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Metaplectic forms and Gauss sums I

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§1. Introduction

This paper is the first of a series devoted to the explication of the general theory of metaplectic forms [6] for number theory. The objective of this paper is the definition and study of certain Dirichlet series which give expression to this arithmetic content. The coefficients of the simplest class of these Dirichlet series are Gauss sums. The residues of these Dirichlet series at the principal pole give the coefficients of the next class, and so on.

One of the fundamental concepts of the theory of exceptional metaplectic forms is the 'distribution' $\mathbf{c}$ introduced in [6], Theorem II.2.2. This is associated, over an $\mathbf{A}$-field k containing the nth roots of unity, to a split torus of the n-fold metaplectic cover of GL_r. More generally one can associate to a split torus of an n-fold cover of $GL_{r_1} \times \cdots \times GL_{r_s}$ ($\subset GL_{r_1+\cdots+r_s}$) an analogous distribution. We shall construct Dirichlet series associated with the n-fold cover of $GL_{r_1} \times GL_{r_2}$ which are essentially Fourier coefficients of the Eisenstein series obtained by inducing an exceptional automorphic representation of $GL_{r_1} \times GL_{r_2}$ (over $k_{\mathbf{A}}$) to an n-fold cover of $GL_{r_1+r_2}$. The coefficients of these Dirichlet series involve the $\mathbf{c}$-distribution for $GL_{r_1} \times GL_{r_2}$, or, what turns out to be the same, the product of the $\mathbf{c}$-distributions for n-fold covers of GL_{r_1} and GL_{r_2}. The residues at the principal poles are the given by the values of the $\mathbf{c}$-distribution for an n-fold cover of $GL_{r_1} \times GL_{r_2}$. The precise statement is given in Theorem 5.1. These results should be understood in conjunction with the formal properties of the $\mathbf{c}$-distribution given in §3.

There are two weaknesses in Theorem 5.1 as it is formulated here. Firstly, rather than being defined by a series the 'Dirichlet series' is given as a rather inconvenient limit. This is merely a technical point since the convergence of the Dirichlet series can be shown to be convergent; this requires estimates on the values of the $\mathbf{c}$-distributions which will be the subject of the next paper in this series. The other weakness is that Theorem 5.1 still has to be 'deciphered' in any special case. The reason for this is that the function

τ (or $\hat{\tau}$) which can be computed by a local analysis at a finite number of places does not have a simple form. This problem does not arise in the case when k is a function field and this case will be discussed in detail in a later paper.

The notations of this paper have been chosen to agree with those of [6] and although most are recalled here there are some that are not.

§2. Eisenstein series

Let k be a global field such that Card $\mu_n(k) = n$. Let G be an algebraic group of the form $GL_{r_1} \times \cdots \times GL_{r_s}$ and let $G_k = GL_{r_1}(k) \times \cdots \times GL_{r_s}(k)$, $G_{\mathbf{A}} = GL_{r_1}(k_{\mathbf{A}}) \times \cdots \times GL_{r_s}(k_{\mathbf{A}})$. As in [6] §0.2 we form a metaplectic extension

$$1 \longrightarrow \mu_n(k) \xrightarrow{i} \tilde{G}_{\mathbf{A}} \xrightarrow{p} G_{\mathbf{A}} \longrightarrow 1;$$

it would suffice to embed G in $GL_{r_1+\cdots+r_s}$ by the obvious representation and to restrict an extension as described in [6] §0.2. Let s: $G_k \to \tilde{G}_{\mathbf{A}}$ be the lift of the standard embedding constructed as in [6] §0.2.

Let H be the diagonal subgroup of G; let $Z^0(G)$ be the centre of G and let $\tilde{Z}_{\mathbf{A}}(G)$ be the centre of $\tilde{G}_{\mathbf{A}}$. Let H_n be the subgroup of those elements which are nth powers. Let $\tilde{H}_{n,\mathbf{A}} = p^{-1}(H_n(k_{\mathbf{A}}))$, $\tilde{H}_{\mathbf{A}} = p^{-1}(H(k_{\mathbf{A}}))$. Let N be the unipotent subgroup with zero entries under the diagonal; then $N_{\mathbf{A}} = N(k_{\mathbf{A}})$ lifts to a subgroup $N_{\mathbf{A}}^*$ of $\tilde{G}_{\mathbf{A}}$ and this lift is unique. Let $B = HN$, $B_n = H_nN$, $B_{\mathbf{A}} = \tilde{H}_{\mathbf{A}}N_{\mathbf{A}}$, $\tilde{B}_{n,\mathbf{A}} = \tilde{H}_{n,\mathbf{A}}N_{\mathbf{A}}^*$.

For a place v of k let $\tilde{G}_v$ be the corresponding factor of $\tilde{G}_{\mathbf{A}}$. If $|n|_v = 1$ let K_v be the standard maximal compact subgroup of $G_v = p(G_v)$); let K_v^* be the standard lift of K_v to $\tilde{G}_v$([6] §0.1).

Let Φ be the set of roots of H relative to G and let Φ^+ be that set of positive roots defined by N. Let $\mu_{\mathbf{A}}$ be the modulus quasicharacter of $\tilde{H}_{\mathbf{A}}$ defined by $\frac{1}{2}\Sigma_{\alpha\in\Phi^+}\,\alpha$ (cf. [6] §I.1).

Let ε be an injective character of $\mu_n(k)$. Let $\Omega_\varepsilon(G)$ be the set of quasicharacters ω of $\tilde{H}_{n,\mathbf{A}}$ trivial on $\tilde{H}_{n,\mathbf{A}} \cap s(G_k)$ and such that $\omega \cdot i = \varepsilon$. Let $X_\varepsilon(G)$ be the set of quasicharacters χ of $\tilde{Z}_{\mathbf{A}}(G) \cap \tilde{H}_{n,\mathbf{A}}$, trivial on $\tilde{Z}_{\mathbf{A}}(G) \cap \tilde{H}_{n,\mathbf{A}} \cap s(G_k)$ and such that $\chi \cdot i = \varepsilon$. Let $\varrho\colon \Omega_\varepsilon(G) \to X_\varepsilon(G)$ be the restriction map, and for $\chi \in X_\varepsilon(G)$ let $\Omega_\chi(G) = \varrho^{-1}(\chi)$. As usual $\Omega_\varepsilon(G)$, $X_\varepsilon(G)$ and $\Omega_\chi(G)$ have structures of complex manifolds, and ϱ is analytic.

We now shall define a holomorphic vector bundle F over $\Omega_\varepsilon(G)$. The fibre of F at ω, denoted by $F(\omega)$, is the space of functions f: $\tilde{G}_{\mathbf{A}} \to \mathbf{C}$ such that

$$f(hng) = \omega\mu(h) \cdot f(g)$$

if $h \in \tilde{H}_{n,\mathbf{A}}$, $n \in N^*_{\mathbf{A}}$, $g \in \tilde{G}_{\mathbf{A}}$. Moreover f is to be a finite linear combination of 'primitive' functions $\otimes f_v$ where f_v is smooth (resp. locally constant) if v is Archimedean (resp. non-Archimedean). For almost all v such that $|n|_v = 1$ we demand finally that $f_v = f_v^0$ where f_v^0 is the function defined by

$$f_v^0(g) = 0 \qquad g \in \tilde{G}_v - \tilde{B}_{n,v} \cdot K_v^*$$

$$f_v^0(k) = 1 \qquad k \in K_v^*.$$

Note that $F(\omega)$ is a $\tilde{H}_{\mathbf{A}} \times \tilde{G}_{\mathbf{A}}$-module by

$$((\eta, \gamma)f)(g) := f(\eta^{-1} \cdot g \cdot \gamma) \quad (\eta \in \tilde{H}_{\mathbf{A}}, \gamma, g \in \tilde{G}_{\mathbf{A}}).$$

For $\alpha \in \Phi$ let $H_\alpha: GL_1 \to GL_r$ be the corresponding coroot. As in [6] §I.2 we obtain a homomorphism

$$H_\alpha^n: k_{\mathbf{A}}^\times \to \tilde{G}_{\mathbf{A}}$$

such that

$$p(H_\alpha^n(x)) = H_\alpha(x^n).$$

For $\omega \in \Omega_\varepsilon(G)$ let $\omega_\alpha^n = \omega \cdot H_\alpha^n$. Recall that one says that ω is dominant if ω_α^n is of the form $\|\ \|_{\mathbf{A}}^{\sigma_\alpha(\omega)}$ where $\sigma_\alpha(\omega) > 0$ for $\alpha \in \Phi_+$ and $\|\ \|_{\mathbf{A}}$ denotes the idele norm.

Let $H_k^* = s(G_k) \cap \tilde{H}_{\mathbf{A}}$.

