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Introduction

Recall that a partially ordered set $\mathscr{P}$ has the *countable chain condition* (ccc) if every collection of pairwise incompatible elements of $\mathscr{P}$ is at most countable. Martin's Axiom (MA) is the following familiar statement:

For every ccc poset $\mathscr{P}$ and every $\mathscr{D}$, a family of fewer than $2^{\aleph_0}$ dense subsets of $\mathscr{P}$ there exists a filter G in $\mathscr{P}$ such that $G \cap D \neq \emptyset$ for every D in $\mathscr{D}$.

For an infinite cardinal κ, MA_κ is the version of MA in which the cardinality of $\mathscr{D}$ is taken to be at most κ. MA was introduced and proved relatively consistent with ZFC + $\neg$CH by Solovay and Tennenbaum in [ST]. It was then studied by Martin and Solovay in [MS]. The original motivation for the introduction of $MA_{\aleph_1}$ was that it implied Suslin's hypothesis, i.e.

Every ccc linearly ordered space is separable.

It was then realized by Hajnal and Juhász [HJ], and Kunen (unpublished), that the only property of linearly ordered spaces used was that ccc linearly ordered spaces have π-weight at most $\aleph_1$. Thus, MA_κ implies the following statement Σ_κ:

Every ccc compact space with a π-basis of size at most κ is separable.

Thus, Σ_κ can be considered as a strong form of Suslin's hypothesis. The equivalent partial order version is that:

Every ccc poset of size at most κ is σ-centered.

It is proved here that Σ_κ is, in fact, equivalent to MA_κ. For $\kappa = \aleph_1$, a stronger result is obtained: $MA_{\aleph_1}$ is equivalent to the following statement $\mathscr{H}$:

Every uncountable ccc poset has an uncountable centered subset.

Note that $\mathscr{K}$ is equivalent to the following familiar topological statement (see [Sh], [KT]):

Every compact ccc space has caliber $\aleph_1$,

i.e. every uncountable collection of open sets has an uncountable subcollection with the non-empty intersection.

Our approach is to associate ccc destructible partitions to certain combinatorial objects. It can be considered as the beginning of a general program of formulating forcing axioms in terms of the Ramsey properties of the uncountable. To explain this point, let us say that a partition of the form

$$[S]^n = K_0 \cup K_1, \quad \text{or} \tag{1}$$

$$[S]^{<\omega} = K_0 \cup K_1 \tag{2}$$

is ccc *destructible* if there is a ccc poset $\mathscr{P}$ and a $\mathscr{P}$-name $\dot{X}$ for a 0-homogeneous set (i.e. $[\dot{X}]^n \subseteq K_0$ or $[\dot{X}]^{<\omega} \subseteq K_0$ respectively) such that any element of S is forced by some condition to be in $\dot{X}$. It is easily seen that $\mathscr{K}$ is equivalent to the following Ramsey-type property of the uncountable:

If S is an uncountable set then every ccc *destructible partition of the form* (2) *has an uncountable* 0-*homogeneous set.*

So, this paper shows that $MA_{\aleph_1}$ is nothing more nor less than this Ramsey-type statement. As to the full MA, the above mentioned equivalence of MA_κ and Σ_κ yields the following reformulation of MA:

If S has size $< 2^{\aleph_0}$, then for every ccc *destructible partition of the form* (2), *S can be covered by countably many* 0-*homogeneous sets.*

Thus, it seems reasonable to consider the following Ramsey-type forcing axioms, for integers $n \geqslant 2$, RFA^n:

If S is an uncountable set and if

$$[S]^n = K_0 \cup K_1$$

is a given partition for which there exists a poset forcing an uncountable 0-*homogeneous, then such a homogeneous set in fact exists.*

Axioms of this form (in particular, RFA^2) were first considered by the first named author in connection with a partition relation on ω_1 which is now known to be false [Tol]. In §2 we show that RFA^n is false for all $n \geqslant 3$, but the status of RFA^2 remains open. The quantification over arbitrary posets in RFA^n thus appears to be too liberal. By requiring the poset to preserve stationary subsets of ω_1, we arrive at the axiom SRFA^n, which is consistent being a consequence of the familiar Semi Proper Forcing Axiom (SPFA). It is open whether SRFA^n or even $\mathrm{SRFA}^{<\omega}$ (in the obvious notation) is equivalent to SPFA. However it can be shown that SRFA^n for $n \geqslant 4$ has roughly the same consistency strength as SPFA.

In Section 1, starting from a tower $\{a_\xi : \xi < \mathfrak{t}\}$ we define a ccc destructible partition:

$$[\mathfrak{t}]^{<\omega} = K_0 \cup K_1$$

without 0-homogeneous sets of size $\mathfrak{t}$. This is then used to define a ccc nonseparable, compact, Hausdorff space of size $2^{\aleph_0}$, thus answering question 9 of Arhangel'skii [Ar].

In Section 2, starting from a non-σ-linked poset $\mathscr{P}$ of size $\aleph_1$, we define a ccc destructible partition:

$$[\omega_1]^3 = K_0 \cup K_1$$

without uncountable 0-homogeneous sets. Partitons with similar properties are also constructed under assumptions such as: $2^{\aleph_0} < 2^{\aleph_1}$; there is a non-special Aronszajn tree, etc.

Section 3 contains the aforementioned reformulations of Martin's Axiom. The main result of this paper was proved in August 1985 and a version of the whole paper was first presented as Chapter 3 in [Ve].

§1. Centered subsets of ccc posets

Recall the definition of the following three uncountable cardinals associated with the continuum (see [vD]): $\mathfrak{p}$ is the least cardinal κ such that there exists a family $\{a_\xi : \xi < \kappa\} \subseteq [\omega]^\omega$ with the finite intersection property (fip) such that there is no $a \in [\omega]^\omega$ such that $\forall \xi < \kappa\ a \subseteq_* a_\xi$. $\mathfrak{t}$ is defined similarly but the family $\{a_\xi : \xi < \kappa\}$ in addition has to be a tower, i.e., $\xi < \eta \rightarrow a_\eta \subset_* a_\xi$. Clearly, $\mathfrak{p} \leqslant \mathfrak{t}$. Whether in fact $\mathfrak{p} = \mathfrak{t}$ is an open problem. $\mathfrak{b}$ is the least cardinality of an unbounded family in ω^ω ordered under eventual dominance. We shall later need the following well-known result which says that $\mathfrak{b}$ is bigger than or equal to $\mathfrak{t}$.

