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The scheme of morphisms from an elliptic curve to a Grassmannian

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0. Introduction

In this article, a condensed version of [B1], we study the scheme of morphisms of given degree d from an elliptic curve X to a Grassmannian Gr (p, E), and we obtain some results about coherent sheaves over an elliptic curve which may seem interesting for their own sake. The prime motivation resides in the theory of σ-models, which led specialists to consider the 'manifold' of harmonic maps $S^2 \to S^{2n}$. By means of the Calabi lifting, this 'manifold' is interpreted as a subscheme of the scheme of algebraic maps from $\mathbb{P}^1(\mathbb{C})$ to the quadratic Grassmannian QG $(\mathbb{C}^{2n+1})$. Hence a study of algebraic maps from $\mathbb{P}^1(\mathbb{C})$ to a Grassmannian; J.-L. Verdier showed that for a fixed degree they constitute a smooth, connected quasi-projective variety. It is natural to replace S^2 by a torus $S^1 \times S^1$, that is to consider maps from a genus 1-curve to a Grassmannian.

After introducing a few notations in §0 we recall some results about indecomposable sheaves over an elliptic curve X which we shall use later on (§1). Those results were proved by Atiyah for the most part [A], but they are more readily understood with the help of the notion of semi-stability. They are given a relatively short and self-contained proof in Appendix A. In §2, we associate with any matrix $M \in SL_2(\mathbb{Z})$ a so-called Fourier transform functor F_M: $D(X) \to D(X)$, $D(X)$ denoting the derived category of Coh (X). Those functors generalize the 'usual' Fourier transform functor introduced by Mukai [M], and define an action of SL_2 $(\mathbb{Z})$ on $D(X)$ modulo the shift, which is evoked in [M]. The Fourier transform functors F_M help understand the structure of Coh (X), and they allow one to 'juggle with ranks and degrees', an indispensable trick in §4. The proof of our Theorem 3 about Fourier transforms is given in Appendix B. Our §3 is devoted to the study of topological properties of the scheme $M_d(X; p, n)$ of morphisms of degree d from the elliptic curve X to the Grassmannian Gr (p, n) of rank

p-subspaces of a rank n-vector space. By means of a rather simplistic stratification of that scheme, we find that, although it is always connected, it may have many irreducible components if $d < n$. In any case, we can list them and compute their dimensions (Theorem 5, Corollaries 1 and 2). Lastly, in §4 we consider the following problem. Any closed point $f \in M_d(X; p, n)$ defines a short exact sequence:

$$0 \to V_f \to E \to Q_f \to 0,$$

where E is a rank n trivial bundle over X. Given two sheaves V, Q over X, does there exist f such that $V_f \simeq V$ and $Q_f \simeq Q$? Over $\mathbb{P}^1(\mathbb{C})$, Verdier proved that this is always the case under obvious assumptions. Over an elliptic curve, the problem is harder to tackle, so we restrict our attention to the case where V and Q are semi-stable (but we allow E to be any semi-stable sheaf over X). Even in that case, the results we obtain are only partial (Theorem 6, Corollary 3) but they give further insight into the structure of $M_d(X; p, n)$. The gist of Corollary 3 is that almost any reasonable V and Q co-occur, provided that $d > n$, but if $d = n$, there is a one-to-one correspondence between co-occurring V and Q's.

0. Notations

Let k denote an algebraically closed field, X an elliptic curve over k (unless otherwise specified), Coh (X) the category of coherent sheaves over X, $D(X)$ its derived category, Pic (X) the Picard group of X, $\mathrm{Pic}_d(X)$ its degree d-component (for any $d \in \mathbb{Z}$), CH (X) the Chow ring of X.

We consider $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ as a set of vectors $\binom{r}{d}$, where r is to be seen as a rank, and d as a degree. We denote by $\mathscr{H}$ the set of all vectors $\binom{r}{d}$ such that r is the rank of a non-zero coherent sheaf over X, and d its degree; so we have:

$$\binom{r}{d} \in \mathscr{H} \Leftrightarrow r \geqslant 0, \quad \text{and} \quad d > 0 \quad \text{if} \quad r = 0.$$

For any $\binom{r}{d} \in \mathscr{H}$ we shall as a rule denote by h the g.c.d. of r and d, and let $\binom{\varrho}{\delta} = 1/h\binom{r}{d}$.

The reason for this notation is that we can state the basic properties of indecomposable sheaves over X in a rather neat way.

1. Indecomposable sheaves over an elliptic curve

Any coherent sheaf over X is isomorphic to a finite direct sum of indecomposable sheaves, which are uniquely determined up to order [A1]. For any $\binom{r}{d} \in \mathscr{H}$, denote by $I\binom{r}{d}$ the set of classes of indecomposable sheaves of rank r and degree d over X. Indecomposable sheaves are described by the following theorem, due to Atiyah (although it is stated here in a slightly different form).

THEOREM 1. (ATIYAH). *For any* $\binom{r}{d}$ *let* $h = (r, d)$ *be the g.c.d. of* r *and* d, *and let* $\varrho = r/h$, $\delta = d/h$.

There exists a unique way of associating with any $\binom{r}{d} \in \mathscr{H}$ *and any* $v \in \operatorname{Pic} X$ *a class* ${}_vE\binom{r}{d} \in I\binom{r}{d}$ *so that the following hold:*

1. *for any* $d > 0$, ${}_vE\binom{r}{d}$ *has support at* v *(considered as a point of* X*);*
2. *for any* $l \in \mathbb{Z}$ *and any* $\mathrm{L} \in \operatorname{Pic}_l(X)$,

$$ {}_vE\binom{r}{d} \otimes L = {}_{v+\varrho L}E\binom{r}{d+lr}; $$

3. *for any* d, r *such that* $d \geqslant r > 0$, *there is a short exact sequence of sheaves over* X*:*

$$ 0 \longrightarrow {}_vE\binom{r}{d} \longrightarrow \Gamma\{{}_vE\binom{d-r}{d}\}_X \xrightarrow{ev} {}_vE\binom{d-r}{d} \longrightarrow 0, $$

where ev *is the evaluation morphism.*

Furthermore, the mapping from $\operatorname{Pic} X$ *to* $I\binom{r}{d}$ *which sends* v *to* ${}_vE\binom{r}{d}$ *is one-to-one.*

A relatively self-contained proof of this theorem, as well as the one that follows, may be found in Appendix A.

Semi-stability for sheaves over an elliptic curve

For any non-zero coherent sheaf F over X, we define the *slope of F* to be the number $\mu(F) = \deg(F)/rk(F)$ (take the slope of a torsion sheaf to be $+\infty$).

A coherent sheaf F over X is said to be *semi-stable* (resp. *stable*) if for any subsheaf G such that $0 \neq G \neq F$ we have $\mu(G) \leqslant \mu(F)$ (resp. $\mu(G) < \mu(F)$).

The following theorem describes semi-stable sheaves over X, and gives a simpler characterization of the sheaves ${}_vE\binom{r}{d}$.

THEOREM 2. *Let X be an elliptic curve over k.*

1. *All indecomposable sheaves over X are semi-stable.*
2. *Stable sheaves are those indecomposable sheaves with coprime rank and degree. Furthermore, if $\binom{r}{d} \in \mathscr{H}$ and $(r, d) = 1$, there exists a universal sheaf $\boldsymbol{E}\binom{r}{d}$ over $\mathrm{Pic}_d X \times X$ such that $\boldsymbol{E}\binom{r}{d}_{|v \times X} \simeq {}_vE\binom{r}{d}$ for any $v \in \mathrm{Pic}_d X$. In other terms, $\mathrm{Pic}_d X$ is the good moduli space for stable sheaves of coprime rank r and degree d over X.*
3. *Let $\binom{r}{d} \in \mathscr{H}$ and $v \in \mathrm{Pic}\, X$. If $h = (r, d) = 1$, then ${}_vE\binom{r}{d}$ is the only stable sheaf with rank r, degree d, and first Chern class v. In general, ${}_vE\binom{r}{d}$ is the only indecomposable sheaf with rank r, degree d, containing a copy of ${}_vE\binom{\varrho}{\delta}$.*

The existence of a universal sheaf $\boldsymbol{E}\binom{r}{d}$ for $(r, d) = 1$ has been proved by Oda [O]. It follows from the first assertion of theorm 2 that semi-stable sheaves coincide with *isocline* sheaves, i.e. direct sums of indecomposable sheaves all of whose slopes are equal.

2. Fourier transform for sheaves over an elliptic curve

We fix a point A in X. This allows us to identify any component of Pic (X) with X itself, which we consider as an Abelian group with identity element A. Let $D(X)$ be the derived category of Coh (X), and for any $a \in \mathbb{Z}$, denote by $[a]$ the functor "shift a places to the left": $D(X) \to D(X)$.

For any coherent sheaf P over $X \times X$, flat with respect to projections, we denote by S_P the functor

$$S_P = pr_{2*}(P \otimes pr_1^*(?)) : \mathrm{Coh}\,(X) \to \mathrm{Coh}\,(X)$$

(following Mukai). This functor is left-exact, and induces a derived functor RS_p: $D(X) \to D(X)$. [M].

Lastly, we denote by $SL_2(\mathbb{Z})^+$ the set of matrices $M \in SL_2(\mathbb{Z})$ such that $M\binom{0}{1} \in \mathscr{H}$. We have the following theorem.

THEOREM 3.

1. *Let M be a matrix in $SL_2(\mathbb{Z})^+$. There exists a coherent sheaf P_M over $X \times X$, flat with respect to projections, and unique up to isomorphism, such that the following hold: (P) for any $\binom{r}{d} \in \mathscr{H}$ and $v \in X$,*

$$S_{P_M}({}_vE\tbinom{r}{d}) = {}_vEM\tbinom{r}{d} \quad \text{if} \quad M\tbinom{r}{d} \in \mathscr{H} \quad = 0 \text{ otherwise;}$$

and

$$R^1 S_{P_M}({}_vE(\begin{smallmatrix} r \\ d \end{smallmatrix})) = 0 \quad \text{if} \quad M(\begin{smallmatrix} r \\ d \end{smallmatrix}) \in \mathscr{H} \quad = \ {}_{-v}E(\begin{smallmatrix} r \\ d \end{smallmatrix}) \text{ otherwise.}$$

For $i = 0, 1$, *denote by* F^i_M *the functor* $R^i S_{P_M}$.