We can now define a $H_k^* \times s(G_k)$-invariant linear form $E(\omega): F(\omega) \to \mathbf{C}$ when $\omega\mu^{-1}$ is dominant by

$$\langle E(\omega), f\rangle := \sum_{\gamma \in B^*_{n,k}\backslash G^*_k} f(\gamma)$$

where $G_k^* = s(G_k)$, $B_{n,k}^* = G_k^* \cap \tilde{B}_{n,\mathbf{A}}$.

We can rephrase this by extending ω to $H_k^*\tilde{H}_{n,\mathbf{A}}$ with $\omega|H_k^* = 1$ so that for $\eta \in H_k^*\tilde{H}_{n,\mathbf{A}}$

$$\langle E(\omega), (\eta, 1)f\rangle = \omega\mu_{\mathbf{A}}(\eta)^{-1} \langle E(\omega), f\rangle.$$

The linear form E which is defined on an open subset of $\Omega_\varepsilon(G)$ has an analytic continuation as a meromorphic function to all of $\Omega_\varepsilon(G)$, and its analytic properties can be described ([4], [5]).

Let us call $\omega \in \Omega_\varepsilon(G)$ *exceptional* if for every positive simple root α one has $\omega_\alpha^n = \| \ \|_{\mathbf{A}}$. When ω is exceptional the representation $F(\omega)$ becomes reducible and has a unique irreducible quotient which we denote by $V_0(\omega)$. There is a residue $\Theta(\omega)$ of E at ω which is a $H_k^* \times G_k^*$-invariant linear form $\Theta(\omega)$: $V_0(\omega) \to \mathbb{C}$ with analogous properties to E. Details, in the case of GL_r are given in [6] §II.1 and the general case involves no new ideas.

We shall next use the $V_0(\omega)$ to construct further Eisenstein series. Let $r = r_1 + r_2 + \ldots + r_s$ and let Π be the parabolic subgroup of GL_r containing B (the standard Borel subgroup of GL_r) and with Levi component isomorphic to G. Let U_Π be the unipotent radical of Π.

We let $\Omega_\varepsilon^{\mathrm{exc}}(G)$ be the set of exceptional quasicharacters in $\Omega_\varepsilon(G)$. Let $W(G)$ be the Weyl group of H in G, that is the normaliser of H modulo the centralizer of H. This can be identified with a subgroup of G_k^*. If ω_1 and ω_2 are in $\Omega_\varepsilon^{\mathrm{exc}}(G)$ then $\omega_1 \cdot \omega_2^{-1}$ is $W(G)$-invariant and trivial on $p^{-1}(H_{n,k})$. The set of such quasicharacters form a complex manifold $\tilde{\Omega}(G)$. A quasicharacter in $\tilde{\Omega}(G)$ can be identified with a quasicharacter of $G_{n,\mathbf{A}}$ where G_n is the algebraic subgroup of G such that every algebraic character takes values in $\mathrm{Im}\,(GL_1 \to GL_1;\ x \mapsto x^n)$. We shall make this identification henceforth.

Let $\alpha \in \tilde{\Omega}(G)$ and form any one-dimensional representation $\alpha^\#$ of $H_k^* \tilde{H}_{n,\mathbf{A}} \times \tilde{G}_{\mathbf{A}}$ which when restricted to $\tilde{H}_{n,\mathbf{A}} \times \{1\}$ yields α^{-1}, and when to $\{1\} \times \tilde{G}_{n,\mathbf{A}}$ yields α. Then

$$V_0(\omega \cdot \alpha) \cong V_0(\omega) \otimes \alpha^\#$$

It is convenient to regard V_0 as a holomorphic vector bundle over $\Omega_\varepsilon^{\mathrm{exc}}(G)$, which, being a coset of $\tilde{\Omega}(G)$, has a complex structure.

We consider next the vector bundle $F^\#$ over $\Omega_\varepsilon^{\mathrm{exc}}(G)$ of which the fibre $F^\#(\chi)$ at $\chi \in \Omega_\varepsilon^{\mathrm{exc}}(G)$ consists of a certain space of maps

$$f\colon GL_{r,\mathbf{A}} \to V_0(\chi)$$

such that

$$f(\gamma u g) \;=\; v(\gamma) \cdot (\gamma^{-1} f)(g) \quad (\gamma \in \tilde{G}_{\mathbf{A}},\ u \in U_{\mathbf{A}}^*,\ g \in GL_{r,\mathbf{A}}).$$

Here v is the quasicharacter of $G_{\mathbf{A}}$ defined as the square root of the modulus of the action of G on the Lie algebra of U. Let μ_1 be the modulus quasicharacter of $GL_{r,\mathbf{A}}$, and μ, as above, that of $G_{\mathbf{A}}$; then for $g \in H$ one has $\mu_1(g) = \mu(g) v(g)$.

We have now to describe the space $F^\#(\chi)$. Each $f \in V_0(\chi)$ is a finite sum of primitive elements uniformly in χ. A primitive element is of the form $\otimes f_v$

under the identification

$$V_0(\chi) \cong \otimes V_{0,v}(\chi_v)$$

where $V_{0,v}(\chi_v)$ is the image of the corresponding intertwining operator from $V_v(\chi_v)$, the corresponding induce representation – see [6] §I.2.

Let us write f_v^0 for the image of the element $f_v^0 \in V_v(\chi_v)$, the standard K_v^*-invariant vector. Then, locally uniformly in χ, we require that amost all the f_v are f_v^0. We also demand that f_v is smooth (resp. locally constant) if v is Archimedean (resp. non-Archimedean).

Note that $F^\#(\chi)$ is a $\tilde{H}_{\mathbf{A}} \times \widetilde{GL}_{r,\mathbf{A}}$-module and that if $\eta \in \tilde{H}_{n,\mathbf{A}}$ then

$$(\eta, 1) \cdot f = (\omega\mu_1)^{-1}(\eta) \cdot f.$$

On an open subset of $\Omega_\varepsilon^{\mathrm{exc}}(G)$ one can define the linear form $E^\#(\chi)$: $F^\#(\chi) \to \mathbf{C}$ by

$$\sum_{\gamma \in \Pi_k^* \backslash GL_{r,k}^*} \langle \Theta(\chi), \gamma f \rangle$$

which satisfies

$$\langle E^\#(\chi), (\eta, \gamma) f \rangle = (\chi\mu)^{-1}(\eta) \langle E^\#(\chi), f \rangle$$

when $\eta \in H_k^* \tilde{H}_{n,\mathbf{A}}$, $\gamma \in GL_{r,k}^*$ and ω is extended as before. In view of [4] Lemma 4.1 the series defining $E^\#(\chi)$ converges when $\underset{\sim}{\sigma(\chi_{\alpha_0}^n)} > n$.

Let $F^{(1)}$ be the space corresponding to $F(\omega)$ but for $\widetilde{GL}_{r,\mathbf{A}}$ in place of $\tilde{G}_{\mathbf{A}}$. An element $f \in F^{(1)}(\omega)$ yields by the quotient map an element of $V_0(\omega)$, and so also an element of $F^\#(\omega)$ when $\omega \in \Omega_\varepsilon^{\mathrm{exc}}(G)$. Denote this map by I: $F^{(1)}(\omega) \to F^\#(\omega)$; one has if $E^{(1)}$ denotes the Eisenstein series operator for $GL_{r,\mathbf{A}}$ then for $\omega \in \Omega_\varepsilon^{\mathrm{exc}}(G)$

$$\operatorname*{Res}_{\omega} \langle E^{(1)}, f \rangle = \langle E^\#(\omega), If \rangle;$$

for this see [5] §10. Here '$\operatorname*{Res}_{\omega}$' is to be understood in the same sense as in [6] §II.2.

From this one can derive all the analytic properties of E which we shall need. We shall not describe these in detail here but we shall formulate the functional equation as it shall later be needed.

Let G be as above and $G' = GL_{r_2} \times GL_{r_1}$. Let $G^{(1)} = GL_r$. Let us write

$$s(G) = \begin{pmatrix} 0 & I_{r_1} \\ I_{r_2} & 0 \end{pmatrix}$$

where the matrix is in $(r_1 + r_2) \times (r_2 + r_1)$-block form, and I_ϱ denotes the $(\varrho \times \varrho)$-identity matrix. Thus

$$s(G) \cdot s(G') = I.$$

We shall also write $s(G)$ for the image of $s(G)$ under $G_k \to G_k^* \subset \tilde{G}_{\mathbf{A}}$. For a quasicharacter χ of $\tilde{H}_{n,\mathbf{A}}$ define χ' by

$$\chi'(h) = \chi(s(G)^{-1} \cdot h \cdot s(G));$$

Thus $\chi' \in \Omega_\varepsilon^{\mathrm{exc}}(G')$ precisely when $\chi \in \Omega_\varepsilon^{\mathrm{exc}}(G)$. Let v' be the analogue of v for G'.

