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Ergodic Z_2-extensions over rational pure point spectrum, category and homomorphisms

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Abstract. Let $\mathscr{W}$ be the space of all Z_2-extensions over a fixed automorphism with a rational pure point spectrum, endowed with the uniform topology. We prove that a set of "typical" points of $\mathscr{W}$ coincides with the class of those Z_2-extensions which are measure-theoretically isomorphic to Morse sequences. The factor problem is studied.

Introduction

Skew products were introduced to ergodic theory by Anzai [2] in connection with the problem of isomorphism. It was shown in [1] that any ergodic automorphism can be represented as a skew product of one of its factor-automorphisms with a family of automorphisms.

However, up to now, a great deal of the attention has been devoted to the study of some simpler forms of skew products, namely to the so called G-extensions over automorphisms with pure point spectra (for instance [2], [10], [11], [19], [21]).

In the present paper we deal with Z_2-extensions over automorphisms with rational pure point spectra. Recall some well-known examples of them: generalized Morse sequences [12], continuous substitutions on two symbols [4], Mathew-Nadkarni's examples [18] with partly continuous Lebesgue spectrum of multiplicity two, Helson-Parry's constructions [8].

Morse sequences are very useful for constructing counterexamples in ergodic theory (for further details see [16]). We hope that our present paper emphasizes once more the peculiar role of the class of Morse dynamic systems in ergodic theory. Our main result here seems to confirm that the combinatorial approach to ergodic theory (in Jacob's sense [9]) is of a rather general nature. In 1981 J. Kwiatkowski posed the question whether any "typical" Z_2-extension over a rational pure point spectrum is a Morse sequence. The first part of the paper positively answers this question. We prove that in the space of Z_2-extensions of a fixed ergodic automorphism with rational pure point spectrum, the set of extensions isomorphic to

Morse sequences is of the second category (in the weak topology). To prove this we use Katok-Stepin's theory of odd approximation of Z_2-extensions [11]. But what seems most important is that by analyzing the proof one can observe that in fact we show more: if an ergodic Z_2-extension over T is oddly approximated with speed $o(1/n^2)$ then it is measure-theoretically isomorphic to a Morse sequence. So in this case, Katok-Stepin's theory has quite a combinatorial nature.

In the second part of the paper we treat the factor problem. It is not hard to see that the class of ergodic Z_2-extensions of automorphisms with pure point spectra is closed under taking factors. It turns out that if $Sp(T_\theta)$ is the group of all p^k-roots of unity, $k \geqslant 1$, and p is a prime number, then the only factors of T_θ are those with discrete spectrum. The main result of the sections states that if $x = b^0 \times b^1 \times \dots$ is a regular Morse sequence and the set of blocks $\{b^0, b^1, \dots\}$ is finite then the only proper factors of x are those with discrete spectrum. In particular all continuous substitutions on two symbols enjoy this property.

I. Definitions and remarks

Let $(X, \mathscr{B}, \mu)$ be a Lebesgue space and $T: (X, \mathscr{B}, \mu)\circlearrowleft$ be an automorphism. By Sp (T) we denote the group of all eigenvalues of T, i.e. $\lambda \in$ Sp (T) iff $fT = \lambda f$ for some $0 \neq f \in L^2(X, \mu)$. We recall here the result of [3] stating that exp $(2\pi i/m) \in$ Sp (T) iff there is a T-stack of height m which is a partition of X, i.e. there is a partition of X of the form $\{A, TA, \dots, T^{m-1}A\}$, for some $A \in \mathscr{B}$. Obviously the ergodicity of T implies that there is only one T-stack of height m filling up X (recordering its elements if necessary). We will denote it by $D^m = (D_0^m, \dots, D_{m-1}^m)$. Moreover, if $k|m$ then $D^k \leqslant D^m$, i.e. D^m is finer than D^k.

Let $\{n_t: t \geqslant 0\}$ be a sequence of natural numbers such that $n_t | n_{t+1}$, $t \geqslant 0$ and $\lambda_{t+1} = n_{t+1}/n_t$, $t = 0, 1, \dots$, $\lambda_0 = n_0 \geqslant 2$. Denote by $G\{n_t: t \geqslant 0\}$ the group of all roots of unity generated by $\{\exp(2\pi i/n_t): t \geqslant 0\}$ (every infinite group of roots of unity can be obtained in this way).

DEFINITION 1. An ergodic automorphism T with discrete spectrum has *rational pure point spectrum* if Sp $(T) = G\{n_t: t \geqslant 0\}$.

These automorphisms were characterized in [17] as follows

LEMMA 1. *T has rational pure point spectrum with* Sp $(T) = G\{n_t: t \geqslant 0\}$ *iff there is a sequence of partitions $\{D^{n_t}\}$ of X, where D^{n_t} are T-stacks of heights n_t resp. and $D^{n_t} \nearrow \varepsilon$.*

By $Z_2 = \{0, 1\}$ we mean the group of all integers mod 2 equipped with Haar measure $m(0) = m(1) = 1/2$.

Let $\theta: X \to Z_2$ be any measurable map.

Let $\tilde{\mu}$ be the product measure $\mu \times m$ on $Y = X \times Z_2$.

DEFINITION 2. By *Z_2-extension of T with respect to θ* we mean the automorphism $T_\theta: Y\circlearrowleft$ defined by the formula

$$T_\theta(x, i) = (Tx, \theta(x) + i), \quad x \in X, \quad i \in Z_2. \tag{1}$$

Following [11], [19], T_θ is ergodic iff there is no measurable function $f: X \to K$ (K denotes the unit circle) such that

$$f(Tx) = (-1)^{\theta(x)} f(x).$$

Let $\sigma: Y\circlearrowleft$, $\sigma(x, i) = (x, i + 1)$ for each $(x, i) \in Y$. Then σ is a $\tilde{\mu}$-preserving automorphism and $\sigma T_\theta = T_\theta \sigma$.

It is not hard to see that

$$L^2(Y, \tilde{\mu}) = \mathscr{C} \oplus \mathscr{D} \tag{2}$$

where $\mathscr{D} = \{f \in L^2(Y, \tilde{\mu}): f\sigma = f\} = L^2(X, \tilde{\mu}), \mathscr{C} = \mathscr{D}^\perp = \{f \in L^2(Y, \tilde{\mu}): f\sigma = -f\}$.

From now on T will be an ergodic automorphism with rational pure point spectrum, Sp $(T) = G\{n_t: t \geqslant 0\}$, and T_0 be a Z_2-extension of T.