LEMMA 1.1 *Let $\mathcal{F} \subseteq \omega^\omega$ be of size less than $\mathfrak{t}$. Then there exists $g \in \omega^\omega$ such that $\forall f \in \mathcal{F}\, f <_* g$.*

Proof. Enumerate $\mathcal{F} = \{f_\xi : \xi < \kappa\}$ for $\kappa < \mathfrak{t}$. For $a \in [\omega]^\omega$ let g_a be the increasing enumeration of a. Choose resursively infinite sets $a_\xi : \xi < \kappa$ such that:

i) $\forall \xi, \eta < \kappa\, [\xi < \eta \rightarrow a_\eta \subseteq_* a_\xi]$

ii) $\forall \xi < \kappa\, f_\xi <_* g_{a_{\xi+1}}$

At a limit stage $\alpha \leqslant \kappa$ use the fact that card(α) $< \mathfrak{t}$ to pick $a_\alpha \in [\omega]^\omega$ such that $\forall \xi < \alpha\, a_\alpha \subseteq_* a_\xi$. Finally, set $g = g_{a_\kappa}$. Then g works.

A subset X of a partially ordered set $\mathcal{P}$ is *centred (k-linked)* if

$$\forall F \in [X]^{<\omega}\ (\forall F \in [X]^k)\ \exists p \in \mathcal{P}\ \forall q \in F\ p \leqslant q.$$

Let *linked* denote 2-linked. A poset $\mathcal{P}$ is *σ-centered (σ-k-linked)* if it is the union of countably many centered (k-linked) subsets. A poset $\mathcal{P}$ has *precaliber κ* if

$$\forall X \in [\mathcal{P}]^\kappa\ \exists Y \in [X]^\kappa\ Y \text{ is centered.}$$

In this section we continue the work of Todorcevic [To2] where among other things the following is proved:

THEOREM 1.2

a) *There is a productively ccc poset of size* $\mathfrak{b}$ *without linked subsets of size* $\mathfrak{b}$.

b) *For each n there is a σ-n-linked poset of size* $\mathfrak{b}$ *without $n + 1$-linked subsets of size* $\mathfrak{b}$.

c) *There is a poset of size* $\mathfrak{b}$ *which is σ-n-linked for each n but which has no centered subsets of size* $\mathfrak{b}$.

The following results, which say that similar posets exist for cardinals $\mathfrak{t}$ and $\mathfrak{p}$, are of additional interest since they are used in §3 to establish the above equivalent formulations of MA.

THEOREM 1.3. *There is a σ-linked poset $\mathcal{P}$ of size* $\mathfrak{t}$ *without centered subsets of size* $\mathfrak{t}$.

Proof. Let us fix a tower $\{a_\xi : \xi < \mathfrak{t}\}$. For $x, y \subseteq \omega$ such that $x \neq y$, define

$$\Delta(x, y) = \min (x \,\Delta\, y),$$

i.e., $\Delta(x, y)$ is the least point of the symmetric difference of x and y. For $F \in [\mathfrak{t}]^{<\omega}$ define

$$\Delta_F = \{\Delta(a_\xi, a_\eta) : \xi, \eta \in F \,\&\, \xi \neq \eta\}$$

$$a_F = \cap \{a_\xi : \xi \in F\}.$$

Define the poset $\mathscr{P}$ by $F \in \mathscr{P}$ iff $F \in [\mathfrak{t}]^{<\omega}$ and

$$\forall k < \omega \text{ card } (a_F \cap k) \geqslant \text{card } (\Delta_F \cap k).$$

The order is reverse inclusion.

Claim 1. $\mathscr{P}$ is σ-linked.

Proof. For $F \in [\mathfrak{t}]^{<\omega}$ define

$$l_F = \text{card } (F),$$

$$m_F = \sup (\Delta_F) + 1,$$

$$n_F = \min \{n \in \omega : \text{card } (a_F \cap (m_F, n)) \geqslant l_F\},$$

and

$$\tau_F = \{a_\xi \cap n_F : \xi \in F\}.$$

Let I be $\omega \times \omega \times \omega \times [[\omega]^{<\omega}]^{<\omega}$. Define for $i \in I$

$$\mathscr{P}_i = \{F \in \mathscr{P} : \langle l_F, m_F, n_F, \tau_F \rangle = i).$$

Let us show that $\mathscr{P}_i$ is linked $\forall i \in I$. Suppose $F, F' \in \mathscr{P}_i$. Then $n_F = n_{F'} = n$, and $\Delta_F = \Delta_{F'} = \Delta_{F \cup F'} \cap n$. Also, $a_F \cap n = a_{F'} \cap n$. Therefore

$$\forall k \leqslant n \text{ card } (a_{F \cup F'} \cap k) \geqslant \text{card } (\Delta_{F \cup F'} \cap k).$$

Also, for $k > n$.

$$\text{card}\,(a_{F\cup F'} \cap k) \geqslant \text{card}\,(a_{F\cup F'} \cap m) + l \geqslant \text{card}\,(\Delta_{F\cup F'} \cap m) + l$$

$$\geqslant \text{card}\,(\Delta_{F\cup F'}),$$

where $m = m_F = m_{F'}$, and $l = l_F = l_{F'}$. This shows that $\mathscr{P}$ is σ-linked. Indeed it can be shown that $\mathscr{P}$ is σ-k-linked for every $k \in \omega$.

Claim 2. $\mathscr{P}$ does not have centered subsets of size $\mathfrak{t}$.

Proof. Let $X \in [\mathfrak{t}]^{\mathfrak{t}}$ be such that $[X]^{<\omega} \subseteq \mathscr{P}$. Let

$$a = \cap \{a_\xi : \xi \in X\}$$

and

$$\Delta = \{\Delta(a_\xi, a_\eta) : \xi \neq \eta \in X\}.$$

Then we have:

$$\forall k < \omega\ \text{card}\,(a \cap k) \geqslant \text{card}\,(\Delta \cap k).$$

Since Δ is infinite, so is a. Then $\forall \xi < \mathfrak{t}\ a \subseteq_* a_\xi$, a contradiction.