One can define an intertwining operator

$$I_{s(G)}\colon F^{\#}(\chi, G) \to F^{\#}(\chi', G')$$

by passage to the quotient from the map $V(\chi) \to V(\chi')$ defined in [6] §II.1 and denoted there by $I_{s(G)}$.

Let α_0 be the root $(r_1, r_1 + 1)$ and $\alpha_0' = (r_2, r_2 + 1)$. Then $I_{s(G)}$ has been regularized by

$$\prod_{j=0}^{r_1-1} L(\chi_{\alpha_0}^n \cdot \| \ \|_{\mathbf{A}}^{r_2+j+1}) L(\chi_{\alpha_0}^n \| \ \|_{\mathbf{A}}^{j})^{-1} \, \varepsilon(\chi_{\alpha_0}^n \| \ \|_{\mathbf{A}}^{r_2+j+1})$$

and is given by an Euler product so that $I_{s(G)}$ acts on almost all the standard vectors f_v^0 by sending them identically to the corresponding standard vector. Here L, ε have the same meanings as in [6] §II.1.

For a quasicharacter $\theta\colon k_{\mathbf{A}}^\times \to \mathbf{C}^\times$ trivial on $k^\times$ we define $\sigma(\theta) \in \mathbb{R}$ by $|\theta| = \| \ \|_{\mathbf{A}}^{\sigma(\theta)}$. Then the only singularity of

$$\left\{ \prod_{1 \leqslant i \leqslant r_1} \ \prod_{1 \leqslant j \leqslant r_2} L(\chi_{\alpha_0}^n \| \ \|_{\mathbf{A}}^{i+j-1} \right\} \cdot E^{\#}(\chi)$$

in $\{\chi \in \Omega_\varepsilon^{\mathrm{exc}}(G)\colon \sigma(\chi_{\alpha_0}^n) > 0\}$ is along $\chi_\alpha^n = \| \ \|_{\mathbf{A}}$ and is a 'simple pole'.

One has the functional equation

$$\prod_{1 \leqslant i \leqslant r_1} \prod_{1 \leqslant j \leqslant r_2} L(\chi^n_{\alpha_0} \| \ \|_{\mathbf{A}}^{i+j-1}) \, E^{\#}_{G}(\chi, f)$$

$$= \prod_{1 \leqslant i \leqslant r_1} \prod_{1 \leqslant j \leqslant r_2} L((\chi')_{a'_0} \| \ \|_{\mathbf{A}}^{1-i-j}) E^{\#}_{G'}(\chi'(v')^{-2+4/(r_1+r_2)}, I_{s(G)} f).$$

This can be derived by the general principles of [4] from the results of [6] §II.1.

§3. The c-distribution

We recall here, in the context of $\tilde{G}_{\mathbf{A}}$, the basic properties of the **c**-distribution introduced in [6] Theorem II.2.2. and also discussed in [7] §4. We shall need some notation. Let for a finite subset S of $\Sigma(k)$ the following expressions

$$\tilde{G}_S, \tilde{H}_{n,S}, \tilde{H}_S, \tilde{B}_{n,S}, \tilde{B}_S, \text{ etc.}$$

denote the S-factors of

$$\tilde{G}_{\mathbf{A}}, \tilde{H}_{n,\mathbf{A}}, \tilde{H}_{\mathbf{A}}, \tilde{B}_{n,\mathbf{A}}, \tilde{B}_{\mathbf{A}}, \text{ etc.}$$

Likewise we can form the S-factor of a representation of $\tilde{G}_{\mathbf{A}}$, or of $\tilde{H}_{\mathbf{A}} \times \tilde{G}_{\mathbf{A}}$ etc., likewise of a quasicharacter of $\tilde{H}_{n,\mathbf{A}}$. Then will also be denoted by a subscript $-$ S.

Let G be as above; let $e\colon \tilde{N}_{\mathbf{A}} \to \mathbf{C}$ be a non-degenerate character trivial on $\tilde{N}_k = p^{-1}(N_k)$. We form

$$\lambda_{\mathbf{A}}(e)\colon V_0(e) \to \mathbf{C}$$

by

$$\langle \lambda_{\mathbf{A}}(e), v \rangle = \int_{N^*_k \backslash N^*_{\mathbf{A}}} \bar{e}(n) \Theta(nv) \, \mathrm{d}n.$$

This a Whittaker functional, [6] §II.2.

We shall now define another family of linear functionals.

Let $\tilde{M}_{\mathbf{A}}$ be the normalizer of $\tilde{H}_{\mathbf{A}}$ in $\tilde{G}_{\mathbf{A}}$ and let $\tilde{M}_{0,\mathbf{A}}$ be the subset of elements of maximal length. Then for $\eta \in \tilde{M}_{0,S}$ we define the linear functional $\lambda_{\eta,S}$ on $V(\omega)_S$ by

$$\langle \lambda_{\eta,S}, v \rangle = \int_{N^*_S} \bar{e}_S(n) v(\eta^{-1} n) \, \mathrm{d}n.$$

When certain linear conditions on c_S: $\tilde{M}_{0,S} \to \mathbf{C}$ satisfying $c(h\eta) = (\omega\mu)_S(h) \cdot c(\eta)$ $(h \in \tilde{H}_{n,S},\ \eta \in \tilde{M}_{0,S})$ are satisfied then $\Sigma_{\eta \in \tilde{H}_{n,S}\backslash\tilde{M}_{0,S}} c_s(\eta) \cdot \lambda_{\eta,S}$ factors through $V_0(\omega)_S$. Let us denote the space of such functions by $U_S(\omega, e)$; one has a natural restriction map

$$\text{res: } U_{S'}(\omega, e) \to U_S(\omega, e) \quad (S' \supset S).$$

The space $U_S(\omega, e)$ is finite-dimensional and one can make certain statements as to its dimension – cf. [6] §§I.4, I.6, II.2. Let

$$U(\omega, e) = \varinjlim U_S(\omega, e)$$

which is a vector space endowed with morphism $U(\omega, e) \to U_S(\omega, e)$ compatible with the restriction map, and universal with respect to this property. An element $c \in U(\omega, e)$ can be regarded as a function c: $\tilde{M}_{0,\mathbf{A}} \to \mathbf{C}$ satisfying

$$c(h\eta) = (\omega\mu)(h) \cdot c(\eta) \quad (h \in \tilde{H}_{n,\mathbf{A}},\ \eta \in \tilde{M}_{0,\mathbf{A}})$$

and the further linear conditions indicated above. Let

$$T(S) = \prod_{v \in S - \Sigma_\infty(k)} \prod_{i=1}^{s} \prod_{j=1}^{r_i - 1} (1 + q_v^{-1} + \cdots + q_v^{-j}).$$

Then one has that there exists $\mathbf{c} \in U(\omega, e)$ such that $\lambda_{\mathbf{A}}(e)$ can be represented as follows. For $v \in V_0(\omega)$ we choose $S \supset \{v \in \Sigma(k) \,|\, |n|_v \neq 1\}$ such that ω is unramified on $\Sigma(k) - S$, and such that in the representation

$$V_0(\omega) \cong V_0(\omega)_S \underset{\mu_n(k)}{\otimes} \bigotimes_{v \in \Sigma(k) - S} V_{0,v}(\omega)$$

the vector v can be represented as $v_S \otimes v_0^S$ where v_0^S is the image under the intertwinning operator of the standard $\Pi_{v \in \Sigma(k) - S} K_v^*$-invariant vector of $\otimes_{v \in \Sigma(k) - S} V_v(\omega)$. Then from [6] Theorem II.2.2, one of the main results of that paper, one has

$$\langle \lambda_{\mathbf{A}}(e), v \rangle = T(S)^{-1} \sum_{\eta \in \tilde{H}_{n,S}\backslash\tilde{M}_{0,S}} \mathbf{c}(\eta) \langle \lambda_{\eta,S}, v_S \rangle$$

where we have written $\mathbf{c}$ also for the restriction of $\mathbf{c}$ to $\tilde{H}_S$. Note that in [6] Theorem II.2.4 this result was given incorrectly and should be altered as in the Corrigendum.

