m+A+k-r'''}\, \mathrm{d}\theta\, \mathrm{d}w\, \mathrm{d}x.$$

Here $\varrho \geqslant 0$, $\varrho' \geqslant 0$, $\varrho'' \geqslant -m$ and $\varrho''' \geqslant 0$. In any case we can verify that the above integral has a meromorphic continuation to all of $\mathbb{C}$ (in s) and has the required continuity properties in f_s. We recall here the differentiable and analytic properties of the map $(s, w, x, \theta) \rightsquigarrow f_s(K(1/w, 1/x, \theta))$!

On the other hand, the second term (the Remainder term) has the form

$$\int_{A\times[0,\infty]\times[0,\infty]} f_s\left(K\left(\frac{1}{w}, \frac{1}{x}, \theta\right)\right)\tilde{\Phi}_1\left(\lambda \frac{w^2}{1 + w^2}\right)\tilde{\Phi}_2\left(\lambda \frac{1}{1 + w^2}\cos^2\theta\right)$$

$$\times\tilde{\Phi}_3\left(\lambda \frac{1}{1 + w^2}\sin^2\theta\right)\mathrm{e}^{2\pi\sqrt{-1}(\lambda/x)}\,\mathrm{e}^{-2\pi\sigma}(\lambda + i\sigma x)^{(s/2)+m+A}(\sin\theta)^{\varrho}$$

$$\times\ (\cos\theta)^{\varrho''}|w|^{s+2m+2A+\varrho''}(1 + w^2)^{-(s/2)-(m+A)-\varrho'''}|x|^{(3s/2)+m+A+N-r'''}$$

$$\times\ \sigma^N\, \mathrm{d}w\, \mathrm{d}\theta\, \mathrm{d}x\, \mathrm{d}\lambda\, \mathrm{d}\sigma.$$

Here ϱ, ϱ', ϱ'', and ϱ''' are as above. In any case we have that for m and N large enough the above integral is absolutely convergent and the required continuity properties of the integral follow automatically!

We note here that if in the inner integral in (3-17) we have a term of the form $(\log|\lambda|)^m$, then a calculation similar to the above works to prove meromorphicity and continuity (in terms of f_s) of (3-17).

Also the same arguments as above will work in the cases of other singularities (i.e., r large, x large, and θ near 0 or $\pi/2$ etc.).

Thus we have established the required analyticity of the integral (3-1) and the continuity properties (dependence on f_s).

REMARK 1. We note that from *Proposition* 3.3 and the above proof we can find a function $F_s \in I_{1,1,s}$ *which is* $U(3)$ *finite* and $F \in \Pi$ so that the zeta integral $Z_\infty(F_s, F)$ has a meromorphic continuation to $\mathbb{C}$ and $Z_\infty(F_s, F)$ is nonvanishing (with a possible pole) at the point $s = s_0$. Here F_s and F depend on s_0. Indeed there exists (by Proposition 3.3) a function $h_s \in I_{1,1,s}$ so that $Z(h_s, F)$ is nonvanishing at $s = s_0$. However there exists a sequence $(h_n)_s$ of $U(3)$ finite functions such that $(h_n)_{s_0}$ converges to h_{s_0} in the C^∞ topology on I_s. But this implies that there exists an n_0 such that $Z((h_{n_0})_s, F)$ is nonvanishing at $s = s_0$! We use here the fact that $Z(f_s, F)$ depends continuously on f_s. This means in particular that each coefficient of the Laurent expansion of $Z(f_s, F)$ at $s = s_0$ defines a continuous linear functional on $I_{1,1,s_0}$ (in the C^∞ topology).

REMARK 2. If we assume only that the function f_s is C^∞ and the Whittaker functions W_i are C^∞ (not K finite), then it is possible to extend the above arguments to show that the zeta integral (3-1) has a meromorphic continuation to $\mathbb{C}$ (for such C^∞ data). Moreover, a given term in the Taylor expansion of (3-1) at a value $s = s_0$ has the property that it defines a continuous linear functional on $\mathrm{ind}_{P_3}^{GSp_3}(\ldots) \otimes \Pi_1 \otimes \Pi_2 \otimes \Pi_3$ (taken relative to the C^∞ topologies in these spaces). The key step in proving such a point is to analyze the dependence of $W((\begin{smallmatrix}\lambda & 0\\ 0 & 1\end{smallmatrix})k)$ as k varies in $SO(2, \mathbb{R})$. Indeed we know it is possible to choose $\phi \in S(\mathbb{R}^2)$ so that

$$W\left(\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}k\right) = |\lambda|^{1/2-\gamma_0}\int \phi[[\alpha, t]\cdot k]\, e^{2\pi\sqrt{-1}(\lambda/t)\alpha}\, |t|^{2\gamma_0}\, d\alpha\, dt.$$

Then the Mellin transform

$$\int W\left(\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}k\right)|\lambda|^s\, d^\times(\lambda) = \int F(\phi\cdot k)[\lambda, t]\, |\lambda|^{s+1/2-\gamma_0}|t|^{s+1/2+\gamma_0}\, d^\times(\lambda)\, d^\times(t)$$

with F a certain partial Fourier transform on $S(\mathbb{R}^2)$. Thus we choose Schwartz functions $\{\phi_i\}$ and $\{\psi_i\}$ so that the function

$$A_N(\lambda, k) = |\lambda|^{1/2+\gamma_0} \sum_{i=1}^{i=N} \phi_i(\lambda)\sigma_i(k) + |\lambda|^{1/2-\gamma_0} \sum_{i=1}^{i=N} \psi_i(\lambda)\beta_i(k)$$

has a Mellin transform whose poles at the values $s = -(\frac{1}{2} - \gamma_0) - r$, $s = -(\frac{1}{2} + \gamma_0) - r$ $(r \geqslant 0)$ agree with the poles of $\int W((\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix})k)|\lambda|^s \, d^\times(\lambda)$. We let N be very large here. Then $W((\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix})k) - A_N(\lambda, k)$ has all its derivatives at $\lambda = 0$ (for each $k \in SO(2)$) vanishing up to a large order. Then we can apply the above proof to A_N and $W - A_N$ separately. We note that a similar proof works in the case where we replace $|\lambda|^{2\gamma_0}$ by $|\lambda|^{2\gamma_0}$ sgn (λ) in the above.

Appendix 2 to §3

At this point we consider any representation ϱ of $G(\mathbb{C})$ (with G, a reductive group) on a finite dimensional space V_ϱ. We consider the corresponding representation of $G(\mathbb{C})$, on $S(V_\varrho)$ = the symmetric algebra of V_ϱ. We know that the Poincaré polynomial

$$\sum_{m=0}^{\infty} \operatorname{tr}(\varrho_{S^m}(g))X^m \equiv \det(I - \varrho(g)X)^{-1}$$

where ϱ_{S^m} = the representation of $G(\mathbb{C})$ on $S^m(V_\varrho)$.

We consider the case where $G = GL_2 \times GL_2 \times GL_2$ and ϱ the representation given by $\sigma(g_1) \otimes \sigma(g_2) \otimes \sigma(g_3)$ $((g_1, g_2, g_3) \in G)$ with σ, the unique irreducible two dimensional representation of $GL_2(\mathbb{C})$.

We must determine ϱ_{S^m} in this case. For this we require the low dimensional isogeny between $GL_2 \times GL_2$ and $O(2, 2)$.

Explicitly we consider the space $M_{2,2}$, 2×2 matrices, provided with the form $X \to \det X$. This defines a $(2, 2)$ form. Then we consider the map $\lambda(g_1, g_2): X \to g_1^{-1}Xg_2$ with $(g_1, g_2) \in GL_2 \times GL_2$ and $X \in M_{2,2}$. Then $\det(g_1^{-1}Xg_2) = (\det g_2/\det g_1)\det X$ implies that $\lambda(GL_2 \times GL_2) \subseteq GO(2, 2)$ (GO = similitude group). By dimension and connectedness arguments we have that $\lambda(GL_2(\mathbb{C}) \times GL_2(\mathbb{C})) = GO(2, 2)(\mathbb{C})$. Thus we note that the *standard* 4 dimensional representation V^4 of $GO(2, 2)(\mathbb{C})$ is equivalent (via λ) to the tensor product $\sigma \otimes \sigma$ (we note that Kernel(λ) = $(\lambda \cdot I_2, \lambda I_2)$).

Thus we apply the above considerations to the group $G = GO(2, 2) \times GL_2$ and the representation $\varrho = V^4 \otimes \sigma$.

However in this case we note that such a task can be accomplished by use of the oscillator representation.

Namely we consider the dual reductive pair $\mathrm{O}(4) \times Sp_2$ over $\mathbb{R}$, the reals. We assume $\mathrm{O}(4)$ is a compact group. In such a case $\mathrm{O}(4) \times Sp_2$ embeds into Sp_8 and we consider the corresponding oscillator representation π acting on $L^2(M_{42}(\mathbb{R}))$ ($M_{42} = 4 \times 2$ matrices). To decompose π restricted to $\mathrm{O}(4) \times Sp_2$ we use the trick of the Fock model $\mathscr{H}$, i.e., a unitarily equivalent representation to π. In such an instance we take the K-finite vectors $\mathscr{H}_K$ in $\mathscr{H}$ and note that the corresponding infinitesimal representation of $LA(\mathrm{O}(4) \times Sp_2)$ on $\mathscr{H}_K$ is equivalent to a certain algebra of polynomial coefficient differential operators acting on the space $\mathscr{P}$ of polynomials on $M_{42}(\mathbb{C})$.

In particular we note $\mathrm{O}(4) \times GL_2(\mathbb{C})$ acts linearly on $M_{42}(\mathbb{C})$ via (g_1, g_2): $Z \to g_1^{-1} Z g_2$ with $Z \in M_{42}(\mathbb{C})$. When we differentiate this action on the space we deduce that this action is equivalent to the action of the $LA(\mathrm{O}(4) \times U(2))(\mathbb{C})$ acting *through the oscillator representation on* $\mathscr{P}$ (note that $U(2)$ is the maximal compact subgroup of Sp_2 and $LA(U(2) \otimes \mathbb{C}) = LA(GL_2(C)))$.

Thus the problem becomes one of decomposing ϱ explicitly under $LA(\mathrm{O}(4) \times U(2))$. This has been done in [K-V]. In particular we decompose $LA(Sp_2) \otimes \mathbb{C} = LA(U(2)) \otimes (\mathbb{C}) \oplus p_+ \oplus p_-$ (as complex spaces) where p_+ and p_- span the holomorphic and antiholomorphic tangent spaces to the symmetric space $U_2 \backslash Sp_2$ (at the point $U_2\{e\}$). Then we consider $U(p_+)$ and $U(p_-)$, the associated enveloping algebras of p_+ and p_-. We let $H = \mathscr{P}^{p_+}$ = space of functions in ϱ annihilated by p_+ ("pluriharmonics"). Then H is $\mathrm{O}(4) \times U(2)$ stable (via the action defined above). Let δ be an irreducible module of $\mathrm{O}(4)$. Then if $\mathscr{P}(\delta) \neq 0$ we have that $H(\delta) \neq 0$. Moreover $H(\delta)$ is GL_2 irreducible and the space $\mathscr{P}(\delta)$ is a free $U(p_-)$ module. That is, $\mathscr{P}(\delta) \cong U(p_-) \otimes H(\delta)$ (as $GL_2(\mathbb{C})$ modules). Also since $U(p_-)$ acts as multiplication by polynomials on $\mathscr{P}$ then the equivalence $\mathscr{P}(\delta) \cong U(p_-) \otimes H(\delta)$ is preserved relative to degrees ("degree" defined in $\mathscr{P}$).

We can make the above statements more explicit by recalling the results of [K-V]. In particular an irreducible representation δ of $\mathrm{O}(4)$ occurring in ϱ is parametrized by a pair of integers (m_1, m_2) with $m_1 \geqslant m_2 \geqslant 0$. We know that this representation occurs in $H^{m_1+m_2}$ and the corresponding infinitesimal representation of $GL_2 \times GL_2$ has highest weight $(0, -(m_1 + m_2)) \otimes (m_1, m_2)$. We note that the highest weight of $H(m_1, m_2)$ as a $GL_2(\mathbb{C})$ module is $(-m_2, -m_1)$.

Thus we note that $\mathscr{P}(m_1, m_2)$ (the isotypic component of $GL_2 \times GL_2$ of the form $(0, -(m_1 + m_2)) \otimes (m_1, m_2))$ must be decomposed under GL_2. This is equivalent to decomposing the $GL_2(\mathbb{C})$ tensor product $U(p_-) \otimes H(m_1, m_2)$. First we must determine the GL_2 representation on the space $U(p_-)$. This again is a separation of variables problem. The action of GL_2 on p_- is the unique three dimensional representation (adjoint representation) of GL_2. Then the representation of GL_2 on $S^t(p_-)$ is equivalent to

$$\sum_{\lambda=0}^{\lambda=[t/2]} (2t - 2\lambda, 2\lambda).$$

Then we deduce that as a GL_2 module $U(p_-) \otimes H(m_1, m_2)$ has the form

$$X^{m_1+m_2} \sum_{t=0}^{t=\infty} \left\{ \sum_{\lambda=0}^{\lambda=[t/2]} (2t - 2\lambda, 2\lambda) \otimes (-m_2, -m_1) \right\} X^{2t}$$

Thus the next problem is to decompose the tensor product $(2t - 2\lambda, 2\lambda) \otimes (-m_2, -m_1)$.

However we note that any representation of $GL_2(\mathbb{C})$ of the form (ℓ_1, ℓ_2) is equivalent to $(\ell_1 - \ell_2, 0) \otimes |\det *|^{\ell_2}$. Thus $(2t - 2\lambda, 2\lambda) \otimes (-m_2, -m_1)$ is equivalent to $(2t - 4\lambda, 0) \otimes (m_1 - m_2, 0) \otimes |\det *|^{2\lambda - m_1}$.

We can analyze $(2t - 4\lambda, 0) \otimes (m_1 - m_2, 0)$ via the Clebsch Gordan formalism. Namely we have that

$$(\alpha, 0) \otimes (\beta, 0) = \sum_{\{j \mid |\alpha - \beta| \leqslant j \leqslant \alpha + \beta,\, j \text{ has the same parity as } \alpha + \beta\}} (j, 0)$$

At this point we are interested in restricting the representation on the space $\mathscr{P}(m_1, m_2)$ to $SL_2(\mathbb{C})$. Moreover we want to compute the "modified" Poincaré polynomial of the $SL_2(\mathbb{C})$ action on $\mathscr{P}(m_1, m_2)$, i.e.,

$$\sum_{\ell=0}^{\infty} \operatorname{tr}\, (\varrho|_{\mathscr{P}^{\ell}(m_1, m_2)}(x))\, Y^{\ell} \qquad (*)$$

(with $x \in SL_2(\mathbb{C})$).