2. *For any* $M \in SL_2(\mathbb{Z})$ *we define a functor* F_M: $D(X) \to D(X)$, *which we call* "Fourier transform associated with M", *in the following way: if* $M \in SL_2(\mathbb{Z})^+$, *let* $F_M = RF^0_M$; *otherwise let* $F_M = F_{-I} \circ F_{-M}$, *where* $F_{-I} = (-Id_X)^* \circ [-1]$.
   Then for any two $M, M' \in SL_2(\mathbb{Z})$ *there is an isomorphism of functors:*

$$F_M \circ F_{M'} \simeq [a] \circ F_{MM'},$$

where $a = 0$ *or* -2; *furthermore,* $a = 0$ *if* $M, M' \in SL_2(\mathbb{Z})^+$.

The Fourier transform functors F_M generalize the "usual" Fourier transform introduced by Mukai, which is obtained for $M = (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$. The second assertion of Theorem 3 gives a concrete interpretation of the action of $SL_2(\mathbb{Z})$ on $D(X)$ modulo the shift, which is evoked in [M]. A proof of Theorem 3 may be found in Appendix B.

Now we introduce a few notations. Let $\mu \in Q$, and let α, β, ϱ, δ be the integers determined by the following conditions: $\varrho > 0$, $\mu = \delta/\varrho$, $\alpha\delta - \beta\varrho = 1$. Denote by M_μ the matrix $(\begin{smallmatrix} \alpha & \varrho \\ \beta & \delta \end{smallmatrix})$, and let $M_\infty = I$. We define the functors $F^i_\mu = F^i_M$ and $F_\mu = F_{M_\mu}$.

The interesting fact is that for any $\mu \in Q \cup \{\infty\}$ the functor F^0_μ induces an equivalence of categories between torsion sheaves over X and semi-stable sheaves of slope μ over X. We give an example of application of this.

Unidecomposable sheaves over an elliptic curve

We say that a coherent sheaf F over X is *unidecomposable* if its decomposition as a direct sum of indecomposable subsheaves is unique up to the order of the terms.

THEOREM 4. *Let* $(\begin{smallmatrix} r \\ d \end{smallmatrix}) \in \mathscr{H}$, *and let* F *be a coherent sheaf of rank* r *and degree* d *over the elliptic curve* X. *Let* $h = (r, d)$ *and* $\mu = d/r$.

1. *The sheaf* F *is unidecomposable if and only if it is semi-stable, and* dim End $(F) = h$.
2. *Unidecomposability is an open condition, and* S^h Pic X *is a good moduli space for unidecomposable sheaves of rank* r *and degree* d *over* X.

Proof of the theorem. Let $F = E_1 + E_2 + \cdots + E_p$, each E_i being indecomposable, and assume that F is unidecomposable. Then for any $i \neq j$ we have hom $(E_i, E_j) = \dim \operatorname{Hom}(E_i, E_j) = 0$, and by Rieman-Roch and semi-stability of the E_i's this implies: $\mu(E_1) = \mu(E_2) = \cdots = \mu(E_p) = \mu(F)$, so F is semi-stable. Now let μ be its slope. Using the functor F^0_μ, we are reduced to studying torsion unidecomposable sheaves; so we assume: $r = 0$. For any $v, w \in X$ we have: hom $({}_vE(\begin{smallmatrix}0\\m\end{smallmatrix}), {}_wE(\begin{smallmatrix}0\\n\end{smallmatrix})) = \inf(m, n)$ if $v = w$, $= 0$ otherwise. So F is of the form:

$$F \simeq {}_{v_1}E\begin{pmatrix}0\\m_1\end{pmatrix} \oplus \ldots \oplus {}_{v_p}E\begin{pmatrix}0\\m_p\end{pmatrix},$$

with $m_1 + m_2 + \cdots + m_p = d = h$, and $v_1, \ldots, v_p$ are distinct because F is unidecomposable. This is equivalent to saying that end $(F) = h$, or that F is a torsion quotient of O_X of degree $d = h$. So S^hX is a good moduli space for unidecomposable torsion sheaves of degree h over X, the universal sheaf being: $\boldsymbol{U}(\begin{smallmatrix}0\\h\end{smallmatrix}) = O_H$, where $H \subset S^hX \times X$ is the incidence hypersurface.

Now the theorem follows immediately. To obtain the universal sheaf $\boldsymbol{U}(\begin{smallmatrix}r\\d\end{smallmatrix})$ it is enough to identify $\operatorname{Pic}_\delta X$ and X, and to apply the relative version of the functor F^0_μ to the sheaf $\boldsymbol{U}(\begin{smallmatrix}0\\h\end{smallmatrix})$.

For any $v \in S^h\operatorname{Pic}_\delta X$ made up of distinct points $v_1, \ldots, v_p$ taken with multiplicities $m_1, \ldots, m_p$, we have:

$$\boldsymbol{U}\begin{pmatrix}r\\d\end{pmatrix}_{v,.} \simeq \bigoplus_1^p {}_{v_i}E\,\frac{m_i}{h}\begin{pmatrix}r\\d\end{pmatrix}.$$

REMARK. The second assertion of Theorem 2 shows that $S^h\operatorname{Pic}_\delta X$ is the coarse moduli space for semi-stable sheaves of rank r and degree d on X; so, an open subset of this coarse moduli space bears a universal sheaf. This fails on a curve of higher genus [S].

3. The scheme of maps from an elliptic curve to a Grassmannian

Let E be a k-vector space with finite dimension n, and let p be an integer such that $0 < p < n$. The Grassmannian $G = \operatorname{Gr}(p, E)$ of p-dimensional subspaces of E is the smooth projective variety, and we have the well-known Plücker imbedding:

$$\text{pluc}: G \hookrightarrow \mathbb{P}(\Lambda^p E).$$

On the other hand, there is a universal short exact sequence of vector bundles over G:

$$0 \to V \to E_G \to Q \to 0,$$

where E_G denotes the trivial bundle with fiber E, V, the universal rank p subbundle, and Q, the universal rank $n - p$ quotient bundle; the tangent bundle of G is canonically isomorphic to the bundle ***Hom*** (V, Q).

Let X be an elliptic curve over k, and d an integer. We are interested in the scheme of morphisms of degree d from X to G, that is a scheme M, together with an evaluation morphism ev: $M \times X \to G$, of degree d with respect to X, such that for any scheme N and any morphism φ: $N \times X \to G$ of degree d with respect to X, there is a unique morphism Φ: $N \to M$ making the following diagram commutative:

$$\begin{array}{ccc} N \times N & \xrightarrow{\ \varphi\ } & G \\ {\scriptstyle \Phi \times Id_X}\downarrow & & \| \\ M \times X & \xrightarrow{\ \text{ev}\ } & G \end{array} .$$

Such a scheme exists. We denote it by $M_d(X; p, E)$ and let $M_d(X; p, n) = M_d(X; p, k^n)$. The scheme $M_d(X; p, E)$ is quasi-projective and any closed point $f \in M_d(X; p, E)$ identifies with a morphism f: $X \to \mathrm{Gr}\,(p, E)$ of degree d. Let $V_f = f^*V$ and $Q_f = f^*Q$, so that we have a short exact sequence of vector bundles over X:

$$0 \to V_f \to E_X \to Q_f \to 0.$$

The tangent space of $M_d(X; p, E)$ at f is canonically isomorphic to $H^0(X; f^*T\,\mathrm{Gr}\,(p, E)) \simeq \mathrm{Hom}\,(V_f, Q_f)$, and if $H^1(X; f^*T\,\mathrm{Gr}\,(p, E)) \simeq \mathrm{Ext}^1(V_f, Q_f)$ is trivial, then f is a smooth point in $M_d(X; p, E)$. (For the existence and the differential properties of schemes of morphisms, see [G].)

To a closed point f we associate two integers $\alpha(f) = h^0(V_f)$ and $\beta(f) = h^0(Q_f^*)$. As functions of f, $\alpha(f)$ and $\beta(f)$ are uppersemicontinuous by the semicontinuity theorem [H]. Therefore, we can define the locally closed, reduced subschemes $M_d^{a,b}(X; p, E) \subset M_d(X; p, E)$ whose closed points f are those morphisms such that $\alpha(f) = a$ and $\beta(f) = b$, a and b being fixed integers.

With those notations, we have the following theorem.

THEOREM 5. *Let X be an elliptic curve over k, E a vector space of finite dimension n over k, p an integer such that* $0 < p < n$, *and* $d \in \mathbb{N}^*$. *Then:*

1. *$M_d^{a,b}(X; p, E)$ is non-empty if and only if $0 \leqslant a < p, 0 \leqslant b < n - p$ and $a + b \geqslant n - d$;*
2. *non-empty $M_d^{a,b}(X; p, E)$'s form a stratification of $M_d(X; p, E)$, the ordering on indices (a, b) being induced by the usual partial ordering of $\mathbb{N}^2$;*
3. *for any $f \in M_d(X; p, E)$, $T_f M_d(X; p, E)$ has dimension $nd + \alpha(f) \cdot \beta(f)$.*

The second assertion of the theorem means that non-empty $M_d^{a,b}(X; p, E)$'s are smooth, connected, locally closed subschemes of $M_d(X; p, E)$; furthermore, they are disjoint and there union is the whole of $M_d(X; p, E)$; lastly, if $M_d^{a,b}(X; p, E)$ is non-empty, its Zarisky closure is the union of all $M_d^{r,s}(X; p, E)$'s with $r \geqslant a$ and $s \geqslant b$.