Let $\tilde{G}_{i,\mathbf{A}} = GL_{r_i}(k_{\mathbf{A}})$ for $1 \leqslant i \leqslant s$. Let $\tilde{H}^i_{\mathbf{A}} = \tilde{H}_{\mathbf{A}} \cap \tilde{G}_{i,\mathbf{A}}$. Let D_i: $\tilde{G}_{i,\mathbf{A}} \xrightarrow{p} GL_{r_i}(k_{\mathbf{A}}) \xrightarrow{\det} k^{\times}_{\mathbf{A}}$. Then one has:

THEOREM 3.1. *With the notations above* $\mathbf{c}$ *is supported on* $\tilde{H}_{n,\mathbf{A}} \cdot (\Pi^s_{i=1} D_i^{-1}(k^{\times}))$. *Let* $\mathbf{c}^i$ *be the corresponding function for* $GL_{r_i}(1 \leqslant i \leqslant s)$ *with restriction of* e *to* $G_{i,\mathbf{A}}$. *Then if* $\eta \in H_{\mathbf{A}}$ *is of the form* $\eta = \eta_1 \ldots \eta_s$ *where* $\eta_i \in D_i^{-1}(k^{\times}) \cdot (\tilde{H}_{n,\mathbf{A}} \cap \tilde{H}^i_{\mathbf{A}})$ *we have*

$$\mathbf{c}(\eta) = \mathbf{c}^1(\eta_1) \ldots \mathbf{c}^s(\eta_s).$$

Note that in this theorem the η_i commute with one other. A primitive version of this theorem is to be found in [7].

Proof. Let S be a finite subset of $\Sigma(k)$ containing $\{v \mid |n|_v \neq 1\}$. Let N^S be the complementary factor to N^*_S in $N^*_{\mathbf{A}}$. Let $e^S = e|N^S$. Suppose $f \in F(\omega)$ can be written as

$$f_S \otimes F^0_s \colon \tilde{G}_S \times_{\mu_n(k)} \left(\tilde{\prod_{v \in \Sigma(k)-S}} \tilde{G}_v \right) \to \mathbf{C}$$

and

$$f^0_S = \bigotimes_{v \in \Sigma(k)-S} f^0_v.$$

Then as in [6] p. 120 one has

$$\int_{N^*_k} \bar{e}(n) \langle E(\omega), nf \rangle \, \mathrm{d}n = \sum_{\eta \in H^*_{n,k} \backslash M^*_{0,k}} \int_{N^*_S} \bar{e}_S(n) f_S(\eta n) \, \mathrm{d}n \cdot \int_{N^S} \overline{e^S}(n) f^0_S(\eta n) \, \mathrm{d}n.$$

The final factor here is not relevant to our considerations and we denote it by $J_S(\omega, e, \eta)$; it can be evaluated fairly explicitly.

The factor

$$\int_{N^*_S} \bar{e}_S(n) f_S(\eta n) \, \mathrm{d}n$$

depends only on the restriction of f_S to a certain subgroup of $\tilde{G}_S$ which we shall now describe, at least for sufficiently large S.

Let $r^s = \Pi_{v \in \Sigma(k)-S} r_v$ where r_v is the ring of integers of k_v. Let U_S be the topological semidirect product of the $k_v^{\times n} r_v^{\times}$ for $v \in \Sigma(k) - S$. This is a subgroup of $k^{\times}_{\mathbf{A}}$.

The η which occur in the sum above and which give a non-zero factor $J_S(\omega, e, \eta)$ are restricted to have

$$D_i(\eta) \in U_S$$

by definition of f_v^0. Thus as $D_i(\eta) \in k^\times$ we also have to have

$$D_i(\eta) \in U_S \cap k^\times.$$

We assume now that S is so large that r^s is a principal ideal domain. Then

$$U_S \cap k^\times \quad = \quad \{x \in k^\times \,|\, \mathrm{ord}_v\,(x) \equiv 0(\mathrm{mod.}n),\ v \in \Sigma(k) \;-\; S\}$$

so that the fractional ideal generated by x in r^s is an nth power. Consequently one has

$$U_S \cap k^\times \quad = \quad k^{\times n} \cdot (k \cap r^S)^\times.$$

Hence the sum depends only on the restriction of f_S to the subgroup

$$\{g \in \tilde{G}_S | D_i(g) \in k_S^{\times n} \cdot (k \cap r^S)^\times, \quad 1 \leqslant i \leqslant s\}.$$

and the same holds for the residues of $E(\omega)$ since this is an open subgroup. Hence

$$\sum_{\eta \in \tilde{H}_{n,S} \backslash \tilde{H}_S} \mathbf{c}(\eta) \cdot \langle \lambda_{\eta,S}, f_S \rangle$$

depends only on the restriction of f_S (at an exceptional ω). This means that we have to have in the sum $D_i(\eta) \in k_S^{\times n} \cdot (k \cap r^S)^\times$ whenever $\mathbf{c}(\eta) \neq 0$. If we now let S increase it follows that $\mathbf{c}$ is supported on

$$\{\eta \in \tilde{M}_{0,\mathbf{A}} | D_i(\eta) \in k_{\mathbf{A}}^{\times n} \cdot k^\times\}.$$

The same argument can be applied to

$$\tilde{G}_{1,\mathbf{A}} \times_{\mu_n(k)} \tilde{G}_{2,\mathbf{A}} \times_{\mu_n(k)} \cdots \times_{\mu_n(k)} \tilde{G}_{S,\mathbf{A}}$$

and we obtain an analogous formula. Since the subgroups of each of these as above are isomorphic the computation of $\mathbf{c}$ cannot distinguish between this group and $\tilde{G}_{\mathbf{A}}$. Thus the resulting $\mathbf{c}$'s are also equal, and the '$\mathbf{c}$' for

$G_{1,\mathbf{A}} \times_{\mu_n(k)} \cdots \times_{\mu_n(k)} \tilde{G}_{S,\mathbf{A}}$ is simply $\mathbf{c}^1(\eta_1) \ldots \mathbf{c}^s(\eta_s)$. This completes the proof of the theorem.

It will be useful to write $\mathbf{c}(r_1, r_2, \ldots, r_s; e, \chi, \eta)$ where we make the dependence on the additive character e and the exceptional quasicharacter explicit. One can easily verify that for $\theta \in H_k^*$

$$\mathbf{c}(r_1, \ldots, r_s; \theta e, \chi, \theta\eta) = \mathbf{c}(r_1, \ldots, r_s; e, \chi, \eta).$$

We shall find it useful to define $\mathbf{c}(1)$ to be the function defined on $H_k^* \tilde{H}_{n,\mathbf{A}}$ such that

$$\mathbf{c}(1; e, \chi, h) = 1 \quad h \in H_k^*$$

and $H \cong GL_1$. This is then consistent with all the results given above.

Moreover if χ_1 and χ_2 are exceptional for G then $\chi_1 = \chi_2\psi$ where ψ is $W(G)$-invariant, where $W(G)$ is the Weyl group of G (relative to H). We extend this $H_k^* \tilde{H}_{n,\mathbf{A}}$ by demanding that it be trivial on $H_k^* \cdot \tilde{H}_{n,\mathbf{A}}$. As it is trivial on $i(\mu_n(k))$ it can be extended to a $W(G)$-invariant quasicharacter on $\tilde{H}_{\mathbf{A}}$ (not uniquely). One can then verify that

$$\mathbf{c}(r_1, \ldots, r_s; e, \chi_1, \eta) = \mathbf{c}(r_1, \ldots, r_s; e, \chi_2, \eta)\psi(\eta)^{-1}.$$

Note that this also yields the assertion of Theorem 3.1 as to the support of $\mathbf{c}$.

§4. The Fourier coefficients

In this section we shall give an expression for

$$\int_{N_k^* \backslash N_{\mathbf{A}}^*} \langle E^*(\chi), nf \rangle \, \bar{e}(n) \, \mathrm{d}n$$

where e is a non-degenerate character of $N_{\mathbf{A}}^*$ trivial on N_k^*.

Let $S \subset \Sigma(k)$ be finite; for any adelic object $X_{\mathbf{A}}$, $\tilde{X}_{\mathbf{A}}$ (here an algebraically defined subset of $GL_{r,\mathbf{A}}$ or a metaplectic cover of such a set) we write X_S and $\tilde{X}_S$ for the component corresponding to S, and X^S or $\tilde{X}^S$ for the complementary factor. For a function f on $X_{\mathbf{A}}$ which can be written as $f(x_S \times x^S) = g_1(x_S)g_2(x^S)$ where $x_S \in X_S$ or $\tilde{X}_S$, $x^S \in X^S$ or $\tilde{X}^S$ we shall write $f_S = g_1$, $f^S = g_2$. In general f_S and f^S are not uniquely determined, but the notation will in general be used only when S is so large that f^S is a 'standard function'. In particular we can define e_S, e^S, χ_S, χ^S and all the components are quasicharacters.