THEOREM 1.4 *There is a* ccc *non-separable, compact Hausdorff space of size continuum.*

Proof. Extend the notation to define a_F and Δ_F for all subsets of $\mathfrak{t}$. Identifying $\mathscr{P}(\mathfrak{t})$ and $2^{\mathfrak{t}}$, let

$$X = \{F \in \mathscr{P}(\mathfrak{t}) : \forall k < \omega\ \text{card}\,(a_F \cap k) \geqslant \text{card}\,(\Delta_F \cap k)\}$$

Then by Claim 2 above $X \subseteq 2^{<\mathfrak{t}}$ and hence card $(X) = 2^{\aleph_0}$. Note that X is a closed subset of $2^{\mathfrak{t}}$, hence is compact. That X is ccc follows by Claim 1 in Theorem 1.3.

THEOREM 1.5. *There is a poset $\mathscr{P}$ of size* $\mathfrak{p}$ *which is σ-linked but not σ-centered.*

Proof. Assume by way of contradiction that such $\mathscr{P}$ does not exist. By Theorem 1.3 we have that $\mathfrak{p} < \mathfrak{t}$. Let $\mathscr{U} = \{u_\alpha : \alpha < \mathfrak{p}\} \subseteq [\omega]^\omega$ be closed under finite intersections such that $\neg \exists a \in [\omega]^\omega \forall \alpha < \mathfrak{p}\ a \subseteq_* u_\alpha$.

Following Rothberger [Ro], recursively construct a decreasing (mod fin) 1–1 sequence $a_\xi : \xi < \mathfrak{p}$ such that
i) $\forall \xi < \mathfrak{p}\ a_{\xi+1} \subseteq u_\xi$
(ii) $\forall \alpha, \xi < \mathfrak{p}\ u_\alpha \cap a_\xi$ is infinite.
Step $\xi = \eta + 1$ for some $\eta < \mathfrak{p}$ is trivial. Step $\operatorname{cof}(\xi) = \omega$ is the same as in [Ro]. That is, fix an increasing sequence of ordinals $\langle \xi_n : n < \omega \rangle$ converging to ξ and let $b_n = a_{\xi_0} \cap a_{\xi_1} \cap \ldots \cap a_{\xi_n}$, for $n < \omega$. For $\alpha < \mathfrak{p}$ let $f_\alpha : \omega \to \omega$ be defined recursively by

$$f_\alpha(n) = \min\left((u_\alpha \cap b_n) - (f_\alpha(n-1) + 1)\right).$$

By Lemma 1.1 and the fact that $\mathfrak{p} < \mathfrak{t}$, there exists a $g : \omega \to \omega$ such that $\forall \alpha < \mathfrak{p}\ f_\alpha <_* g$. Alternatively we can use Theorem 8 of [To2]. Let then

$$a_\xi = \cup \{ b_n \cap g(n) : n \in \omega \}.$$

Assume now $\xi < \mathfrak{p}$ and $\operatorname{cof}(\xi) > \omega$. We want to construct a_ξ. Define the poset $\mathscr{P}$ by: $\langle F, G \rangle \in \mathscr{P}$ iff $F \in [\xi]^{<\omega}$, $G \in [\mathfrak{p}]^{<\omega}$ and

$$\forall k < \omega \forall \alpha \in G \operatorname{card}(a_F \cap u_\alpha \cap k) \geqslant \operatorname{card}(\Delta_F \cap k).$$

The order is coordinatewise reverse inclusion.

Claim 1. $\mathscr{P}$ is σ-k-linked for every k.

Proof. Similar to Claim 1 in Theorem 1.3.

By our assumption $\mathscr{P}$ is σ-centered. Let $\mathscr{P} = \cup\{\mathscr{P}_n : n < \omega\}$ be the required decomposition. Since $\operatorname{cof}(\xi) > \omega$ we may assume that for every n

$$A_n = \cup\{F : \exists G \langle F, G \rangle \in \mathscr{P}_n\}$$

is cofinal in ξ. Let, for $n < \omega$,

$$b_n = \cap\{a_\eta : \eta \in A_n\}.$$

Note that $\forall n < \omega \forall n < \xi\ b_n \subseteq_* a_\eta$.

Claim 2. If $\alpha \in G$, $n \in \omega$ and for some F, $\langle F, G \rangle \in \mathscr{P}_n$, then $u_\alpha \cap b_n$ is infinite.

Proof. Same as Claim 2 in Theorem 1.3.

By [Ro] or an argument similar to Step cof $(\xi) = \omega$ above, pick $a_\xi \subseteq \omega$ such that

$$\forall n < \omega \forall \eta < \xi \; b_n \subseteq_* a_\xi \subseteq_* a_\eta .$$

Then a_ξ works.

Thus, we have produced a decreasing (mod fin) sequence $\{a_\xi : \xi < \mathfrak{p}\}$ with no infinite a such that $\forall \xi < \mathfrak{p} \; a \subseteq_* a_\xi$. This contradicts the fact that $\mathfrak{p} < \mathfrak{t}$.

Question 1.6. Does there exist a σ-linked poset without precaliber $\mathfrak{p}$?

§2. CCC destructible partitions

Recall that a poset $\mathscr{P}$ has *property* K_n iff

$$\forall X \in [\mathscr{P}]^{\aleph_1} \; \exists Y \in [X]^{\aleph_1} \; Y \text{ is } n\text{-linked.}$$

Let $\mathscr{K}_n$ denote the statement that every ccc poset has property K_n. Recall that a coloring $[\omega_1]^n = K_0 \cup K_1$ is *ccc destructible* iff there is a ccc poset which adds an uncountable 0-homogeneous set. Observe that $\mathscr{K}_n$ is equivalent to:

Every ccc *destructible partition of* $[\omega_1]^n$ *has an uncountable* 0-*homogeneous set.*

Our goal is to produce, under various weak assumptions, ccc destructible partitions without uncountable 0-homogeneous sets. We use the work of Todorcevic [ToI] on negative partition relations on ω_1. Let us start by describing the definitions and results from [ToI] that we need. We refer the reader to [ToI] for the motivation behind.