Before proceeding to prove this theorem, we give a description of the strata $M_d^{a,b}(X; p, E)$. For $(a, b) \in \mathbb{N}^2$ we denote by $D^{a,b}(E)$ the space $Dr(a, n - b, E)$ of linear flags $(O \subset A \subset B' \subset E)$ with $\dim A = a$ and $\dim B' = n - b$. Then we have

PROPOSITION 1. *For any $(a, b) \in \mathbb{N}^2$ there is a natural, locally trivial fibration map*

$$M_d^{a,b}(X; p, E) \to D^{a,b}(E)$$

with smooth fiber isomorphic to $M_d^{0,0}(X; p - a, n - a - b)$.

Proof of the proposition. Consider the universal vector bundle $\boldsymbol{E}$ over $D^{a,b}(E)$ whose fiber at $D = (O \subset A \subset B' \subset E)$ is isomorphic to B'/A. By glueing, one constructs a locally trivial bundle $\boldsymbol{G}$ over $D^{a,b}(E)$ whose fiber at D is $\mathrm{Gr}\,(p - a, B'/A)$ and a locally trivial bundle $\boldsymbol{M}$ over $D^{a,b}(E)$ whose fiber at D is $M_d(X; p - a, B'/A)$. Notice that $\boldsymbol{M}$ is the scheme of morphisms of degree d from X to fibers of $\boldsymbol{G} \to D^{a,b}(E)$.

We denote by $\boldsymbol{M}^{0,0}$ the open subscheme of $\boldsymbol{M}$ whose fiber at any $D = (O \subset A \subset B' \subset E)$ is $M_d^{0,0}(X; p - a, B'/A)$, so that $\boldsymbol{M}^{0,0}$ is a locally trivial bundle over $D^{a,b}(E)$ with fiber isomorphic to $M_d^{0,0}(X; p - a, n - a - b)$. We shall prove that $\boldsymbol{M}^{0,0}$ is isomorphic to $M_d^{a,b}(X; p, E)$, but we show the smoothness of the fiber first.

LEMMA 1. *For any n, d, p, $M_d^{0,0}(X; p, n)$ is smooth of dimension nd, or empty.*

Proof of the lemma. By upper semicontinuity of α and β, $M_d^{0,0}(X; p, n)$ is an open subscheme of $M_d(X; p, n)$, so we only have to prove that whenever $\alpha(f) = \beta(f) = 0$, $\mathrm{ext}^1(V_f, Q_f) = 0$. By Riemann-Roch, it will follow that $\hom\,(V_f, Q_f) = \deg \boldsymbol{Hom}\,(V_f, Q_f) = nd$, hence the lemma. Now, the short exact sequence $0 \to V_f \to k_X^n \to Q_f \to 0$ induces an epimorphism $\mathrm{Ext}^1(V_f, k_X^n) \to \mathrm{Ext}^1(V_f, Q_f)$. Since $\mathrm{ext}^1(V_f, k_X^n) = n \cdot h^1(V_f^*) = n \cdot h^0(V_f)$ (by Serre duality) $= n\alpha(f) = 0$, it follows that $\mathrm{ext}^1(V_f, Q_f) = 0$.

To show that $M_d^{a,b}(X; p, E)$ and $\boldsymbol{M}^{0,0}$ are isomorphic, we associate with any f in $M_d^{a,b}(X; p, E)$ a point g in $\boldsymbol{M}$ which we define in the following way. Consider the flag $D_f = (O \subset A_f \subset B_f' \subset E)$ where $A_f = H^0(V_f)$, and $B_f' = \mathrm{Vect}\,(V_f)$ is the linear subspace of E spanned by the fibers of V_f. Since B_f' is the orthogonal of $H^0(Q_f^*)$ in E^*, this flag is in $D^{a,b}(E)$. The short exact sequence

$$0 \to V_f/A_f \to B_f'/A_f \to B_f'/V_f \to 0$$

defines a morphism g: $X \to \mathrm{Gr}\,(p - a, B_f'/A_f)$ of degree d. We have $\beta(g) = 0$, because V_f/A_f spans B_f'/A_f, and $\alpha(g) = 0$ by duality. So g is a point in the fiber of $\boldsymbol{M}^{0,0}$ above D_f.

Now clearly the map sending f to g is one-to-one between closed points of $M_d^{a,b}(X; p, E)$ and $\boldsymbol{M}^{0,0}$; by the universal properties of $\boldsymbol{M}$, it is induced by a canonical morphism; and since both $M_d^{a,b}(X; p, E)$ and $\boldsymbol{M}^{0,0}$ are reduced, the latter being smooth, this morphism is an isomorphism, hence the proposition.

Proof of Theorem 5

Third assertion. By Riemann-Roch we have $\mathrm{hom}\,(V_f, Q_f) = nd + \mathrm{ext}^1(V_f, Q_f)$ so it is enough to show: $\mathrm{ext}^1(V_f, Q_f) = \alpha(f)\beta(f)$. This was proved already when $\alpha(f) = 0$ (see the proof of Lemma 1); in general, we have a short exact sequence:

$$0 \to A_f \to V_f \to V_f/A_f \to 0$$

where $A_f = H^0(V_f)$ and $H^0(V_f/A_f) = 0$, as we noticed in the course of the proof of Proposition 1. Hence an exact sequence:

$$\mathrm{Ext}^1(V_f/A_f, Q_f) \to \mathrm{Ext}^1(V_f, Q_f) \to \mathrm{Ext}^1(A_f, Q_f) \to 0.$$

The short exact sequence $O \to V_f/A_f \to E/A_f \to Q_f \to O$ arises from a morphism $g \in M_d(X; p - \alpha(f), E/A_f)$ with $\alpha(g) = 0$, so $\mathrm{ext}^1(V_f/A_f, Q_f) = 0$, and $\mathrm{ext}^1(V_f, Q_f) = \mathrm{ext}^1(A_f, Q_f) = \alpha(f)\, h^1(Q_f) = \alpha(f)\, h^0(Q_f^*) = \alpha(f)\beta(f)$.

First assertion. Assume that $M_d^{a,b}(X; p, E) \neq 0$. Since $d > 0$, any morphism $X \to \mathrm{Gr}\,(p, E)$ of degree d is non-constant, hence $0 \leqslant a < p$ and

$0 \leqslant b < n - p$. Now it follows from Proposition 1 that $M_d^{0,0}(X; p - a, n - a - b)$ is non-empty. Let $g \in M_d^{0,0}(X; p - a, n - a - b)$. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_g & \longrightarrow & k_X^{n-a-b} & \longrightarrow & Q_g & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle\varphi} & & \Big\| & & \\ & & & & \Gamma(Q_g)_X & \xrightarrow{\text{ev}} & Q_g & \longrightarrow & 0 \end{array}$$

where φ is injective because $h^0(V_g) = \alpha(g) = 0$; hence $n - a - b \leqslant d + h^0(Q_g^*) = d$, because $\beta(g) = 0$. So the conditions in 5.1 are necessary. To show that they are sufficient, we use the following lemma.

LEMMA 2. *Let $f \in M_d^{a,b}(X; p, E)$ and assume: $b > 0$, $a + b > n - d$. Then f lies in the Zarisky closure of $M_d^{a,b-1}(X; p, E)$.*

Proof of the lemma. Assume first that $b = 1$. Let $B_f' = \text{Vect } V_f$, and choose a line $B \subset E$ such that $E = B \oplus B_f'$. There is a morphism

$$\text{Hom}\,(V_f, B) \times X \to \text{Gr}\,(p, E)$$

$$(\varphi, x) \mapsto \{y + \varphi(y);\ y \in (V_f)_x\}$$

hence a morphism Φ: Hom $(V_f, B) \to M_d(X; p, E)$. If $\varphi \in$ Hom (V_f, B) then $V_{\Phi(\varphi)} \simeq V_f$ so $\alpha\,(\Phi(\varphi)) = a$. But $\beta(\Phi(\varphi)) = b = 1$ if and only if φ belongs to the image of the morphism

$$\text{Hom}\,(B_f', B) \hookrightarrow \text{Hom}\,(V_f, B);$$

otherwise $\beta(\Phi(\varphi)) = 0$. Now hom (V_f, B) $-$ hom $(B_f', B) = \alpha(f) + d - n + 1 > 0$; so $f = \Phi(0)$ lies in the closure of $M_d^{a,b-1}(X; p, E)$.

In the general case, we reduce trivially to $b = 1$ by choosing a linear space C such that $B_f' \subset C \subset E$ and dim $C = n - b + 1$, and by considering the natural inclusion $M_d(X; p, C) \hookrightarrow M_d(X; p, E)$.

By this lemma, which we can also use in its dual form (interchanging the roles of a and b) it is enough to prove that $M_d^{p-1,n-p-1}(X; p, E)$ is no-empty as soon as $d \geqslant 2$. Now by Proposition 1 this is the same as showing that $M_d(X; 1, 2)$ is non-empty, a well-known result: there are morphisms of degree d: $X \to \mathbb{P}^1$ as soon as $d \geqslant 2$.

Second assertion. We know already that the strata are smooth, and that the Zarisky closure of any stratum is as announced (by Lemma 2). There only remains to prove the following proposition.

PROPOSITION 2. *The scheme $M_d^{a,b}(X; p, E)$ is connected.*

Proof. Thanks to Proposition 1, we can assume $a = b = 0$. Denote by $Q\binom{n-p}{d}(E)$ the scheme of quotients of rank $n - p$ and degree d of E_X, so that $M_d(X; p, E)$ appears as an open subscheme of $Q\binom{n-p}{d}(E)$. Now let U be the open subset of $Q\binom{n-p}{d}(E)$ of those f such that $h^0(V_f) = 0$. That condition implies that $\mathrm{ext}^1(V_f, Q_f) = 0$, so U is smooth. Since U contains $M_d^{0,0}(X; p, E)$ it is enough to show that it is connected. The connectedness of U has been proved by Hernandez [H] in a more general setting. We give a sketch of his proof adapted to our special case.