REMARK 1. If T_θ is ergodic and has any point spectrum on $\mathscr{C}$, then T_θ has discrete spectrum.

Proof. Let $H \subseteq L^2(Y, \tilde{\mu})$ be the space generated by all eigenfunctions of T_θ. Then

$$H \supsetneq \mathscr{D} = L^2(X, \mu), \tag{3}$$

$$H \text{ is a unitary subring [23].} \tag{4}$$

Consider the partition η of Y, where

$$A \in \eta \text{ iff } A = \{(x, 0), (x, 1)\}, \quad x \in X. \tag{5}$$

Then η is T_0-invariant and measurable and the corresponding factor-automorphism T_0/η is isomorphic to T. Moreover, the unitary subring corresponding to η is contained in H, so that ξ is the partition corresponding to H [23], then

$$\eta \leqslant \xi, \tag{6}$$

$$T_0/\xi \text{ has discrete spectrum [23].} \tag{7}$$

But the ergodicity of T_0 implies that the number of elements in any atom of ξ is constant and common for a.e. atom of ξ. Therefore from (3), (5) and (6) it follows that $\xi = \varepsilon$. Thus by (7) T_0 has discrete spectrum.

Note that Remark 1 may be reformulated as follows: if T_0 is ergodic, then

$$T_0 \textit{ has partly continuous spectrum iff } \mathrm{Sp}\,(T_0) = \mathrm{Sp}\,(T). \tag{8}$$

If this is the case, we call T_0 a continuous Z_2-extension.

Let us consider a class of well-known examples of ergodic Z_2-extensions usually called generalized Morse sequences [12]. We briefly recall their definition referring for further properties to [12], [14], [16].

Let $B = (b_0, \ldots, b_{n-1})$, $C = (c_0, \ldots, c_{m-1})$ be blocks (finite sequences of 0 and 1) with lengths $|B| = n$ and $|C| = m$. By $B \times C$ we mean the juxtaposition of blocks $B \times C = B^{c_1} \ldots B^{c_{m-1}}$, where $B^0 = B, B^1 = \tilde{B} = (b_0 + 1, \ldots, b_{n-1} + 1)$. By fr (B, C) we denote the cardinality of the set $\{i\colon\ 0 \leqslant i \leqslant |C| - |B|,\ \ B = C[i, i + |B| - 1]\}$, where $C[r, s] = (c_r, c_{r+1}, \ldots, c_s)$. If $|B| = |C| = n$, then $d(B, C) = \text{card}\,\{i\colon 0 \leqslant i \leqslant n - 1, B[i] \neq C[i]\}/n$.

Now, let $b^0, b^1, b^2, \ldots$ be blocks starting with zero, $|b^i| = \lambda_i \geqslant 2$ and let $x = b^0 \times b^1 \times b^2 \times \ldots$.

DEFINITION 3. x is said to be a *Morse sequence* if
i) infinitely many of the b^i's are different from $0 \ldots 0$,
ii) infinitely many of the b^i's are different from $01 \ldots 010$,
iii) $\Sigma_{i \geqslant 0}\, r_i = \infty$, $r_i = \min\,(1/\lambda_i \text{ fr } (0, b^i), 1/\lambda_i \text{ fr } (1, b^i))$, $\ i \geqslant 0$.

We extend x to two-sided sequence $\omega \in \{0, 1\}^Z$ [12] preserving the almost periodicity condition. Let $\mathcal{O}_x = \{\tau^i\omega\colon i \in Z\}^{cl}$, where the closure is taken in $\{0, 1\}^Z$ and τ is the shift transformation. It is known [12] that $(\mathcal{O}_x, \tau)$ is strictly ergodic. The unique (ergodic) τ-invariant measure we denote by μ_x. Let $P = (P_0, P_1)$ be the zero-time partition, $P_i = \{y \in \mathcal{O}_x\colon y[0] = i\}$. Then P is a generator of τ on $\mathcal{O}_x$. Let σ be the mirror map on $\mathcal{O}_x$, i.e. $\sigma(y) = \tilde{y}$, $\tilde{y}[i] = \widetilde{y[i]}$. Then σ is an automorphism of $(\mathcal{O}_x, \mu_x)$ $\sigma\tau = \tau\sigma$. Let

$c_t = b^0 \times \cdots \times b^t$, $n_t = |c_t| = \lambda_0 \times \cdots \times \lambda_t$, $t \geqslant 0$ and let (T, X, μ) be an ergodic automorphism with discrete spectrum and $\mathrm{Sp}\,(T) = G\{n_t: t \geqslant 0\}$.

REMARK 2 [14]. For every Morse sequence $x = b^0 \times \ldots$ there is a measurable $\theta: X \to Z_2$ such that x is isomorphic to T_θ (more precisely, the τ associated with x is isomorphic to T_θ). If no confusion can arise we will speak about properties of x instead of properties of $(\mathcal{O}_x, \tau, \mu_x)$.

A Morse sequence $x = b^0 \times b^1 \times \ldots$ is called *continuous* if $\mathrm{Sp}\,(x) = G\{n_t: t \geqslant 0\}$ [12].

REMARK 3 [12]. x is continuous iff either
a) infinitely many of the λ_i's are even, or
b) $\Sigma_{t \geqslant 0}\, \omega(b^t) = \infty$.

Notice that a) can be strengthened as follows: if infinitely many of the λ_i's are even, then every ergodic Z_2-extension is continuous (see [12] p. 348).

Let us observe that any constant Morse sequence $x = b \times b \times \ldots$ is continuous. The class of all continuous substitutions on two symbols [4] coincides with the class of all constant Morse sequences. A larger subclass of continuous Morse sequences is the class of regular Morse sequences [14], where $x = b^0 \times b^1 \times \ldots$ is *regular* provided that there is $\varrho > 0$ such that

$$\tfrac{1}{2} - \varrho > \max\,(\mu_{x_t}(00), \mu_{x_t}(01)) > \varrho; \quad t \geqslant 0, \tag{9}$$

where $x_t = b^t \times b^{t+1} \times \ldots$.

