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Differentiable structures of elliptic surfaces with cyclic fundamental group

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1. Introduction

Recently two very different results on differentiable structures of elliptic surfaces have been proved. On one hand, there is the following theorem of Ue [U]:

THEOREM *Let X, X' be relatively minimal elliptic surfaces over smooth curves S, S' such that the Euler number $e(X)$ is positive and $\pi_1(X)$ is not cyclic. Then X and X' are oriented diffeomorphic if and only if $e(X) = e(X')$ and $\pi_1(X) \cong \pi_1(X')$.*

In particular, the diffeomorphism type of such a surface is already determined by its homeomorphism type.

The elliptic surfaces not covered by Ue 's result are elliptic surfaces over $\mathbb{P}^1$ with at most two multiple fibres F_p, F_q of multiplicities p and q; their fundamental group is isomorphic to $\mathbb{Z}/k$ where $k =$ g.c.d. (p, q) ([Dv], [U]). If $k = 1$ and $p_g = 0$ these are the so-called Dolgachev surfaces $X_{p,q}$, which are all homeomorphic to $\mathbb{P}^2$ blown up in nine points [F]. For these surfaces Friedman and Morgan [FM] resp. Okonek and Van de Ven [OV] proved the following theorem which is in sharp contrast to Ue's result.

THEOREM *The Dolgachev surfaces $X_{2,q}$ with $q \equiv 1 \bmod 2$ are pairwise differentiably inequivalent.*

In [FM] it is furthermore proved that the mapping $(p, q) \mapsto X_{p,q}$ which associates to a pair (p, q) of relatively prime integers the diffeomorphism type of a Dolgachev surface $X_{p,q}$ is finite-to-one.

In this paper we tackle the case $p_g = 0$ and g.c.d. $(p, q) = k \geqslant 1$. We will show, that for every fixed k there are infinitely many differentiably inequivalent surfaces $X_{p,q}$. For odd k all $X_{p,q}$ are homeomorphic by a result of Hambleton and Kreck [HK], whereas for even k the topological classification is still incomplete [HK];* the case $k = 2$ has been treated in [O].

The main tool in the proof of this result is Donaldson's invariant introduced in [D1], [D2]. Most of the techniques which we will use have already been developed in [OV] and [LO], so we refer to these papers for some details.

2. Precise statement of results

To describe the surfaces we are dealing with, let Y_0, Y_1, Y_2 resp. x_0, x_1 be homogeneous coordinates in $\mathbb{P}^2$ resp. $\mathbb{P}^1$ and Q_0, Q_1 two irreducible homogeneous cubic polynomials in Y_0, Y_1, Y_2. Let $X \subset \mathbb{P}^1 \times \mathbb{P}^2$ be the zero-set of the polynomial $x_0 Q_0 + x_1 Q_1$. For generic Q_0, Q_1 the surface X is smooth. The induced projection to $\mathbb{P}^1$ defines an elliptic fibration with irreducible fibres and without multiple fibres. Applying logarithmic transformations of multiplicities p and q along two smooth fibres, we obtain the surface $X_{p,q}$ with multiple fibres F_p and F_q. The fundamental group of this surface is $\pi_1(X_{p,q}) \cong \mathbb{Z}/k$ where k = g.c.d. (p, q).

If we write $p = kp'$, $q = kq'$, then there are uniquely determined integers β, γ with $\gamma p' + \beta q' = 1$ and $0 \leqslant \gamma \leqslant q' - 1$; we define two divisors as follows:

$$C_{p,q} = \beta F_p + \gamma F_q$$

$$T_{p,q} = p' F_p - q' F_q .$$

Clearly $T_{p,q}$ is a torsion element in H^2 is a torsion element in $H^2(X_{p,q}, \mathbb{Z})$ because

$$k T_{p,q} = p F_p - q F_q \sim 0$$

(here $\sim$ denotes linear equivalence). Every vertical divisor D on $X_{p,q}$ can be written in the form

$$D \sim aF + bF_p + cF_q$$

where $a, b, c \in \mathbb{Z}$ and F is a generic fibre. We define

$$N(D) = ap'q'k + bq' + cp' .$$

* See: Note added in proof.