We shall need some definitions and results concerning Whittaker functionals of representations of $(G^{(1)})_S$. We define, in a notation more precise than the previous $\lambda_{w,S}$

$$L_S(e_S, w, \omega_S)\colon V(\omega_S) \to \mathbf{C}$$

as

$$\langle L_s(e_S, w, \omega_S), f\rangle = \int_{N_S^*} f(w^{-1}n)\bar{e}_S(n)\,\mathrm{d}n$$

and

$$w \in \tilde{M}_{0,S}.$$

This can be expressed as a product over the places in S. Likewise we can define for $w \in \tilde{M}_S$ the intertwining operator

$$I_{w,S}(\omega_S)\colon V(\omega_S) \to V({}^{w}\omega_S)$$

as in [6] §I.2. Define the scalar $\theta_{S,w_1,w_2}(\omega_S)$ by

$$I_{w_1,S}({}^{w_2}\omega_S)I_{w_2,S}(\omega_S) = \theta_{S,w_1,w_2}(\omega_S)I_{w_1w_2,S}(\omega_S)$$

and $\tau_S(e_S, \omega_S, w; w_1, w_2)$ for $w \in \tilde{M}_S$ and $w_1, w_2 \in \tilde{M}_{0,S}$ by

$$\langle L_S(e_S, w\, {}^{w}\omega_S), I_{w,S}(\omega_S)f\rangle$$

$$= \sum_{w \in \tilde{M}_{0,S}/\tilde{H}_{n,S}} \tau_S(e_S, \omega_S, w; w_1, w_2)\,\langle L_S(e_S, w_2, \omega_S), f\rangle$$

valid for all $f \in V(\omega_S)$ as in [6] p. 75 or in [7]. One has the following properties of θ_S and τ_S:

PROPOSITION 4.1. *One has*

i) $\theta_{S,w_1,w_2w_3}(\omega_S)\theta_{S,w_2,w_3}(\omega_S) = \theta_{S,w_1,w_2}({}^{w_3}\omega_S)\theta_{S,w_1w_2,w_3}(\omega_S)$,

ii) *if for each $v \in S$ one has $l_v(w_1) + l_v(w_2) = l_v(w_1w_2)$ where $l_v(w)$ denotes the length of the* vth *component of w in $\tilde{M}_v/\tilde{H}_v \subset \tilde{M}_S/\tilde{H}_S$ then* $\theta_{S,w_1,w_2}(\omega_S) = 1$

iii) *if* $h_1, h_2 \in \tilde{H}_{n,S}$ *then*

$$\tau_S(e_S, \omega_S, w; w_1h_1, w_2h_2)$$

$$= ({}^w\omega_S\mu_S)(h_1)^{-1}(\omega_S\mu_S)(h_2)\tau_S(e_S, \omega_S, w; w_1, w_2),$$

iv) *for* $\eta \in \tilde{H}_S$ *one has*

$$\tau_S(\eta e_S, \omega_S, w; w_1, w_2) = \tau_S(e_S, \omega_S, w; \eta^{-1}w_1, \eta^{-1}w_2),$$

$$\tau_S(e_S, \omega_S, \eta w; w_1, w_2) = \mu_S(\eta)\mu_S(\eta^w)^{-1}\tau_S(e_S, \omega_S, w; w_1\eta, w_2)$$

and

$$\tau_S(e_S, \omega_S, w; w_1\eta, w_2\eta^w) = \mu_S(\eta^w)\mu_S(\eta)^{-1}\tau_S(e_S, \omega_S, w; w_1, w_2),$$

v) $\tau_S(e_S, \omega_S, w'w''; w_1, w_2)\theta_{S,w',w''}(\omega_S)$

$$= \sum_{w \in \tilde{M}_{0,S}/\tilde{H}_{n,S}} \tau_S(e_S, {}^{w''}\omega_S, w'; w_1, w)\, \tau_S(e_S, \omega_S, w''; w, w_2)$$

vi) $\tau_S(e_S, \omega_S, I; w_1, w_2) = 1$ *if* $w_1 = w_2$
$= 0$ *if* $w_1^{-1}w_2 \notin \tilde{H}_{n,S}$.

The properties listed here follow directly from the definitions and can be left to the reader. It is worth noting that θ_S can be computed explicitly using [6] Theorem I.2.6 but we shall not need that here. For a technique for computing $\tau_S(e_S, \omega_S, w; w_1, w_2)$ when w is a simple reflection see [7] §3.4 and [6] Lemma I.3.3. Note that all the necessary information about the complex places is given in [6] §I.6; we shall not need to discuss real places as we shall concentrate on the case $n > 2$.

We now define the subset W^+ of $W(G^{(1)})$ by demanding that $w(\alpha) > 0$ for every positive root in $G \cap N$. One has then W^+ is a set of representatives for $W(G^{(1)})/W(G)$. Next let M^+ be the lift of W^+ back to M. Let $\tilde{M}^+ = p^{-1}(M^+)$.

Let w_0 be the element

$$\kappa\left(\begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ 1 & 0 & \dots & 0 \end{bmatrix}\right) \in \tilde{M}_0(G^{(1)})$$

representing the longest element of $W(G^{(1)})$. Let $s(G)$ be as in §2. Then the maps

$$\tilde{M}^0(G) \to \tilde{M}^0(G^{(1)});\ w \mapsto s(G)^{-1}w \quad \text{and} \quad w \mapsto ws(G)$$

are bijections. These sets, in which we have specified the reference group, can be understood globally, locally or semilocally. In the latter two cases $s(G)$ is the standard lift, [6] §0.1.

We now need a purely local theorem dealing with the generic places; it is a generalization of the main theorem of [3]. Suppose F is a local field in which $|n|_F = 1$ and that Card $\mu_n(F) = n$. We call a quasicharacter ω of $\tilde{H}_{n,F}$ *unramified* if $\omega|\tilde{H}_{n,F}K^* = 1$. Let e be a non-degenerate character of N^*. We call e *unramified* if $e|N^* \cap K^* = 1$ but $\eta e|N^* \cap K^* \neq 1$ for any $\eta \in \tilde{H}_F$ for which there exists a positive root α with $|\eta^\alpha|_F < 1$. Concepts defined over k_S above can be transferred to the case of F without any trouble.

THEOREM 4.2. *Suppose that ω and e are unramified. Then one has for $f \in V(\omega)^{K^*}$*

$$\begin{aligned} \langle L(e, w, \omega), \eta f\rangle &= \sum_{w' \in \tilde{M}/\tilde{H}_n} \tau(e, {}^{w'}\omega, w'^{-1}, w, w) I_{\eta^{-1}ww'} f(I) \\ &\qquad \textit{if } |\eta^\alpha|_F \leqslant 1 \textit{ for all positive } \alpha, \\ &= 0 \quad \textit{otherwise.} \end{aligned}$$

The proof of Theorem 4.2 will be based on the following proposition:

PROPOSITION 4.3. *Suppose that ω is unramified and dominant. Then for $f \in V(\omega)^{K^*}$ one has that $\langle L(e, w, \omega), f\rangle$ can be written as*

$$\sum_{w' \in W} \langle L^*_{w'}(e, w, \omega), f\rangle$$

if $e|N^* \cap K^* = 1$, where, for $h \in H_n$

$$L^*_{w'}(he, w, \omega) = (\omega/{}^{w'}\omega)(h^{w_0}) L^*_{w'}(e, w, \omega).$$

Proof. The proof follows [3] and is based on an idea of Casselman [2]. Let $B_1 \subset K$ be the Iwahori subgroup

$$B_I = \{k \in K\colon k \equiv I\ (\mathrm{mod}.P_F)\} \cdot (B_n \cap K).$$

Then $B_n \backslash G/B_I$ can be identified set-theoretically with $\tilde{H}_n \backslash \tilde{M}$. Let B_I^* be B_I considered as a subgroup of K^*.

Let $V(\omega)$ be as above and let $V(w) \to V(\omega)_N$ be the map induced by the Jacquet functor. Let $V(\omega)^{B_I^*}$ be the space of B_I^*-invariants in $V(\omega)$ and consider the composite map

$$j \colon V(\omega)^{B_I^*} \hookrightarrow V(\omega) \longrightarrow V(\omega)_N.$$

Casselman proves in [2] §2 that this map is an isomorphism of vector spaces when $n = 1$. Both proofs in [2] can be adapted to the case of covering groups using [1] Theorem 5.2. Here we shall give a fairly elementary proof of this isomorphism.