The calculation of $(*)$ is given in the following lemma.

LEMMA (i). $(*)$ *equals the series*

$$V_\nu(Y) = \frac{1}{1-Y^4}\left[\sum_{j=0, j\equiv \nu \bmod 2}^{j=2\nu-2} \frac{1-Y^{2(j+1)}}{1-Y^2} Y^{\nu-j}(j,0)\right.$$

$$\left. + \left(\frac{1-Y^{2(\nu+1)}}{1-Y^2}\right) \sum_{k=0, k\equiv 0 \bmod 2}^{k=\infty} Y^k(\nu+k,0)\right]\cdot Y^{m_1+m_2}$$

where $\nu = m_1 - m_2$.

Proof. The proof proceeds by induction on ν. We let $\tilde{V}_\nu(Y) = V_\nu(Y)/Y^{m_1+m_2}$ and $(**) = (*)/Y^{m_1+m_2}$.

Indeed let $\nu = 0$. Then we see easily that $(**)$ equals

$$\frac{1}{1-X^4}\sum_{k=0}^{\infty} X^{2k}(2k,0).$$

Then let $\nu = 1$. Again it is direct to verify that $(**)$ equals

$$\frac{1\cdot(1-X^4)}{(1-X^4)(1-X^2)}\sum_{k=0}^{\infty} X^{2k}(2k+1,0).$$

Thus to prove the general case we assume that the validity of the identity in the above Lemma for $\nu = 2\ell$. We then consider the case $\nu = 2\ell + 2$.

Indeed we consider the tensor product $U(p_-) \otimes (2\ell, 0) \otimes (2, 0)$ in two different ways. On one hand by induction hypothesis $U(p_-) \otimes (2\ell, 0) \otimes (2, 0) = \tilde{V}_{2\ell}(X) \otimes (2, 0)$. On the other hand $U(p_-) \otimes (2\ell, 0) \otimes (2, 0) = U(p_-) \otimes ((2\ell + 2, 0) \oplus (2\ell, 0) \oplus (2\ell - 2, 0)) = (U(p_-) \otimes (2\ell + 2, 0)) \oplus \tilde{V}_{2\ell}(X) \oplus \tilde{V}_{2\ell-2}(X)$. Thus we must prove the identity:

$$\tilde{V}_{2\ell}(X) \otimes (2,0) = \tilde{V}_{2\ell+2}(X) \oplus \tilde{V}_{2\ell}(X) \oplus \tilde{V}_{2\ell-2}(X).$$

This can be checked directly!

The proof when ν is odd is similar and is omitted! Q.E.D

COROLLARY TO LEMMA (i). *The* $SL_2 \times SL_2 \times SL_2$ *module* $\mathscr{P}(m_1, m_2)$ *decomposes via the Poincaré polynomial*

$$(m_1 + m_2, 0) \otimes (m_1 - m_2, 0) \otimes V_{m_1-m_2}(X).$$

At this point we consider certain other Poincaré polynomials. In particular let t and t' be the permutations given by $t(v_1 \otimes v_2 \otimes v_3) = v_2 \otimes v_3 \otimes v_1$ and $t'(v_1 \otimes v_2 \otimes v_3) = v_2 \otimes v_1 \otimes v_3$. We are interested in determining the Poincaré polynomials

$$(1) \quad = \quad \sum_{m=0}^{\infty} \operatorname{tr} (\varrho|_{S^m}((g_1, g_2, g_3) \circ t))X^m$$

and

$$(2) \quad = \quad \sum_{m=0}^{\infty} \operatorname{tr} ((\varrho|_{S^m}((g_1, g_2, g_3) \circ t')))X^m$$

where $(g_1, g_2, g_3) \in SL_2 \times SL_2 \times SL_2$ (embedded in $GL_2 \times GL_2 \times GL_2$).

We observe that the only possible $SL_2 \times SL_2 \times SL_2$ irreducible components that give nonzero contributions to (i) have the form $(\ell, 0) \otimes (\ell, 0) \otimes (\ell, 0)$. Hence, using *Lemma* (i) and *Corollary to Lemma* (i), (1) equals

$$\frac{1}{1 - X^4} \sum_{\ell=0}^{\infty} \frac{1 - (X^2)^{\ell+1}}{1 - X^2} X^{\ell}((\ell, 0) \otimes (\ell, 0) \otimes (\ell, 0))$$

Similarly we note that the only $SL_2 \times SL_2 \times SL_2$ irreducible components that give nonzero contributions to (2) have the form $(\ell, 0) \otimes (\ell, 0) \otimes (v, 0)$ where v is arbitrary. Thus (2) equals

$$\frac{1}{1 - X^4} \sum_{\ell=0}^{\infty} (\ell, 0) \otimes (\ell, 0) \otimes V_{\ell}(X)$$

Appendix 3 to §3

At this point we prove the rationality of the local zeta integrals in the case of the Rankin triple product (i.e., $\mathbb{K} = K \oplus K \oplus K$). Here we make the assumption that $\omega = \omega' = 1$ and $\Phi_{1,1,s} = f_s$. Moreover we have that $\omega_{\Pi_1}\omega_{\Pi_2}\omega_{\Pi_3} = 1$.

We express the zeta integrals (3-1) in such a case as follows:

$$Z_v(f_s, F) = \int_{(K^\times)^2\backslash K^\times} \int_{M\backslash SL_2 \times SL_2 \times SL_2(K)} f_s\left(g_1\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}, g_2\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}, g_3\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}\right)$$

$$\times \left(\prod_{i=1}^{i=3} W_{F_i}^{\psi}\left(g_i\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}\right)\right) \mathrm{d}g_1\, \mathrm{d}g_2\, \mathrm{d}g_3\, \mathrm{d}\lambda.$$

But we know that $(K^\times)^2\backslash K^\times$ is a finite set. Thus the zeta integral becomes a finite sum of terms of the form

$$\int_{M\backslash SL_2 \times SL_2 \times SL_2} f_s\left(g_1\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}, g_2\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}, g_3\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}\right)$$

$$\times \left(\prod W_{F_i}^{\psi}\left(g_i\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}\right)\right) \mathrm{d}g_1\, \mathrm{d}g_2\, \mathrm{d}g_3.$$

But then we can express the above integral as an iterated integral of the form

$$\int_{N \times N\backslash SL_2 \times SL_2} W_{F_1}^{\psi}(x)\, W_{F_2}^{\psi}(y)$$

$$\left\{\int_{SL_2} f_s(x, y, z) W_{F_3}^{\psi}(z)\, \mathrm{d}z\right\} \mathrm{d}x\, \mathrm{d}y.$$

Thus our first problem is to compute the integral

$$\int_{SL_2} f_s(x, y, z) W_{F_3}^{\psi}(z)\, \mathrm{d}z.$$

We consider the decomposition of the space $K^2 \times K^2 \times K^2$ (see *Lemma* 1.1) relative to the form $A_1' \oplus A_2' \oplus A_3'$. In this context we then consider $Sp(A_1') \times Sp(A_2' \oplus A_3') \cong Sp_1 \times Sp_2 \cong SL_2 \times Sp_2$ embedded in $Sp(A_1' \oplus A_2' \oplus A_3') \cong Sp_3$.

The problem then becomes one of determining

$$\int_{SL_2} f_s(g_1, G) W_F^{\psi}(g_1)\, \mathrm{d}g_1$$

as a function of $G \in Sp_2$.

We recall that $f_s \in \mathrm{Ind}_P^{GSp_3}(\ldots)$ where P is the parabolic subgroup of GSp_3 stabilizing the 3 dimensional isotropic subspace M spanned by $\{(e_0, e_0, e_0), (e_1, -e_1, 0), (0, e_1, -e_1)\}$. Then we let $Q = \mathrm{Stab}_{Sp_2}\{\lambda(0, e_1, -e_1)\}$. Then Q has the form $GL_1 \times Sp_1 \times U_3$. In such an instance we consider the basis $\{X_1, X_2, X_3, X_4\}$ of $\{0\} \times K^2 \times K^2$ where $X_1 = (0, e_0, -e_0)$, $X_2 = (0, e_0, e_0)$, $X_3 = (0, e_1, -e_1)$, $X_4 = (0, e_1, e_1)$. Then relative to this basis

$$U_3 = \left\{ \left[\begin{array}{cc|cc} 1 & x & y & z \\ 0 & 1 & z & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{array}\right] \middle| x, y, z \in K \right\},$$

$$GL_1 = \left\{ \left[\begin{array}{cc|cc} \lambda & & & \\ & 1 & 0 & \\ \hline & 0 & \lambda^{-1} & \\ & & & 1 \end{array}\right] \middle| \lambda \in K^{x} \right\},$$

and

$$SL_2 \cong Sp_1 = \left\{ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ \hline 0 & 0 & 1 & 0 \\ 0 & \gamma & 0 & \delta \end{array}\right] \middle| \alpha\delta - \beta\gamma = 1 \right\}.$$

We can show that

$$\mathrm{Stab}_{Sp_1 \times Sp_2}(M) = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ \hline 0 & 0 & 1 & 0 \\ 0 & \gamma & 0 & \delta \end{array}\right] \middle| \alpha\delta - \beta\gamma = 1 \right\}$$

$$\times \{(1, u \cdot z) | u \in U_3, z \in GL_1\}.$$

Then we consider an element q of Q having the form $q = u_3 \cdot t \cdot z$ with $u_3 \in U_3$, $t \in GL_1$ and $z \in Sp_1$. From the invariance property of f_s we have that

$$\int_{SL_2} f_s(g, u_3 tzG) W_F^{\psi}(g)\, dg \;=\; |t|^{s+2} \int_{SL_2} f_s(z^{-1}g, G) W_F^{\psi}(g)\, dg$$

$$= \;|t|^{s+2} \int_{SL_2} f_s(g, G) W_F^{\psi}(zg)\, dg.$$

On the other hand, we proved in [P-R-(II)] and [P-R-(III)] that

$$\int_{SL_2} f_s(g, 1)\, \langle gv|w\rangle_{\Pi_v}\, dg \;=\; H_v(\Pi_v, s) \cdot U_v(s)$$

where $H_v(\Pi_v, s)$ is a fixed meromorphic function in s (with H_v of the form $1/P(q_v^{-s})$ if $v < \infty$) and U_v an entire function of s ($U_v \in \mathbb{C}[q_v^s, q_v^{-s}]$ if $v < \infty$). We note that if Π_v is an unramified principal series of GL_2, that $H_v(\Pi_v, s)$ coincides with the symmetric square L_v factor of Π_v.

Thus we have for $F \in \Pi_v$

$$\int_{SL_2} f_s(g, 1) W_F^{\psi}(xg)\, dg \;=\; H_v(\Pi_v, s)\xi_s(x)$$

where $\xi_s \in \mathbb{C}[q_v^s, q_v^{-s}] \otimes W(\Pi_v, \psi)$ if $v < \infty$ or $\xi_s \in R_s \otimes W(\Pi_v, \psi)$ with $R_s =$ space of entire functions in s if $v = +\infty$.

Thus we deduce that the integral

$$\int_{SL_2} f_s(g_1, g_2, g_3) W_F^{\psi}(g_1)\, dg_1$$

equals a finite linear combination of terms of the form

$$H_v(\Pi_{3,v}, s)|(g_2, g_3)_{GL_1}|^{2+s}\xi_s((g_2, g_3)_{SL_2})$$

where $(g_2, g_3)_{GL_1}$ and $(g_2, g_3)_{SL_2}$ denote the GL_1 and SL_2 part of the element (g_2, g_3) in Sp_2 relative to the decomposition $Sp_2 = U_3 \cdot GL_1 \cdot SL_2 \cdot Sp_2(\mathcal{O}_v) = Q \cdot Sp_2(\mathcal{O}_v)$.

It suffices to make the calculation for $g_2 = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{smallmatrix}\right)$ and $g_3 = \left(\begin{smallmatrix} t_2 & 0 \\ 0 & t_2^{-1} \end{smallmatrix}\right)$. In such an instance we compute that in terms of the basis $\{X_1, X_2, X_3, X_4\}$ the matrix (g_2, g_3) is represented by

$$\frac{1}{2} \cdot \left[\begin{array}{cc|cc} t_1 + t_2 & t_1 - t_2 & & \\ t_1 - t_2 & t_1 + t_2 & \multicolumn{2}{c}{0} \\ \hline & & t_1^{-1} + t_2^{-1} & t_1^{-1} - t_2^{-1} \\ \multicolumn{2}{c|}{0} & t_1^{-1} - t_2^{-1} & t_1^{-1} + t_2^{-1} \end{array}\right].$$

Then we consider the Iwasawa decomposition of the element

$$\begin{bmatrix} t_1 + t_2 & t_1 - t_2 \\ t_1 - t_2 & t_1 + t_2 \end{bmatrix} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} k_{(t_1,t_2)}.$$

Here α and β are uniquely determined elements in $\Pi^{\mathbf{Z}}$; thus $k_{(t_1,t_2)}$ is uniquely determined mod $\left(\begin{smallmatrix} \alpha & u \\ 0 & 1 \end{smallmatrix}\right)$, $u \in \mathcal{O}$ and $\alpha \in \mathcal{O}^x$.

We have that

$$\beta = \begin{cases} \max(|t_1|, |t_2|) & \text{if} \quad v < \infty \\ \sqrt{|t_1|_\infty^2 + |t_2|_\infty^2} & \text{if} \quad v = \infty. \end{cases}$$

and

$$\alpha = \begin{cases} \min(|t_1|, |t_2|) & \text{if} \quad v < \infty \\ \dfrac{|t_1 t_2|_\infty}{\sqrt{|t_1|_\infty^2 + |t_2|_\infty^2}} & \text{if} \quad v = +\infty. \end{cases}$$

We note that the function on $Sp_1 \times GL_2$ given by $(g, g') \rightsquigarrow$

$$f_s\left(g, \left[\begin{array}{c|c} g' & 0 \\ \hline 0 & {}^t(g')^{-1} \end{array}\right]\right)$$

has the property of being invariant under $(g, g') \rightsquigarrow (g\left(\begin{smallmatrix} \alpha & z \\ 0 & 1 \end{smallmatrix}\right)g')$ for all $z \in K$ and $\alpha \in \mathcal{O}^x$. Thus it follows there exist matrix coefficients $\mathscr{I}_i$ on the maximal compact subgroup of GL_2 so that $f_s(g, k) = \Sigma f_s^i(g, 1)\mathscr{I}_i(k)$ and $\mathscr{I}_i(\left(\begin{smallmatrix} \alpha & u \\ 0 & 1 \end{smallmatrix}\right)k) = \mathscr{I}_i(k)$ for all $u \in \mathcal{O}$, $\alpha \in \mathcal{O}^x$, and all $k \in$ the maximal compact subgroup of GL_2 (this last condition holds if $v < \infty$).