Let N be a $n - p$ dimensional subspace of E, and $E' = E/N$. Any $f \in U$ defines a map $V_f \to E'$. Denote by U_N the dense open subscheme of U of those f such that $V_f \to E'$ is injective; there is a natural morphism:

$$\Phi\colon U_N \to Q\binom{0}{d}(E').$$

It is well-known that $Q\binom{0}{d}(E')$ is smooth, connected, so it is enough to show that Φ is dominant, with connected fibers. Consider a point h in $Q\binom{0}{d}(E')$, defined by the short exact sequence

$$0 \longrightarrow V \xrightarrow{\ i\ } E' \longrightarrow Q \longrightarrow 0.$$

Now h can be in the image of Φ only on the open condition that $h^0(V) = 0$. Assuming that this holds, there is a surjective map

$$\sigma\colon \mathrm{Hom}(V, E) \to \mathrm{Hom}(V, E')$$

because $\mathrm{ext}^1(V, N) = \hom(N, V) = 0$. So $\sigma^{-1}(i)$ is a non-empty affine space; any $j \in \sigma^{-1}(i)$ is an injection $V \hookrightarrow E$, hence a morphism $\sigma^{-1}(i) \to U_N$ whose image is the fiber of Φ at h; so the fiber is non-empty, connected, hence the proposition.

Corollaries of Theorem 5

Let E be a k-vector space of finite dimension $n \geqslant 2$, and let p, d be integers such that $0 < p < n$ and $d \geqslant 2$.

COROLLARY 1. (smooth points). *The smooth points of $M_d(X; p, E)$ are the points f such that $\alpha(f)\beta(f)\,(n - d - \alpha(f) - \beta(f)) = 0$. In particular, $M_d(X; p, E)$ is smooth if and only if $p = 1$, or $p = n - 1$*, or $d = 2$.

COROLLARY 2 (Topology, dimensions)
1. *If $d \geqslant n$, $M_d(X; p, E)$ is irreducible of dimension nd.*
2. *If $d < n$, the irreducible components of $M_d(X; p, E)$ are the Zarisky closures $C^{a,b}$ of the strata $M_d^{a,b}(X; p, E)$ for all (a, b)'s such that* $0 \leqslant a < p$, $0 \leqslant b < n - p$, and $a + b = n - d$.
   The dimension of the component $C^{a,b}$ is $nd + ab$.
   In particular, $M_d(X; p, E)$ is irreducible if and only if $p = 1$, or $p = n - 1$, or $d = 2$ (ie, when it is smooth).
3. *In any case, $M_d(X; p, E)$ is non-empty and connected.*

The corollaries are immediate, except maybe the part concerned with dimensions; but, thanks to Proposition 1, it is easy to compute the dimension of a stratum; namely

$$\begin{aligned}\dim M_d^{a,b}(X; p, E) &= \dim M_d^{0,0}(X; p - a, k^{n-a-b}) + \dim D^{a,b}\\ &= (n - a - b)(d + a + b) + ab.\end{aligned}$$

4. Short exact sequences of semi-stable sheaves over an elliptic curve

We consider the following problem: given three non-zero semi-stable sheaves E', E, E'' over the elliptic curve X, such that $\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{Ch}(E'')$ and $\mu(E') < \mu(E'')$, does there exist a short exact sequence:

$$0 \to E' \to E \to E'' \to 0 \quad ? \qquad (\xi)$$

We shall see that the non-unidecomposability of the sheaves involved entails certain restrictions on the existence of such an exact sequence. For simplicity, we shall assume that E' and E'' are unidecomposable. Of course, we are especially interested in the case where E is a free sheaf. In this case, the restrictions are in a sense maximal.

Let E be a semi-stable sheaf of slope μ over X. We define the *ambiguousness* of E, which we denote by amb (E), to be the integer

$$\mathrm{amb}(E) = \sup \hom(V, E)$$

for V ranging over the class of stable sheaves of slope μ (with the convention amb $(0) = 0$). Clearly amb $(E) = 1$ if and only if E is non-zero and unidecomposable; and amb (E) is maximal (for given rank and degree) when E is of the form $k^h \otimes V$, with V stable and $h \in \mathbb{N}$. So amb (E) measures the non-unidecomposability of E. We have the following theorem.

THEOREM 6. *Let* $\binom{r'}{d'}, \binom{r''}{d''} \in \mathscr{H}$ *such that* $\mu' = d'/r' < \mu'' = d''/r''$ *and let* $\binom{r}{d} = \binom{r'}{d'} + \binom{r''}{d''}$. *Let E be a semi-stable sheaf over X of rank r and degree d, and denote by E', E'' any two unidecomposable sheaves of respective ranks and degrees r', d', r'', d'' such that* ch (E) = ch (E') + ch (E''). *Then*

1. *There exists a short exact sequence* (ξ) *for a suitable choice of* E' and E'' *if and only if*

   (C_1) amb $(E) \leqslant 1/h \det \begin{pmatrix} r & r'' \\ d & d'' \end{pmatrix}$, *where* $h = (r, d)$.
2. *If* (C_1) *holds, then for any choice of* E' *one can find* E'' *such that there exists an exact sequence* (ξ) *(and conversely).*
3. *Consider the condition*

   (C_3) *E is of the form* $k^n \otimes U$, *where U is unidecomposable and* $n =$ amb $(E) = 1/h \det \begin{pmatrix} r & r'' \\ d & r'' \end{pmatrix}$.

   If (C_3) *holds, there is a one-to-one correspondence between E' and E'' such that there exists an exact sequence* (ξ).
4. *If* (C_1) *holds, but* (C_3) *doesn't, then for a general choice of* E' and E'' *there exists an exact sequence* (ξ).

   Furthermore, such an exact sequence exists for any choice of E' and E'', *provided that the following stronger condition holds*:

   (C_4) amb $(E) \leqslant \varepsilon(h', h'') \cdot 1/h \det \begin{pmatrix} r & r'' \\ d & d'' \end{pmatrix}$
   where $h' = (r', d')$, $h'' = (r'', d'')$, *and*

$$\varepsilon(h', h'') = \begin{cases} 1 \text{ \textit{if} } h' = 1 \text{ \textit{or} } h'' = 1 \\ 1 - 1/\sup(h', h'') \quad \textit{otherwise.} \end{cases}$$

REMARK. The formulation of this theorem is rather complicated, because it contains a large number of assertions; however, we can infer from it the following simple facts.

1. If an exact sequence of semi-stable sheaves

$$0 \to E' \to E \to E'' \to 0 \qquad (\xi)$$

exists, with $\mu(E') < \mu(E)$ then in fact we have

$$\mu(E') \leqslant \mu(E) - \frac{h}{rr'} \text{amb}(E).$$

(This is (C_1)). Conversely, if this inequality holds, as well as the obvious compatibility relations between the ranks and first Chern classes of our sheaves, then an exact sequence (ξ) can be found, at least if one allows E' (or E'') to move inside its moduli space.

2. When amb (E) is minimal, that is when E is unidecomposable, we get the following result: for any unidecomposable sheaves, E, E', E'' over X such that $\mu(E') < \mu(E)$, there exists an exact sequence (ξ), provided that the ranks and first Chern classes are compatible.
3. In the other extreme case, that is when the ambiguousness of E is maximal, the upper bound for $\mu(E')$ is lowest and there are additional obstructions to the existence of an exact sequence (ξ). This happens when E is the direct sum of a number of copies of the same stable sheaf.

In the special case where E is free, we obtain the following corollary.

COROLLARY 3. *Let X be an elliptic curve over k, E a k-vector space of dimension n, p an integer such that $0 < p < n$, and $d \geqslant 2$.*

Denote by V and Q any two unidecomposable sheaves over X of respective ranks and degrees p, $-d$, $n-p$, d, such that $c_1(V) + c_1(Q) = 0$, and let $M(V, Q)$ be the set of morphisms $f \in M_d(X; p, E)$ such that $V_f \simeq V$ and $Q_f \simeq Q$. Then:

1. *If $d < n$, $M(V, Q)$ is always empty.*
2. *If $d = n$, then $M(V, Q)$ is non-empty if and only V is the kernel of the evaluation morphism of Q.*
3. *If $d > n$, then $M(V, Q)$ is non-empty for a general choice of V and Q. For any V there exists Q such that $M(V, Q) \neq \varnothing$ (and conversely).*
4. *Furthermore, $M(V, Q)$ is always non-empty, provided that $d \geqslant c(h_1, h_2)n$, where $h_1 = (p, d)$, $h_2 = (n-p, d)$ and*

$$c(h_1, h_2) = \begin{cases} 1 \text{ if } h_1 = 1 \text{ or } h_2 = 1 \\ 1 + 1/\sup(h_1, h_2) - 1 \quad \text{otherwise.} \end{cases}$$

Proof of Theorem 6 Assertions 1 and 2. Assume first that there exists a short exact sequence

$$0 \to E' \to E \to E'' \to 0. \qquad (\xi)$$

Let $\mu = \mu(E)$ and consider the matrix $M = -M_\mu^{-1} \in SL_2(\mathbb{Z})^+$. We have $M\binom{r}{d} = \binom{0}{-h}$ so, applying the functor F_M to (ξ), we obtain the exact sequence:

$$0 \to F_M^0 E'' \to F_M^1 E' \to F_M^1 E \to 0. \qquad (M\xi)$$

Now, $F_M^1 E$ is a rank-zero sheaf, and we have amb (E) = amb $(F_M^1(E))$ = $\sup_{v \in X}$ dim $F_M^1 E \otimes k(v)$. So $(M\xi)$ implies: amb $(E) \leqslant rk\ F_M^1 E' = 1/h$ $(r'd - rd') = 1/h \det \begin{pmatrix} r' & r \\ d' & d \end{pmatrix} = 1/h \det \begin{pmatrix} r & r'' \\ d & d'' \end{pmatrix}$, hence (C_1) holds.

Conversely, assume that (C_1) holds, and fix E''. Does there exist E' such that there is a short exact sequence (ξ)? Applying the Fourier transform functor $(F_{\mu''})^{-1}$, we can assume that $rk(E'') = 0$.