II. Theorem on category

In the class $\mathcal{W}$ of all Z_2-extensions of T we introduce some topology. Namely

$$\varrho(T_\theta, T_{\theta'}) = \mu(\theta^{-1}(1) \,\Delta\, \theta'^{-1}(1)) \quad [11]. \tag{10}$$

Simultaneously, we have the uniform topology

$$d(T_\theta, T_{\theta'}) = \tilde{\mu}\{(x, i): T_\theta(x, i) \neq T_{\theta'}(x, i)\} \quad [7]. \tag{11}$$

It is clear that these topologies coincide. The class $\mathcal{W}$ is completely metrizable in the uniform topology because ϱ is a complete metric. In other words $\mathcal{W}$ is a closed subspace of the class of all automorphisms of $(Y, \tilde{\mu})$ endowed with the uniform topology, so it has Baire property.

Our goal is to prove the following:

THEOREM 1. *The class of all Z_2-extensions which are isomorphic to Morse sequences is of the second category in $\mathscr{W}$.*

Notice that from Theorem 1 it follows that the class of all ergodic Z_2-extensions is residual in $\mathscr{W}$ (cf. [20]).

The proof of Theorem 1 goes by steps.

Step 1. The concept of odd approximation [11].

We say $F \subseteq X$ is oddly approximated with respect to $D^{n_t} \nearrow \varepsilon$ with a speed $o(g(n))$ if for some subsequence $\{n_{t_k}\}$ there exist sets F_k consisting of an odd number of atoms of $D^{n_{t_k}}$ such that

$$\mu(F \,\Delta\, F_k) \;=\; o(g(n_{t_k})) \tag{12}$$

(we recall here that n_{t_k} is the number of atoms of the T-stack $D^{n_{t_k}}$ of height n_{t_k}).

It is known [11] that the collection of all measurable functions $\theta\colon X \to Z_2$ such that $\theta^{-1}(1)$ is oddly approximated with a fixed speed contains a dense G_δ-set. Therefore the set of all ergodic Z_2-extensions is residual since if θ is oddly approximated with speed $o(1/n)$, then it has simple spectrum [11].

REMARK. The proof of density of ergodic G-extensions can also be found in [10] for a more general situation.

We would like to briefly explain the notion of odd approximation with 'sufficiently high speed' in our situation.

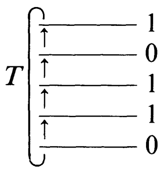

Fig. 1

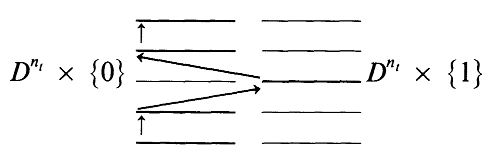

Fig. 2

If the speed of approximation is sufficiently high then on each level $D_i^{n_t}$ of D^{n_t} the function θ is constant apart from a set of a small measure (Fig. 1). Thus for T_0, two T_0-stacks arise (one of them is denoted by fat dashes on Fig. 2). If we want to assure that T_0 admits a cyclic approximation, we must show that the top of the first T_0-stack is carried by T_0 on the base of the second one. To show this we must show that there is an odd number of 1's on Fig. 1.

Step 2. Let $\theta\colon X \to Z_2$ be measurable and oddly approximated (i.e. $\theta^{-1}(1)$ is oddly approximated) with speed $o(1/n^2)$. Then there is a sequence of two T_θ-stacks $\bar{C}^{n_t}(0)$ and $\bar{C}^{n_t}(1)$, $\bar{C}^{n_t}(j) = \{\bar{C}_i^{n_t}\colon i = 0, 1, \ldots, n_t - 1\}$, $j = 0, 1$ such that

$$\sigma\bar{C}_i^{n_t}(0) = \bar{C}_i^{n_t}(1), \quad i = 0, 1, \ldots, n_t - 1, \tag{13}$$

$$\bar{C}^{n_t} \to \varepsilon \quad \text{(not necessarily monotonically).} \tag{14}$$

Indeed, let $E_i^{n_t}$ be the subset of $D_i^{n_t}$ consisting of all $x \in D_i^{n_t}$ such that $\mu(E_i^{n_t}) > 1/2$ and $\theta\,|\,E_i^{n_t}$ is constant. Consider the set

$$E_t = \bigcup_{i=0}^{n_t-2} T^{-i}(D_i^{n_t} \backslash E_i^{n_t}).$$

Thus $E_t \subseteq D_0^{n_t}$ and

$$\mu(E_t) \leqslant \sum_{i=0}^{n_t-2} \mu(D_i^{n_t} \backslash E_i^{n_t}). \tag{15}$$

From our assumption

$$\begin{aligned} &n_t^2\mu(F_t \,\Delta\, \theta^{-1}(0)) \underset{t}{\longrightarrow} 0, \quad F_t \leqslant D^{n_t}, \\ &n_t^2\mu(F_t' \,\Delta\, \theta^{-1}(1)) \underset{t}{\longrightarrow} 0, \quad F_t' \leqslant D^{n_t}. \end{aligned} \tag{16}$$

In particular (16) implies

$$n_t^2\mu(D_i^{n_t} \,\Delta\, E_i^{n_t}) \underset{t}{\longrightarrow} 0, \quad i = 0, \ldots, n_t - 2. \tag{17}$$

From (15) and (17) it follows that

$$n_t\mu(E_t) \leqslant n_t(n_t - 1) \max_{0 \leqslant i \leqslant n_t - 2} \mu(D_i^{n_t} \backslash E_i^{n_t}) \underset{t}{\longrightarrow} 0. \tag{18}$$

So, the sets $(D_0^{n_t} \backslash E_t) \times \{i\}$, $i = 0, 1$ can be taken as the bases of two T_θ-stacks of heights n_t. Putting $\bar{C}_0^{n_t}(i) = (D_0^{n_t} \backslash E_t) \times \{i\} \cup E_t \times \{i\}$, $i = 0, 1$, we obtain two T_θ-stacks with the property (13). It is clear that (18) implies the property (14) because the partitions

$$B^{n_t} = \{D_i^{n_t} \times \{j\}: i = 0, 1, \ldots, n_t - 1, j = 0, 1\}, \quad t \geq 0,$$

generate all measurable sets in Y.

REMARK. For simplicity we have assumed in the proof above that the subsequence $\{n_{t_k}\}$ from (12) is equal to the sequence $\{n_t\}$.