Fix, for each countable α, a 1–1 function $e_\alpha : \alpha \to \omega$ such that $\alpha < \beta \to e_\beta \restriction \alpha =_* e_\alpha$. For $\alpha < \beta < \omega_1$ let

$$\sigma(\alpha, \beta) = \min \{\xi : e_\alpha(\xi) \neq e_\beta(\xi)\}$$

$(\sigma(\alpha, \beta) = \alpha$, if this set is empty). Consider the partition $c : [\omega_1]^2 \to \omega_1$, defined by

$$c(\alpha, \beta) = \min \{\xi > \alpha : e_\beta(\xi) \leqslant e_\beta(\sigma(\alpha, \beta))\}$$

if this set is nonempty, otherwise $c(\alpha, \beta) = \beta$. Note that $\alpha < \beta \to \alpha < c(\alpha, \beta) \leqslant \beta$.

The following is proved in [To 1; §4.2]; we reproduce the argument for completeness.

THEOREM 2.1. *Let $X \subseteq \omega_1$ be uncountable, and M a countable elementary submodel of $H_{\aleph_2}$ such that $X, \langle e_\alpha : \alpha < \omega_1 \rangle \in M$. Let $\delta = M \cap \omega_1$. Then, for every $\beta \in X$ with $\beta > \delta$, there is an $\alpha \in X \cap \delta$ such that $c(\alpha, \beta) = \delta$ and $\alpha \leqslant \gamma < \delta \to e_\beta(\gamma) = e_\delta(\gamma)$.*

Proof. Let X, M, and δ be as stated, and fix a $\beta \in X$ such that $\beta > \delta$. Consider the tree

$$T = \{e_\alpha \restriction \xi : \alpha \in X \;\&\; \xi \leqslant \alpha\}.$$

Let $n = e_\beta(\delta)$, and fix $\xi < \delta$ such that $\xi \leqslant \gamma < \delta \to e_\beta(\gamma) = e_\delta(\gamma) \geqslant n$. Since T is an Aronszajn tree, there must be a $t \in T \restriction \delta$ such that

$$e_\beta \quad \xi \subseteq t \nsubseteq e_\beta$$

and

$$C = \{\alpha \in X : t \subseteq e_\alpha\}$$

is uncountable. Let

$$\varepsilon = \min\{\eta : t(\eta) \neq e_\beta(\eta)\}$$

Then for all α in $C \cap M$, $\sigma(\alpha, \beta) = \varepsilon$. Let $m = e_\beta(\varepsilon)$ ($\geqslant n$). If α is any member of $C \cap M$ above $e_\beta^{-1}[m]$, it follows that

$$c(\alpha, \beta) = \delta \text{ and } \alpha \leqslant \gamma < \delta \to e_\beta(\gamma) = e_\delta(\gamma),$$

as required.

We shall need the following two lemmas about the partition c (see [To 1; §6]).

LEMMA 2.2. *Let X and Y be uncountable subsets of ω_1. Then there exist uncountable $X' \subseteq X$, uncountable $Y' \subseteq Y$, and ordinals σ_β, for $\beta \in Y'$, such that*

$$\forall \alpha \in X' \forall \beta \in Y' \; \alpha < \beta \to c(\alpha, \beta) = \sigma_\beta.$$

Proof. First find $\bar{X} \in [X]^{\aleph_1}$, $\bar{Y} \in [Y]^{\aleph_1}$, and $\sigma < \omega_1$, such that $\forall \alpha \in \bar{X} \forall \beta \in \bar{Y}$ $\sigma(\alpha, \beta) = \sigma$. Then let $D = \{\delta < \omega_1 : \sup(\bar{X} \cap \delta) = \delta\}$. For each $\delta \in D$, pick $\beta_\delta \in \bar{Y}\backslash\delta$. Define

$$\sigma_{\beta_\delta} = \min\{\xi \geqslant \delta : e_{\beta_\delta}(\xi) \leqslant e_{\beta_\delta}(\sigma)\}$$

($\sigma_{\beta_\delta} = \beta_\delta$ if this set is empty), and

$$g(\delta) = \min\{\xi : \forall\eta < \delta\, [\xi \leqslant \eta \rightarrow e_{\beta_\delta}(\sigma) < e_{\beta_\delta}(\eta)]\}.$$

Then $g: D \rightarrow \omega_1$ is regressive and

$$\forall\alpha \in \bar{X}\, \forall\delta \in D\, [g(\delta) \leqslant \alpha < \delta \rightarrow c(\alpha, \beta_\delta) = \sigma_{\beta_\delta}].$$

By the Pressing Down Lemma, find an uncountable $E \subseteq D$ and $\gamma < \omega_1$, such that $\forall\delta \in E\, g(\delta) = \gamma$. Finally, find an uncountable $F \subseteq E$ and uncountable $X' \subseteq \bar{X}\backslash\gamma$ such that $\forall\delta \in F\, X' \cap [\delta, \beta_\delta) = \emptyset$. Set $Y' = \{\beta_\delta : \delta \in F\}$. Then X' and Y' work.

Fix a function $s: \omega_1 \rightarrow \omega$ such that $s^{-1}(n)$ is stationary for all n. Define $p: [\omega_1]^2 \rightarrow \omega_1$ by

$$p(\alpha, \beta) = e_\beta^{-1}(s(c(\alpha, \beta)))$$

if this makes sense, otherwise set $p(\alpha, \beta) = 0$.

LEMMA 2.3 *For all $X \in [\omega_1]^{\aleph_1}$ there exists $\delta < \omega_1$ such that for any $\xi < \omega_1$ there exist $\alpha \in X \cap \delta$ and $\beta \in X$ such that $p(\alpha, \beta) = \xi$.*

Proof. For each $n < \omega$, fix a countable elementary submodel M_n of $H_{\aleph_2}$ containing everything relevant such that and $s(\delta_n) = n$, where $\delta_n = M_n \cap \omega_1$. Define then $\delta = \sup\{\delta_n : n \in \omega\}$. We claim that this δ works. So, let $\xi < \omega_1$. Fix $\beta \in X$ such that $\beta > \xi, \delta$. Let $n = e_\beta(\xi)$. By Theorem 2.1 there is $\alpha \in X \cap M_n$ such that $c(\alpha, \beta) = \delta_n$. Thus, $s(c(\alpha, \beta)) = n$, and therefore $p(\alpha, \beta) = e_\beta^{-1}(n) = \xi$.