Thus it follows that $(t_1, t_2) \rightsquigarrow \mathscr{I}_i(k_{(t_1,t_2)})$ is a well defined function and moreover is locally constant on $K^x \times K^x$. ($v < \infty$ here).

If $v = +\infty$ then $(t_1, t_2) \to \mathscr{I}_i(k_{(t_1,t_2)})$ has the form $(|t_1|_\infty^2 + |t_2|_\infty^2)^{-r/2} P(t_1, t_2)$ with P a homogeneous polynomial of degree r.

On the other hand, if we assume that t_1 and t_2 are both close to zero (i.e. $|t_1| \leqslant A$, $|t_2| \leqslant A$ with A small) then $\mathscr{I}_i(k_{(t_1,t_2)})$ can be expressed as a linear combination of the form

$$\chi_{K_{(\varepsilon)}}((t_1 + t_2) \pm \sqrt{\varepsilon}(t_1 - t_2))$$

where $\chi_{K_{(\varepsilon)}}$ is a unitary character on the group $K(\sqrt{\varepsilon})^x K(\sqrt{\varepsilon})$, the unique unramified extension of K of degree 2).

Thus we have that for $(g_2, g_3) = (\left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} t_2 & 0 \\ 0 & t_2^{-1} \end{smallmatrix}\right))$, the integral (for $v < \infty$)

$$\int f_s(g_1, g_2, g_3) W_{F_3}^{\psi}(g_3)\, \mathrm{d}g_3$$

is given as a linear combination of terms of the form

$$H_v(\Pi_v, s)|\alpha|^{s+2}\xi_s \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \mathscr{I}_i(k_{(t_1, t_2)}).$$

More specifically if both t_1 and t_2 are small, then we may assume that $\mathscr{I}_i(k_{(t_1,t_2)}) = \chi_{K_{(\varepsilon)}}((t_1 + t_2) \pm \sqrt{\varepsilon}(t_1 - t_2))$.

We recall that any Whittaker function $W_{F_i}^{\psi}\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$ = a linear combination of terms $\phi(x)\chi_1(x)$ ($\phi \in S(K)$ and χ_1, a character on $K^{\times}$). Thus when we compute the above data we have a sum of terms of the form $H_v(\Pi_{3,v}, s)$ times

$$\int_{\{(t_1,t_2)\,|\,|t_1| \leqslant A, |t_2| \leqslant A\}} W_{F_1}^{\psi}\left(\begin{pmatrix} t_1^2 & 0 \\ 0 & 1 \end{pmatrix}\right) W_{F_2}^{\psi}\left(\begin{pmatrix} t_2^2 & 0 \\ 0 & 1 \end{pmatrix}\right) |\alpha|^{s+2}\xi_s \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$$

$$\chi_{K_{(\varepsilon)}}((t_1 + t_2) \pm \sqrt{\varepsilon}(t_1 - t_2)) \frac{\omega_{\Pi_1}^{-1}(t_1)\omega_{\Pi_2}^{-1}(t_2)}{|t_1 t_2|^2}\, \mathrm{d}^{\times}(t_1)\, \mathrm{d}^{\times}(t_2)$$

$$+ \int_{\{(t_1,t_2)\,|\,|t_1| > A \text{ or } |t_2| > A\}} W_{F_1}^{\psi}\left(\begin{pmatrix} t_1^2 & 0 \\ 0 & 1 \end{pmatrix}\right) W_{F_2}^{\psi}\left(\begin{pmatrix} t_2^2 & 0 \\ 0 & 1 \end{pmatrix}\right) |\alpha|^{s+2}\, \xi_s \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$$

$$\mathscr{I}_i(k_{(t_1,t_2)}) \frac{\omega_{\Pi_1}^{-1}(t_1)\omega_{\Pi_2}^{-1}(t_2)}{|t_1 t_2|^2}\, \mathrm{d}^{\times}(t_1)\, \mathrm{d}^{\times}(t_2).$$

We recall that $\xi_s = \Sigma\, P_i(q^{-s})W_{h_i}^{\psi}$ with $W_{h_i}^{\psi} \in W(\Pi_3, \psi)$ and $P_i \in \mathbb{C}[x]$. Moreover, we recall that support $(W_{F_i}\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right))$ is compact in K. We observe here that any integral of the form

$$\int_{\{(t_1,t_2)\,|\,|t_1| = A_1, |t_2| = A_2\}} \cdots$$

with the same integrands as in the above integral is a polynomial in q^{-s}.

Then we analyze the first integral above and decompose the integration as follows:

$$\int_{\{(t_1,t_2)\,|\,|t_1| \leqslant |t_2| \leqslant A\}} \cdots + \int_{\{(t_1,t_2)\,|\,|t_2| < |t_1| \leqslant A\}} \cdots.$$

We analyze both integrals above; the method of calculating the second is similar to the first. Then the first integral becomes a sum of terms of the form

$$\int_{\{t_2 \mid |t_2| \leqslant A\}} W_{F_2}^{\psi}\left(\begin{pmatrix} t_2^2 & 0 \\ 0 & 1 \end{pmatrix}\right) W_h^{\psi}\left(\begin{pmatrix} (t_2^*)^2 & 0 \\ 0 & 1 \end{pmatrix}\right) \left\{ \int_{\{t_1 \mid |t_1| \leqslant |t_2|\}} W_{F_1}^{\psi}\left(\begin{pmatrix} t_1^2 & 0 \\ 0 & 1 \end{pmatrix}\right) \right.$$

$$\left. \times\ |t_1|^s \chi_{K(\varepsilon)}((t_1 + t_2) \pm \sqrt{\varepsilon}(t_1 - t_2))\omega_{\Pi_1}^{-1}(t_1) \right\}$$

$$\times\ \mathrm{d}^\times(t_1)\ \omega_{\Pi_2}^{-1}(t_2)\omega_{\Pi_3}^{-1}(t_2^*) \frac{1}{|t_2|^2}\, \mathrm{d}^\times(t_2).$$

Here $t_2^* = (\pi^{\prime}\omega)^* = \pi^{\prime}$ in the integral above. Since A is small we have that $W_{F_1}^{\psi}\left(\begin{smallmatrix} t_1^2 & 0 \\ 0 & 1 \end{smallmatrix}\right) =$ a sum of terms of the form $\gamma(t_1^2)$ with γ_1 a character on $K^\times$. Thus the inner integral equals

$$\int_{\{t_1 \mid |t_1| \leqslant |t_2|\}} \gamma_1^2 \omega_{\Pi_1}^{-1}(t_1)|t_1|^s \chi_{K(\varepsilon)}((t_1 + t_2) \pm \sqrt{\varepsilon}(t_1 - t_2))\, \mathrm{d}^\times(t_1).$$

However, we note that

$$\chi_{K(\varepsilon)}((t_1 + t_2) \pm \sqrt{\varepsilon}(t_1 - t_2)) = \chi_{K(\varepsilon)}(t_2)\chi_{K(\varepsilon)}\left((1 \mp \sqrt{\varepsilon}) + (1 \mp \sqrt{\varepsilon})\frac{t_1}{t_2}\right)$$

$$= \chi_{K(\varepsilon)}(t_2)\chi_{K(\varepsilon)}(1 \mp \sqrt{\varepsilon})\chi_{K(\varepsilon)}\left(1 + \frac{t_1}{t_2}\frac{1 \mp \sqrt{\varepsilon}}{1 \mp \sqrt{\varepsilon}}\right).$$

We note that $1 \mp \sqrt{\varepsilon}$ is a unit in $\mathcal{O}_{K(\varepsilon)}^\times$. Then if $|t_1/t_2| < T$ (for T small) it follows that

$$\chi_{K(\varepsilon)}\left(1 + \frac{t_1}{t_2}\frac{1 \mp \sqrt{\varepsilon}}{1 \mp \sqrt{\varepsilon}}\right) = 1.$$

Hence the integral above equals the finite sum of terms

$$|t_2|^s \chi_{K(\varepsilon)}(t_2)\gamma_1^2\omega_{\Pi_1}^{-1}(t_2) \left\{ \sum_{\alpha=0}^{\alpha=R} q^{-\alpha s}\gamma_3^2\omega_{\Pi_1}^{-1}(\pi^\alpha) \right.$$

$$\left. \int_{\{t_1 \in \mathcal{O}^\times\}} \gamma_1^2\omega_{\Pi_1}^{-1}(t_1)\chi_{K(\varepsilon)}\left(1 + \pi^\alpha t_1 \frac{1 \mp \sqrt{\varepsilon}}{1 \mp \sqrt{\varepsilon}}\right) \mathrm{d}^\times(t_1) \right\}$$

$$+\ q^{-(R+1)s}|t_2|^s\gamma_1^2\omega_{\pi_1}^{-1}(\pi^{R+1}t_2)\chi_{K(\varepsilon)}(t_2)\xi_v(\gamma_1^2\omega_{\Pi_1}^{-1}, s).$$

Then when we substitute the above expression in the integral

$$\int_{\{t_2 \mid |t_2| \leqslant A\}} W_{F_2}^{\psi}\left(\begin{pmatrix} t_2^2 & 0 \\ 0 & 1 \end{pmatrix}\right) W_h^{\psi}\left(\begin{pmatrix} (t_2^*)^2 & 0 \\ 0 & 1 \end{pmatrix}\right) \{\ldots\}\, \omega_{\Pi_2}^{-1}(t_2)\, \omega_{\Pi_3}^{-1}(t_2^*) \frac{1}{|t_2|^2}\, \mathrm{d}^{\times}(t_2),$$

we find a linear combination of terms of the form

$$\int_{\{t_2 \mid |t_2| \leqslant A\}} W_{F_2}^{\psi}\left(\begin{pmatrix} t_2^2 & 0 \\ 0 & 1 \end{pmatrix}\right) W_h^{\psi}\left(\begin{pmatrix} (t_2^*)^2 & 0 \\ 0 & 1 \end{pmatrix}\right)$$
$$\times\ \gamma_1^2 \omega_{\Pi_1}^{-1} \omega_{\Pi_2}^{-1}(t_2)\, \chi_{K_{(\varepsilon)}}(t_2) \omega_{\Pi_3}^{-1}(t_2^*) |t_2|^{s-2}\, \mathrm{d}^{\times}(t_2).$$

Noting that $\chi_{K_{(\varepsilon)}}$ is trivial in $\pi\mathcal{O}_{K(\sqrt{\varepsilon})}^{\times}$ and that $\omega_{\Pi_1}\omega_{\Pi_2}\omega_{\Pi_3} = 1$ we see easily that the above integral has the form of a polynomial in q^{-s} times

$$\zeta_v(\gamma_1^2\gamma_2^2\gamma_3^2, s - 2)$$

(where γ_1 and γ_2 are characters appearing in the decomposition of $W_{F_2}^{\psi}$ and W_h^{ψ} restricted to $\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$).

On the other hand an integral of the form

$$\int_{\{(t_1, t_2) \mid |t_1| \leqslant A_1, |t_2| = A_1\}} \cdots$$

with A_1 small can be analyzed in a similar but easier fashion.

We note that in the remaining cases a similar proof for rationality of (3-1) can be given.

Thus we have the following consequence of the above calculations.

THEOREM. *Let $v < \infty$. Then in the zeta integrals* (3-1) $Z_v(f_s, F)$ *is rational in q^{-s} and admits a common denominator.*

REMARK. If $\Pi_i = \mathrm{ind}_{B_2}^{GL_2}\left(\left(\begin{smallmatrix} \alpha & x \\ 0 & \beta \end{smallmatrix}\right) \rightsquigarrow \gamma_i^1(\alpha)\gamma_i^2(\beta)|\alpha/\beta|^{1/2}\right)$ we assume that a Whittaker function W_{Π_i} has an asymptotic expansion of the form $c_1 |\;|^{1/2}\gamma_i^1 + c_2 |\;|^{1/2}\gamma_i^2$ near the origin. Then under the assumption that $\omega_{\Pi_i} = 1$ we can deduce from the proof above that $Z_v(f_s, F)$ can be expressed as a rational function with a denominator which divides

$$\sum_{i=1}^{i=3} H_v(\Pi_{i,v}, s) \cdot \sum_{\varepsilon_i \in \{1,2\}} \zeta_v((\gamma_1^{\varepsilon_1}\gamma_2^{\varepsilon_2}\gamma_3^{\varepsilon_3})^2, s + 1).$$

The key point here is to note that $H_v(\Pi_{i,v}, s) \equiv \zeta_v((\gamma_i^1)^2, s + 1)\zeta_v((\gamma_i^2)^2, s + 1)\zeta_v(s + 1)$. We observe that the Rankin triple product for

$\Pi_1 \otimes \Pi_2 \otimes \Pi_3$ is contained in the second term above. That is, if all the data is unramified (i.e. $\gamma_i^{\varepsilon_i}$ is unramified), then $\zeta_v((\gamma_1^{\varepsilon_1}\gamma_2^{\varepsilon_2}\gamma_3^{\varepsilon_3})^2, s+1) = [(1 - \gamma_1^{\varepsilon_1}\gamma_2^{\varepsilon_2}\gamma_3^{\varepsilon_3}(\pi)q^{-((1+s)/2)})(1 + \gamma_1^{\varepsilon_1}\gamma_2^{\varepsilon_2}\gamma_3^{\varepsilon_3}(\pi)q^{-((1+s)/2)})]^{-1}$.

The next case we consider is when $v = +\infty$ and $K_v = \mathbb{R}$.

In such a case the basic zeta integral equals a finite sum of the terms of the form

$$\int_{\mathbf{R}^\times \times \mathbf{R}^\times} W_{F_1}^{\psi}\left(\begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix}\right) W_{F_2}^{\psi}\left(\begin{pmatrix} y^2 & 0 \\ 0 & 1 \end{pmatrix}\right) W_{F_3}^{\psi}\left(\begin{pmatrix} x^2+y^2 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$\times \left(\frac{|xy|}{\sqrt{x^2+y^2}}\right)^{s+2} \omega_{\Pi_1}^{-1}(x)\omega_{\Pi_2}^{-1}(y)\omega_{\Pi_3}^{-1}(\sqrt{x^2+y^2})$$

$$\times \left(\frac{1}{\sqrt{x^2+y^2}}\right)^{r} P_r(x, y)\,\frac{1}{|xy|^2}\,\mathrm{d}^\times(x)\,\mathrm{d}^\times(y).$$

We expect by using the asymptotics of the $W_{F_i}^{\psi}$ that the above integral is a meromorphic function in s. Moreover, we expect that the poles occur in arithmetic progressions: that is, there exist values $\lambda_1, \ldots \lambda_r$ so that s is a pole of the above integral if and only $s = \lambda_i + k$ with k an integer $\geqslant 0$.