By $\bar{C}^{n-1}$ we will mean two stacks of height 1,

$$\bar{C}^{n-1}(i) = \{X \times \{i\}\}, \quad i = 0, 1.$$

Step 3. For θ as in Step 2 there is a sequence of two T_θ-stacks $C_i^{n_t}(j)$, $i = 0, \ldots, n_t - 1, t = -1, 0, 1, \ldots$ such that

$$\sigma C_i^{n_t}(0) = C_i^{n_t}(1), \tag{19}$$

$$C^{n_t} \nearrow \varepsilon. \tag{20}$$

The proof of Step 3 is in some sense a modification of Goodson's considerations from [6] (Theorem 3). Indeed, (20) follows directly from that theorem. Now, fix $t \geq 0$ and consider all 2-T_θ-stacks C'^{n_t} (i.e. C'^{n_t} is a disjoint union of $C'^{n_t}(0)$ and $C'^{n_t}(1)$ where $C'^{n_t}(i) = \{C_0'^{n_t}(i), \ldots, C_{n_t-1}'^{n_t}(i)\}, i = 0, 1$ are T_θ-stacks) satisfying $C'^{n_t} \leq \bar{C}^{n_t+1}$. Then it is not hard to see that if $\bar{C}'^{n_t}$ realizes the minimum of the set

$$d_t = \min\left\{\sum_{i=0}^{n_t-1} \sum_{k=0}^{1} \tilde{\mu}(C_i'^{n_t}(k) \,\Delta\, \bar{C}_i^{n_t}(k)): C'^{n_t} \leq \bar{C}^{n_t+1}, C'^{n_t} \text{ is a 2-}T_\theta\text{-stack}\right\}$$

then $\sigma\bar{C}_i'^{n_t}(0) = \bar{C}_i'^{n_t}(1)$ and therefore (19) holds.

REMARK 4. Our considerations are restricted by the conditions:

$$\sigma C_i'^{n_t}(0) = C_i'^{n_t}(1), C_i'^{n_t}(0) \cup C_i'^{n_t}(1) = D_i^{n_t}, d_t < \delta_t \quad \text{and} \quad \sum_{t \geq 0} \delta_t < \infty.$$

This last condition may be obtained by passing to a subsequence if necessary. Observe also that the condition $C_i^{n_t}(0) \cup C_i^{n_t}(1) = D_i^{n_t}$ allows us to assume $C_0^{n_t+1}(i) \subseteq C_0^{n_t}(i)$, $t \geq 0$, $i = 0, 1$.

Step 4. Let θ be as in Step 2. Denote $Q_i = C^{n_{-1}}(i)$, $i = 0, 1$. Then there is a sequence $x = b^0 \times b^1 \times \ldots$ such that Q-s-names of a.e. $z \in Y$ are sectors of x.

From the preceding step we have

$$C_i^{n_t}(j) = 1/2n_t,\ t = -1, 0, 1, \ldots, i = 0, \ldots, n_t - 1, j = 0, 1, n_{-1} = 1. \tag{21}$$

We define a sequence of blocks $b^0, b^1, b^2, \ldots$ satisfying $b^i[0] = 0$, $|b^i| = \lambda_i = n_{i+1}/n_i$, as follows

DEFINITION OF b^0. We divide Q_i into λ_0 pieces $C_j^{n_0}(k)$ of measure $1/2n_0$ using the condition $Q \leqslant C^{n_0}$ (Fig. 3).

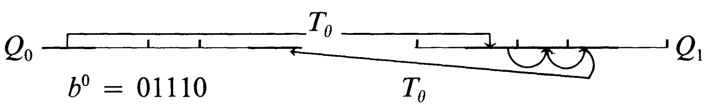

Fig. 3

We look at the trajectory of $C_0^{n_0}(0)$ and put $b^0[i] = b^0[i+1]$ iff $T_0(C_0^{n_0}(0))$ and $C_i^{n_0}(0)$ are contained in the same atom Q_s, $s = 0, 1$.

Let us suppose $b^0, \ldots, b^t$ are already defined.

DEFINITION OF b^{t+1}

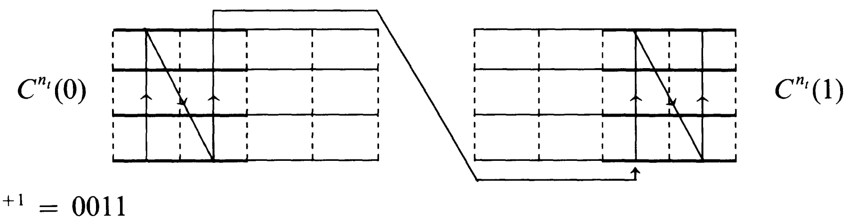

Fig. 4

We divide $C_0^{n_t}(i)$ into λ_{t+1} pieces $C_{jn_t}^{n_{t+1}}(k_j)$, $j = 0, 1, \ldots, \lambda_{t+1} - 1$ and we look at the trajectory of $C_0^{n_{t+1}}(0)$ (Fig. 4). We put $b^{t+1}[i] = b^{t+1}[i+1]$ iff $T_0^{n_t}(C_{in_t}^{n_{t+1}}(0))$ and $C_{in_t}^{n_{t+1}}(0)$ are contained in the same atom $C_0^{n_t}(s)$, $s = 0, 1$.

From the definition of $x = b^0 \times b^1 \times \ldots$ it follows that

– Q-n_0-name of y from $C_0^{n_0}(0)$ is equal to b^0,

– Q-n_1-name of y from $C_0^{n_1}(0)$ is equal to $b^0 \times b^1 = c_1$,

. . .

– Q-n_t-name of y from $C_0^{n_t}(0)$ is equal to $b^0 \times b^1 \times \cdots \times b^t = c_t$, $t \geqslant 0$.

Moreover, since $\sigma Q_0 = Q_1$, Q-n-name of $\sigma y = (Q\text{-}n\text{-name of } y)^{\sim}$, $n \geqslant 0$.

This implies that Q-n-names of a.e. $y \in Y$ are sectors of $x = b^0 \times b^1 \times \dots$, unless $b^t = 0 \dots 0$, $t \geqslant t_0$. But this situation is excluded since T_0 is ergodic (if $b^t = 0 \dots 0$, $t \geqslant t_0$, then $C_0^{n_{t_0}}(0) \cup \dots \cup C_{n_{t_0}-1}^{n_{t_0}}(0)$ is T_0-invariant).

Step 5. Either $b^t = 01 \dots 010$, $t \geqslant t_0$ or $x = b^0 \times b^1 \times \dots$ is a Morse sequence (continuous or not).