First of all we show that $V(\omega)^{B_I^*}$ and $V(\omega)_N$ have the same dimension. This is so as by [6] Proposition I.2.1. one has

$$\dim (V(\omega)_N) \;=\; n^{2r} \cdot r!,$$

and, since $\tilde{G}$ is the disjoint union of the open cosets $\tilde{B}_n \cdot w \cdot B_I^*$ where $w \in \tilde{H}_n \backslash \tilde{M}$ one also has

$$\begin{aligned} \dim (V(\omega)^{B_I^*}) &= \operatorname{Card} (\tilde{H}_n \backslash \tilde{M}) \\ &= n^{2r} \cdot r!. \end{aligned}$$

This proves the identity of the dimensions. Thus it is only necessary to show that j is injective. To do this we need some geometrical facts. Let W be the Weyl group of GL_r and write $w_1 \leqslant w_2$ ($w_1, w_2 \in W$) if $l(w_1) + l(w_2 w_1^{-1}) = l(w_2)$, and $w_1 < w_2$ if $w_1 \leqslant w_2$, $w_1 \neq w_2$. We lift these relations to $\tilde{M}$. Then we have firstly that

i) $w_1 N^* \cap (\tilde{B}_n \cdot w \cdot B_I^*) \neq \varnothing$ only if $w_1 \geqslant w$; this follows directly from [2] Proposition 1.3(a). Moreover the proof of [2] Proposition 1.3 (b) can easily be adapted to show that

ii) if $\eta \in \tilde{H}$ and $(\eta w N^*) \cap (\tilde{B}_n \cdot w \cdot B_I^*) \neq \varnothing$ then $\eta \in \tilde{H}_n$,

and

iii) $(\tilde{B}_n w N^*) \cap (\tilde{B}_n w B_I^*) = \tilde{B}_n w (N^* \cap K^*)$.

Let now χ_w be that function in $V(\omega)^{B_I^*}$ which is supported on $\tilde{B}_n w B_I^*$ with $\chi_w(w) = 1$. Then the χ_w form a base of $V(\omega)^{B_I^*}$ as w runs through a set of representatives of $\tilde{H}_n \backslash \tilde{M}$. Suppose now that $\Sigma\ c(w)\chi_w$ were a non-trivial linear combination of the χ_w where the w lie in a fixed set of representative of $\tilde{H}_n / \tilde{M}$, such that it maps to zero in $V(\omega)_N$ under j. By Jacquet's Lemma there exists an open compact subgroup U of N^* so that

$$\int_U \sum c(w)\chi_w(gu)\, \mathrm{d}u \;=\; 0.$$

Without loss of generality we may assume that $U \supset N^* \cap K^*$. Let w_1 be minimal so that $c(w_1) \neq 0$. Set $g = w_1$ and by i), ii) we have

$$c(w_1) \int_U \chi_{w_1}(w_1 u) \, \mathrm{d}u = 0.$$

By iii) the intergral is non-zero and hence $c(w_1) = 0$. This is a contradiction; hence no such linear combination exists, and j is injective, as claimed.

We shall now consider the representations $V(\omega)$ in Bernstein's generic sense described in [6] §I.2. There exists, as j is an isomorphism, a basis of $V(\omega)^{B_I^*}$ dual to the linear forms $f \mapsto (I_w f)(I)$ induced by the intertwinning operators. Denote the elements of this basis by $f_{w'}$. If $f \in V(\omega)^{K^*}$ then $f \in V(\omega)^{B_I^*}$ and hence there is a representation of f as $\Sigma c(w_1, f) \cdot f_{w_1}$.

It now follows that

$$\langle L(e, w, \omega) f \rangle = \sum_{w_1} c(w_1, f) \cdot \langle L(e, w, \omega), f_{w_1} \rangle.$$

We now observe that

$$g \mapsto \int_{N^* \cap K^*} f_{w'}(gna) \, \mathrm{d}n \qquad (*)$$

for a satisfying $|a^\alpha|_F \leqslant 1$ for every positive root α is itself in $V(\omega)^{B_I^*}$ and can therefore be represented as $\Sigma\ t(w_1) f_{w_1}$. Apply the functional $f \mapsto (I_{w_2} f)(I)$ the element of $V(\omega)$ defined by $(*)$; it is clear that since this functional factors through the Jacquet functor $V(\omega) \to V(\omega)_{N^*}$ that the result is

$$(I_{w_2} f_{w'})(a) = \mu(a)^2 (I_{a^{-1} w_2} f_{w'})(I).$$

On the other hand, by the definition of the $t(w_1)$ this is also equal to $t(w_2)$. Thus we have shown that

$$\int_{N^* \cap K^*} f_{w'}(gna) \, \mathrm{d}n = \mu(a)^2 \cdot f_{w'}(a).$$

We now replace g by $w'^{-1} \cdot n'$, multiply by $\bar{e}(n')$ and integrate over n' in N^*. Suppose that $e|N^* \cap K^* = 1$ and that a is as above; then we obtain

$$\langle L(e, w', \omega), a f_{w'} \rangle = \mu(a)^2 \langle L(e, w', \omega), f_{aw'} \rangle.$$

Thus if η satisfies the condition that $|\eta^{\alpha}|_F \geqslant 1$ for all positive roots α we have that

$$\begin{aligned}\langle L(\eta e, w', \omega), f\rangle &= \sum_{w} I_w f(I)\, \langle L(\eta e, w', \omega), f_w\rangle \\ &= \sum_{w} I_w f(I)\, \langle L(e, \eta^{-1} w', \omega), \eta^{-1} f_w\rangle \mu(\eta)^2 \\ &= \sum_{w} I_w f(I)\, \langle L(e, \eta^{-1} w', \omega), f_{\eta w}\rangle.\end{aligned}$$

The assertion of the proposition follows if we take

$$\langle L^*_{w^*}(\eta e, w', \omega), f\rangle$$

to be

$$\sum \langle L(e, \eta^{-1} w', \omega), f_w\rangle\, I_{\eta w} f(I)$$

where the summation is taken over those w in $\tilde{H}_n \backslash \tilde{M}$ which project to w^* in $H \backslash M$.

We shall next identify the $\langle L^*_{w'}(e, w, \omega), f\rangle$ of Proposition 4.3. Suppose first that ω is dominant. Then we see that for $w' \neq 1$

$$L^*_{w'}(he, w, \omega) \to 0$$

if h runs through a sequence such that for each positive α one has $|(h^w)^{\alpha}| \to 0$. If this is so he converges boundedly and locally uniformly to 1 and hence in the limit we have

$$I_w(\omega)(f)(I) = \langle L^*_I(e, w, \omega), f\rangle.$$

This identifies L^*_I. By regularization this will remain valid for all ω.

Next we have for $w_1 \in \tilde{M}$ that

$$\langle L(e, w, {}^{w_1}\omega), I_{w_1} f\rangle = \sum_{w_2 \in \tilde{M}_0/\tilde{H}_n} \tau(e, \omega, w_1; w, w_2) \cdot \langle L(e, w_2, \omega), f\rangle.$$

This yields

$$\langle L^*_{w'}(e, w, {}^{w_1}\omega), I_{w_1} f\rangle = \sum_{w_2 \in \tilde{M}_0/\tilde{H}_n} \tau(e, \omega, w_1; w, w_2)\, \langle L^*_{w_1 w'}(e, w_2, \omega), f\rangle.$$

This we apply with $w_1 = w'^{-1}$. We obtain

$$\langle L^*_{w'}(e, w, w'^{-1}\omega), I_{w'^{-1}}f\rangle = \sum_{w_2 \in \tilde{M}_0/\tilde{H}_n} \tau(e, \omega, w'^{-1}; w, w_2) I_{w_2}(f)(I).$$

We now replace f by $I_{w'}f$ and ω by ${}^{w'}\omega$. This yields

$$\langle L^*_{w'}(e, w, \omega), f\rangle = \sum_{w_2 \in \tilde{M}_0/\tilde{H}_n} \tau(e, {}^{w'}\omega, w'^{-1}; w, w_2) I_{w_2 w'}(f)(I)$$

$$\times\ \theta_{w_2, w'}(\omega) \cdot \theta_{w'^{-1}, w'}(\omega)^{-1}.$$

Observe that

$$\theta_{w_2, w'}(\omega) \cdot \theta_{w_2 w', w'^{-1}}({}^{w'}\omega) = \theta_{w', w'^{-1}}({}^{w'}\omega),$$

$$= \theta_{w'^{-1}, w'}(\omega).$$

This means that if we sum over w' in W and set $w_2 = w \cdot \eta$ we obtain, as w is of maximal length, from Proposition 4.1.ii)

$$\langle L(e, w, \omega), f\rangle = \sum_{w'} \sum_{\eta \in \tilde{H}/\tilde{H}_n} \tau(e, {}^{w'}\omega, w'^{-1}, w, w\eta) I_{w\eta w'} f(I)$$

$$= \sum_{w'} \sum_{\eta} \tau(e, {}^{w'}\omega, w'^{-1}\eta^{-1}; w, w) I_{w\eta w'} f(I)$$

This we regard as a sum over $\tilde{M}/\tilde{H}_n$ and we obtain now

$$\langle L(e, \omega, w), f\rangle = \sum_{w' \in \tilde{M}/\tilde{H}_n} \tau(e, {}^{w'}\omega, w'^{-1}; w, w) I_{ww'} f(I)$$

The assertion of the theorem now follows from the elementary relation

$$\langle L(e, \eta, w), \eta f\rangle = \mu(\eta)^2 \langle L(\eta^{-1}\varepsilon, \omega\eta^{-1} w), f\rangle.$$

and Proposition 4.1.iv).