§4. Intertwining operators

The basic object of study is the analytic properties of the Eisentein series $E(\Phi_{\omega.\omega',s})$ on $GSp_3(\mathbb{A})$ constructed in §2. Indeed we analyze the constant term of $E(\Phi_{\omega,\omega',s})$ relative to the unipotent radical of the Borel subgroup of GSp_3.

First we consider the more general case of $GSp_n(\mathbb{A})$. In such a case we consider the induced representation

$$I_{\omega,s} = \mathrm{ind}_{P_n}^{GSp_n}\left[\left[\begin{array}{c|c} A & Z \\ \hline 0 & \lambda^t A^{-1} \end{array}\right]\right.$$

$$\left.\rightsquigarrow |\det A|^{s+(n+1)/2}\omega^2(\det A)\omega^{-n}(\lambda)|\lambda|^{-(n/2)(s+(n+1)/2)}\right].$$

Then we let $f_{\omega,s} \in I_{\omega,s}$ be an entire section. Then we form the associated Eisenstein series

$$E(f_{\omega,s}, g) = \sum_{\gamma \in (P_n)_K \backslash (GSp_n)_K} f_{\omega,s}(\gamma g).$$

We recall (6) in the Notation Section for the hypotheses placed on $f_{\omega,s}$.

The constant term of E along the unipotent radical of the Borel subgroup of GSp_n is constructed in [P-R-(II)]. We have the formula

$$E_{U_\varnothing}(f_{\omega,s}, g) = \sum_{w \in \Omega} M_w(f_{\omega,s})(g)$$

where M_w is a certain intertwining operator defined in $I_{\omega,s}$ (see [P-R-(II)] and below).

We analyze the analytic properties of each M_w separately. Basically we know that M_w can be factorized into a product of local intertwining operators

$$M_w \doteq \otimes_v (M_w)_v$$

where $M_w)_v$ is given by the corresponding local integral.

Following the methods given in [P-R-(II)] we recall the construction of such M_w.

We let $U_\varnothing$ be the unipotent radical of a Borel subgroup $B_\varnothing$ of GSp_n. Then we have a Bruhat decomposition in the form

$$GSp_n = \bigcup_{w \in W_H} PwU_\varnothing$$

where W_H runs thru a certain set of representatives of Weyl group elements in H. Then we have that

$$M_w(f_{\omega,s})(g) = \int_{U_w^-} f_{\omega,s}\,[wug]\,du$$

where U_w^- is a subgroup which satisfies

$$\{wuw^{-1} \mid u \in U_w^-\} = \left\{ \left[\begin{array}{cc|c} \multicolumn{2}{c|}{I} & 0 \\ \hline 0 & X & \\ {}^tX & S & I \end{array}\right] \;\middle|\; S \in \mathrm{Sym}_j(\mathbb{A}) \text{ and } X \text{ has the form} \right.$$

$$\left. \left[\begin{array}{ccccc} 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & & * \\ 0 & 0 & & * & * \\ 0 & & \cdots & & \\ 0 & * & * & & * \end{array}\right] \;\middle|\; * \text{ arbitrary} \right\}$$

We note that M_w is an intertwining operator from $I_{\omega,s}$ to the space

$$\mathrm{Ind}_{B_\varnothing}^{GSp_n}((w^{-1}\chi_{\omega,s}) \otimes \delta_{B_\varnothing}^{1/2})$$

where $\chi_{\omega,s}$ is the character on the Borel group $B_\varnothing$ given by

$$\left[\begin{array}{c|c} \begin{matrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_n \end{matrix} & 0 \\ \hline 0 & \begin{matrix} \lambda u_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda u_n^{-1} \end{matrix} \end{array}\right] \left[\begin{array}{c|c} \begin{matrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{matrix} & * \\ \hline 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{matrix} \end{array}\right]$$

$$\rightsquigarrow \left[\prod_{i=1}^{i=n} |u_i|^{s-(n-1)/2+i}\omega^2(u_i)\right] \omega^{-n}(\lambda)|\lambda|^{-(ns)/2+n(n+1)/4}$$

At this point we restrict to the case where $\omega = 1$. In such an instance $I_{1,s} = I_s, f_{1,s} = f_s$, etc. Let Φ_s be a factorizable function of the form $\otimes_{v\notin S} \Phi_{K_v,s} \otimes (\otimes_{v\in S} f_{v,s})$ where $\Phi_{K_v,s}$ is the unique $GSp_n(\mathcal{O}_v)$ fixed vector in $I_v(s)$ (so that $\Phi_{K_v,s}(e) = 1$) and $f_{v,s}$ is an *arbitrary* "*entire*" element in $I_v(s)$.

We let

$$d_{H_n}(s) = \left(\prod_{\substack{\ell=2 \\ \ell=0 \bmod 2}}^{\ell=n} \zeta_v(2s + n + 1 - \ell)\right) \zeta_v\left(s + \frac{n+1}{2}\right)$$

Then there exists a factor $c_{w_v}(s)$ (a ratio of factors of the form $(\zeta_v(k_i s + \alpha_i)\ldots)/(\zeta_v(m_i s + \beta_i)\ldots)$ so that $M_w(\Phi_{K_v,s}) = c_{w_v}(s)\Phi_{K_v,s}$. Again we note that $c_{w_v}(s)$ coincides with the calculation for the Sp_n case in [L].

Then we have that

$$d_{H_n}^S(s)M_w(f_s) \equiv [d_{H_n}^S(s)c_w^S(s)]\left[\prod_{v\in S} M_{w_v}(f_{v,s})\right]$$

$$\equiv [d_{H_n}(s)c_w(s)]\left[\prod_{v\in S} \frac{1}{d_{H_{n,v}}(s)c_{w_v}(s)} M_{w_v}(f_{v,s})\right].$$

We know from [P-R-(II)] that $d_H(s)c_w(s)$ has a finite number of poles.

Our goal is to show that the term

$$f_{v,s} \rightsquigarrow \frac{1}{d_{H_{n,v}}(s)c_{w_v}(s)} M_{w_v}(f_{v,s})$$

is *entire*. This will then show that the set of poles of $M_w = \otimes (M_w)_v$ is of a global nature. Explicitly this means that the poles of M_w are determined by the poles of $d_H(s)c_w(s)$!

Thus starting with an arbitrary function f_s in I_s we must determine the poles of $M_w(f_s)$ (a local problem). We observe that such a calculation can be facilitated by use of the following Lemma.

LEMMA 4.1. *Let* $S = \{\varphi \in I_s |$ support $(\varphi) \subset Pw_nP\}$ *(recall* Pw_nP = *open cell in* $P\backslash GSp_n$*). Then the analytic properties of the family*

$$\{M_w(\varphi)[w_n] | \varphi \in S\}$$

coincide with the analytic properties of the family

$$\{M_w(f_s) | f_s \text{ varies as analytic "sections" in } I_s\}$$

Proof. We consider the Laurent expansion of $M_w(f_s)$ at $s = s_0$, i.e.

$$M_w(f_s) = \sum_{k \geqslant A} \ell_k(f_{s_0})(s - s_0)^k.$$

We know that there is a smallest integer A so that the map $f_{s_0} \rightsquigarrow \ell_A(f_{s_0})$ defines a nonzero intertwining operator from I_{s_0} to $\mathrm{ind}_{B_\varnothing}^{GSp_n}(w^{-1}\chi_{s_0})$ (see above).

We must show that there exists a function $\varphi \in S$ so that $\ell_A(\varphi)[w_n] \neq 0$. Thus we assume that $\ell_A(\varphi)[w_n] = 0$ for all $\varphi \in S$.

In particular this implies that the function $\ell_A(\varphi)$ vanishes on the open cell in $B_\varnothing\backslash GSp_n$. Indeed the open cell is given by $B_\varnothing w_n B_\varnothing$ which is included in the set $B_\varnothing w_n P_n$. Thus since the open cell $B_\varnothing w_n B_\varnothing$ is dense in $B_\varnothing\backslash GSp_n$ this implies that $\ell_A(\varphi) = 0$ in the space $\mathrm{ind}_{B_\varnothing}^{GSp_n}(w^{-1}\chi_{s_0})$!

Hence we have that $\mathrm{Kernel}(\ell_A) \supseteq S$.

But this implies that $\mathrm{Kernel}(\ell_A) = I_{s_0}$. Otherwise we can find a *proper* GSp_n *invariant subspace* X in I_{s_0} which contains S (if K is Archimedean, then we assume X is a closed subspace). By using the duality between I_s and I_{-s} we see that this implies there exists a nonzero space $X^\perp$ in I_{-s_0} so that if

$\xi \in X^{\perp}$, then

$$0 = \int_{\mathrm{Sym}_n(K_v)} \xi \left[w_n \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \right] \varphi_1(X)\, \mathrm{d}X$$

for *all* $\varphi_1 \in C_c^{\infty}(\mathrm{Sym}_n(K_v))$.

But this says that $\xi = 0$. Q.E.D.

Thus the basic problem will be to analyze the analytic properties of the functional $\varphi \rightsquigarrow M_w(\varphi)[w_n]$ for φ belonging to the space S. To analyze the calculation of $M_w(\varphi)[w_n]$ we first observe that by the support properties it simply suffices to compute $(wnw^{-1})(ww_n)$ in terms of the Bruhat decomposition of the open cell Pw_nP (here $n \in U_w^-$). But we know that $\{wnw^{-1} | n \in N_w^-\}$ has the form

$$\left\{ \left[\begin{array}{c|c} I & 0 \\ \hline Z_w & I \end{array} \right] \middle| Z_w \text{ a subspace of } \mathrm{Sym}_n(K_v) \right\}.$$

Thus we arrive at the following integrals to analyze for the different $w \in \Omega$. We compute $M_w(\varphi)[w_n]$ for φ having the explicit form

$$\varphi \left[\left(\begin{array}{c|c} A & 0 \\ \hline 0 & {}^tA^{-1}\lambda \end{array} \right) \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array} \right) \left(\begin{array}{c|c} I & W \\ \hline 0 & I \end{array} \right) \right]$$
$$= |\det A|^{s+(n+1)/2} |\lambda|^{-(ns)/2} |\lambda|^{-(ns)/2-(n(n+1))/2} \tilde{\varphi}(W)$$

with $\tilde{\varphi} \in C_c^{\infty}(\mathrm{Sym}_n(K_v))$.

We first choose special w. Namely we let $w = w_j$ be the element in W_{GSp_n} having the form

$$\left[\begin{array}{cc|cc} I_{n-j} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_j \\ \hline 0 & 0 & I_{n-j} & 0 \\ 0 & I_j & 0 & 0 \end{array} \right].$$

In such an instance $Z_{w_j} = \{[\begin{smallmatrix} 0 & 0 \\ 0 & s \end{smallmatrix}] | s \in \mathrm{Sym}_j(K)\}$. Thus we have that for $\varphi \in s$,

$$M_{w_j}(\varphi)[w_n] = \int_{\mathrm{Sym}_j(K)} \tilde{\varphi} \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} |\det(s)|^{s+(n+1)/2-(j+1)}\, \mathrm{d}s$$

where $\tilde{\varphi}$ is defined above.

Then we consider the functions

$$b_{w_j}(s) = \left(\prod_{\substack{\ell=2 \\ \ell \equiv 0 \bmod 2}}^{\ell=j} \zeta_v(2s + n + 1 - \ell) \right) \zeta_v\left(s + \frac{n+1}{2}\right)$$

and

$$a_{w_j}(s) = \left\{ \prod_{\ell=\{[j/2]+\varepsilon_j,\, \varepsilon_j = \left\{\substack{1, j \text{ even} \\ 2, j \text{ odd}}\right.}}^{\ell=j} \zeta_v(2s + n - 2\ell + 2) \right\} \zeta_v\left(s + \frac{n+1}{2} - j\right).$$

In particular we note that $c_{w_j}(s) = (a_{w_j}(s))/(b_{w_j}(s))$. Moreover we deduce from the Appendix to §4 that the function

$$s \rightsquigarrow \frac{1}{a_{w_j}(s)} M_{w_j}(f_s)$$

is entire. Indeed we need only note the relationship

$$a^n_{w_j}(s) = a^j_{w_j}\left(s + \left(\frac{n-j}{2}\right)\right)$$

(here we use the notation a^t to refer to which Sp_t is being used).

Now we consider a general element w in W_{GSp_n}. Using the element w_j above, it is possible to express M_w as

$$M_w(f_{\omega,s})(g) \equiv \int_{u \in U_w^-} f_{\omega,s}[wuw^{-1}wg]\, du$$

$$\equiv \int f_{\omega,s}\left[w_j \left[\begin{array}{c|c} I & \begin{matrix} 0 & 0 \\ 0 & S \end{matrix} \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c|c} \begin{matrix} I & 0 \\ {}^tX & 1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 1 & -X \\ 0 & I \end{matrix} \end{array}\right] w_j wg \right] dS\, dX.$$

Here S and X run over the sets described above.

But in simple terms M_w has the form

$$\int M_{w_j}\left(\left[\begin{array}{cc|cc} I & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ {}^tX & I & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & I & -X \\ & & 0 & I \end{array}\right] w_j w g\right) \mathrm{d}X \tag{4-1}$$

where M_{w_j} is the intertwining operator associated to the element w_j in W_H.

First we note that M_{w_j} maps the space I_s to

$$\operatorname{ind}_{GL_{n-j}\times GL_j\times U'}^{GSp_n}\left[\left[\begin{array}{cc|cc} A & * & \multicolumn{2}{c}{\multirow{2}{*}{$*$}} \\ 0 & B & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & {}^tA^{-1} & 0 \\ & & * & {}^tB^{-1} \end{array}\right] \rightsquigarrow |\det A|^{s+(n+1)/2}|\det B|^{-s+j+1-((n+1)/2)}\right].$$

Here $A \in GL_{n-j}$, $B \in GL_j$ and U' = the appropriate unipotent radical of a parabolic P' of GSp_n with $GL_{n-j} \times GL_j$ as Levi factor.

The next point is that the integration in the X variable also represents an intertwining operator on Sp_n. This requires some care to explain in a precise manner.

Before we consider such a point we make certain preliminary remarks concerning the operator M_w. We recall that our goal is to prove that

$$s \to \frac{1}{d_{H_n}(s)c_w(s)} M_w(f_s)$$

is an entire function in s. We consider certain special w and apply an induction assumption to prove the validity of the above statement.

Indeed let w satisfy the assumption that X has the form where the first row consists entirely of zeroes. Then in such an instance the functional $\varphi \rightsquigarrow M_w(\varphi)[w_n]$ ($\varphi \in S$) can be viewed as a functional on the group Sp_{n-1}. By restricting $\varphi \in S$ to Sp_{n-1} (embedded in Sp_n via the map

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & A & 0 & B \\ \hline 1 & 0 & 0 & 0 \\ 0 & C & 0 & D \end{array}\right])$$

we obtain an element of the corresponding S space for the group Sp_{n-1} (relative to the corresponding induced representation of Sp_{n-1}).