THEOREM 2.4 *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Then there exists a* ccc *destructible partition of $[\omega_1]^3$ without uncountable* 0*-homogeneous sets.*

Proof. The following weak diamond principle was shown to be equivalent to $2^{\aleph_0} < 2^{\aleph_1}$ by Devlin and Shelah in [DS]:

$$\forall F: 2^{<\omega_1} \rightarrow 2\ \exists h: \omega_1 \rightarrow 2\ \forall g: \omega_1 \rightarrow 2\ \{\alpha : F(g \restriction \alpha) = h(\alpha)\} \textit{ is stationary.}$$

To each $h: \omega_1 \rightarrow 2$ we associate a ccc destructible partiton of $[\omega_1]^3$, and then use weak diamond to choose h such that the associated partition has no uncountable 0-homogeneous sets.

For each countable limit ordinal α, fix a strictly increasing cofinal sequence $s_\alpha : \omega \to \alpha$, and for a successor ordinal $\alpha = \beta + 1$ define $s_\alpha : \omega \to \alpha$ to be constantly equal to β. Define the partition $[\omega_1]^3 = K_0 \cup K_1$ by $\{\alpha, \beta, \gamma\}_< \in K_0$ if

$$h(c(\alpha, \beta)) \neq h(c(\alpha, \gamma)) \to s_{c(\alpha,\beta)}(e_\beta(\alpha)) \neq s_{c(\alpha,\gamma)}(e_\gamma(\alpha)).$$

Let $\mathscr{P}$ be the poset of 0-homogeneous finite sets, i.e. $F \in \mathscr{P}$ iff $F \in [\omega_1]^{<\omega}$ and $[F]^3 \subseteq K_0$. The order is reverse inclusion.

Claim. $\mathscr{P}$ satisfies the ccc.

Proof. Let $\langle F_\alpha : \alpha < \omega_1 \rangle$ be a Δ-system of elements of $\mathscr{P}$ each of size n, and let F be the root. We have to find $\alpha, \beta < \omega_1$ such that $\alpha \neq \beta$ and $F_\alpha \cup F_\beta$ is in $\mathscr{P}$. We first get rid of the root.

For each $\xi \in F$, $\alpha < \omega_1$ and $i \in \{0, 1\}$ let

$$S^i_\xi(\alpha) = \{s_{c(\xi,\eta)}(e_\eta(\xi)) : \eta \in F_\alpha \;\&\; h(c(\xi, \eta)) = i\}$$

Then, by the homogeneity of F_α, $S^0_\xi(\alpha) \cap S^1_\xi(\alpha) = \emptyset$. Using the fact that the usual poset for uniformizing ladder systems has property K_2 (see [DS]) we can find an uncountable $X \subseteq \omega_1$ such that

$$\forall \alpha, \beta \in X \, \forall \xi \in F \, S^0_\xi(\alpha) \cap S^1_\xi(\beta) = \emptyset.$$

This implies that if $F_\alpha \cup F_\beta$ is not 0-homogeneous, then neither is $(F_\alpha \cup F_\beta)\backslash F$. We can thus assume, by subtracting F, that the F_α for $\alpha \in X$ are pairwise disjoint. For simplicity assume also that $X = \omega_1$. Let the increasing enumeration of F_α be $\{a^0_\alpha, \ldots, a^{n-1}_\alpha\}$. Using Lemma 2.2 repeatedly n^2 times find uncountable X, $Y \subseteq \omega_1$ and ordinals σ^{ij}_β for $\beta \in Y$ and $(i, j) \in n^2$ such that

$$\forall \alpha \in X \, \forall \beta \in Y \, \forall (i, j) \in n^2 [\alpha < \beta \to c(a^i_\alpha, a^j_\beta) = \sigma^{ij}_\beta]$$

For $\alpha \in X$ let $Z_\alpha = \{s_\delta(n) : \exists \xi, \eta \in F_\alpha \; c(\xi; \eta) = \delta \;\&\; e_\eta(\xi) = n\}$. We may assume that the Z_α for $\alpha \in X$ form a Δ-system with root Z, and that $\alpha < \beta \to \sup (Z_\alpha \backslash Z) < \inf (Z_\beta \backslash Z)$. Choose $\delta < \omega_1$ such that $\forall \alpha < \delta \sup (Z_\alpha) < \delta$, and pick $\beta \in Y$ such that $\min (F_\beta) \geqslant \delta$. Let $\Sigma = \{\sigma^{ij}_\beta : (i, j) \in n^2\}$. From the definition of c it follows that $\min (\Sigma) \geqslant \delta$. Let $U = \cup\{s''_\xi[\omega] : \xi \in \Sigma\}$. Let $k \in \omega$ be large enough such that $\forall \xi \in \Sigma \, \forall m \geqslant k \; s_\xi(m) \notin Z$. Finally choose $\gamma < \delta$ such that

$$\forall \xi \in \Sigma \, \forall \nu < \delta \, [\nu \geqslant \gamma \to e_\xi(\nu) \geqslant k].$$

Since U has order type $\leqslant \omega n^2$ and $\mathrm{ot}(X \cap \delta) = \delta > \omega n^2$, there exists $\alpha \in X \cap \delta$ such that $(Z_\alpha \backslash Z) \cap U = \emptyset$ and $\inf(Z_\alpha \backslash Z) > \gamma$. This implies $F_\alpha \cup F_\beta$ is in $\mathscr{P}$.

Let us now assume that weak diamond holds and define $F: 2^{<\omega_1} \times 2^{<\omega_1} \to 2$ as follows.

Fix a limit ordinal δ, a subset X of δ, and a function $f: \delta \to 2$. We describe how to define $F(\chi, f)$, for $\chi: \delta \to 2$ the characteristic function of X.