Let S Hom (E, E'') denote the open subset of Hom (E, E'') of epimorphisms $s: E \to E''$. It is non-empty, provided that amb $(E'') \leqslant rk(E)$, which is the case, since E'' is unidecomposable; and it parametrizes a flat family of subsheaves N_s = Ker (s) of E. Denote by $\mathscr{U}$ the open Shatz stratum of S Hom (E, E''), that is the set of all $s \in S$ Hom (E, E'') for which the Harder-Narasimhan polygon of N_s takes its generic value. Let $\mu_0 = \delta_0/\varrho_0$ denote the maximal slope of any stable subsheaf of N_s for $s \in \mathscr{U}$, and consider the scheme Σ of all stable subsheaves of E of slope μ_0 (ie of rank ϱ_0 and degree δ_0); with any $\sigma \in \Sigma$ is associated a subsheaf $S_\sigma \subset E$. Denote by $Z \subset \Sigma \times \mathscr{U}$ the incidence variety, so that we have a diagram

$$\begin{array}{ccc} Z & \xrightarrow{\quad q \quad} & \mathscr{U} \\ {\scriptstyle p}\big\downarrow & & \\ \Sigma & & \end{array}$$

where p and q are the restrictions of the projections; notice that q is an epimorphism, and that the fiber $p^{-1}(\sigma) \simeq qp^{-1}(\sigma)$ is isomorphic to an open subset of S Hom $(E/S_\sigma, E'')$.

Assume $\mu_0 = \mu$; then Σ has dimension amb $(E) - 1$ and the fibers of p have constant dimension hom $(E/S_\sigma, E'') = d''(r - r/h)$, so we have: rd'' = dim $\mathscr{U} \leqslant$ dim $Z \leqslant$ amb $(E) - 1 + d''(r - r/h)$. Finally, amb $(E) > d''r/h$, which contradicts (C_1).

So we have $\mu_0 < \mu$. For any $\sigma \in \Sigma$ we have ext^1 $(S_\sigma, E/S_\sigma) = 0$, so Σ is smooth of dimension $d\varrho_0 - r\delta_0$. Denote by β the dimension of the general fiber of q, so that we have: dim $Z = rd'' + \beta$. Now for any $\sigma \in \Sigma$ let $\lambda(\sigma)$ be the degree of the torsion part of E/S_σ; it is upper-semicontinuous as a function of σ. For any $\lambda \in \mathbb{N}$, let Σ^λ be the locally closed subvariety of Σ of those σ such that $\lambda(\sigma) = \lambda$. There exists $\lambda_0 \in \mathbb{N}$ such that dim $p^{-1}(\Sigma^{\lambda_0})$ = dim Z. Now assume that $S_\sigma \subset N_S$ for some $s \in \mathscr{U}$. Then N_s/S_σ is locally free, and the torsion part of E/S_σ injects in E''. Since E'' is torsion unidecomposable, there are finitely many subsheaves of E'' of degree λ_0, and all of them are unidecomposable. If Q is one of them, denote by Σ^Q the subvariety of Σ of σ's such that the torsion part of E/S_σ is isomorphic to Q. There exists

Q such that $\dim p^{-1}(\Sigma^{\varrho}) = \dim Z$, and for any $\sigma \in \Sigma^{\varrho}$, $p^{-1}(\sigma)$ has dimension $\hom(E/S_\sigma, E'') = d''(r - \varrho_0) + \lambda_0$ if it is non-empty. So we obtain:

$$\dim \Sigma^{\varrho} + d''(r - \varrho_0) + \lambda_0 \geqslant rd'' + \beta.$$

Now it is an easy computation to show that if Σ^{ϱ} is non-empty, its codimension in Σ^{λ_0} is λ_0. Therefore we have:

$$\dim \Sigma^{\lambda_0} \geqslant \beta + d''\varrho_0,$$

that is $d\varrho_0 - r\delta_0 - d''\varrho_0 \geqslant \beta + \operatorname{codim}_\Sigma \Sigma^{\lambda_0} \geqslant 0$. That implies $\mu_0 \leqslant (d - d'')/r = \mu(N_s)$ so N_s is semi-stable for any $s \in \mathscr{U}$; equalities hold everywhere, so $\beta = 0$, which means that N_s is generically unidecomposable, and so there exist E' and a short exact sequence (ξ).

If we fix E' instead of E'', a suitable Fourier transform functor brings us to the case we have just proved. Hence assertions 1 and 2.

Assertion 3. Assume that (C_3) holds, and fix E''. There exist E' and a short exact sequence:

$$0 \to E' \to k^n \otimes U \to E'' \to 0. \qquad (\xi)$$

We have $\operatorname{Hom}(k^n \otimes U, E') = 0$, hence an injection

$$\operatorname{End}(k^n \otimes U) \xrightarrow{i} \operatorname{Hom}(k^n \otimes U, E'').$$

Now $\operatorname{end}(k^n \otimes U) = n^2 \operatorname{end}(U) = \hom(k^n \otimes U, E'')$ so i is an isomorphism, and E' is uniquely determined by (ξ). A dual argument shows that the correspondance between E'' and E' is one-to-one (one can even show that it is induced by a Fourier transform [B1]). Hence assertion 3.

Assertion 4. Denote by $Q\binom{r''}{d''}(E)$ the scheme of quotients of rank r'' and degree d'' of E; for any $f \in Q\binom{r''}{d''}(E)$ there is a short exact sequence

$$0 \to V_f \to E \to Q_f \to 0;$$

let $\mathscr{U}$ be the open subscheme of $Q\binom{r''}{d''}(E)$ whose closed points are those f such that V_f and Q_f are unidecomposable. Now recall that $S^h \operatorname{Pic}_\delta X$ is a good moduli space for unidecomposable sheaves of rank r and degree d, so we have a natural morphism

$$\Psi: \mathcal{U} \to S^{h'} \operatorname{Pic}_{\delta'} X \times S^{h''} \operatorname{Pic}_{\delta''} X$$

$$f \mapsto ([V_f], [Q_f])$$

with image in the smooth hypersurface H defined by the condition $c_1(V_f) + c_1(Q_f) = c_1(E)$. Denote by Φ the restriction: $\mathcal{U} \to H$. We only have to show that if (C_1) holds, but not (C_3), then Φ is dominant, and that it is onto if (C_4) holds as well.

As usual, we assume $r'' = 0$. Suppose that (C_1) holds, but not (C_3), and consider a unidecomposable sheaf E' of rank r' and degree d', and a subsheaf E'_1 of E' of rank $r' - \varrho'$ and degree $d' - \delta'$. Let $\alpha = \det \left(\begin{smallmatrix} \varrho' & \varrho \\ \delta' & \delta \end{smallmatrix}\right)$ and $n = \operatorname{amb}(E)$. We have $n \leqslant h'\alpha$, and if $n = h'\alpha$ then E is not of the form $k^n \otimes U$ with U unidecomposable. By grouping indecomposable summands of E is a suitable way, we can write $E = U_1 \oplus \cdots \oplus U_n$, where each U_i is unidecomposable and $rk\ U_1 \geqslant \cdots \geqslant rk\ U_n$. Furthermore, if $n = h'\alpha$ we can assume: $rk\ U_1 < rk\ U_n$. Now let $A = (h' - 1)\ \alpha$ and $E_1 = U_1 \ldots U_A$. Then we have $\operatorname{amb}(E_1) = A$ and in any case $rk\ E_1 > (A/h'\alpha) rk(E) = rk\ E'_1$. So by assertion 2 the general morphism $E'_1 \to E_1$ is injective with positive-rank, unidecomposable, hence locally free cokernel. On the other hand, the restriction map $\operatorname{Hom}(E', E) \to \operatorname{Hom}(E'_1, E)$ is onto (because $\operatorname{ext}^1(E'/E'_1, E) = 0$) so, putting these facts together, there exists an $f \in \mathcal{U}$ such that $E' \simeq V_f$ and E/E'_1 is locally free. We shall show that such a point f is smooth for Φ (hence Φ is dominant).

By the Kodaira-Spencer maps, we have the following isomorphisms

$$T_{[V_f]} S^{h'} \operatorname{Pic}_{\delta'} X \xrightarrow{\sim} \operatorname{Ext}^1(V_f, V_f)$$

and

$$T_{[Q_f]} S^{h''} \operatorname{Pic}_{\delta''} X \xrightarrow{\sim} \operatorname{Ext}^1(Q_f, Q_f),$$

and $T_f\Phi$ identifies with the morphism

$$\operatorname{Hom}(V_f, Q_f) \xrightarrow{\partial'_1 \oplus \partial''_1} \operatorname{Ext}^1(V_f, V_f) \oplus \operatorname{Ext}^1(Q_f, Q_f)$$

where ∂'_1 and ∂''_1 are connectants arising from the exact sequence

$$0 \to V_f \to E \to Q_f \to 0. \qquad (E^1)$$

Now consider the short exact sequence

$$0 \to V_f/E'_1 \to E/E'_1 \to Q_f \to 0. \qquad (E^2)$$

We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}\,(V_f, Q_f) & \xrightarrow{\partial_1' \oplus \partial_1''} & \mathrm{Ext}^1\,(V_f, V_f) \oplus \mathrm{Ext}^1\,(Q_f, Q_f) \\
\uparrow\cup & i \quad \uparrow\cup & \| \\
\mathrm{Hom}\,(V_f/E_1') & \xrightarrow{\partial_2' \oplus \partial_2''} & \mathrm{Ext}^1\,(V_f/E_1', V_f/E_1') \oplus \mathrm{Ext}^1\,(Q_f, Q_f)
\end{array}$$

where ∂_2' and ∂_2'' are connectants arising from (E^2) and i is the map

$$\mathrm{Ext}^1\,(V_f/E_1', V_f/E_1') \xleftarrow{\sim} \mathrm{Ext}^1\,(V_f/E_1', V_f) \hookrightarrow \mathrm{Ext}^1\,(V_f, V_f).$$

Since E/E_1' is locally free, $\mathrm{ext}^1\,(E/E_1', Q_f) = 0$ so ∂_2'' is onto; similarly ∂_2' has rank 1, so the image of $\partial_2' \oplus \partial_2''$, and a fortiori that of $\partial_1' \oplus \partial_1''$, contains a hypersurface of $\{0\} \oplus \mathrm{Ext}^1\,(Q_f, Q_f)$. Since ∂_1' is onto, we conclude: $rk(\partial_1' \oplus \partial_1'') \geqslant h' + h'' - 1$, hence the smoothness of Φ at f.