REMARK. As in [3] the proof here uses no features of the series GL_r which are special to this series. Therefore the results will remain valid also for the metaplectic covers of other Chevalley groups. It is also worth noting that we have also not used the non-degeneracy of e so that this result can also be applied in cases where e is degenerate. This is useful in other applications.

We are now in a position to carry out the computation which is the main purpose of this section.

THEOREM 4.4. *Suppose* $\chi \in \Omega_\varepsilon^{\mathrm{exc}}(G)$ *is such that*

$$\sigma(\chi_{\alpha_0}^n) > n.$$

Suppose that $S \subset \Sigma(k)$ *is such that* ω *and* e *are unramified outside* S *and that* $f \in F^{\#}(\chi)$ *can be represented by* $f_S \otimes f^S$ *where* $f^S = \otimes_{v \in \Sigma(k)-S} f_v^0$. *Let* U *be the unipotent subgroup of the standard parabolic subgroup of* $G^{(1)}$ *with Levi component* G. *Let* $\Phi_+(U)$ *be the set of positive roots associated with* U. *Then*

$$\int_{N_k^*\backslash N_{\mathbf{A}}^*} \bar{e}(n) \, \langle E^{\#}(\chi), nf \rangle \, \mathrm{d}n$$

is equal to

$$\lim_{S_1 \nearrow, S_1 \supset S} \sum_{\eta \in \tilde{M}_{0,S}/\tilde{H}_{n,S}} \mathbf{c}(r_1, r_2; e', \chi, \eta) \cdot \int_{U_S^*} \bar{e}(u) \langle \lambda\eta_S, s(G)^{-1} u f_S \rangle \, T(S)^{-1} \, \mathrm{d}u$$

$$\cdot \prod_{v \in S_1 - S} \sum_{w \in (M_v^+ \cap K_v^*)} \tau(e_v, {}^{w}\chi_v, w^{-1}; (s(G)\eta)_v, w_0) \prod_{\substack{\alpha \in \Phi_+(U) \\ w\alpha > 0}} L((\chi_\alpha^n)_v) L(|\ |_v \chi_\alpha^n)^{-1}$$

Proof. We have that

$$\langle E^{\#}(\chi), f \rangle = \sum_{\gamma \in \Pi_k^* \backslash G_k^{(1)*}} \langle \Theta(\chi), \gamma f \rangle.$$

Since e is non-degenerate a standard computation using the Bruhat decomposition shows that

$$\int_{N_k^*\backslash N_{\mathbf{A}}^*} \bar{e}(n) \langle E^{\#}(\chi), nf \rangle \, \mathrm{d}n$$

$$= \int_{U_{\mathbf{A}}} \bar{e}(u) \int_{N_k'^*\backslash N_{\mathbf{A}}'^*} \overline{e'}(n) \, \langle \Theta(\chi), ns(G)^{-1} uf) \rangle \, \mathrm{d}n \, \mathrm{d}u$$

where $N' = N \cap G$ and if Π' is the standard parabolic subgroup associated with G' then U' is its unipotent radical. Here $e'(n) = e(n^{s(G)})$ for $n \in N_{\mathbf{A}}'^*$. The intergral on the right has remained absolutely convergent, and so we can rewrite it as

$$\lim_{S_1 \nearrow} \int_{U_{S_1}^{\prime *}} \overline{e(u)} \int_{N_k^{\prime *} \backslash N_{\mathbb{A}}^{\prime *}} \bar{e}^{\prime}(n) \langle \Theta(\chi), ns(G)^{-1} uf \rangle \, \mathrm{d}n \, \mathrm{d}u.$$

If now we restrict S_1 by $S_1 \supset S$ then we have by the results recalled in §3 that this is equal to

$$\lim_{S_1 \nearrow} \int_{U_{S_1}^{\prime *}} \bar{e}(u) \cdot \sum_{\eta \in \tilde{H}_{n,S_1} \backslash \tilde{M}_{0,S_1}} \mathbf{c}(r_1, r_2; e^{\prime}, \chi, \eta) \langle \lambda_{\eta, S_1}, s(G)^{-1} uf_{S_1} \rangle \, \mathrm{d}u \cdot T(S)^{-1}.$$

Since $\tilde{H}_{n,S_1} \backslash \tilde{H}_{S_1}$ is finite the intergral and sum are absolutely convergent ([6] §I.3).

We separate the places in S from the others and obtain

$$\lim_{S_1 \nearrow} \sum_{\eta \in \tilde{H}_{n,S_1} \backslash \tilde{M}_{0,S_1}} \mathbf{c}(r_1, r_2; e^{\prime}, \chi, \eta) \cdot \int_{U_S^{\prime *}} \cdot \, \bar{e}(u) \cdot \langle \lambda_{\eta_S S}, s(G)^{-1} uf_S \rangle \, \mathrm{d}u T(S)^{-1}$$

$$\times \prod_{v \in S_1 \ S} \left\{ \int_{U_v^*} \langle \lambda_{\eta_v}, s(G)^{-1} uf_v^0 \rangle \bar{e}_v(u) \, \mathrm{d}u \cdot T_v^{-1} \right\}$$

where λ_η is formed with respect to $e^{\prime}$ and

$$T_v = \left\{ \prod_{1 \leqslant j \leqslant r_1 - 1} (1 + q_v^{-1} + \cdots + q_v^{-j}) \right\}$$

$$\times \left\{ \prod_{1 \leqslant k \leqslant r_2 - 1} (1 + q_v^{-1} + \cdots + q_v^{-k}) \right\}.$$

By Fubini's theorem the vth factor in the latter integral is precisely what is denoted by

$$\langle L(e_v, (s(G)\eta)_v, \chi_v), f_v^0 \rangle \cdot T_v^{-1}.$$

We can now apply Theorem 4.2 since e_v is unramified at v. We obtain then for this factor:

$$\sum_{w^{\prime} \in \tilde{M}_v / \tilde{H}_{n,v}} \tau(e_v, {}^{w^{\prime}}\chi_v, w^{\prime -1}, (s(G)\eta)_v, (s(G)\eta)_v) T_v^{-1} (I_{(s(G)\eta)_v w^{\prime}} f^0)(I)$$

By [6] Theorem I.2.4 we can restrict the summation in w' to w' of the form $(s(G)\eta)_v^{-1} w_0 \cdot w''$ where w'' runs through $(\tilde{M}_v \cap K_v^*)/(\tilde{H}_{n,v} \cap K_v^*)$. We note further by Proposition 4.1.iv) that

$$\tau(e_v, {}^{w'}\chi_v, w'^{-1}, (s(G)\eta)_v, (s(G)\eta)_v)$$

$$= \tau(e_{v'}\, {}^{w''}\chi_v, w''^{-1}, (s(G)\eta)_v, w_0).$$

From [6] Theorem I.2.4 we now obtain

$$\sum_{w''} \tau(e_{v'}, {}^{w''}\chi_{v'} w''^{-1}, (s(G)\eta)_v, w_0) \cdot \prod_{\substack{\alpha>0 \\ w''\alpha>0}} \frac{L((\chi_\alpha^n)_v)}{L(|\ |_v(\chi_\alpha^n)_v)} \cdot T_v^{-1}$$

for the vth factor above.

In view of the linear relations satisfied by the $\mathbf{c}$ we see that any w'' for which there exists an $s \in W(G) \subset W(G^{(1)})$, a simple reflection, with

$$l(w''s) < l(w'')$$

can be omitted from the summation. But this means precisely that w'' should represent an element of W^+.