For $\xi < \delta$ and $e \in T_\xi(X)$ let:

$$T_\xi(X) = \{e_\alpha \restriction \xi : \alpha \in X\},$$

$$X_e = \{\alpha \in X : e_\alpha \restriction \xi = e\},$$

$$R_\xi(X) = \{e \in T_\xi(X) : \sup(X_e) = \delta\},$$

and

$$R(X) = \cup\{R_\xi(X) : \xi < \delta\}.$$

Define $F(\chi, f)$ to be 1 if

$$\forall e \in R(X)\, \exists \alpha, \beta \in X_e\, [f(c(\alpha, \beta)) = 0 \;\&\; s_{c(\alpha,\beta)}(e_\beta(\alpha)) = s_\delta(e_\delta(\alpha))]$$

In any other case let $F(\chi, f)$ to be equal to 0.

Let now $h: \omega_1 \to 2$ be such that

$$\forall \chi, f \in 2^{\omega_1} \{\alpha < \omega_1 : F(\chi \restriction \alpha, f \restriction \alpha) = h(\alpha)\} \text{ is stationary.}$$

Claim. The partition $[\omega_1]^3 = K_0 \cup K_1$ associated to h has no uncountable 0-homogeneous sets.

Proof. Let χ be the characteristic function of X, an uncountable 0-homogeneous subset of ω_1. Since $E = \{\alpha < \omega_1 : F(\chi \restriction \alpha, h \restriction \alpha) = h(\alpha)\}$ is stationary, we can find a countable elementary submodel N of $H_{\aleph_2}$ containing X, h, and c such that $\delta = N \cap \omega_1 \in E$.

Case 0. $h(\delta) = 0$. Let $e \in R(X \cap \delta)$ be arbitrary. Then $(X \cap \delta)_e$ is unbounded in δ, and hence by elementary of N, X_e is uncountable. Fix $\beta \in X_e \backslash \delta$. Then as in the proof of Theorem 2.1 we can find $\alpha \in X \cap \delta$ such that $c(\alpha, \beta) = \delta$ and $e_\beta(\alpha) = e_\delta(\alpha) = n$ for some $n \in \omega$. Let $\xi = s_\delta(n)$. Let $\varphi(\xi, \alpha, \beta)$ be the following formula:

$$\alpha, \beta \in X_e \;\&\; h(c(\alpha, \beta)) = 0 \;\&\; e_\beta(\alpha) = n \;\&\; \xi = s_{c(\alpha,\beta)}(n)$$

Then, by what we have just said, $H_{\aleph_2} \models \varphi(\xi, \alpha, \beta)$. By elementarity choose $\beta' < \delta$ such that $N \models \varphi(\xi, \alpha, \beta')$. Then we have

$$\alpha, \beta' \in (X \cap \delta)_e \;\&\; h(c(\alpha, \beta')) = 0 \;\&\; s_{c(\alpha,\beta')}(e_{\beta'}(\alpha)) = s_\delta(e_\delta(\alpha))$$

Since $e \in R(X \cap \delta)$ was arbitrary this shows that $F(\xi \restriction \delta, h \restriction \delta) = 1$. Now, $1 = F(\chi \restriction \delta, h \restriction \delta) = h(\delta) = 0$. Contradiction.

Case 1. $h(\delta) = 1$. Fix $\gamma \in X \backslash \delta$. As in the proof of Theorem 2.1 we can find $e \in R(X \cap \delta)$ such that

$$\forall \alpha \in (X \cap \delta)_e [c(\alpha, \gamma) = \delta \;\&\; e_\gamma(\alpha) = e_\delta(\alpha)].$$

Fixing such an e and applying the fact that $F(\chi \restriction \delta, h \restriction \delta) = 1$ find $\alpha, \beta \in (X \cap \delta)_e$ such that

$$h(c(\alpha, \beta)) = 0 \;\&\; s_{c(\alpha,\beta)}(e_\beta(\alpha)) = s_\delta(e_\delta(\alpha)).$$

It then follows that $\{\alpha, \beta, \gamma\} \in K_1$, contradicting the fact that X is 0-homogeneous.

Using partitions similar to the one in the previous argument $\mathscr{K}_4$ can be shown to imply that every ladder system on ω_1 can be uniformized, every set of reals of size $\aleph_1$ is a Q-set, etc.

Let $\mathscr{P}$ be a poset of size $\aleph_1$ and let $\{q_\alpha : \alpha < \omega_1\}$ be an enumeration of $\mathscr{P}$. Fix an ω_1-sequence $\langle r_\alpha : \alpha < \omega_1 \rangle$ of distinct reals. For $F \in [\omega_1]^{<\omega}$ and $s \in 2^{<\omega}$ let

$$\mathscr{P}_F^s = \{q_\gamma : \exists \alpha, \beta \in F\,[p(\alpha, \beta) = \gamma \;\&\; r_\beta \restriction e_\beta(\alpha) = s]\}.$$

Define the poset $\mathscr{Q} = \mathscr{Q}(\mathscr{P})$ by $F \in \mathscr{Q}$ iff $F \in [\omega_1]^{<\omega}$ and

$$\forall s \in 2^{<\omega}\; \mathscr{P}_F^s \text{ is centered.}$$

The order is reverse inclusion.

The idea is that uncountable centered subsets of $\mathscr{Q}$ should yield decompositions of $\mathscr{P}$ into countably many centered sets. Thus, it is natural to consider a function

$$f : [\omega_1]^2 \to \mathscr{P}$$

such that for every uncountable $X \subseteq \omega_1$,

$$f''[X]^2 = \mathscr{P}$$

Then $\mathscr{Q}$ can be the set of $F \in [\omega_1]^{<\omega}$ which in some canonical way code a decomposition of $f''[F]$ into centered subsets. The reals r_α are used to make $\mathscr{Q}$ ccc. The partition p is employed since it gives a rather economical decomposition of $\mathscr{P}$ into k-linked sets from uncountable $k+1$-linked subsets of $\mathscr{Q}$.