Moreover we note that the elements $\{u^- ww_n | u^- \in U_w^-\}$ are of the type

$$\left[\begin{array}{c|c|c|c} 0 & 0 & -1 & 0 \\ \hline 0 & A' & 0 & B' \\ \hline 1 & 0 & 0 & 0 \\ \hline 0 & C' & 0 & D' \end{array}\right]$$

where $\left[\frac{A'}{C'}\middle|\frac{B'}{D'}\right]$ has the form $\{u^- ww_{n-1} | u^- \in U_w^- \cap Sp_{n-1}\}$ (here we can view w as an element of Sp_{n-1} since it has the form

$$\left[\begin{array}{c|c|c|c} 1 & & 0 & 0 \\ \hline & X & 0 & Y \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & Z & 0 & W \end{array}\right]).$$

Thus we can conclude that for w satisfying the above assumption, the functional $\varphi \rightsquigarrow M_w(\varphi)[w_n]$ coincides with the functional $\varphi' \to M_w(\varphi')[w_{n-1}]$ on Sp_{n-1} (here φ' belongs to the corresponding S space for the group Sp_{n-1}).

Thus we are in a position to apply an induction assumption. Indeed we note that $c_w^{n-1}(s+\frac{1}{2}) = c_w^n(s)$ and that

$$d_{H_{n-1}}(s+\tfrac{1}{2}) = d_{H_n}(s)\begin{cases} 1 & \text{if } n \text{ odd} \\ \zeta_v(2s+1) & \text{if } n \text{ even} \end{cases}.$$

Thus in any case we deduce by *induction* that $s \to (1/d_{H_n}(s)c_w(s))\, M_w(f_s)$ is *entire in* s. In fact if n is even then the above function has a zero where $1/(\zeta_v(2s+1))$ has a zero!

Now we analyze in (4-1) the integration in the X variable. Indeed we first consider how to construct an arbitrary element in the space

$$\operatorname{Ind}_{GL_{n-j}\times GL_j\times U'}^{GL_n}\left(\begin{bmatrix} A & * \\ 0 & B \end{bmatrix} \rightsquigarrow |\det A|^{s+(n+1)/2}|\det B|^{-s+(j+1)-(n+1)/2)}\right).$$

For this we let $\varphi \in S\,[M_{n-j,n}(K)]$ and consider the zeta integral ($G \in GL_n$)

$$Z(\varphi, G, s) = |\det G|^{-s+(j+1)-[(n+1)/2]} \int_{GL_{n-j}} \varphi([X|0]^t G^{-1})|\det X|^{2s+(n-j)}\, d^{\times}(X).$$

Then an easy calculation shows that $Z(\varphi, G, s)$ defines an element in the above induced representation of GL_n. In order to assume that $Z(\varphi, s)$ determines an "entire" section we must have that

$$\text{support}(\varphi) \subset \{Z|\text{rank}\,(Z) = n - j\}.$$

Thus by such a construction $Z(\varphi, s)$ is an arbitrary "entire" section in

$$\text{Ind}_{GL_{n-j}\times GL_j \times U'}^{GL_n}\left(\left[\begin{array}{c|c} A & * \\ \hline 0 & B \end{array}\right] \to |\det A|^{s+(n+1)/2}|\det B|^{-s+j+1-[(n+1)/2]}\right).$$

REMARK 4.1. We can construct in another fashion an arbitrary "entire" section in $\text{Ind}_{GL_{n-j}\times\cdots}^{GL_n}(\ldots)$ by taking

$$\tilde{Z}(\varphi, G, s) = |\det G|^{s+(n+1)/s} \int_{GL_j} \varphi([0|X]G)|\det X|^{2s+(n-j)}\, d^{\times}(X).$$

with support $(\varphi) \subset \{Z|\ \text{rank}\,(Z) = j\}$.

Then we must compute the integral

$$\int Z\left(\varphi, \left[\begin{array}{c|c} I & 0 \\ \hline {}^tX & I \end{array}\right], s\right) dX = \int\left\{\int_{GL_{n-j}} \varphi(Y|Y\cdot X)|\det Y|^{2s+(n-j)}\, dY\right\} dX. \tag{4-2}$$

The main problem that we must consider is to determine the poles of the above integral.

For this calculation we decompose GL_{n-j} as the maximal compact of GL_{n-j} times the lower triangular matrices. In such an instance (4-2) equals a sum of integrals of the form

$$\int\left\{\int \varphi\left[\begin{bmatrix} a_{11} & & & 0 & \\ & \ddots & & & \\ & & a_{ii} & & \\ & & & \ddots & \\ b_{ij} & & & & \ddots \end{bmatrix}\right]\left|\begin{bmatrix} a_{11} & & & 0 & \\ & \ddots & & & \\ & & a_{ii} & & \\ & & & \ddots & \\ b_{ij} & & & & \ddots \end{bmatrix}\right.\right.$$

$$\times\left.\left.\begin{bmatrix} & 0 & \\ & & \\ & * & \end{bmatrix}\right]\prod_i |a_{ii}|^{2s+i}\, \mathrm{d}^{\times}(a_{ii})\, \mathrm{d}b_{ij}\right\}\mathrm{d}*. \tag{4-3}$$

Since the set $\{Z \in M_{n-j,n}(k) | \operatorname{rank}(Z) = n - j\}$ is stable under the $GL_{n-j} \times GL_n$ action (given by $T \xrightarrow{(g_1,g_2)} g_1^{-1}Tg_2$), then we may assume that the φ above satisfy support$(\varphi) \subseteq \{Z| \operatorname{rank}(Z) = n - j\}$. On the other hand when we write

$$X = \begin{bmatrix} 0 & 0 & * \\ 0 & * & \\ * & * & * \end{bmatrix} = \begin{bmatrix} A_1 \\ \vdots \\ A_{n-j} \end{bmatrix}$$

(where A_i are the rows of X), then

$$\begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ b_{i\ell} & & a_{n-j,n-j} \end{bmatrix} X = \begin{bmatrix} a_{11}A_1 \\ \vdots \\ b_{1i}A_1 + \cdots + a_{ii}A_i \\ \vdots \end{bmatrix}.$$

In particular when we make the appropriate change of variables in $dX = dA_1 dA_2 \ldots dA_{n-j}$ we deduce that (4-3) equals

$$\int \left\{ \int \varphi \begin{bmatrix} a_{11} & & & & \left| {}^{\prime}A_1 \right. \\ & \ddots & & 0 & \left| A_2 \right. \\ & & \ddots & & \left| \vdots \right. \\ b_{i\ell} & & & a_{n-j,n-j} & \left| A_{n-j} \right. \end{bmatrix} \prod_{i=1}^{n-j} |a_{ii}|^{2s+i-n_i} \, d^{\times}(a_{ii}) \, db_{ij} \right\} \Pi \, dA_i. \tag{4-4}$$

Here n_i represents the number of nonzero entries in the ith row, A_i, of X given above (recall $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_{n-j}$).

Thus we deduce easily that (4-2) has the form

$$\prod_{i=1}^{i=n-j} \zeta_v(2s + i - n_i)$$

times an entire function in s (if $v < \infty$, then we get a polynomial in q^{-s}).

Now we note that if $n_1 = 0$ and $v < \infty$, we can omit the term $\zeta_v(2s + 1 - n_1)$ $(i = 1, n_1 = 0)$ above because of the support properties of the function φ. If $v = +\infty$ prime, then we must replace $\zeta_\infty(2s + 1 - n_1)$ by $\zeta_\infty(2s + 2 - n_1)$.

Now we are at the point of determining the analyticity properties of a general intertwining operator M_w. Moreover we can assume that X (given above) has the form where $n_1 > 0$ (otherwise we use the inductive step given above). We use the decomposition of M_w given above to get a first estimate on the possible poles of M_w. Indeed we see that

$$\prod_{i=1}^{i=n-j} (\zeta_v(2s + i - n_i)^{-1})(a_{w_j}(s))^{-1} M_w$$

is entire in s. Finally when we normalize M_w by $d_{H_n}(s) c_w(s)$ we note that

$$d_{H_n}(s) c_w(s) = \frac{d_{H_n}(s)}{b_{w_j}(s)} a_{w_j}(s) \prod_{i=1}^{i=n-j} \frac{\zeta_v(2s + i - n_i)}{\zeta_v(2s + i)}.$$

Thus we have that (for $f_s \in I_s$)

$$\frac{1}{d_{H_n}(s)c_w(s)} M_w(f_s)$$

equals

$$\frac{b_{w_j}(s)}{d_{H_n}(s)}\left[\prod_{i=1}^{i=n-j} \zeta_v(2s + i)\right]$$

times an entire function in s.

But the above product equals

$$\prod_{i \in X_{n,j}} \zeta_v(2s + i)$$

where $X_{n,j}$ is the set

$$\{a \in [1, n - j] \mid a \equiv n \bmod 2\} \quad \text{if} \quad j \text{ even}$$

and

$$\{a \in [1, (n - j) - 1] \mid a \equiv n \bmod 2\} \quad \text{if} \quad j \text{ odd}$$

(note that $X_{n,n-1} = \varnothing$ when n is even and $X_{n,n} = \varnothing$ in general).

REMARK 4.2. We let v be a finite prime. Then we observe that at those values of s where $2s \in X_{n,j}$ (when $X_{n,j} \neq \varnothing$) the representation I_s is irreducible (see [Gu]). However we note that for any w

$$\frac{1}{d_{H_n}(s)c_w(s)} M_w(\varphi_{K,s} * \xi) = \frac{1}{d_{H_n}(s)} \varphi_{K,w(s)} * \xi$$

($\xi \in \mathscr{H}(Sp_n)$, Hecke algebra of Sp_n). Thus for those values of s where $2s \in X_{n,j}$, the normalized intertwining map $\varphi_{K,s} * \xi \rightsquigarrow [1/d_{H_n}(s)c_w(s)]\, M_w(\varphi_{K,s} * \xi)$ is *entire* in the s variable. However if $f_s \rightsquigarrow [1/d_{H_n}(s)c_w(s)]\, M_w(f_s)$ admits a pole (for s satisfying $2s \in X_{n,j}$), then we get a contradiction to the irreducibility of I_s (i.e. on the subspace $\{\varphi_{K,s} * \xi \mid \xi \in \mathscr{H}(G)\}$ the normalized interwining operator given above is *entire*).

Thus the basic problem is to show that for those s such that $2s \in X_{n,j}$ the map $s \rightsquigarrow [1/d_{H_n}(s)c_w(s)]M_w(f_s)$ is *entire*. (We note if $K_v = \mathbb{R}$ or $\mathbb{C}$ we must show that the statement is true for all $2s \in X_{n,j} + \mathbb{Z}_-$ with $\mathbb{Z}_-$ the non-positive integers.)

At this point we consider specific values of n. Namely we consider the cases $n = 1, 2$ and 3.

The cases $n = 1$ and 2 follow from what has already been proved. We thus consider the case $n = 3$. Using the above data we see that the relevant w fall into three cases. Namely (i) $j = 2, n_1 = 2$, (ii) $j = 2, n_1 = 1$, and (iii) $j = 1, n_1 = 1, n_2 = 1$.

We observe first that case (iii) can be treated directly by using $\tilde{Z}$ instead of Z in *Remark* 4.1 above and then following similar ideas as above.

Then we consider (i) and (ii) above. Indeed we know that M_{w_2} has a pole at $s = -\frac{1}{2} + \varepsilon_q(\pi\sqrt{-1})/[\log(q)]$ if $v < \infty$ and at $s = -\frac{1}{2} - k/2$ (with k a nonnegative integer) if $v = +\infty$. Now if $v < \infty$ then the pole is simple and in fact we have (see Appendix) with $\varepsilon_q = 0$ or 1

$$\underset{s = -\frac{1}{2} + \varepsilon_q \frac{\pi\sqrt{-1}}{\log(q)}}{\text{residue}} M_{w_2}(\varphi)\left[w_n\begin{bmatrix} I & Z \\ 0 & I\end{bmatrix}\right] \equiv c\varphi\left[w_n\begin{bmatrix} I & Z \\ 0 & I\end{bmatrix}\right]$$

where $\varphi \in S$ and c, a nonzero constant independent of φ. On the other hand we note that the Sp_3 module

$$\mathrm{Ind}_{GL_1 \times GL_2 \times U'}^{Sp_3}\left(\left[\begin{array}{cc|c} \begin{matrix} a & * \\ \begin{matrix}0\\0\end{matrix} & B \end{matrix} & & * \\ \hline 0 & & \begin{matrix} a^{-1} & 0\ 0 \\ * & {}^tB^{-1} \end{matrix} \end{array}\right] \rightsquigarrow |a|^{s+2}|\det B|^{-(s-1)}\right)$$

contains the space I_s where

$$s = -\tfrac{1}{2} + \varepsilon_q \frac{\pi\sqrt{-1}}{\log(q)}$$

Indeed this is true since the GL_3 module

$$\mathrm{Ind}_{GL_1\times GL_2\times U''}^{GL_3}\left(\begin{bmatrix} a & * \\ 0 & \\ 0 & B \end{bmatrix} \to |a|^{2+s}|\det B|^{-(s-1)}\right)$$

contains the one dimensional space $\{|\det G|^{2+s} | G \in GL_3\}$ when

$$s = s_0^* = -\tfrac{1}{2} + \varepsilon_q \frac{\pi\sqrt{-1}}{\log(q)}.$$

But from above, $\mathrm{res}_{s=s_0^*} M_{w_2}$ carries the space S isomorphically to S. And since $I_{s_0^*}$ is the Sp_3 span of S (this follows from *Lemma* 4.1), then $\mathrm{res}_{s=s_0^*} M_{w_2}$ carries $I_{s_0^*}$ *Sp_2-equivariantly* into $I_{s_0^*}$! Thus a given h belonging to Image $\mathrm{res}_{s=s_0^*} M_{w_2}(\ldots)$ has the property that h restricted to GL_3 has the form

$$|\det x|^{-3/2+\varepsilon_q(\pi\sqrt{-1})/[\log(q)]}, \quad x \in GL_3.$$

Then we consider the unique $GL_3(\mathcal{O}_v)$ invariant vector in

$$\mathrm{Ind}_{GL_1\times GL_2\times U''}^{GL_3}\left(\begin{bmatrix} a & * & * \\ 0 & & \\ 0 & B & \end{bmatrix} \to |a|^{2+s}|\det B|^{-(s-1)}\right)$$

given by $\tilde{\varphi}_s$ normalized so that $\tilde{\varphi}_s(e) = 1$. For such a vector we have that

$$\int \tilde{\varphi}_s\left(\begin{bmatrix} 1 & 0 & 0 \\ t_1 & & \\ t_2 & I_2 & \end{bmatrix} h\right) \mathrm{d}t_1\, \mathrm{d}t_2 = \frac{\zeta_v(2s-1)}{\zeta_v(2s+1)}\, \tilde{\varphi}_{s'}(\tilde{w}h)$$

where $\tilde{\varphi}_{s'}$ is the corresponding $GL_3(\mathcal{O}_v)$ invariant in $M_{\tilde{w}^{-1}}(\mathrm{Ind}_{GL_1\times GL_2\times U''}^{GL_3}(\ldots))$ and $\tilde{w}$ is the Weyl group element in GL_3 which maps the positive roots into negative roots ($M_{\tilde{w}^{-1}}$ is the corresponding intertwining operator associated to the integral given above).