Now we assume that (C_4) holds. If $h' = 1$ or $h'' = 1$, Φ is onto by assertion 2, so we can assume for instance $h'' \geqslant h' > 1$, $r'' = 0$ and amb $(E) \leqslant (d'' - 1)\varrho$. Consider any E', E'' such that $([E'], [E'']) \in H$, and let $v \in \mathrm{supp}\,(E'')$. There is a short exact sequence:

$$0 \to k(v) \to E'' \to E_1'' \to 0,$$

with E_1'' torsion unidecomposable of degree $d'' - 1$. By (C_4) there exists a short exact sequence

$$0 \to F \to E \to E_1'' \to 0$$

with F unidecomposable of rank r' and degree $d' + 1$. Consider $E_1' \subset E'$ as above. Then there exists an injection $E' \hookrightarrow F$ such that F/E_1' is locally free. We obtain the following diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & k(v) & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & E' & \to & E & \to & E/E' & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & F & \to & E & \to & E_1'' & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & k(v) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Now we have $\text{ext}^1(E_1'', k(v)) = 0$ or 1 so E/E' is isomorphic to E'' or $k(v) \oplus E_1''$. In the first case, we are done. Otherwise, consider the short exact sequence

$$0 \to E'/E_1' \to E/E_1' \to k(v) \oplus E_1'' \to 0,$$

and the point f_0 it defines in Quot (E/E_1'). It is enough to show that there exist points f arbitrarily close to f_0 such that $Q_f \simeq E''$, because for such an f we shall have: $V_f \simeq E'/E_1'$ (by stability) hence a short exact sequence

$$0 \to E' \to E \to E'' \to 0.$$

Since E'' is the general extension of $k(v)$ by E_1'', it is enough to show that s_0: $E/E_1' \to k(v) \oplus E_1''$ deforms to a morphism s: $E/E_1' \to E_1''$. Now the torsion part of E/E_1' injects into E_1'' because F/E_1' is locally free. Therefore we can write $E/E_1' \simeq L \oplus T$, where L is locally free and $s_0(T) \subset E_1''$. Then $s_{0|L}$ deforms to a morphism $L \to E''$because L is locally free, and $S_{0|T}$ deforms also because it takes its values in the fixed part E_1''. Hence assertion 3 and the theorem.

Appendix A

We begin with a few remarks.

1. Let X be a smooth curve, and $E \in \text{Coh}(X)$ an indecomposable sheaf. Then E is either a torsion sheaf, or a locally free sheaf. This is because we have a short exact sequence

$$0 \to E_{\text{tors}} \to E \to E' \to 0,$$

where E_{tors} is the torsion of E, and E' is torsion-free, and therefore locally free. Now $\text{Ext}^1(E', E_{\text{tors}}) = 0$, so the exact sequence splits.

2. Indecomposable torsion sheaves over X can be described as follows. Let $v \in X$ and $d \in \mathbb{N}^*$. There is a natural short exact sequence

$$0 \to O_X(-d \cdot v) \to O_X \to O_X/O_X(-d \cdot v) \to 0.$$

We denote by ${}_vE\binom{0}{d}$ the quotient $O_X/O_X(-d \cdot v)$. Then ${}_vE\binom{0}{d}$ is indecomposable, and all indecomposable torsion sheaves are obtained in this way. We are going to show that, in the case of an elliptic curve, all indecomposable sheaves can be constructed starting from indecomposable torsion sheaves. *From now on, X is an elliptic curve.*

3. Any indecomposable sheaf over the elliptic curve X is semi-stable. Let E be an indecomposable sheaf over X. Any torsion sheaf is semi-stable, so we can assume that E is locally free.

There is a Harder-Narasimhan flag

$$0 = E_0 \subset \ldots \subset E_l = E,$$

where, for any $i = 1, 2, \ldots l$, $G_i = E_i/E_{i-1}$ is a non-zero semi-stable locally free sheaf, and $\mu(G_1) > \mu(G_2) > \ldots > \mu(G_l)$. For any i, j such that $1 \leqslant i < j \leqslant l$, we have $\mu(G_i) > \mu(G_j)$ and this implies hom $(G_i, G_j) = 0$ because G_i and G_j are semi-stable. By Serre duality, $\mathrm{Ext}^1(G_j, G_i) = 0$, so the Harder-Narasimhan flag splits completely; since E is indecomposable, this implies $l = 1$, hence E is semi-stable. (For elementary properties of semi-stable vector bundles, see [Sh] or [B]). Notice that if the rank and the degree of E are coprime, E is stable.

4. Let E, E' be two indecomposable sheaves over X, and let $D(E, E') = rk(E)\deg(E') - rk(E')\deg(E)$. Then if $D(E, E') \neq 0$ we have:

$$\hom(E, E') = D(E, E')_+$$

and

$$\mathrm{ext}^1(E, E') = D(E, E')_-.$$

This is because by Riemann-Roch we have $D(E, E') = \hom(E, E') - \mathrm{ext}^1(E, E')$. Now if $D(E, E') < 0$, we have $\mu(E) > \mu(E')$ so $\hom(E, E') = 0$; if $D(E, E') < 0$, then $\mu(E) < \mu(E')$ so $\mathrm{ext}^1(E, E') = \hom(E', E)$ (by Serre duality) $= 0$.

We are interested in sheaves generated by sections. They are described in the following lemma.

LEMMA A1. *Let E be an indecomposable sheaf over X. Then E is generated by sections if and only if*

$$E \simeq O_X \quad \text{or} \quad \mu(E) > 1.$$

Apart from those cases, E is generically generated by sections if and only if $\mu(E) = 1$.

Proof. Let $r = rk(E)$ and $d = \deg(E)$. Then if $d \neq 0$, $h^0(E) = d_+$ and $h^1(E) = d_-$. Clearly O_X is generated by sections. Now assume $d > r$; for any $x \in X$ we have $h^1(E \otimes O_X(-x)) = (d - r)_- = 0$, so E is generated by sections. Conversely, consider the evaluation morphism ev: $H^0(E)_X \to E$. If $d < r$, ev is injective on fibers because for any $x \in X$ we have $h^0(E \otimes O_X(-x)) = (d - r)_+ = 0$. In that case, E can be generated by sections only if ev is an isomorphism, and that implies: $E \simeq O_X$. Now there remains to check that for $r = d$, E is generically generated by sections, that is: for general $x \in X$, $h^0(E \otimes O_X(-x) = 0$. In other words, for a general $L \in \mathrm{Pic}_1(X)$, Hom $(L, E) = 0$. Now that is a special case of the well-known fact that if E is a semi-stable locally free sheave, there is only a finite number of classes of stable locally free sheaves F with the same slope, such that Hom $(F, E) \neq 0$ (see [B], for instance).

If E is a sheaf generated by sections, we denote by $P(E)$ the dual of the kernel of the evaluation morphism for E, so that there is a short exact sequence:

$$0 \longrightarrow P(E)^* \xrightarrow{\ i\ } H^0(E)_X \xrightarrow{\mathrm{ev}} E \longrightarrow 0. \qquad (*)$$

Notice that $P(E)$ is locally free, and that $P(E) = 0$ if and only if E is free. We have the following proposition.

Proposition A1. *Let* $\binom{r}{d} \in \mathscr{H}$ *and assume* $d > r$. *Then*
1. *for any* $E \in I\binom{r}{d}$, $P(E) \in I\binom{r-d}{d}$;
2. *the map* $P: I\binom{r}{d} \to I\binom{r-d}{d}$
   $E \mapsto P(E)$
   is one-to-one;
3. *let* E, F *be two irreducible sheaves of slope* > 1. *There is a natural isomorphism* Hom $(E, F) \xrightarrow{\sim}$ Hom $(P(F), P(E))$.

Proof of the proposition. Let $E \in I\binom{r}{d}$ with $d > r$, and consider the short exact sequence $(*)$. By Riemann-Roch and Serre duality, we have $h^0(P(E)) = d + h^0(P(E)^*) = d$. Now if $r > 0$, E is locally free and we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^* & \longrightarrow & H^0(E)_X & \xrightarrow{{}^t i} & P(E) & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle\varphi} & & \Big\| & & \\ & & & & H^0(P(E))_X & \longrightarrow & P(E) & & \end{array}$$

The morphism φ is injective (because $h^0(E^*) = 0$), so it is a canonical isomorphism $H^0(E)^* \xrightarrow{\sim} H^0(P(E))$. Therefore, $P(E)$ is generated by sections, and $P(P(E))$ is canonically isomorphic to E. Since E is indecomposable, so is $P(E)$. If $r = 0$, i is generically an isomorphism, hence a commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & H^0(E)^*_X & \xrightarrow{{}^t i} & P(E) \\ & & \Big\downarrow{\scriptstyle\varphi} & & \Big\| \\ & & H^0(P(E))_X & \xrightarrow{\text{ev}} & P(E) \end{array}$$

so φ remains a canonical isomorphism $H^0(E)^* \xrightarrow{\sim} H^0(P(E))$. We can re-write $(*)$ as:

$$0 \longrightarrow P(E)^* \xrightarrow{{}^t\text{ev}} H^0(P(E))^* \longrightarrow E \longrightarrow 0,$$

so $P(E)$ is indecomposable.

So much for assertion 1. For assertion 2, we construct the inverse of P in the following way. Let $F \in I\binom{d-r}{d}$. The evaluation morphism ev: $H^0(F)_X \to F$ is injective, hence a short exact sequence:

$$0 \to F^* \to H^0(F)^*_X \to E \to 0,$$

where $E = \text{coker}\,({}^t\text{ev})$. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & F^* & \to & H^0(F)^* & \to & E & \to & 0 \\ & & \Big\downarrow & & \Big\downarrow{\scriptstyle\varphi} & & \Big\| & & \\ 0 & \to & P(E) & \to & H^0(E) & \to & E & \to & 0. \end{array}$$

As above, one checks easily that φ is an isomorphism, so $F \simeq P(E)$, and E is indecomposable. Assertion 2 follows immediately.