Next, note that if α is a root of G then for exceptional χ the L-factors are explicitly computable. These factors together yield T_v. Thus our expression reduces to

$$\sum_{w'' \in (M_v^+ \cap K_v^*)/(H_n \cap K_v^*)} \tau(e_v, {}^{w''}\chi_v, w''^{-1}, (s(G)\eta)_v, w_0) \cdot \prod_{\substack{a \in \Phi_+(U) \\ w''\alpha>0}} \frac{L((\chi_\alpha^n)_v)}{L(|\ |_v(\chi_\alpha^n)_v)}$$

§5. The main theorem

We can now assemble all that we have done to prove the main theorem. To formulate it we introduce a semilocal space of functions, $\mathscr{H}_S(\chi_S)$ where S is a finite subset of $\Sigma(k)$ as in §4 and χ exceptional. This space consists of functions $\varphi\colon \tilde{M}_{0,S} \to \mathbf{C}$ satisfying

$$\varphi(h\eta) = (\chi\mu)_S^{-1}(h)\varphi(\eta) \quad (h \in \tilde{H}_{n,S}, \eta \in \tilde{M}_{0,S}).$$

It is finite dimensional. We can regard this as a fibre of a holomorphic vector bundle over the complex manifold of exceptional quasicharacters $\tilde{H}_{n,S} \to \mathbf{C}^\times$ which we denote by $\Omega_\varepsilon^{\mathrm{exc}}(G, S)$. This is one-dimensional but not connected (if $n > 1$).

In particular we have an operator

$$\tau_S(e_S, \chi_S)\colon \mathscr{H}_S(\chi_S) \to \mathscr{H}_S(\chi_S^t)$$

defined by

$$(\tau_S(e_S, \chi_S)\varphi)(\eta) = \sum_{\eta' \in \tilde{M}_{0,S}/\tilde{H}_{n,S}} \tau_S(e_S, \chi_S, s(G); \eta, \eta')\varphi(\eta')$$
$$\times \prod_{v \in S} \prod_{\alpha \in \Phi_+(U)} \varepsilon_v(\chi_\alpha^n, e_v)L_v(\| \ \|\chi_\alpha^{-n})/L_v(\chi_\alpha^n)$$

where τ_S is the semilocal analogue of the τ defined in §4 and ε is as in [6] §II.1, e being a character of $k_{\mathbb{A}}$ trivial on k unramified outside S.

We define next a local modification $\hat{\tau}$ as follows:

$$\hat{\tau}(e, {}^w\chi, w^{-1}; \eta, \eta') = \tau(e^w, \chi, w^{-1}; \eta, \eta')$$
$$\times \prod_{\substack{\alpha \in \Phi_+(U) \\ w\alpha > 0}} L(\chi_\alpha^n) \cdot \prod_{\substack{\alpha \in \Phi_+(U) \\ w\alpha < 0}} L(| \ |_F \chi_\alpha^n)$$

where χ is in $\Omega_\varepsilon^{\mathrm{exc}}(G)$ and $w \in \tilde{M}^+$.

For $S \subset \Sigma(k)$ we define $L_S(\omega)$ for a quasicharacter ω of $k_{\mathbb{A}}^\times$ trivial on $k^\times$ to be the product taken over all $v \in S$ of the local L-functions.

We can now define for $\varphi \in \mathscr{H}_S(\chi_S)$, $\sigma(\chi_{\alpha_0}^n) > n$

$$\psi_S(\varphi, e, \chi) = \lim_{S_1 \nearrow, S_1 \supset S} \left\{ \sum_{\eta \in M_{0,S_1}/\tilde{H}_{n,S_1}} \mathbf{c}(r_1, r_2; e^t, \chi, \eta) \cdot (\eta_S) \right.$$
$$\left. \times \prod_{v \in S_1 - S} \left[\sum_{w \in (\tilde{M}_0^+ \cap K_v^*)/(\tilde{H}_{n,v} \cap K_v^*)} \hat{\tau}(e_v, {}^w\chi_v, w^{-1}; (s(G)\eta)_v, w_0) \right] \right\}.$$

The corresponding construction with respect to $G(r_2, r_1)$ will be denoted by ψ^t.

THEOREM 5.1. *The functional $\psi_S(e)$ has a meromorphic continuation to $\Omega_\varepsilon^{\mathrm{exc}}(G, S)$ as an integral function of finite order. It satisfies the functional equation*

$$\psi_S^t(\tau_S\varphi, e^t, \chi^t v^{-2+4/(r_1+r_2)}) = \psi_S(\varphi, e, \chi).$$

The only singularity of $\psi_S(\varphi, e, \chi)$ *in*

$$\{\chi \colon \sigma(\chi^n_{\alpha_0}) > 0\}$$

is on $\{\chi \colon \chi^n_{\alpha_0} = \|\ \|_{\mathbf{A}}\}$. *Let*

$$T^*(S, r_1, r_2) = \prod_{v \in S - \Sigma_\infty(k)} \prod_{\substack{1 \leqslant i \leqslant r_1 \\ 1 \leqslant j \leqslant r_2}} (1 - q_v^{-i-j})$$

$$\cdot \prod_{1 \leqslant k \leqslant r_1} (1 - q_v^{-k}) \prod_{1 \leqslant l \leqslant r_2} (1 - q_v^{-l}) \Big/ \prod_{1 \leqslant m \leqslant r_1 + r_2} (1 - q_v^{-m}).$$

Then if χ_0 *is such that* $(\chi_0)^n_{\alpha_0} = \|\ \|_{\mathbf{A}}$ *one has*

$$\lim_{\chi \to \chi_0} L(\chi^n_{\alpha_0})^{-1} \cdot \psi_S(\varphi, e, \chi)$$

$$= \sum_{\eta \in M_{0,S}/\tilde{H}_{n,S}} \mathbf{c}(r_1 + r_2; e, \chi_0, \eta)\varphi(\chi_0, \eta)\, T^*(S, r_1, r_2) \prod_{\substack{1 \leqslant i \leqslant r_1 \\ 1 \leqslant j \leqslant r_2}} \zeta(i + j).$$

REMARKS.

1. The analytic continuation, functional equation and, at the moment, hypothetical bounds for $\mathbf{c}(r_1, r_2)$ would yield via the Phragmén-Lindelöf theorem (in the number-field case) or the maximum principle (in the function-field case) bounds for $\mathbf{c}(r_1 + r_2)$. These bounds would have amongst other consequences for large enough $\sigma(\chi^n_{\alpha_0})$ that one could define $\psi_S(\varphi, e, \chi)$ by following expression which would be more satisfactory than the one we have had to use:

$$\Bigg\{ \sum_{\eta \in \tilde{M}_{0,\mathbf{A}}/\tilde{H}_{n,\mathbf{A}}} \mathbf{c}(r_1, r_2; e', \chi, \eta)\varphi(\eta_S)$$

$$\cdot \prod_{v \in \Sigma(k) - S} \Bigg[\sum_{w \in M_0^+ \cap K_v^*/\tilde{H}_{n,v} \cap K_v^*} \hat{\tau}(e_v, {}^w\chi, w^{-1}, (s(G)\eta)_v w_0) \Bigg] \Bigg\}.$$

   This estimate would have other useful consequences. It will be give in the later papers of this series.
2. In the case of a function-field an integral function of finite order is to be understood as a 'rational function'.

3. One could investigate the further singularities of $\psi_S(e)$ by related methods. We shall not undertake this investigation here, as it plays no role in the further investigation of the **c**.

Proof. The meromorphic continuation follows more or less directly from Theorem 4.1 and [6] Lemma I.3.1 which shows that pointwise in χ any $\varphi \in \mathscr{H}_S(\chi_S)$ can be represented by a suitable expression of the form:

$$\int_{U_S^*} \bar{e}_S(u) \langle \lambda_{\eta_S}, s(G)^{-1} \cdot u \cdot f_S \rangle \, du \; T(S)^{-1}.$$

That the function is integral of finite order follows from the estimates of the truncated Eisenstein series given by the Maaß-Selberg relations – cf. [5] §6.

The functional equation follows from the functional equation for the Eisenstein series ([5] §§6,7) and the definition of τ_S.

Finally we can derive the final formula by substitution in the definition of $\mathbf{c}(r_1 + r_2)$ combined with the identification of $E^{\#}$ with a residue of the full Eisenstein series.

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References

1. I.N. Bernstein and A.V. Zelevinsky: Induced representations of reductive p-adic groups, I. *Ann. scient. Ec. Norm. Sup.* 10 (1977) 441–472.
2. W. Casselman: The unramified principal series of p-adic groups, I. The spherical function. *Comp. Math.* 40 (1980) 387–406.
3. W. Casselman and J. Shalika: The unramified principal series of p-adic groups, II. The Whittaker function. *Comp. Math.* 41 (1980) 207–231.
4. R.P. Langlands: *Onf the functional equations satisfied by Eisenstein series. Lecture notes*, 544. Springer (1976).
5. R.P. Langlands: Eisenstein series. *Proc. Symp. Pure Math.* IX (1966) 235–251.
6. D.A. Kazhdan and S.J. Patterson: Metaplectic forms. *Publ. Math. IHES* 59 (1984) 35–142 (corrigendum, *Publ. Math. IHES* 62 (1985) 149).
7. S.J. Patterson: *Whittaker models of generalized theta series. Sém.Théorie des Nombres de Paris, 1982–1983*. Birkhauser (1984) 199–232.