On the other hand we also have that

$$\int \tilde{\varphi}_s\left(\begin{bmatrix} 1 & 0 & 0 \\ t_1 & & \\ 0 & I_2 & \end{bmatrix} h\right) \mathrm{d}t_1 \equiv \frac{\zeta_v(2s)}{\zeta_v(2s+1)}\, \tilde{\varphi}_{s''}(\tilde{w}'h)$$

where $\tilde{\varphi}_{s''}$ and $\tilde{w}'$ are similar objects as in the previous case.

The upshot of the above calculations is that the two intertwining operators $M_{\tilde{w}}$ and $M_{\tilde{w}'}$ map $\tilde{\phi}_s$ to zero when $s = s_0^*$(moreover these operators are holomorphic at such a point also).

Finally we note that

$$\tilde{\phi}_{s_0^*} \equiv |\det|^{-1+s_0^*}.$$

Thus we deduce that for cases (i) $j = 2, n_1 = 2$ and (ii) $j = 2, n_1 = 1$, the operator $M_w(f_s)$ is entire at $s = s_0^*$ (M_w is the *composition* of M_{w_2} and either $M_{\tilde{w}}$ or $M_{\tilde{w}'}$). From this we see that (for $v < \infty$)

$$s \to \frac{1}{d_H(s)c_w(s)} M_w(f_s)$$

is analytic at $s = s_0^*$ (for cases (i) and (ii) above).

The reason for this is as follows. For $f_s \in I_s = \operatorname{ind}_{P_3}^{GSp_3}(\ldots)$ which is K finite, we note that $(1/(s - s_0^*))M_{w_2}(f_s) = \Sigma\, a_{\Lambda_i}(s)h_i$ where h_i is a finite family of linearly independent K finite functions belonging to $\operatorname{Ind}_{GL_1 \times GL_2 \times U''}^{Sp_3}(\ldots)$ (h_i are independent of s), and the a_{Λ_i} are analytic functions in s. Moreover we note that for $s = s_0^*$, the a_{Λ_i} will be nonzero according to whether h_i belongs to $I_{s_0^*}$ (when embedded in $\operatorname{Ind}_{GL_1 \times GL_2 \times U''}^{Sp_3}(\ldots)$). But then we note by the comments above, the intertwining operators $M_{\tilde{w}}$ and $M_{\tilde{w}'}$ when evaluated at $s = s_0^*$annihilate $\tilde{\phi}_{s_0^*}$ (recall that both operators are analytic at that value of s). This implies that $M_{\tilde{w}}$ and $M_{\tilde{w}'}$ annihilate h_i when $h_i \in I_{s_0^*}$. On the other hand $a_{\Lambda_i}(\ )$ vanishes at $s = s_0^*$ if $h_i \notin I_{s_0^*}$. Thus the composite operators $M_{\tilde{w}} \circ M_{w_2}$ and $M_{\tilde{w}'} \circ M_{w_2}$ are analytic at $s = s_0^*$!

We now assume that v is an Archimedean prime. Indeed we note that from the Appendix to §4

$$\operatorname{residue}_{s=-1/2-k/2} M_{w_2}(\varphi)\left[w_N\begin{bmatrix}1 & Z\\ 0 & 1\end{bmatrix}\right] \equiv c\,(\Box_Z^2)^k * \varphi\left[w_N\begin{bmatrix}I & Z\\ 0 & I\end{bmatrix}\right]$$

where $\Box^2$ represents the differential operator on the space $\operatorname{Sym}_3(\mathbb{R}) = \{(x_{ij}) | x = {}^t x\}$ given by $\det[\partial^2/(\partial x_{23}) - \partial/(\partial x_{22})\partial/(\partial x_{33})]$. In any case this shows that there exists an element W_k of the enveloping algebra $U(\overline{\operatorname{Sym}_3})$ (here we identify $\overline{\operatorname{Sym}_3}$ as the Lie algebra of the group $\{(\frac{I}{W}\vert\frac{0}{I}) | W = {}^t W\}$ so that

$$\operatorname{residue}_{s=-1/2-k/2} M_{w_2}(\varphi)\left[w_n\begin{bmatrix}I & Z\\ 0 & I\end{bmatrix}\right] = c(W_k \bowtie \varphi)\left[w_N\begin{bmatrix}I & Z\\ 0 & I\end{bmatrix}\right].$$

We note here W_k is acting on the left side of the function (that is, a degree one element ξ in $U(\overline{\mathrm{Sym}_3})$ acts via

$$\xi \bowtie f_s(g) \;=\; \frac{\mathrm{d}}{\mathrm{d}t} f_s(\exp(-t\xi)g|_{t=0}).$$

From the above calculation we deduce that for *all* $g \in Sp_3$

$$\mathrm{residue}_{s=-1/2-k/2} M_{w_2}(\varphi)(g) \equiv c(W_k \bowtie \varphi)(g)$$

(for $\varphi \in S$).

We note here that we can show that $\mathrm{residue}_{s=-1/2-k/2}(\ldots)$ vanishes away from the open Bruhat cell Pw_nP.

Then by restricting $W \bowtie f_{-1/2-k/2}$ to GL_3 we have

$$W_k \bowtie f_{-1/2-k/2}(x) \;=\; (Ad(x)\cdot W_k) \bowtie (f_{-1/2-k/2}(e))|\det x|^{3/2-k/2}.$$

But as x varies in GL_3 we know that $Ad(x)\cdot W_k$ lies in $U(\overline{\mathrm{Sym}_3})$ and in fact generates a *finite* dimensional representation of GL_3.

The upshot of the above statements is that

$$\{\mathrm{residue}_{s=-1/2-k/2} M_{w_2}(f_s)(g) | g \in GL_3\}$$

generates a finite dimensional subspace (GL_3 stable) of

$$\mathrm{Ind}_{GL_1 \times GL_2 \times U''}^{GL_3}\left(\begin{bmatrix} a & * & * \\ 0 & & \\ 0 & & \end{bmatrix} \!\!\begin{matrix} \\ B \end{matrix} \to |a|^{3/2-k/2}|\det B|^{3/2+k/2}\right).$$

Hence it follows that $\mathrm{residue}_{s=-1/2-k/2} M_{w_2}(f_s)$ generates an Sp_3 submodule equivalent to $\mathrm{ind}_{GL_3 \otimes U}^{Sp_3}(V \times 1)$ where V is the unique, finite-dimensional GL_3 submodule (given above) of $\mathrm{ind}_{GL_1 \times GL_2 \times U''}^{GL_3}(\ldots)$.

The last point that we must verify is that the operators $M_{\tilde{w}}$ and $M_{\tilde{w}'}$ (defined above) annihilate the finite dimensional spaces given above. In fact we show that the GL_3 modules

$$\mathrm{Ind}_{GL_1 \times GL_2 \times U''}^{GL_3}\left(\begin{bmatrix} B & & * \\ & & * \\ 0 & 0 & a \end{bmatrix} \to |\det B|^{(5+k)/2}|a|^{-[(1+k)/2]}\right)$$

and

$$\mathrm{Ind}_{B_3}^{GL_3}\left(\begin{bmatrix} a_1 & * & * \\ 0 & a_2 & * \\ 0 & 0 & a_3 \end{bmatrix} \to |a_1|^{5/2+k/2}|a_2|^{1/2-k/2}|a_3|^{3/2+k/2}\right)$$

do not contain any finite dimensional GL_3 submodules. These latter GL_3 module representations are the ones which contain the image of $M_{\tilde{w}}(\ldots)$ and $M_{\tilde{w}'}(\ldots)$ respectively. Indeed we restrict each of the above GL_3 spaces to

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{g}} \\ 0 & & \end{pmatrix} \,\middle|\, g \in GL_2\right\} \quad \text{and} \quad \left\{\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{g}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, g \in GL_2\right\}.$$

We note that *any* GL_3 stable finite dimensional subspace of these GL_3 modules must restrict to either one of the above subgroups in a nonzero fashion. We then obtain the GL_2 induced modules

$$\mathrm{Ind}_{B_2}^{GL_2}\left(\begin{bmatrix} \alpha & * \\ 0 & \beta \end{bmatrix} \to |\alpha|^{(5+k)/2}|\beta|^{-[(1+k)/2]}\right)$$

and

$$\mathrm{Ind}_{B_2}^{GL_2}\left(\begin{bmatrix} \alpha & * \\ 0 & \beta \end{bmatrix} \to |\alpha|^{(5+k)/2}|\beta|^{(1-k)/2}\right).$$

Then it is an easy exercise to see that such GL_2 modules cannot contain any finite dimensional GL_2 subrepresentations.

Then we deduce that (for $v = +\infty$)

$$s \rightsquigarrow \frac{1}{d_{H_n}(s)c_w(s)} M_w(f_s)$$

is analytic at those values of $s = -1/2 - k/2$ (k a nonnegative integer). Thus we use the same type of argument as in the p adic case to conclude the proof in the Archimedean case.

Thus we have established the following theorem.

THEOREM 4.2. *The function*

$$s \rightsquigarrow \frac{1}{d_{H_n}(s)c_w(s)} M_w(f_{v,s})$$

is entire in s in the following cases:
(a) *if* $v < \infty$ *and arbitrary* n.
(b) *if* $v = \infty$ *and* $n = 1, 2$ *and* 3 (here $K_v = \mathbb{R}$).

The arguments above thus imply that the poles of the global intertwining operator $f_s \to [1/d_{H_n}(s)c_w(s)]M_w(f_s)$ are of a global nature if $n \leqslant 3$. We expect that this also is the case if $n > 3$. Indeed from *Remark* 4.3, it suffices basically to prove the validity of *Theorem* 4.2 in the Archimedean cases when $n > 3$. The essential problem is apparently of a technical nature. Namely we must determine the exact nature of the operator M_{w_j} at the points $s = -m/2$, m a positive integer.

We now consider the case $n = 3$. By simple calculation (using [P-R-(II)] we deduce that $d_{H_3}(s)c_w(s)$ has possible poles in the set of s belonging to $\{-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2\}$.

Thus the possible poles of the function

$$s \rightsquigarrow d^S_{H_3}(s)E_{U_\varnothing}(f_s, \ )$$

occur at the values of s given above.

We note that some of the above values of s can be ruled out (as poles) for other reasons.

For instance let $s = 0$. Then we know that the map $s \rightsquigarrow E_{U_\varnothing}(f_s, \ )$ is holomorphic for all s such that $\operatorname{Re}(s) = 0$. Moreover we know that $d^S_{H_3}(s) = \zeta_S(s + 2)\zeta_S(2s + 2)$ is nonvanishing at $s = 0$! Thus $d^S_{H_3}(s)E_{U_\varnothing}(f_s, \ )$ is entire at $s = 0$.

On the other hand we consider the point $s = -2$. In such an instance the only term $d^S_{H_3}(s)M_w(f_s)$ which has a possible pole at $s = -2$ is when $w =$ identity. Then we note that $d^S_{H_3}(s)f_s = \zeta_S(s + 2)\zeta_S(2s + 2)f_s$ has no pole at $s = -2$.

Thus we can summarize our results so far in the following statement.

THEOREM 4.3. *The possible poles of the Eisenstein series* $d^S_{H_3}(s)E(f_s, \ )$ *occur at those values of* $s = \pm\frac{1}{2}, \pm 1$ *and* $s = 2$.

REMARK 4.4. We know that $E(f_s, \ )$ admits a nonzero pole at $s = 2$. Indeed if we choose $f_s = \otimes_v \Phi_{K_{v,s}}$ (with $\Phi_{K_{v,s}}$, the unique spherical vector in $(I_v)_s$), then from [J]

$$\mathrm{res}_{s=2}[d^S_{H_3}(s)E(f_s, \ )] \ = \ c \cdot 1$$

On the other hand it also follows from [Ca] that if we take a function of the form

$$f_s \ = \ (\bigotimes_{v \notin S} \Phi_{K_v,s}) \otimes (\bigotimes_{v \in S} (f_v)_s)$$

where $(f_v)_2$ lies in a GSp_3 proper submodule of $(I_v)_2$ (for all $v \in S$), then

$$\mathrm{res}_{s=2}[d^S_{H_3}(s)E(f_s, \ )] \ \equiv \ 0.$$

Appendix I

We let K be a local field.

We consider the space $\mathrm{Sym}_n(K)$ of $n \times n$ symmetric matrices and the action of $G\ell_n(K)$ on this space via $Z \to gZ^tg$ with $Z \in \mathrm{Sym}_n$ and $g \in G\ell_n$. Moreover we know that such an action has a finite number of orbits. Let $\mathscr{P}_1, \ldots, \mathscr{P}_s$ be representatives of the *open* GL_n orbits.

Let $S(\mathrm{Sym}_n(K))$ be the Schwartz space.

We define a zeta integral associated to the orbit $\mathscr{P}_i$. Namely let

$$Z_{\mathscr{P}_i}(\varphi, s) \ = \ \int_{\{X \in \mathrm{Sym}_n(K) | x \in \mathscr{P}_i\}} \varphi(X)|\det X|^{s-(n+1)/2}\, \mathrm{d}X.$$

This is defined for s such that Re (s) is large.

Moreover let

$$Z(\varphi, s) \ = \ \int_{\mathrm{Sym}_n} \varphi(X)|\det X|^{s-(n+1)/2}\, \mathrm{d}X.$$

Then the following facts can be easily established.

(1) The distribution $\varphi \to Z_{\mathscr{P}_i}(\varphi, s)$ satisfies the homogeneity property:

$$Z_{\mathscr{P}_i}(g^{-1}\varphi, s) \ = \ |\det g|^{-2s} Z_{\mathscr{P}_i}(\varphi, s).$$

(A similar formula holds for $Z(\varphi, s)$.