THEOREM 2.5
a) *If $\mathscr{P}$ is powerfully* ccc, *then $\mathscr{Q} = \mathscr{Q}(\mathscr{P})$ is* ccc.
b) *For every uncountable $X \subseteq \mathscr{Q}$ there is a partition $\mathscr{P} = \cup\{\mathscr{P}_k : k \in \omega\}$ such that if X is $n+1$-linked in $\mathscr{Q}$, then $\forall k \in \omega$ $\mathscr{P}_k$ is n-linked in $\mathscr{P}$.*

Proof a) Let $\langle F_\alpha : \alpha < \omega_1 \rangle$ be an uncountable Δ-system of elements of $\mathscr{Q}$. Let the root be F. For $\alpha < \omega_1$ let:

$$n_\alpha = \max\{e_\eta(\xi) : \xi, \eta \in F_\alpha \ \&\ \xi < \eta\}$$

and

$$m_\alpha = \min\{k \in \omega : \forall \xi, \eta \in F_\alpha\ \xi \neq \eta \rightarrow r_\xi \upharpoonright k \neq r_\eta \upharpoonright k\}.$$

We may assume that all the n_α's are equal to, say, n, and all the m_α's are equal to, say, m. Note that $\mathscr{P}^s_{F_\alpha}$ is nonempty only for $s \in 2^{\leqslant n}$.

Since the F_α's form a Δ-system and since $\mathscr{P}^k$ is ccc for $k = 2^{n+1}$, we can find $\alpha < \beta < \omega_1$ such that:

(1) $\sup(F_\alpha \backslash F) < \inf(F_\beta \backslash F)$,
(2) $\forall \xi \in F_\alpha \backslash F\ \forall \eta \in F_\beta \backslash F\ e_\eta(\xi) > \max(m, n)$, and
(3) $\forall s \in 2^{\leqslant n}\ \mathscr{P}^s_{F_\alpha} \cup \mathscr{P}^s_{F_\beta}$ is centered.

Let us show that $F_\alpha \cup F_\beta \in \mathscr{Q}$. Let $s \in 2^{<\omega}$. If $lh(s) \leqslant n$, then

$$\mathscr{P}^s_{F_\alpha \cup F_\beta} = \mathscr{P}^s_{F_\alpha} \cup \mathscr{P}^s_{F_\beta},$$

and thus is centered. If $lh(s) > n$, then $\mathscr{P}^s_{F_\alpha \cup F_\beta}$ is at most a singleton and thus is, trivially, centered.

Proof b) Let $\delta < \omega_1$, be as in Lemma 2.3. For $\alpha \in X \cap \delta$ let

$$\mathscr{P}^s_\alpha = \{q_\xi : \exists \beta \in X[\alpha < \beta \ \&\ p(\alpha, \beta) = \xi \ \&\ r_\beta \upharpoonright e_\beta(\alpha) = s]\}.$$

By Lemma 2.3

$$U\{\mathscr{P}^s_\alpha : s \in 2^{<\omega};\ \alpha \in X \cap \delta\} \ = \ \mathscr{P}.$$

We claim that this is the required partition. For, assuming $[X]^{n+1} \subseteq \mathscr{Q}$ it follows from the definition of $\mathscr{Q}$ that $\forall s \in 2^{<\omega}\ \forall \alpha \in X \cap \delta\ \mathscr{P}^s_\alpha$ is n-linked.

COROLLARY 2.6 *Let $n \in \omega$. Then $\mathscr{K}_{n+1}$ implies that every* ccc *poset of size $\aleph_1$ is σ-n-linked.*

The following was first proved by Fremlin (see [Fr; Notes 41L]) by a completely different argument.

COROLLARY 2.7. *If every* ccc *poset has precailber $\aleph_1$, then every* ccc *poset of size $\aleph_1$ is σ-centered.*

COROLLARY 2.8. *Assume there exists a nonspecial Aronszajn tree. Then there exists a* ccc *destructible partition $[\omega_1]^3 = K_0 \cup K_1$ without uncountable 0-homogeneous sets.*

CONJECTURE 2.9. *$\mathscr{K}_2$ does not imply $\mathscr{K}_3$.*

THEOREM 2.10. RFA^3 *is false.*

Proof. Fix a stationary costationary subset S of ω_1. Let $c:[\omega_1]^2 \to \omega_1$ be as usual and fix, for each limit ordinal $\alpha < \omega_1$, a cofinal sequence $s_\alpha : \omega \to \alpha$. For a successor $\alpha = \beta + 1$, let s_α be constantly equal to β. Define the partition

$$[\omega_1]^3 \ = \ K_0^S \cup K_1^S$$

by: $\{\alpha, \beta, \gamma\}_< \in K_0^S$ if

$$[\beta' = c(\alpha, \beta) \in S \ \&\ \gamma' = c(\alpha, \gamma) \in S \ \&\ \beta' \neq \gamma'] \to s_{\beta'}(e_\beta(\alpha)) \neq s_{\gamma'}(e_\gamma(\alpha)).$$

It follows by a pressing down argument and some facts about c that there are no uncountable 0-homogeneous sets. Define the poset $\mathscr{P}$ by: $p \in \mathscr{P}$ if $p = \langle H_p, F_p \rangle$ where

(1) $H_p, F_p \in [\omega_1]^{<\omega}$
(2) $[H_p]^3 \subseteq K_0^S$
(3) $\forall \alpha, \beta \in H_p\ \forall \gamma \in F_p\ [\xi = c(\alpha, \beta)\ \&\ n = e_\beta(\alpha)] \to \gamma \notin (s_\xi(n), \xi)$

To prove $\mathscr{P}$ preserves $\aleph_1$, let $\dot{f}$ be a name for a function $\omega \to \omega_1$ and $p \in \mathscr{P}$. Fix a countable $M \prec H_\theta$ such that $p, \mathscr{P}, \dot{f} \in M$ and such that

$$\delta = M \cap \omega_1 \in \omega_1 \backslash S.$$

Let $q = \langle H_p, F_p \cup \{\delta\}\rangle$. By a standard argument, it follows that

$$q \Vdash \operatorname{ran}(\dot{f}) \subseteq \delta.$$

Define the name $\dot{H}$ for a subset of ω_1 by

$$\dot{H} = \cup\{H_p : p \in \dot{G}_p\}.$$

Then $\dot{H}$ is forced to be 0-homogeneous. To ensure that $\dot{H}$ be uncountable fix a countable $M \prec H_\theta$ containing everything relevant and such that $\delta = M \cap \omega_1 \in \omega_1 \backslash S$. Then force below $\langle \varnothing, \{\delta\}\rangle$.