In particular, since the canonical divisor of $X_{p,q}$ is

$$K_{p,q} \sim -F + (p-1)F_p + (q-1)F_q \sim F - F_p - F_q,$$

we have $N(K_{p,q}) = p'q'k - p' - q'$. An easy calculation shows:

LEMMA *For every vertical divisor* $D \sim aF + bF_p + cF_q$ *we have*

$$D \sim N(D)C_{p,q} + (b\gamma - c\beta)T_{p,q}.$$

In other words, the group of vertical divisors modulo torsion is isomorphic to $\mathbb{Z}$ with generator $C_{p,q}$.

If L is any ample divisor on $X_{p,q}$, then

$$N(D) = \deg_L(D)/\deg_L(C_{p,q}),$$

so every vertical divisor of degree 0 is torsion.

What we need to know about the topology of $X_{p,q}$ is the following. First of all, $c_1^2(X_{p,q}) = 0$ since $K_{p,q}$ is vertical. Then, since the geometric genus and the topological Euler characteristic are invariant under logarithmic transformations, we have $p_g(X_{p,q}) = 0$, $e(X_{p,q}) = 12$. Therefore the signature of $X_{p,q}$ is $\sigma(X_{p,q}) = -8$. The intersection form

$$S_{X_{p,q}}: H^2(X_{p,q}, \mathbb{Z})/\text{torsion} \times H^2(X_{p,q}, \mathbb{Z})/\text{torsion} \to \mathbb{Z}$$

is even if and only if $k \equiv 0 \bmod 2$ and $p' + q' \equiv 0 \bmod 2$ [O]. Thus

$$S_{X_{p,q}} \triangleq \begin{cases} -E_8 \oplus H & \text{if } k \equiv 0 \bmod 2,\ p' + q' \equiv 0 \bmod 2, \\ \langle 1 \rangle \oplus 9\langle -1 \rangle & \text{otherwise.} \end{cases}$$

We will use the following result of Hambleton and Kreck [HK]:

THEOREM *Let M be a closed, oriented, differentiable manifold of real dimension* 4 *with* $\pi_1(M) \cong \mathbb{Z}/k$. *If k is odd, then the oriented homeomorphism type of M is determined by the intersection form on* $H^2(M, \mathbb{Z})$/torsion.

COROLLARY *For fixed odd* k, all surfaces $X_{p,q}$ with* g.c.d. $(p, q) = k$ *are homeomorphic.*

Since the surfaces $X_{p,q}$ are algebraic with $p_g(X_{p,q}) = 0$, we have $b_+(X_{p,q}) = 1$ and the Donaldson invariant Γ is defined for every $X_{p,q}$. For the definition

* See: Note added in proof.

and the properties of Γ see [D1], [D2], [FM], [OV]. Now we state our main result.

THEOREM: *For every pair of integers $p, q \geqslant 1$ there exists an ample divisor $L_{p,q}$ on $X_{p,q}$ and an integer $n_{p,q} \geqslant N(K_{p,q})$ such that*

$$\Gamma(L_{p,q}) \equiv n_{p,q} C_{p,q}$$

in $H^2(X_{p,q}, \mathbb{Z})$/torsion. If the surfaces $X_{p,q}$ and $X_{r,s}$ are diffeomorphic, then $n_{p,q} = n_{r,s}$.

COROLLARY *Given $p_0, q_0 \geqslant 1$ there exist only finitely many pairs p, q such $X_{p,q}$ is diffeomorphic to X_{p_0,q_0}.*

3. Sketch of proofs

Choose an ample divisor $L^0_{p,q}$ on $X_{p,q}$ and let $L_{p,q} = L^0_{p,q} + nK_{p,q}, n \gg 0$. The main ingredient for calculating the Donaldson invariant is the moduli space of $L_{p,q}$-stable 2-bundles E with Chern classes $c_1(E) = 0$, $c_2(E) = 1$ on $X_{p,q}$. We will denote this space by $M_{p,q}$. It can be determined by the same methods as in [LO] and [OV]: Each stable bundle E is given by an extension

$$0 \to \mathcal{O}(D - K_{p,q}) \to E \to \mathcal{I}_z \otimes \mathcal{O}(K_{p,q} - D) \to 0 \qquad (*)$$