Indeed, suppose infinitely many of the b^t's are different from $01 \dots 010$ (and $0 \dots 0$ by Step 4). All we have to show is that

$$\sum_{t \geqslant 0} r_t = \infty \quad \text{(see Definition 3).}$$

Let us observe that for the sequence $\{C^{n_t}\}$ we have defined in Step 3,

$$\varrho(\bar{C}^{n_t}, C^{n_t}) \xrightarrow[t]{} 0 \tag{22}$$

holds (see the proof of Theorem 3 in [6]). Hence

$$2n_t \tilde{\mu}(\bar{C}_0^{n_t}(i) \,\Delta\, C_0^{n_t}(i)) \xrightarrow[t]{} 0, \quad i = 0, 1, \tag{23}$$

since $\bar{C}^{n_t}$ and C^{n_t} are 2-T_0-stacks. Moreover, $\{\bar{C}^{n_t}\}$ has the property

$$2n_t \tilde{\mu}(T_0^{n_t}(\bar{C}_0^{n_t}(0)) \,\Delta\, \bar{C}_0^{n_t}(1)) \xrightarrow[t]{} 0. \tag{24}$$

Since

$$\tilde{\mu}(T_0^{n_t}(\bar{C}_0^{n_t}(0)) \,\Delta\, C_0^{n_t}(1)) < \tilde{\mu}(T_0^{n_t}(\bar{C}_0^{n_t}(0)) \,\Delta\, \bar{C}_0^{n_t}(1)) + \tilde{\mu}(\bar{C}_0^{n_t}(1) \,\Delta\, C_0^{n_t}(1))$$

and

$$\tilde{\mu}(T_0^{n_t}(C_0^{n_t}(0)) \,\Delta\, C_0^{n_t}(1)) < \tilde{\mu}(T_0^{n_t}(\bar{C}_0^{n_t}(0)) \,\Delta\, C_0^{n_t}(1))$$

$$+ \tilde{\mu}(T_0^{n_t}(\bar{C}_0^{n_t}(0)) \,\Delta\, T_0^{n_t}(C_0^{n_t}(0))),$$

we have

$$2n_t \tilde{\mu}(T_0^{n_t}(C_0^{n_t}(0)) \,\Delta\, C_0^{n_t}(1)) \xrightarrow[t]{} 0. \tag{25}$$

Now, we show that

$$\frac{1}{\lambda_{t+1}}(\text{fr}\,(00, b^{t+1}) + \text{fr}\,(11, b^{t+1})) \leqslant 2n_t\tilde{\mu}(T_\theta^{n_t}(C_0^{n_t}(0)) \,\Delta\, C_0^{n_t}(1)). \quad (26)$$

Indeed, if $b^{t+1}[i] = b^{t+1}[i+1]$, then $T_\theta^{n_t}(C_{in_t}^{n_{t+1}}(j)) \subseteq C_0^{n_t}(j)$, $j = 0, 1$. With any such a pair $(i, i+1)$ we can assign the level $C_{in_t}^{n_{t+1}}(0)$ with the measure equal to $1/2n_t$ and $C_{in_t}^{n_{t+1}}(0) \subseteq T_\theta^{n_t}(C_0^{n_t}(0)) \,\Delta\, C_0^{n_t}(1)$. So, $1/2n_{t+1}(\text{fr}\,(00, b^{t+1}) + \text{fr}\,(11, b^{t+1})) \leqslant \tilde{\mu}(T_\theta^{n_t}(C_0^{n_t}(0)) \,\Delta\, C_0^{n_t}(1))$.

From (25) and (26) it follows that

$$\frac{1}{\lambda_{t+1}}(\text{fr}\,(00, b^{t+1}) + \text{fr}\,(11, b^{t+1})) \xrightarrow[t]{} 0$$

and therefore x is a Morse sequence.

Step 6. Q is a generator for T_θ as soon as x is a Morse sequence.

We will show that for a.e. $y, y' \in Y$ there is an $n \in N$ such that Q-n-names of y and y' are different. Indeed, suppose that y, y' have the property from Step 4, i.e. their Q-n-names are sectors of x. Moreover

$$y = \bigcap_{t\geqslant 0} C_{i_t}^{n_t}(j_t), \quad y' = \bigcap_{t\geqslant 0} C_{i_t'}^{n_t}(j_t'), \quad \text{since } C^{n_t} \nearrow \varepsilon.$$

Now, let us take $\mathcal{O}_x$. Then we have also a sequence of two τ-stacks $E^{n_t}(j)$,

$$E_i^{n_t}(j) = \{y \in \mathcal{O}_x : y[-i + kn_t, -i + (k+1)n_t - 1] = c_t \text{ or}$$

$$\tilde{c}_t, y[-i, -i + n_t - 1] = \sigma^j c_t\}$$

and the corresponding partitions $E^{n_t} \nearrow \varepsilon$ [14]. Let $z = \bigcap_{t\geqslant 0} E_{i_t}^{n_t}(j_t)$ and $z' = \bigcap_{t\geqslant 0} E_{i_t'}^{n_t}(j_t')$. Since $y \neq y'$, there is t such that $(i_t, j_t) \neq (i_t', j_t')$ and therefore $z \neq z'$. Moreover, any Q-n-name of y (y') is equal to P-n-name of z (z'). But P is a generator of τ, so P-n-names of z and z' are different for some n and therefore for that n, the Q-n-names of y and y' are different.

Step 7. If $x = b^0 \times b^1 \times \ldots$ determined by the θ is a Morse sequence then T_θ and x are isomorphic as dynamical systems.

By Step 4 and 6 we can find a point $y \in Y$ which is generic for T_θ and Q-s-name of y is always a sector of x, $s \geqslant 0$. Next, we select $z \in \mathcal{O}_x$ such that τ-P-s-name of z is equal to Q-s-name of y, $s \geqslant 0$. Hence, there are two generic points y for T_0 and z for x such that Q-∞-name of y = P-∞-name of z. Since Q and P are generators, T_0 and x must be isomorphic.

Step 8, proof of Theorem 1. As we have seen in Step 5 it is possible $x = b^0 \times b^1 \times \ldots, b^t = 01 \ldots 010\ t \geqslant t_1$. If this is the case we see that $T_0^{n_t}(C_0^{n_t}(0)) = C_0^{n_t}(1)$ and therefore we can define a T_0-stack of height $2n_t$, so $\exp(2\pi i/2n_t) \in \mathrm{Sp}(T_0)$, $t \geqslant t_1$. Thus, from Remark 1, T_0 has pure point spectrum and $\mathrm{Sp}(T_0) = G\{n_t' \colon t \geqslant 0\}$ where $n_t' = n_t$ for $t < t_0$, $n_t' = 2n_t$ for $t \geqslant t_0$, where t_0 is the smallest natural number such that λ_{t_0} is odd. It is a consequence of the fact that if $\exp(2\pi i/2n_t) \in \mathrm{Sp}(T_0)$ then $\exp(2\pi i/2n_{t-1}) = \exp(2\pi i\lambda_t/2n_t) \in \mathrm{Sp}(T_0)$.