Now let E, F be as in assertion 3. Any morphism: $E \to F$ induces a morphism $H^0(E) \to H^0(F)$, hence a morphism $P(E)^* \to P(F)^*$, and a morphism $P(F) \to P(E)$. Conversely, any morphism $P(F) \to P(E)$ induces a morphism $H^0(P(F)) \to H^0(P(E))$. By the natural identification between $H^0(P(E))$ and $H^0(E)^*$, we get a morphism $H^0(E) \to H^0(F)$ which factors to a morphism $E \to F$.

This defines a natural isomorphism Hom $(E, F) \simeq$ Hom $(P(F), P(E))$ and proves assertion 3.

Now we can construct the indecomposable sheaves ${}_vE(\binom{r}{d})$ of Theorem 1. We fix a point A in X and we proceed by induction on r. The sheaves ${}_vE(\binom{0}{d})$ have already been constructed so as to satisfy the first assertion of Theorem 1. Now assume that ${}_vE(\binom{r'}{d'})$ has been constructed for $r' < r$. Let d be such that $r < d \leqslant 2r$; then ${}_vE(\binom{d-r}{d})$ has been constructed already, so by the third assertion of Theorem 1 we must set

$$ {}_vE(\tbinom{r}{d}) \quad = \quad P({}_vE(\tbinom{d-r}{d})). $$

For a general d, there exists a unique $\lambda \in \mathbb{Z}$ such that $r \leqslant d + \lambda r < 2r$. By the second assertion, we must set

$$ {}_vE(\tbinom{r}{d}) \quad = \quad \mathrm{O}_X(-\lambda A) \otimes {}_{v+\lambda A}E(\tbinom{r}{d+\lambda r}). $$

So we can see by induction that there is only one way of defining sheaves ${}_vE(\binom{r}{d})$. This construction works out, and by Proposition A1, it is one-to-one.

Furthermore, by construction, assertion 3 holds at least when $d < 2r$. Now if $d > 2r$, we have

$$ {}_vE(\tbinom{d-r}{d}) \quad = \quad P({}_vE(\tbinom{d}{r})) $$

so $P({}_vE(\binom{d-r}{d})) = {}_vE(\binom{r}{d})$ and assertion 3 holds in that case as well.

There only remains to prove:

- assertion 2, which is known so far only when $\Lambda = \mathrm{O}_X(\lambda A)$. That is because our construction is not 'intrinsic' – it relies on the choice of a point A in X;
- assertion 3 when $r = d$: we must show that for any $E \in I(\binom{r}{2r})$ we have $P(E) \simeq E$.

The first thing to do is to show that our construction is intrinsic. It is enough to prove the second assertion of Theorem 2, whose first assertion is known already. All we have to show is the following property (property (P)): let $\binom{r}{d} \in \mathscr{H}$; let $h = (r, d)$ and $\binom{\varrho}{\delta} = 1/h\binom{r}{d}$. Let v, $w \in \mathrm{Pic}_\delta\, X$ and $E = {}_vE(\binom{\varrho}{\delta})$, $F = {}_wE(\binom{r}{d})$. Then $c_1(E) = v$, $c_1(F) = hw$, and hom $(E, F) =$ hom $(F, E) = 1$ if $v = $ w, 0 otherwise.

We prove (P) by induction on r. It is obvious for $r = 0$. Now to reduce to a smaller value of r, we can assume that $r \leqslant d < 2r$ (by tensoring by some $\mathrm{O}_X(\lambda A)$). Let $E' = {}_vE(\binom{\delta-\varrho}{\delta})$ and $F' = {}_wE(\binom{d-r}{d})$ so that $E = P(E')$ and $F = P(F')$. Then we have $c_1(E) = c_1(E') = v$, $c_1(F) = c_1(F') = hw$ and hom $(E, F) =$ hom $(P(F), P(E)) =$ hom $(F', E') = 1$ if $v = w$, 0 otherwise, the same holding for hom (F, E). Hence the property.

Let $E \in I(\binom{r}{2r})$. For $r = 1$ at least, $P(E) \simeq E$. Now for a general r, there is a unique $F \in I(\binom{1}{2})$ such that hom $(F, E) \neq 0$. We have hom $(F, P(E)) =$ hom $(P(F), E) =$ hom $(F, E) \neq 0$, so $P(E) \simeq E$. We have proved Theorem 1.

For Theorem 2, there remains only to show the existence of universal sheaves $\boldsymbol{E}(\binom{r}{d})$. As usual we proceed by induction on r. For $r = 0$, let I be the ideal sheaf of the diagonal $\Delta \subset X \times X$.

Then $\boldsymbol{E}\binom{0}{d} = O_{X\times X}/I^d$ works. For $r > 0$, we can always assume $r \leqslant d < 2r$. (by tensoring with a suitable line bundle $\in$ Pic X). Now by induction there exists a universal sheaf $\boldsymbol{E}\binom{d-r}{d}$ over $\mathrm{Pic}_\delta X \times X$. Consider the evaluation morphism

$$EV\colon pr_1^* pr_{1*}\boldsymbol{E}\binom{d-r}{d} \to \boldsymbol{E}\binom{d-r}{d}.$$

The restriction of EV to a fiber $\{v\} \times X$ is the ordinary evaluation morphism ev: $H^0({}_vE\binom{d-r}{d})_X \to {}_vE\binom{d-r}{d}$, whose kernel is ${}_vE\binom{r}{d}^*$, so $\boldsymbol{E}\binom{r}{d} = (\ker EV)^*$ works.

Appendix B – proof of Theorem 3

Denote by $CH(X)$ the Chern group of X, which we identify with $\mathbb{Z}^2 \times X$. The Chern character of any $F \in$ Coh (X) is the element

$$ch(F) = \left(\binom{rk(F)}{\deg(F)},\ c_1(F)\right) \in \mathbb{Z}^2 \times X.$$

With any left-exact functor $\boldsymbol{F}$ from Coh (X) to itself we can associate functorially a group endomorphism $CH(\boldsymbol{F})$ of $CH(X)$, so that for any $F \in$ Coh (X) we have:

$$CH(\boldsymbol{F})(ch(F)) = ch\,(\boldsymbol{F}(F)) - ch(R^1\boldsymbol{F}(F)) + \cdots.$$

Now let P be a coherent sheaf over $X \times X$, flat with respect to projections, and let

$$M = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in SL_2(\mathbb{Z})^+.$$

We consider the following properties:
- $(P1)$ $CH(S_P)\colon \mathbb{Z}^2 \times X \to \mathbb{Z}^2 \times X$ is the left multiplication by M on the first factor;
- $(P2)$ for any $v \in X$, $P_{v,} \simeq {}_vE\binom{y}{t}$;
- $(P3)$ for any $v \in X$, $P_{,v} \simeq {}_vE\binom{y}{x}$.

We have the following lemma.

Lemma B1. *If P satisfies the property (P) of Theorem 3, then it satisfies also $(P1)$, $(P2)$ and $(P3)$. As a consequence, it is uniquely determined up to isomorphism.*

Proof. Assume that P satisfies (P). Then clearly P satisfies $(P1)$. It also satisfies $(P2)$ because $P_{v,} = pr_{2*}(P \otimes pr_1^* k(v)) = S_P({}_vE\binom{0}{1}) = {}_vE\binom{y}{t}$. This means that P is a universal family of stable sheaves of rank y and degree t over X.

For unicity, assume that P' satisfies $(P1)$ and $(P2)$. Then there exists $L \in$ Pic (X) such that $P' \simeq pr_1^* L \otimes P$, so that we have $S_{P'} = S_P(? \otimes L)$, hence $CH(? \otimes L) = Id_{CH(X)}$, so $L \simeq O_X$ and $P' \simeq P$.

Now to check that $(P3)$ holds as well, assume first $y = 0$. Then $P = pr_1^* O_X(z \cdot A) \otimes O_\Delta$ clearly satisfies (P), $(P1)$, $(P2)$ and $(P3)$. Assume $y > 0$. Then P is a rank y locally free sheaf. Let $v \in X$ and let E be an indecomposable component of $P_{,v}$. By the semi-continuity theorem, there are canonical morphisms

$$k_0\colon S_P(E^*) \otimes k(v) \to H^0(E^* \otimes P_{,v});$$

$$k_1\colon R^1 S_P(E^*) \otimes k(v) \to H^1(E^* \otimes P_{,v});$$

furthermore, k_1 is always an isomorphism, and k_0 is an isomorphism provided that $R^1S_P(E^*)$ is locally free at v (see [H], p. 290).

Now, E being a component of $P_{.,v}$, we have: $h^0(E^* \otimes P_{.,v}) = \hom(E, P_{.,v}) \neq 0$, and $h^1(E^* \otimes P_{.,v}) = \hom(P_{.,v}, E) \neq 0$. This implies that $R^1S_P(E^*)$ is non-zero. In view of (P), $R^1S_P(E^*)$ is indecomposable and $S_P(E^*)$ is trivial; so k_0 is not an isomorphism, and $R^1S_P(E)$ is not locally free at v. Therefore there exists $h \geqslant 1$ such that $R^1S_P(E^*) = {}_vE\binom{0}{h}$. So we have: $E^* = {}_{-v}Eh\binom{y}{-x}$ and because of the rank, we can conclude: $h = 1$ and $E = P_{,v} \simeq {}_vE\binom{y}{x}$, hence $(P3)$.

Before proceeding to show the existence of the sheaves P_M, we make a few remarks about the 'usual' Fourier transform introduced by Mukai.