where $D = bF_p + cF_q$ is a vertical divisor with $0 \leqslant b \leqslant p - 1$, $0 \leqslant c \leqslant q - 1$, and z is a simple point in $F_p \cup F_q$. Since there may be torsion in $H^2(X_{p,q}, \mathbb{Z})$, a vertical curve D is not necessarily determined by its degree, but still there is a *unique* pair (D, z) defining E by $(*)$ if we require D to have maximal degree *and* maximal b. It is not hard to check that given a pair (D, z_0) maximal (in the above sense) for a stable bundle E_0 with $z_0 \in F_p$ (or F_q), then also for every other point $z \in F_p$ (or F_q) the bundle E defined by $(*)$ is stable and D is maximal for E. Hence $M_{p,q}$ is as a set the disjoint union of a finite number of copies of F_p and F_q, but the analytic structure of $M_{p,q}$ is in general non-reduced. If $\mathbb{E}_{p,q}$ is the universal bundle over $M_{p,q} \times X_{p,q}$, which can be constructed as in [OV], [LO], then from [D2] we get

$$\Gamma(L_{p,q}) \equiv 2a_1 [M^{\mathrm{red}}_{p,q}] \backslash c_2(\mathbb{E}_{p,q}) + a_2 K_{p,q}$$

in $H^2(X_{p,q}, \mathbb{Z}/\text{torsion}$, where a_1, a_2 are suitable positive integers (this is the only difference from the formula for Γ used in [FM], [OV]). The coefficient

a_1 comes from the multiplicities of $M_{p,q}$ ([D2], Prop. 3.13), and a_2 depends on the torsion group $H_1(X_{p,q}, \mathbb{Z})$ ([D2], Appendix). As in [OV] the first term on the right hand side of the equation above consists of a certain number of copies of F_p and F_q, so

$$\Gamma(L_{p,q}) \equiv a_2 K_{p,q} + C$$

where C is a vertical divisor with $N(C) \geqslant 0$, or

$$\Gamma(L_{p,q}) \equiv n_{p,q} C_{p,q}$$

with $n_{p,q} \geqslant N(K_{p,q})$.

Now the same arguments as in [OV] show that for large n the chamber in the positive cone in $H^2(X_{p,q}, \mathbb{R})$ containing $L_{p,q} = L^0_{p,q} + nK_{p,q}$ is independent of n; also if $f\colon X_{p,q} \to X_{r,s}$ is an orientation preserving diffeomorphism and $L_{r,s}$ is a suitable ample divisor on $X_{r,s}$, then $L_{p,q}$ and $f^*(L_{r,s})$ are up to sign in the same chamber. From the constance of Γ on the chambers and naturality we get

$$n_{p,q} C_{p,q} \equiv \Gamma(L_{p,q}) \equiv \pm\Gamma(f^*(L_{r,s})) \equiv \pm f^*(\Gamma(L_{r,s})) \equiv \pm n_{r,s} f^*(C_{r,s}).$$

Now let D be a divisor on $X_{p,q}$ representing the class $f^*(C_{r,s}) \in H^2(X_{p,q}, \mathbb{Z})$. Then D is vertical, and modulo torsion we get from our lemma

$$n_{p,q} C_{p,q} \equiv \pm n_{r,s} N(D) C_{p,q}$$

implying $n_{r,s} \mid n_{p,q}$. Since the same argument works also the other way, we conclude $n_{p,q} = n_{r,s}$; this proves the theorem.

Finally for a given $N \in \mathbb{N}$ there are only finitely many pairs (p, q) with $n_{p,q} \leqslant N$, thus the corollary follows immediately.

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Note added in proof: Hambleton and Kreck have meanwhile extended their topological classification to smooth, oriented 4-manifolds with fundamental group $\pi_1 = \mathbb{Z}/k$, $k \equiv 0 \bmod 2$ (I. Hambleton, M. Kreck: Smooth structures on algebraic surfaces with cyclic fundamental group, preprint 1987). Their result implies in particular that the oriented homeomorphism type of the surfaces $X_{p,q}$ with g.c.d. $(p, q) = k \equiv 0 \bmod 2$ is also determined by their intersection form. From the corollary to our main theorem it follows that all these surfaces have infinitely many smooth structures too.

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