Let $y = \beta^0 \times \beta^1 \times \ldots$ be a Morse sequence $|\beta^i| = |b^i|$, $i \geqslant 0$ and $\Sigma_{i\geqslant 0}\,\omega(\beta^i) < \infty$ (see Remark 3). Then y has a discrete spectrum and $\mathrm{Sp}(y) = \mathrm{Sp}(T_0)$. Therefore we can "replace" x by some Morse sequence. Now, our proof is complete by Step 1, Step 5 and Step 7.

III. On the factor problem for Z_2-extensions

A motivation to study the factors problem lies in the following:

PROPOSITION 1. *Let T_0 be a continuous, ergodic Z_2-extension and let U: $(Y', \tilde{\mu}')\circlearrowleft$ be a factor of it with partly continuous spectrum. Then there are T': $(X', \mu')\circlearrowleft$ with discrete spectrum and $\theta'\colon X' \to Z_2$ measurable, such that U is isomorphic to $T_{\theta'}'$.*

Proof. We will use Pickel's and Kushnirenko's theorems [13, 22] concerning sequence entropy. If T_0 is a continuous, ergodic Z_2-extension, then $\sup_{A\subseteq N} h_A(T_0) = \log 2$ [22]. Furthermore, $\sup_{A\subseteq N} h_A(U) \leqslant \sup_{A\subseteq N} h_A(T_0)$. But if U does not have discrete spectrum then $\sup_{A\subseteq N} h_A(U) > 0$ [13]. So using Pitskel's result once more we obtain $\sup_{A\subseteq N} h_A(U) = \log 2$. We recall that 2 is then the number of elements in $\varphi^{-1}(x')$ (a.e. x') where φ: $(Y, T_0, \tilde{\mu}) \to (X', T', \mu')$ establishes a homomorphism between T_0 and its maximal factor with discrete spectrum.

Now, let T_0 be a continuous Z_2-extension and η be the partition described by (5). Then η is T_0-invariant and measurable and T_0/η is isomorphic to T. This factor (and all the factors which are determined by subgroups of $\mathrm{Sp}(T)$) has discrete spectrum.

In the sequel we consider proper factors of a given Z_2-extension.

It was observed in [16] that some Morse sequences have the only factors with discrete spectra. This can be generalized as follows.

PROPOSITION 2. *Let p be a fixed prime number and let* $\mathrm{Sp}(T) = G\{n_t : t \geqslant 0\}$ *have the property* $\lambda_t = p^{k_t}$, $k_t \geqslant 1$, $t \geqslant 0$. *Then every ergodic continuous* Z_2*-extension has no factors with partly continuous spectrum.*

Proof. First, let us observe that if U is a factor with partly continuous spectrum (via φ) of T_0, then $\mathrm{Sp}\,(U)$ is an infinite subgroup of $\mathrm{Sp}\,(T)$. Indeed, otherwise U would be a Z_2-extension of some ergodic transformation defined on a finite space. But there is no ergodic transformation with partly continuous spectrum on a finite space. So, we may assume $\mathrm{Sp}\,(U) = \mathrm{Sp}\,(T)$ since no other infinite subgroup of $\mathrm{Sp}\,(T)$ exists.

Now let $\{\bar{D}^{n_t}\}$ be a sequence of U-stacks of heights n_t corresponding to $\mathrm{Sp}\,(U)$. Then $\varphi^{-1}(\bar{D}^{n_t})$ is again a T_0-stack of height n_t, so it is equal to D^{n_t}, $t \geqslant 0$. Therefore σ-algebra generated by $\{\varphi^{-1}(\bar{D}^{n_t})\}_{t\geqslant 0}$ contains a σ-algebra of η-measurable sets, because the latter σ-algebra is generated by $\{D^{n_t}\}_{t\geqslant 0}$. As a conclusion we have $\eta \leqslant \xi$, where ξ is T_0-invariant measurable partition corresponding to the factor U. Hence, either $\xi = \eta$, a contradiction to continuity of U, or $\xi = \varepsilon$, and U is then isomorphic to T_0.

The above investigations might suggest that the case $\lambda_t = p^{k_t}$, $t \geqslant 0$ is the only case where all factors of T_0 have discrete spectra. However, this is not true as the following theorem shows.

THEOREM 2. *If* $x = b^0 \times b^1 \times \ldots$ *is a regular Morse sequence and* $\lambda_i \leqslant r$, $i \geqslant 0$, *then all factors of x have discrete spectrum.*

Before the proof we establish some auxiliary facts.

Let $x = b^0 \times b^1 \times \ldots$ be a continuous Morse sequence. By η_x we mean the measurable partition (τ-invariant) corresponding to the maximal factor with discrete spectrum. Hence

$$A \in \eta_x \text{ iff } A = \{z, \tilde{z}\}, \quad z \in \mathcal{O}_x. \tag{27}$$

Then by Proposition 1, any proper factor of x with partly continuous spectrum is also a Z_2-extension. Thus it is of the form T_θ: $(X \times Z_2, \tilde{\mu})^\circlearrowleft$, where T: $(X, \mu)^\circlearrowleft$ has discrete spectrum with an infinite group of eigenvalues and $\mathrm{Sp}\,(T) \subsetneqq G\{n_t : t \geqslant 0\}$. Consider now the factor of T_θ generated by the partition $Q = (X \times \{0\}, X \times \{1\})$ (i.e. the σ-algebra corresponding to $\bigvee_{-\infty}^{+\infty} (T_\theta^{-i}Q)$. We assert that this factor has partly continuous spectrum. Indeed, otherwise $\bigvee_{-\infty}^{+\infty} T_\theta^{-i}Q \leqslant \eta$ (see (5)) since any factor with discrete spectrum is canonical [19] and η determines the maximal factor of T_θ with discrete spectrum. But, then $X \times \{0\}, X \times \{1\}$ would be η-measurable, i.e.