Consider the matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The usual Fourier transform is the functor RS_{P_J}, where

$$P_J = O_{X\times X}(\Delta - \{A\} \times X - X \times \{A\}).$$

Clearly P_J satisfies $(P2)$ and $(P3)$ for $M = J$. Now let $P_0 = O_{X\times X}(-\Delta)$. One can check easily that $P_J \simeq (-Id_{X\times X})^*P_0 \otimes pr_1^*O_X(A) \otimes pr_2^*O_X(A)$. Therefore $S_{P_J} \simeq (-Id_X)^*(O_X(A) \otimes S_{P_0}(? \otimes O_X(A))$. Now we know everything about S_{P_0}, which is none other than the 'kernel of the evaluation morphism' functor. To be precise, there is a short exact sequence:

$$0 \to P_0 \to O_{X\times X} \to O_\Delta \to 0; \tag{E^1}$$

for any $F \in \mathrm{Coh}(X)$ we have $\mathrm{Tor}^1(O_\Delta, pr_2^*F) = 0$, so, tensoring by pr_2^*F and applying pr_{1*}, we get a long exact sequence:

$$0 \to S_{P_0}(F) \to H^0(F)_X \xrightarrow{\text{ev}} F \to R^1S_{P_0}(F) \to H^1(F)_X \to 0, \tag{E}$$

so $S_{P_0}(F) = \ker ev$. We can deduce immediately that P_J satisfies $(P1)$.

We have the following proposition.

PROPOSITION B1. *For any $M \in SL_2(\mathbb{Z})^+$ there exists a sheaf P_M satisfying* $(P1)$, $(P2)$ *and* $(P3)$.

Proof of the proposition. We saw this already when $y = 0$ and when $M = J$. So assume $y \geqslant 1$. There exists a locally free sheaf P over $X \times X$ satisfying $(P3)$ (cf. Theorem 2). Let $Y: X \to X$ be the morphism 'multiplication by y', and consider the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{pr_2} & X \\ {\scriptstyle Id_X}\downarrow \quad \downarrow{\scriptstyle Y} & & \downarrow{\scriptstyle Y} \\ X \times X & \longrightarrow & X \end{array}$$

For any $v \in X$ we have $((Id_X \times Y)^*P)_{,v} \simeq {}_{yv}E\binom{y}{x} \simeq O_X(v - A) \otimes {}_AE\binom{y}{x}$, so there exists $L \in \mathrm{Pic}(X)$ such that

$$(Id_X \times Y)^*P \simeq P_J \otimes pr_{1A}^*E\binom{y}{x} \otimes pr_2^*L; \tag{1}$$

therefore $Y^* \circ R^! S_P \simeq R^! S_{P_J}(? \otimes {}_A E(\begin{smallmatrix} y \\ x \end{smallmatrix})) \otimes L$, hence a relation between endomorphisms of $CH(X)$; denote by M' the matrix of the action of $CH(S_P)$ on $\mathbb{Z}^2$, and let $\lambda = \deg L$. By functoriality, we get

$$\begin{pmatrix} 1 & 0 \\ 0 & y^2 \end{pmatrix} M' = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ x & y \end{pmatrix},$$

so $M' = \begin{pmatrix} x & y \\ (\lambda x - y)/y^2 & \lambda/y \end{pmatrix}$, and $\lambda \equiv ty(y^2)$. Let $q = (\lambda - ty)/y^2$; replacing P by $P \otimes pr_2^*(O_X(qA))$ we can assume $M' = M$.

Now let $v \in X$. By (1) we have $Y^* P_{v,} \simeq L \otimes O_X(v - A) \otimes k_X^y$, so $Y^* P_{v,}$ is semi-stable of rank y and degree y; therefore $P_{v,.}$ is semi-stable of rank y and degree $\lambda/y = t$, hence stable. Because of $(P3)$, the determinant bundle of P is a Poincaré bundle; tensoring P by a suitable line bundle, we can assume:

$$c_1(P_{v,}) = v + tA,$$

so $(P2)$ holds. Now tensoring again by a suitable $\Lambda \in \mathrm{Pic}_0(X)$ such that $\Lambda^{\otimes y} \simeq O_X$, we obtain a sheaf P_M which satisfies $(P1)$ as well.

For any $M \in SL_2(\mathbb{Z})^+$, let P_M be the sheaf satisfying the conditions of the proposition, and for simplicity let $F_M^! = R^! S_{P_M}$ and $F_M = RS_{P_M}$. We know already that (P) holds for $y = 0$. So assume $y > 0$. We have the following lemma.

LEMMA B2. *Let* $M' = -M^{-1}$. *Then*

$$F_{M'}^0 \circ F_M^0 = 0,$$

and $$F_{M'} \circ F_M = (-Id_X)^* \circ [-1].$$

Proof. Denote by p_{12}, p_{23} and p_{13} the projection $X^3 \to X^2$. For any two coherent sheaves P, Q over $X \times X$, flat with respect to projections, there is a canonical isomorphism:

$$RS_Q \circ RS_P \simeq RS_{H^\cdot}.$$

where $H^\cdot$ is the complex $R^\cdot p_{13*}(p_{12}^* P \otimes p_{23}^* Q)$ and $RS_{H^\cdot}$ is the functor $Rpr_{2*}(H^\cdot \overset{L}{\underset{=}{\otimes}} pr_1^*(?))$. There is also a canonical isomorphism

$$S_Q \circ S_P \simeq S_{H0}.$$

(The proof is an elementary application of base-change and the adjunction formula.) In order to apply this to $P = P_M$ and $Q = P_{M'}$, we consider the sheaf $R = p_{12}^* P_M \otimes p_{23}^* P_{M'}$. For any v, $w \in X$ we have: $R_{v,.,w} \simeq {}_v E(\begin{smallmatrix} y \\ t \end{smallmatrix}) \otimes {}_w E(\begin{smallmatrix} y \\ -t \end{smallmatrix})$ so $h^0(R_{v,,w}) = \hom({}_{-w}E(\begin{smallmatrix} y \\ t \end{smallmatrix}), {}_v E(\begin{smallmatrix} y \\ t \end{smallmatrix})) = 1$ if $v = -w$, 0 otherwise, and the same holds for $h^1(R_{v,,w})$. By the semi-continuity theorem, we conclude that $\mathrm{H}^0 = p_{13*}R = 0$, and $H^1 = R^1 p_{13*} R$ is supported by the second diagonal $\Delta' \subset X \times X$. So we have $F_{M'}^0 \circ F_M^0 = 0$ and $F_{M'} \circ F_M = [-1] \circ S_{H1}$. Now $(P1)$ holds for P_M and $P_{M'}$, so we have $CH(S_{H1}) = (-Id_X)^*$. Therefore for any $v \in X$ we have $H_{v,}^1 \simeq k(-v)$, and $pr_{2*} H^1 \simeq O_X$; we obtain: $H^1 \simeq O_{\Delta'}$, and since $S_{O_{\Delta'}} = (-Id_X)^*$, we are done.

Now we proceed to prove that P_M satisfies (P). Let $E = {}_vE\binom{r}{d}$ be an indecomposable sheaf, and assume first: $M\binom{r}{d} \in \mathscr{H}$. One can check easily that this condition implies $xr + yd > 0$. Let $w \in X$. We have: $\dim(F_M^1(E) \otimes k(v)) = h^1(E \otimes_w E\binom{y}{x})$ (by the semi-continuity theorem) $= \hom(E, {}_wE\binom{y}{x}^*)$ (by Riemann-Roch) $= 0$ because $\mu({}_wE\binom{y}{x}^*) - \mu(E) = (-1/yr)(xr + yd) < 0$; so $F_M^1(E) = 0$. Now let $E' = F_M^0(E)$. We have $F_{M'}^0 \circ F_M^0(E) = 0$, and since $F_{M'} \circ F_M = (-Id_X)^* \circ [-1]$, we deduce that $F_{M'}^1(E') = F_{M'}^1 \circ F_M^0(E) \simeq (-Id_X)^*E$. Hence, $F_{M'}^1(E')$ and E' are indecomposable. By $(P1)$ we have $ch(E') = Mch(E)$, so $E' \simeq {}_wEM\binom{r}{d}$ with $(r, d) \cdot (v - w) = 0$. So we are done at least when $(r, d) = 1$. Let $h = (r, d)$ and $E_0 = {}_vE\,1/h\binom{r}{d}$. We have $E_0 \hookrightarrow E$, so $F_M^0(E_0) = {}_vEM\,1/h\binom{r}{d} \hookrightarrow E' = {}_wEM\binom{r}{d}$; this implies $w = v$.

If $M\binom{r}{d} \notin \mathscr{H}$ then $M'^{-1}\binom{r}{d} = -M\binom{r}{d} \in \mathscr{H}$; so $E \simeq F_M^0; E''$, where $E'' = {}_vE - M\binom{r}{d}$. We have $F_M^0(E) = (-Id_X)^*E'' \simeq {}_{-v}E - M\binom{r}{d}$, and (P) is satisfied.

Now we prove the second assertion of Theorem 3. Clearly, we can reduce to the case when $M, M' \in SL_2(\mathbb{Z})^+$. The result is known already if $M' = -M^{-1}$ and it is trivial if M or $M' = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ for some $z \in \mathbb{Z}$. So we can assume that $MM'\binom{0}{1} \in \mathbb{Z}^* \times \mathbb{Z}$. Assume for instance that $MM'\binom{0}{1} \in \mathbb{N}^* \times \mathbb{Z}$, so that $MM' \in SL_2(\mathbb{Z})^+$ (the proof is similar in the other case). We have $F_M \circ F_{M'} = RS_{H^\cdot}$, where $H^\cdot = R^\cdot p_{13*}(p_{12}^* P_{M'} \otimes p_{23}^* P_{M'})$. Let $R = p_{12}^* P_{M'} \otimes p_{23}^* P_M$, and let $v, w \in X$. Then $R_{v,,w} \simeq {}_vE\binom{y'}{t'} \otimes {}_wE\binom{y}{x}$. Since $xy' + yt' > 0$, we conclude that $H^1 = 0$, and by base-change, $H^0_{v,} \simeq F_M^0({}_vE\binom{x'}{x'}) \simeq {}_vE\binom{xy'+yt'}{zy'+tt'}$ and similarly, $H^0_{,v} \simeq {}_vE\binom{xy'+yt'}{xx'+yz'}$.

So H^0 is locally free, and satisfies $(P1)$, $(P2)$ and $(P3)$ for the matrix MM'. This implies $H^0 \simeq P_{MM'}$, hence the theorem.

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