§3. Martin's axiom

We shall need the following result of Bell [Be]. For completeness again, we sketch the argument from [Be].

THEOREM 3.1. MA_κ *(σ-centered) is equivalent to* $\kappa < \mathfrak{p}$.

LEMMA 3.2. *Suppose* $A_{\alpha,s} \in [\omega]^\omega$ *for* $\alpha < \kappa$, $s \in \omega^{<\omega}$, *and* $\forall s \in \omega^{<\omega}$ $\{A_{\alpha,s} : \alpha < \kappa\}$ *has the fip. Then* $\exists f \in \omega^\omega$ *such that* $\forall \alpha < \kappa\ \exists n \forall m \geqslant n\ f(m) \in A_{\alpha, f\restriction m}$.

Proof. Using $\kappa < \mathfrak{p}$, choose $A_s \in [\omega]^\omega$ for $s \in \omega^{<\omega}$ such that $\forall \alpha < \kappa\ A_s \subseteq_* A_{\alpha,s}$. Define

$$f_\alpha(s) = \min\{n \in A_s : A_s - n \subseteq A_{\alpha,s}\}.$$

Since $\kappa < \mathfrak{p} \leqslant \mathfrak{t}$ by Lemma 1.1 find $g : \omega^{<\omega} \to \omega$ such that $\forall \alpha < \kappa f_\alpha <_* g$. Moreover make sure that $\forall s \in \omega^{<\omega} g(s) \in A_s$. Finally, define recursively $f : \omega \to \omega$ by $f(n) = g(f \restriction n)$.

Proof of Theorem 3.1. $\to$) is easy and well-known. We prove $\leftarrow$). Let $\mathscr{P}$ be a σ-centered poset (which we may assume is of size $\leqslant \kappa$) and $\{D_\alpha : \alpha < \kappa\}$ a family of dense subsets of $\mathscr{P}$. By a standard argument it is enough to produce a linked subset of $\mathscr{P}$ which intersects each D_α. Fix a partition $\mathscr{P} = \cup\{\mathscr{P}_n : n \in \omega\}$ into centered sets. For a fixed $\alpha < \kappa$ pick recursively $p_{\alpha,s} : s \in \omega^{<\omega}$ and define sets $A_{\alpha,s}$ such that:

i) $lh(s) = n + 1$ implies $p_{\alpha,s} \in \mathscr{P}_{s(n)}$,
ii) $A_{\alpha,s} = \{n < \omega : \exists q \in \mathscr{P}_n \cap D_\alpha q \leqslant p_{\alpha,s}\}$,
iii) if $n \in A_{\alpha,s}$, then $p_{\alpha,s\frown n} \in \mathscr{P}_n \cap D_\alpha$ and $p_{\alpha,s\frown n} \leqslant p_{\alpha,s}$.

It then follows that for $s \in \omega^{<\omega}$, $\{A_{\alpha,s} : \alpha < \kappa\}$ has the fip. As in Lemma 3.2 find $f : \omega \to \omega$ such that

$$\forall \alpha < \kappa \, \exists n_\alpha \forall m \geqslant n_\alpha f(m) \in A_{\alpha, f\restriction m}.$$

Let then

$$q_\alpha = p_{\alpha, f\restriction(n_\alpha+1)}$$

for $\alpha < \kappa$. Then $\{q_\alpha : \alpha < \kappa\}$ is a linked subset of $\mathscr{P}$ meeting all the D_α.

THEOREM 3.3 MA_κ *holds iff every* ccc *poset of size* κ *is* σ*-centered.*

Proof. Follows directly from Theorems 1.4 and 3.1.

THEOREM 3.4. $\mathrm{MA}_{\aleph_1}$ *holds iff every uncountable* ccc *poset has an uncountable centered subset.*

Proof. Follows directly from Corollary 2.7 and Theorem 3.3.

Question 3.5. Is MA_κ equivalent to *every* ccc *poset of size* κ *is* σ*-linked*?

References

[Ar] A. Arhangel'skii: The construction and classification of topological spaces and cardinal invariants. *Russian math. Surveys* 33 : 6 (1978) 33–96.

[Be] M. Bell: On the combinatorial principle P(c) *Fund. Math.* 114 (1981) 149–67.

[DS] K. Delvin and S. Shelah: A weak version of ◇ which follows from $2^{\aleph_0} < 2^{\aleph_1}$. *Israel J. Math.* 29 (1978) 239–247.

[vD] E.K. van Douwen: The Integers and Topology. In: *Handbook of Set-Theoretic Topology*, eds. K. Kunen and J. Vaughan. North Holland (1984) 111–167.

[Fr] D. Fremlin: Consequences of Martin's Axiom. *Cambridge Tracts in Math.* 84, Cambridge University Press (1984).

[HJ] A. Hajnal and I. Juhász: A Consequence of Martin's Axiom. *Indag. Math.* 33 (1971) 457–463.

[KT] K. Kunen and F. Tall: Between Martin's Axiom and Souslin's Hypothesis. *Fund. Math.* 102 (1979) 173–181.

[MS] D. Martin and R. Solovay: Internal Cohen Extensions. *Ann. Math. Logic* 2 (1970) 143–178.

[Ro] F. Rothberger: On some problems of Hausdorff and of Sierpinski. *Fund. Math.* 35 (1948) 29–47.

[Sh] N.A. Shanin: On intersection of open subsets in the product of topological spaces. Comptes Rendus, (Doklady). *Acad. Sci. U.R.S.S.* 53 (1946) 449–501.

[ST] R. Solovay and S. Tennenbaum: Interated Cohen Extensions and Souslin's Problem. *Ann. Math.* 94 (1971) 201–245.

ITo1] S. Todorcevic: Partitioning pairs of countable ordinals. *Acta Math.*, to appear.

[To2] S. Todorcevic: Remarks on cellularity in products. *Compositio Math.* 57 (1986) 357–372.

[Ve] B. Velickovic: Ph.D. thesis, University of Wisconsin, Madison (1986).