(2) The function $s \to Z_{\mathscr{P}_i}(\varphi, s)$ has a meromorphic continuation to $\mathbb{C}$ and there exists a functional equation of the form

$$Z(\check{\varphi}, s) = \sum_{\mathscr{P}_i} c_{\mathscr{P}_i}(\Psi, s) Z_{\mathscr{P}_i}\left(\varphi, \frac{n+1}{2} - s\right)$$

where $\check{\varphi}$ is the Fourier transform of φ taken relative to the character $T \rightsquigarrow \Psi(\operatorname{tr}(T))$ (with Ψ some additive character on K). Moreover $c_{\mathscr{P}_i}(\Psi, s)$ is a rational function in q_v^{-s} if $v < \infty$ (a ratio of factors of the form $(\Gamma(\alpha_i s + \beta_i) \ldots)/(\Gamma(\gamma_i s + \delta_i) \ldots)$ if $v = \infty$).

(3) If we take the Laurent expansion of $Z(\varphi, s)$ at $s = s_0$, we get

$$Z(\varphi, s) = \sum_{k \geqslant A} \ell_k(\varphi)(s - s_0)^k.$$

Here $\varphi \to \ell_k(\varphi)$ defines a distribution on $S(\mathrm{Sym})$. Moreover if A is the smallest integer so that $\ell_A(\varphi) \neq 0$ and $A < 0$, then the distribution $\varphi \to \ell_A(\varphi)$ is supported on the set $\{X \in \mathrm{Sym}_n | \det(X) = 0\}$. *Locally* such a distribution has the following form. Let μ_k be the distribution on the orbit $\{gA_k^t g | g \in G\ell_n\}$ (where A_k has rank $= k < n$) which transforms according to the character $g \to |\det g|^{-k}$. This distribution is unique up to scalar multiple. Then $\ell_A(\varphi) \equiv P[\partial/(\partial\xi)] \otimes \mu_k(\varphi)$ where P is some polynomial in the variables $\partial/(\partial\xi)$ (with $\partial/(\partial\xi)$ the transversal coordinates at a point X_0 on the orbit given above).

The problem is to determine the exact nature of the poles of $Z(\varphi, s)$. We will determine the poles by using an inductive technique and the points mentioned above.

We recall that if $w = w_n$ (the big Weyl group element), then (see §4)

$$M_{w_n}(\varphi_{K,s}) = \frac{a_{H_n}(s)}{d_{H_n}(s)} \varphi_{K,-s}$$

where

$$a_{H_n}(s) = \left\{ \prod_{\ell = \left\{[n/2] + \varepsilon_n, \varepsilon_n = \left\{\begin{smallmatrix} 1 \text{ if } n \text{ even} \\ 2 \text{ if } n \text{ odd}\end{smallmatrix}\right\}\right.}^{\ell = n} \zeta_v(2s + n - 2\ell + 2) \right\}$$
$$\times \left\{ \zeta_v\left(s - \left(\frac{n-1}{2}\right)\right)\right\}$$

and $d_{H_n}(s)$ is given in §4 (here $a_{H_n} = a_{w_n}$).

Then we prove the following Theorem giving the exact analytic behaviour of $Z(\varphi, s)$.

THEOREM. *The function*

$$s \to \frac{1}{a_{w_n}(s)} Z(\varphi, s)$$

is an entire function. Moreover for any given $s = s_0$ *we can find a* φ_0 *so that* $[1/a_{w_n}(s)]Z(\varphi_0, s)$ is nonzero at $s = s_0$!

PROOF. The proof is by induction on n. The case $n = 1$ is just Tate's thesis. That is, $s \to [1/\zeta_v(s)]Z(\varphi, s)$ is entire (for $n = 1$) and nonvanishing for suitable choice of φ.

Then we assume that the Theorem is true for $\ell < n$.

We look at the subspace in $C_c^\infty(\mathrm{Sym}_n)$ given by

$$S_t \equiv \left\{\varphi| \text{ support } (\varphi) \subseteq \left\{\begin{bmatrix} X & Y \\ {}^tY & W \end{bmatrix} \in \mathrm{Sym}_n | X, Y \text{ arbitrary and } W \in GL_t\right\}\right\}.$$

Then we consider the family of zeta integrals $Z(\varphi, s)$ as φ varies in $S_t(\{[\begin{smallmatrix} X & Y \\ {}^tY & Z\end{smallmatrix}]|X, Y$ arbitrary and $W \in GL_t\}$ is an *open* set in Sym_n).

We note that the pole behaviour of $(a_{w_n}(s))^{-1}Z(\varphi, s)$ at $s = s_0$ (as ϕ varies in S_t) is "less than" the pole behaviour of $(a_{w_n}(s))^{-1}Z(\phi, s)$ at $s = s_0$ (as ϕ varies in $C_c^\infty(\mathrm{Sym}_n)$). That is, from (3) above, the smallest k_r so that $\ell_{k_r}(\varphi) \neq 0$ $(\varphi \in S_t)$ satisfies $k_r \geqslant A$.

The main problem is to determine when is it possible that $k_r > A$. In such an instance this means that the distribution $\varphi \to \ell_A(\varphi)$ vanishes in S_t. In other words

$$\mathrm{supp}\ (\varphi \to \ell_A(\varphi)) \subseteq \left\{\begin{bmatrix} U & V \\ {}^tV & W \end{bmatrix} |\mathrm{rank}\ (W) < t\right\}.$$

We also note that by (3) above

$$\mathrm{supp}\ (\varphi \to \ell_A(\varphi)) \subseteq \{X|\det X = 0\}.$$

The set $\{X|\det X = 0\}$ is the union of a finite number of GL_n orbits of the form

$$\left\{g \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right] {}^tg|g \in GL_n\right\}$$

where A is some $\ell \times \ell$ symmetric matrix with $\ell < n$.

We also know that since the distribution $\varphi \to \ell_A(\varphi)$ is GL_n homogeneous we have that supp $(\varphi \to \ell_A(\varphi))$ is the union of a finite number of GL_n orbits of the above type.

We observe that the only possible GL_n orbits of the above type that are contained in $\{[\begin{smallmatrix} U & V \\ {}^tV & W \end{smallmatrix}] | \text{rank}\,(W) < t)$ have the form

$$\left\{ g \left(\begin{array}{c|c} B & 0 \\ \hline 0 & 0 \end{array} \right) {}^t g | g \in G \right\}$$

with B a nondegenerate symmetric matrix of rank $< t$.

We apply the inductive situation to the case when $t = n - 1$. Then the only possible GL_n orbit in $\{[\begin{array}{c|c} U & V \\ \hline {}^tV & 0 \end{array}] | U \in \text{Sym}_{n-1}(K),\ V \text{ arbitrary}\}$ is $\{0\}$ itself.

Thus $\varphi \to \ell_A(\varphi)$ is a homogeneous distribution supported at $\{0\}$.

If v is a finite prime, we deduce that $\ell_A(\varphi) = \varphi(0)$; hence the distribution $\varphi \to Z(\varphi, s)$ has a possible pole at the values of s where $2s \equiv 0 \bmod (2\pi\sqrt{-1})/[\log(q_v)]$ (i.e. when $s = 0$ or $s = (\pi\sqrt{-1}/[\log(q_v)] \bmod (2\pi\sqrt{-1}/[\log(q_v)]$.

If $v = +\infty$, then $\ell_A(\varphi) = [\det(\partial/(\partial x_{ij}))]^m(\varphi)(0)$ where m is some nonnegative integer. We observe that $\ell_A(\varphi)$ transforms according to $\ell_A(g^{-1}\varphi) = (\det g)^{2m}\ell_A(\varphi)$. In particular the distribution $\varphi \to (a_{w_n}(s))^{-1}Z(\phi, s)$ has a possible pole at those values of s where $2s = -2m$ (for all $m \geqslant 0$).

Thus we have located the values of s where the order of pole of $Z(\varphi, s)$ on the set S_{n-1} is "less than" on the set $C_c^\infty(\text{Sym}_n)$.

To analyze these values of s we use the functional equation given in (2) above. In particular we observe that if $Z(\varphi, s)$ has a pole at $s = s_0$ (of the above type), then $Z_{\mathscr{P}_i}(\check{\varphi}, (n+1)/2 - s)$ has no such pole at $s = s_0$ (recall here Re $((n+1)/2 - s_0) \geqslant (n+1)/2$; this implies that for some Ψ and $\mathscr{P}_i$, $c_{\mathscr{P}_i}(\Psi, s)$ has a pole at such s_0.

Let $X_{\mathscr{P}_1}(\ ) = \langle \text{disc}\,(\mathscr{P}_1)|\rangle$ with $\langle 1 \rangle$, the Hilbert symbol of K and disc $(_{\mathscr{P}_1}) =$ discriminant of $\mathscr{P}_1$.

However we can now use the explicit form of $c_{\mathscr{P}_i}(\Psi, s)$. Namely by direct calculation we have that

$$c_{\mathscr{P}_1}(\Psi, s) = \varepsilon_{\mathscr{P}_i}(\Psi, s)\left(\frac{a_H(s)}{d_H(-s)}\right)\begin{cases} \left(\dfrac{\zeta_v(X_{\mathscr{P}_i}, \frac{1}{2} - s)}{\zeta_v(X_{\mathscr{P}_i}, \frac{1}{2} + s)}\right) & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd} \end{cases}$$

where $\varepsilon_{\mathscr{P}_i}(\Psi, s)$ is an entire function in s without poles or zeroes.

Thus in any case we deduce that $s \to [1/a_H(s)]c_{\mathscr{P}_t}(\Psi, s)$ has no pole at the values of $s = s_0$ with $s_0 = -m$, m a nonnegative integer! This implies that the function $s \to (a_{w_n}(s))^{-1}Z(\phi, s)$ cannot have poles at the values of $s = -m$, m a nonnegative integer!

To finish the proof we must analyze the set $\{Z(\varphi, s) | \varphi \in S_{n-1}\}$ (this is the inductive step).

Again we work with $\{Z(\varphi, s) | \varphi \in S_t\}$ for general t.

We observe that the set

$$\left\{\begin{pmatrix} U & V \\ {}^tV & W \end{pmatrix} \middle| U, V \text{ arbitrary and } W \in GL_t\right\}$$

is isomorphic to the product

$$M_{n-t,t}(K) \times \mathrm{Sym}_{n-t} \times \{A \in \mathrm{Sym}_t | \det(A) \neq 0\}$$

via the map

$$\begin{pmatrix} I & U \\ 0 & I \end{pmatrix} \times \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \rightsquigarrow \begin{pmatrix} I & U \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & 0 \\ {}^tU & I \end{pmatrix}$$

$$\equiv \begin{pmatrix} A + UB{}^tU & BU \\ {}^tUB & B \end{pmatrix}$$

(here $U \in M_{n-t,t}(K)$, $A \in \mathrm{Sym}_{n-t}$, $B \in \mathrm{Sym}_t \cap GL_t$).

Then for $\varphi \in S_t$ we can find functions $\varphi_1 \in S[M_{n-t,t}(K)]$, $\varphi_2 \in S[\mathrm{Sym}_{n-t}]$ and $\varphi_3 \in S(\mathrm{Sym}_t \cap GL_t)$ so that $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$. Hence we have that

$$Z(\varphi, s) = \left(\int_{M_{n-t,t}} \varphi_1(u)\, \mathrm{d}u\right) \cdot \left(\int_{\mathrm{Sym}_{n-t}} \varphi_2(a) |\det A|^{s-[(n+1)/2]}\, \mathrm{d}A\right)$$
$$\cdot \left(\int_{\mathrm{Sym}_t} \varphi_3(B) |\det B|^{s-[(n+1)/2]+t}\, \mathrm{d}B\right).$$

Then we note the zeta integral with φ_3 is entire since $\mathrm{supp}(\varphi_3) \subset GL_t$. On the other hand the zeta integral with φ_2 is precisely a zeta integral of the type we are analyzing (on the space Sym_{n-t}). Then by induction we can assume that for $t = 1$ and $\varphi \in S_{n-1}$

$$s \rightsquigarrow \frac{1}{a_{H_{n-1}}(s - 1/2)} Z(\varphi, s)$$

is an *entire function* in s.

Finally we note that

$$a_{H_n}(s) \equiv \begin{cases} a_{H_{n-1}}(s - 1/2) & \text{if } n \text{ is odd} \\ \zeta_v(2s)a_{H_{n-1}}(s - 1/2) & \text{if } n \text{ is even} \end{cases}$$

Thus we have established that for all $\varphi \in S$

$$s \to \frac{1}{a_{w_n}(s)} Z(\varphi, s)$$

is an entire function in s!

It is straightforward to verify that such a function is nonvanishing (for a particular choice of φ).

§5. Properties of Rankin triple L functions

At this point we collect together the results of §2, §3 and §4.

We let ψ be a nontrivial additive character on $\mathbb{A}_K/K$.

Let K be a totally real field, i.e., $K_\infty = \mathbb{R}$ for all Archimedean primes in K.

We let Π be an automorphic cuspidal representation of $GL_2(\mathbb{K})$ (defined in §2). We assume that if (i) $\mathbb{K}$ = cubic extension of K then $\omega_\Pi = 1$, (ii) $\mathbb{K} = K_1 \oplus K$ (K_1 quadratic extension of K) then $\omega_\Pi = \omega_{\Pi_1} \cdot \omega_{\Pi_2} = 1$, and (iii) $\mathbb{K} = K \oplus K \oplus K$ then $\omega_\Pi = \omega_{\Pi_1} \cdot \omega_{\Pi_2} \cdot \omega_{\Pi_3} = 1$.

We consider the decomposition $\Pi = \otimes_v \Pi_v$, when Π is a representation of $GL_2(\mathbb{A}_K)$. We let $S_\Pi = \{v \in K | v$ archimedean and Π_v not an unramified principal series representation of $GL_2(K_v)\}$. If Π is a similar such representation of either $GL_2(\mathbb{A}_{K_1})$ or $GL_2(\mathbb{A}_{\mathbb{K}})$, then we can define the corresponding S_Π set (recall here that the $GL_2(\mathbb{A}_{K_1})$ or $GL_2(\mathbb{A}_{\mathbb{K}})$ are defined by restriction of scalars.)

Then we define the restricted *Rankin* triple L_S function of Π as given in (0-1) and (0-2). Indeed we let the S be as follows: (i) $\mathbb{K}$ = cubic extension then $S = S_\Pi \cup S_1$, (ii) $\mathbb{K} = K_1 \oplus K$ then $S = S_{\Pi_1} \cup S_{\Pi_2} \cup S_1$ and (iii) $\mathbb{K} = K \oplus K \oplus K$ then $S = S_{\Pi_1} \cup S_{\Pi_2} \cup S_{\Pi_3}$. Here $S_1 = \{v \in K|$ either residual char$(K_v) \neq 3$ or $\psi_v(\mathrm{tr}_{\mathbb{K}_w/K_v}(\ ))$ is not of order zero when $w|v\}$.