$\sigma(X \times \{i\}) = X \times \{i\}$, a contradiction. The factor generated by Q is isomorphic to a shift dynamical system (W, τ, ν), where

$$W \subseteq \{0, 1\}^Z, \quad \sigma W = W, \quad \sigma\nu = \nu \quad (\sigma y = \tilde{y}). \tag{28}$$

Moreover if ξ is the partition of W given by

$$B \in \xi \text{ iff } B = \{y, \tilde{y}\}, \quad y \in W \tag{29}$$

then ξ is measurable, τ-invariant and the corresponding factor-automorphism τ/ξ has discrete spectrum. Furthermore, since T_0 is a proper factor of x, so is (W, τ, ν).

Now, suppose that $\psi\colon (\mathcal{O}_x, \tau, \mu_x) \to (W, \tau, \nu)$ establishes a homomorphism. Then

$$\psi^{-1}\xi \leqslant \eta \tag{30}$$

since $\psi^{-1}\xi$ induces a factor of x with discrete spectrum and this factor is canonical.

The condition that (W, τ, ν) has partly continuous spectrum implies ψ cannot identify any pair $(z, \tilde{z}) \in \mathcal{O}_x$, because the set $\{z \in \mathcal{O}_x\colon \psi z = \psi\tilde{z}\}$ is τ-invariant. So

$$\psi z \neq \psi\tilde{z} \quad \text{a.e. } z \in \mathcal{O}_x. \tag{31}$$

This implies

$$\psi\tilde{z} = \widetilde{\psi z} \quad \text{a.e. } z \in \mathcal{O}_x. \tag{32}$$

Indeed, from (29) and (30) it follows that for a.e. pair $\{z, \tilde{z}) \subseteq \mathcal{O}_x$ there is a pair $\{y, \tilde{y}\} \subseteq W$ such that $\{z, \tilde{z}\} \subseteq \psi^{-1}\{y, \tilde{y}\}$. Now, (32) follows directly from (31).

Let us fix $\delta > 0$. Then from the Birkhoff theorem there exists a code φ_δ: $\mathcal{O}_x \to \{0, 1\}^Z$ (i.e. φ_δ is measurable $\varphi_\delta\tau = \tau\varphi_\delta$, $z[-k, k] = z'[-k, k]$ implies $(\varphi_\delta z)[0] = (\varphi_\delta z')[0]$ a.e. $z, z' \in \mathcal{O}_x$, k is the length of the code) such that

$$d(\varphi z, \varphi_\delta z) < \delta \quad \text{a.e. } z \in \mathcal{O}_x, \tag{33}$$

where $d(z, z') = \lim_m d(z[-m, m], z'[-m, m])$ if the limit exists.

In view of (32) and (33) we have

$$\liminf_m d((\varphi_\delta z)[-m, m], (\varphi_\delta\tilde{z})[-m, m]) > 1 - 2\delta. \tag{34}$$

The second kind of argument we use in the proof of Theorem 2 is connected with a property of $\mathcal{O}_x$ which holds for any regular Morse sequence of the form: $x = b^0 \times b^1 \times \ldots, \lambda_i \leqslant r, i \geqslant 0$. Namely

$$\text{There is } \bar{\delta} > 0 \text{ such that for every } y, y' \in \mathcal{O}_x \quad (35)$$
$$y \neq y' \text{ implies } \liminf_m d(y[-m, m], y'[-m, m]) \geqslant \bar{\delta}.$$

This fact is an obvious consequence of Proposition 1 in [15].

Proof of Theorem 2. Let $\delta > 0$ be fixed. Denote the code of c_t and $\tilde{c}_t$ (via φ_δ) by d_t and $\hat{d}_t$, $n_t > 2k - 1$. Of course we cannot assume $\hat{d}_t = \tilde{d}_t$.

From (34) it follows that

$$d(d_t, \hat{d}_t) > 1 - 3\delta \quad \text{for } t \text{ large enough.} \quad (36)$$

Since (W, τ, ν) is a proper factor, ψ is not one-to-one. Let

$$\psi z = \psi z'. \quad (37)$$

Hence from (33)

$$\limsup_m d((\varphi_\delta z)[-m, m], (\varphi_\delta z')[-m, m]) \leqslant d(\varphi_\delta z, \psi z) + d(\varphi_\delta z', \psi z') < 2\delta. \quad (38)$$

Notice that we may assume $\psi z \neq \psi \tau^s z$ a.e. $z \in \mathcal{O}_x$, $s \in Z$, because otherwise a.e. point of W would be periodic, and this is a contradiction since (W, τ, ν) has partly continuous spectrum.

Consequently we have

$$z' \neq \tilde{z}, \tau^s z, \quad s \in Z. \quad (39)$$

Now, fix t satisfying (36) and

$$2k/n_t < \delta. \quad (40)$$

We divide $\varphi_\delta z$ and $\varphi_\delta z'$ into a juxtaposition of d_t, $\hat{d}_t$ and some "holes" of length $2k$ (see Fig. 5).

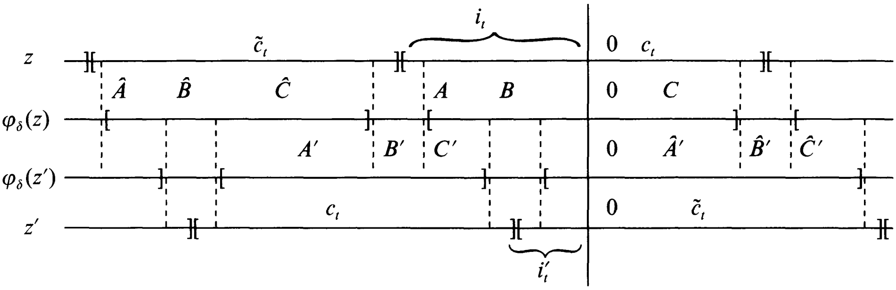

Fig. 5

We obtain a partition of d_t ($\hat{d}_t$) on $\varphi_\delta z$ into three blocks A, B, C ($\hat{A}$, $\hat{B}$, $\hat{C}$) and also d_t ($\hat{d}_t$) on $\varphi_\delta z'$ into three blocks A', B', C' ($\hat{A}'$, $\hat{B}'$, $\hat{C}'$) where $|B| = |B'| = 2k$, $|A| = |\hat{A}| = |C'| = |\hat{C}'|$, $|C| = |\hat{C}| = |A'| = |\hat{A}'|$. Then, from (40) either $|A| \geqslant n_t/4$ or $|C| \geqslant n_t/4$. We will consider the case $|A| \geqslant n_t/4$. Thus, from (36) it follows that

$$d(A, \hat{A}) \geqslant [(1 - 3\delta)|d_t| - \tfrac{3}{4}|d_t|]/\tfrac{1}{4}|d_t| = 1 - 12\delta. \tag{41}$$

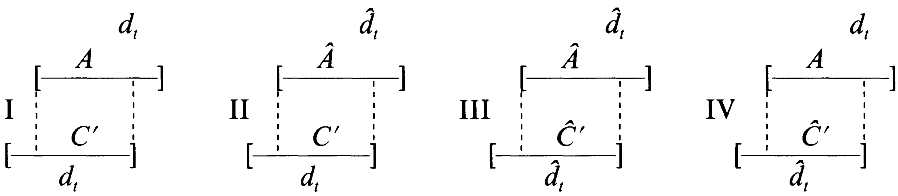

Fig. 6

We have either

$$d(A, C') \geqslant (1 - 12\delta)/2 \quad \text{or} \quad d(\hat{A}, C') \geqslant (1 - 12\delta)/2, \tag{42}$$

since $d(A, C') + d(\hat{A}, C') \geqslant d(A, \hat{A})$ and (41) hold.

Combining (42) with (38) we see that if for instance situation I (Fig. 6) appears on $\varphi_\delta z$ and $\varphi_\delta z'$ then situation II is nearly "excluded". Since the frequency c_t and $\tilde{c}_t$ on z (z') are within δ provided that we consider the places of the form $i_t + sn_t$ ($i'_t + sn_t$), $s \in Z$, we get the following:
if the situation I *appears* (*and situation* II *is nearly "excluded"*) *then situation* III *appears* (*and situation* IV *is nearly "excluded"*).

Let us turn back to z and z' and take into consideration $\tau^{-(i_t - i'_t)}z$ and z'. If situations I and III appear, then it follows that below the c_t's ($\tilde{c}_t$'s) of $\tau^{-(i_t - i'_t)}z$ there are c_t's ($\tilde{c}_t$'s) of z' nearly always

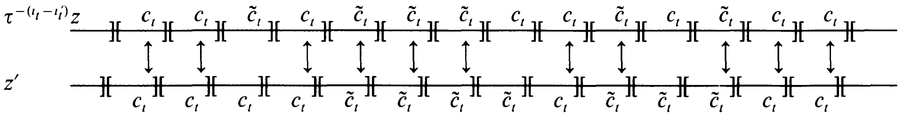

Fig. 7

If situations II and IV appear then it means that below the c_t's ($\tilde{c}_t$'s) of $\tau^{-(i_t - i_t')}z$ there are $\tilde{c}_t$'s (c_t's) of z' nearly "always", i.e. in the first case

$$\liminf_{m} d(\tau^{-(i_t - i_t')}z[-m, m], z'[-m, m]) < 100\delta$$

and in the second

$$\liminf_{m} d(\tau^{-(i_t - i_t')}z[-m, m], \tilde{z}'[-m, m]) < 100\delta.$$

Therefore (31), (35) and (39) give a contradiction for a suitable choice of $\delta > 0$..

COROLLARY 1. *For any continuous substitution on two symbols the only factors are those with discrete spectrum.*

We finish our considerations by giving a class of ergodic Z_2-extensions having some partly continuous factors.

Assume $G\{n_t: t \geqslant 0\}$ has the property that the λ_t's are odd, $t \geqslant 0$ and in addition $\theta^{-1}(1)$ is oddly approximated with speed $o(1/n)$ in such a way that T_θ has partly continuous spectrum.

REMARK. Such a T_θ exists. For instance, if we put

$$b^i = \underbrace{01\ldots01}_{v_i}\underbrace{10\ldots10}_{v_i}1 \qquad \lambda_i = 2v_i + 1,\ i \geqslant 0$$

then $x = b^0 \times b^1 \times \ldots$ is a continuous Morse sequence and admits an odd approximation with a speed depending upon how fast the sequence $\{\lambda_i\}$ tends to infinity.

Let T': (X', μ') be an ergodic dynamical system with discrete spectrum and $\mathrm{Sp}\,(T') = G\{n_t': t \geqslant 0\}$, $n_t' = 3n_t$, $t \geqslant 0$. There is φ: $X' \to X$, $T\varphi = \varphi T'$, $\mu = \mu'\varphi^{-1}$. Now define θ': $X' \to Z_2$ putting

$$\theta' = \theta\varphi. \qquad (43)$$

We assert $\theta'^{-1}(1)$ is oddly approximated with the speed $o(1/n)$. Indeed

$$\mu\left(\theta^{-1}(1)\,\Delta\bigcup_{r=0}^{2k_1} D_{j_r}^{n_t}\right) = \mu'\left(\varphi^{-1}\left(\theta^{-1}(1)\,\Delta\bigcup_{r=0}^{2k_1} D_{j_r}^{n_t}\right)\right)$$

$$= \mu'\left(\theta'^{-1}(1)\,\Delta\bigcup_{r=0}^{2k_1}\varphi^{-1}D_{j_r}^{n_t}\right)$$

But $\varphi^{-1}D^{n_t}$ is a T'-stack of height n_t and $\varphi^{-1}D^{n_t} \leqslant D'^{n'_t}$ because $n_t|n'_t$ and moreover any level $\varphi^{-1}D_{j_r}^{n_t}$ is a union of three levels of $D'^{n'_t}$, so

$$\mu\left(\theta^{-1}(1)\,\Delta\bigcup_{r=0}^{2k_1} D_{j_r}^{n_t}\right) = \mu'\left(\theta'^{-1}\,(1)\,\Delta\bigcup_{s=0}^{2l_1} D_{i_s}^{n'_t}\right).$$

We get $\theta'^{-1}(1)$ is oddly approximated with the required speed. In particular, $T'_{\theta'}$ is ergodic and has partly continuous spectrum. Using (43) it is not difficult to verify that $\varphi \times id$: $X' \times Z_2 \to X \times Z_2$ establishes a homomorphism from T'_θ to T_θ.

Finally, note that if we assume $\theta^{-1}(1)$ is oddly approximated with speed $o(1/n^2)$, then we can obtain a homomorphism between some continuous Morse sequences.

Remark. It would be interesting to characterize all Z_2-extensions having a factor with partly continuous spectrum.

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