THEOREM 5.1. *Let Π satisfy the hypotheses given above (Theorem 2.1.). The L function $L_S(\Pi, \sigma', s)$ admits a meromorphic continuation to all of $\mathbb{C}$ with a finite number of poles. Moreover the possible poles are located at $s = 0, \frac{1}{4}, \frac{3}{4}$ or 1.*

Proof. The idea is to use here *Theorem* 2.1, *Theorem* 3.1, and *Proposition* 3.3. Indeed we let $\omega = \omega' = 1$ and choose $\Phi_{1,1,s} = f_s = \otimes_v f_{v,s}$ in such a way that $Z((f_s)_v, F) = 1$ (see *Proposition* 3.3) for *all* $v \in S_{\text{fin}}$. Moreover, using *Proposition* 3.3 (and *Remark* 2 at the end of Appendix 1 of §3), we can choose $f_{v,s}$ and F both K finite relative to GSp_3 and $GL_2(\mathbb{K})^0$, respectively so that $Z_\infty(f_{\infty,s}, F)$ is nonvanishing at $s = s_0$ (with a possible pole). This implies that (2-2) equals a function $\phi_\infty(s)$ (nonvanishing at $s = s_0$) times

$$\frac{L_S\left(\Pi, \sigma', \dfrac{1+s}{2}\right)}{\zeta_S(s+2)\zeta_S(2s+2)}.$$

But then we note that $d^S_{H_3}(s) = \zeta_S(s+2)\zeta_S(2s+2)$ and we apply the results of §4 to deduce that $\lambda \to L_S(\Pi, \sigma', \lambda)$ has a meromorphic continuation to $\mathbb{C}$ with a finite number of poles. However, from *Theorem* 4.2, $s \rightsquigarrow d^S_{H_3}(s)E(f_s, \ )$ has its possible poles at $s = \pm\frac{1}{2}, \pm 1$ and 2. But we note from *Remark* 4.4 that $\operatorname{res}_{s=2} d^S_{H_3}(s)E(f_s, \ )$ is a multiple of the identity representation of GSp_3 (and hence of $GL_3(\mathbb{K})^0$); this implies $\operatorname{res}_{\lambda=3/2} L_S(\Pi, \sigma', \lambda) \equiv 0$. Q.E.D.

It is possible now to extend the definition of the L_S function above to include the ramified primes ($v \in S_{\text{fin}}$).

For this we need to modify some of the above results.

The first step is to choose a correct family of sections f_s as input data in the family of zeta integrals (3-1). Our original choice of f_s just requires "holomorphic" data. For this we recall the space I_s given in §4 for general n. We assume $v < \infty$. Then we consider a certain faimly of "meromorphic" sections. That is, we say f_s is a "good" section if it belongs to one of the following three families:

(a) $d_{H_n}(s)\{\phi_{K,s} * \xi | \xi \in H(Sp_n(K_v)) \otimes \mathbb{C}[q_v^s, q_v^{-s}]\}$
(b) $f_s|_{Sp_n(\mathcal{O}_v)}$ is independent of s.
(c) $f_s = M^*_{w_n}(g_{-s})$ where g_{-s} belongs to (b).

Here $M^*_{w_n}$ is the normalized intertwining operator

$$M^*_{w_n} = \frac{d_{H_n}(-s)}{a_{H_n}(s)} M_{w_n}.$$

Moreover $H(Sp_n(K_v))$ is the usual Hecke algebra on Sp_n. We observe that

$$c_{w_n}(s)c_{w_n}(-s) = \left[\frac{d_{H_n}(-s)}{a_{H_n}(s)}\frac{d_{H_n}(s)}{a_{H_n}(-s)}\right]^{-1}.$$

Thus $M^*_{w_n}$ satisfies $M^*_{w_n} \circ M^*_{w_n} = I$, the identity operator.

With this data it is easy to verify that if f_s is "good", then $M^*_{w_n}(f_{-s})$ is also a "good" section. Moreover any "good" section $g_s = M^*_{w_n}(f_{-s})$ when f_s is a "good" section.

Then we consider the family of zeta integrals $\{Z(f_s, F) | f_s$ "good" and $F \in \Pi\}$ (see §3). We let I_{Π_v} be the *fractional ideal* of the ring $\mathbf{C}[q_v^s, q_v^{-s}]$ (in the field $\mathbf{C}\langle q_v^s, q_v^{-s}\rangle$) generated by the above family.

We know that any entire section f_s in I_s can be written as a finite linear combination of the form $\Sigma\, P_i(q_v^s)h_i$ where $P_i \in \mathbf{C}[q_v^s, q_v^{-s}]$ and $h_i \in I_s$ has the property that $h_i|_{Sp_n(\mathcal{O}_v)}$ is independent of s.

Thus I_{Π_v} has a generator of the form $1/P(q^{-s})$ with $P \in \mathbf{C}[X]$ and $P(0) = 1$. (Here we use *Proposition* 3.3, where by construction the function $h_{1,1,s}|_{Sp_3(\mathcal{O}_v)}$ is independent of s).

We let $L_v(\Pi_v, \sigma', (1 + s)/2)$ be the local factor associated to Π_v given by $(1/P(q_v^{-s}))$.

Then from *Corollary* 1 to *Proposition* 3.1 we have the identity:

$$\frac{Z_v(M^*_{w_n}(f_s), F)}{L_v\left(\Pi_v, \sigma', \dfrac{1-s}{2}\right)} \equiv \varepsilon_v(\Pi_v, \sigma', s)\, \frac{Z_v(f_s, F)}{L_v\left(\Pi_v, \sigma, \dfrac{1+s}{2}\right)}$$

which holds for all $F \in \Pi$ and f_s "good". Here $\varepsilon_v(\Pi_v, \sigma', s)$ is a rational function of q_v^s which has no zeroes or poles.

With the local $L_v(\Pi_v, \sigma', s)$ defined above we now can define L_{fin} as

$$L_{\text{fin}}(\Pi, \sigma, s) \equiv L_S(\Pi, \sigma, s) \prod_{v \in S_{\text{fin}}} L_v(\Pi_v, \sigma, s).$$

Then we can prove the analogue to *Theorem* 5.1 for L_{fin}.

THEOREM 5.2. *Let* Π *satisfy the hypothesis given in Theorem* 5.1. *Then* $L_{\text{fin}}(\Pi, \sigma', s)$ *admits a meromorphic continuation to* $\mathbf{C}$ *with possible poles at* $s = 0, \frac{1}{4}, \frac{3}{4}$ and 1.

Proof. Following the idea of the proof of *Theorem* 5.1 we must show that the Eisenstein series

$$d^S_{H_n}(s)E(f_s, \ )$$

(with $f_s = (\otimes_{v \notin S} \Phi_{K_v,s})\,(\otimes_{v \in S} f_{v,s})$ with $f_{v,s}$ a "good" section for $v \in S_{\text{fin}}$ and $f_{v,s}$ "entire" if $v = \infty$) has possible poles at $s = \pm 1/2$ and ± 1. Again following §4 it suffices to show that

$$s \rightsquigarrow \frac{1}{d_{H_{n,v}}(s)c_{w_v}(s)} M_{w_v}(f_{v,s})$$

(for the w defined in §4) is *entire* when $f_{v,s}$ is a "good" section.

If f_s ($=f_{v,s}$) satisfies (a) or (b) above then the statement above follows from §4. Thus we can assume that $f_s = M^*_{w_n}(g_{-s})$ with g_{-s} belonging to family (b). We consider

$$s \rightsquigarrow \frac{1}{d_{H_n}(s)c_w(s)} M_w(M^*_{w_n}(g_{-s}))$$

$$\equiv \left[\frac{1}{d_{H_n}(s)c_w(s)}\right]\left[\frac{d_{H_n}(s)}{a_{H_n}(-s)}\right] M_w(M_{w_n}(g_{-s})).$$

However, we know that $f_s \to M_w(M_{w_n}(f_s))$ defines an GSp_n intertwining operator from I_s to $\operatorname{ind}_{B_\phi}^{GSp_n}(w^{-1}(\chi_{-s}) \otimes \delta_{B_\phi}^{1/2})$. But we know that the element $w' = \gamma_n w_n w$ (where γ_n is the unique element of W_{GL_n}, the Weyl group of GL_n, that sends $\Delta^+_{GL_n}$ to $\Delta^-_{GL_n}$) has the property that $(w')^{-1}(\Delta^+_{GL_n}) \subseteq \Delta^+$ and the associated intertwining operator $M_{w'}$ maps I_s to $\operatorname{ind}_{B_\phi}^{GSp_n}(w^{-1}(\chi_{-s}) \otimes \delta_{B_\phi}^{1/2})$. Thus by the general uniqueness principle of intertwining operators there exists a rational function $\tilde{c}(s)$ so that

$$M_w \cdot M_{w_n} = \tilde{c}(s) M_{\gamma_n w_n w}.$$

But if we apply both sides of this identity to $\phi_{K,s}$ we see that

$$\tilde{c}(s) = \frac{c_w(-s)c_{w_n}(s)}{c_{\gamma_n w_n w}(s)}.$$

Thus we have that

$$\frac{\tilde{c}(-s)}{a_{H_n}(-s)c_w(s)} = \frac{c_{w_n}(-s)}{a_{H_n}(-s)c_{\gamma_n w_n w}(s)} = \frac{1}{d_{H_n}(-s)c_{\gamma_n w_n w}(-s)}.$$

Hence we have the identity:

$$\frac{1}{d_{H_n}(s)c_w(s)} M_w(M^*_{w_n}(g_{-s})) \equiv \frac{1}{d_{H_n}(-s)c_{\gamma_n w_n w}(-s)} M_{\gamma_n w_n w}(g_{-s}).$$

But using the results of §4 we know that the function

$$s \rightsquigarrow \frac{1}{d_{H_n}(-s)c_{\gamma_n w_n w}(-s)} M_{\gamma_n w_n w}(g_{-s})$$

is entire in s. Hence

$$s \rightsquigarrow \frac{1}{d_{H_n}(s)c_w(s)} M_w(M^*_{w_n}(g_{-s}))$$

is entire in s.

Then we note that if f_s is a "good" section, that either f_s or $[d_{H_n}(s)]^{-1}f_s$ is entire in s. This is clearly the case if f_s belongs to family (a) or (b) above. If $f_s = M^*_{w_n}(g_{-s})$, then we know that $[d_{H_n}(s)]^{-1}f_s = [1/a_{H_n}(-s)]M_{w_n}(g_{-s})$ is entire in s (from §4).

Thus in any case we have that

$$d^S_{H_n}(s)E(f_s,\ ) = d^{S'}_{H_n}(s)E(f_s',\ )$$

where f_s' is an entire section and $S' = \{v \in S_{\text{fin}} | f_{v,s} \text{ is entire}\} \cup S_\infty$.

With this fact it is now possible to apply the comments preceding *Theorem* 4.3 to obtain the above Theorem. Q.E.D.

REMARK 5.1. We note that the assumption $n = 3$ is not used in an essential way in the above proof. What we have in fact shown is that for *all n* the "good" sections f_s satisfy the property that the function (for each w)

$$s \rightsquigarrow \frac{1}{d_{H_n}(s)c_w(s)} M_w(f_s)$$

is entire (for $v < \infty$).

With the definition of L_{fin} given above it is also possible to give a functional equation for L_{fin}.

THEOREM 5.3. *Let* Π *satisfy the hypothesis of Theorem* 5.1. *Then* $L_{fin}(\Pi, \sigma', s)$ *satisfies the functional equation*:

$$L_{\text{fin}}(\Pi, \sigma', s) = \varepsilon(\Pi, \sigma', s)L_{\text{fin}}(\Pi, \sigma', 1 - s)$$

where

$$\varepsilon(\Pi, \sigma', s) = \left(\prod_{v \in S_{\mathrm{fin}}} \varepsilon_v(\Pi, \sigma', 2s - 1) \right)$$
$$\times \left(\prod_{v \in S_\infty} \frac{d_{H_{3,v}}(1 - 2s)}{a_{H_{3,v}}(2s - 1)} \Gamma_v(\Pi, 1, 1, 2s - 1) \right).$$

Proof. We let $f_s = (\otimes_{v \notin S} \Phi_{K_{v,s}}) \otimes (\otimes_{v \in S} f_{v,s})$ where $f_{v,s}$ is a "good" section for $v \in S_{\mathrm{fin}}$. Then from the general theory of Eisenstein series we have that $E(f_s, \ ) = E(M_{w_n}(f_s), \ )$. This implies that

$$[d_{H_3}^S(s)]^{-1} L_S\left(\Pi, \sigma', \frac{1+s}{2}\right) \left(\prod_{v \in S} Z_v(f_s, F) \right)$$
$$= [d_{H_3}^S(-s)]^{-1} L_S\left(\Pi, \sigma', \frac{1-s}{2}\right) \left[\frac{a_{H_3}^S(s)}{d_{H_3}^S(s)} \right] \left(\prod_{v \in S} Z_v(M_{w_n}(f_{v,s}), F) \right).$$

From this we then deduce

$$L_{\mathrm{fin}}\left(\Pi, \sigma', \frac{1+s}{2}\right) \left(\prod_{v \in S_{\mathrm{fin}}} \frac{Z_v(f_{v,s}, F)}{L_v\left(\Pi_v, \sigma', \frac{1+s}{2}\right)} \right) \left(\prod_{v \in S_\infty} Z_v(f_{v,s}, F) \right)$$
$$\equiv L_{\mathrm{fin}}\left(\Pi, \sigma', \frac{1-s}{2}\right) \left[\frac{a_{H_3}(s)}{d_{H_3}(-s)} \right] \left(\prod_{v \in S_{\mathrm{fin}}} \frac{Z_v(M_{w_n}^*(f_{v,s}), F)}{L_v\left(\Pi_v, \sigma', \frac{1-s}{2}\right)} \right)$$
$$\times \left(\prod_{v \in S_\infty} \frac{d_{H_{3,v}}(-s)}{a_{H_{3,v}}(s)} Z_v(M_{w_n}(f_{v,s}), F) \right).$$

Then we establish the Theorem by using the functional equation of the local zeta integral $Z_v(. . .)$ given in *Proposition* 3.1 and in the above comments (also $a_{H_3}(s) = d_{H_3}(-s)$ globally). Q.E.D.

Remark 5.2. We expect that the techniques used above will apply to the study of the L functions defined in [P-R-(II)] and [P-R-(III)]. In particular the method of defining local factors (using "good" sections) probably coincides with the method given in [P-R-(III)].

REMARK 5.3. The assumption in this section that K be toally real may not be necessary. We need it only to apply *Theorem* 4.2(b). Theorem 4.2(b) should be valid for $K_v = \mathbb{C}$.

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