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1. Introduction

The study of Archimedean f-algebras has received a recent revival because of their intimate relationship with orthomorphisms. The interchange has not been one-sided and the study of orthomorphisms has gained from the renewed interest in f-algebras. There possibly remains a great deal to learn about the structure of an Archimedean Riesz space as a module over some f-algebra of (possibly generalized) orthomorphisms on it. This work arose from consideration of the, almost degenerately, simple case when the original Riesz space is itself an f-algebra. The injective hull of an Archimedean f-algebra, considered as a module over itself, has an algebraic description as its complete ring of quotients. The fact that this description ignores the order structure of the original algebra, and consequently fails to give the injective hull any such structure makes it rather unsatisfactory for us. When we add to this the difficulty of obtaining a concrete description of the complete ring of quotients, the situation cries out for clarification.

De Pagter, in [18], has identified the complete ring of quotients of a uniformly complete f-algebra with identity with the space of extended orthomorphisms on it. We are able to drop the requirement of an identity from this result, but in order to drop the requirement of uniform completeness we must generalize the notion of extended orthomorphism. This extension is a natural one, but is slightly surprising in that it seems only to have an interesting structure when defined on an f-algebra, rather than a general Archimedean Riesz space. The main theorem of the paper, from which the description of the injective hull is derived, is possibly of some interest to algebraists in that it gives an intrinsic description of those f-algebras which are injective as modules over themselves, the two conditions involved comprising one algebraic and one order-theoretic one. Another aspect of our results that may be of general interest is the fact that we have, via our order-theoretic description of it, a representation of the injective hull of any concretely represented *f*-algebra.

After preliminaries on rings and modules in §2, and on Riesz spaces and *f*-algebras in §3, we introduce the notion of a *weak orthomorphism* in §4. The main results of the paper are then presented in §5. Our notation and terminology are fairly standard, except that we have taken care to distinguish between ring- and order-theoretic notions which are normally given the same descriptions.

2. Preliminaries about rings and modules

Our rings will not be assumed to possess an identity. Even if a ring R has an identity we will not assume that it acts as the identity mapping on a module over R. A ring R is *von Neumann regular* if for every $r \in R$ where is $s \in R$ with $rsr = r$. A commutative ring is *semi-prime* if $r \in R$ and $r^k = 0$ for some positive integer k together imply that $r = 0$.

If M and N are modules over a commutative ring R, a mapping $T: M \to N$ is *R-linear* if

i) $T(m_1 \pm m_2) = Tm_1 \pm Tm_2 \quad (\forall m_1, m_2 \in M)$

ii) $T(rm) = rT(m) \quad (\forall m \in M, \forall r \in R)$

Note that even if R is an algebra then an R-linear mapping need not, in general, be linear. An R-module N is *injective* if whenever M_0 is a submodule of an R-module M and $T_0: M_0 \to N$ is R-linear then T_0 extends to an R-linear mapping of the whole of M into N. A ring R is *self-injective* if it is injective as a module over itself. Every R-module M can be embedded in a minimal injective R-module, and this embedding is unique to within an R linear isomorphism that fixes M. Such a minimal injective extension of M is called an *injective hull* of M.

Now suppose that R is a semi-prime commutative ring. A ring ideal D in R is *ring dense* (not the usual terminology, but it prevents confusion with order theoretic density) if the only $r \in R$ with $rd = 0$ for all $d \in D$ is the zero element of R. A *fraction* is an R-linear mapping of a ring dense ring ideal in R into R itself. Identify fractions that coincide on some ring dense ring ideal, and let $\underline{Q(R)}$ denote the resulting set. If D_1 and D_2 are ring dense ring ideals then so is $D_1 \cap D_2$, so that $Q(R)$ may be given a ring structure by defining addition and multiplication pointwise on the intersections of domains. This makes $Q(R)$ a von Neumann regular ring. There is a natural embedding of R into $Q(R)$, so that we may also regard $Q(R)$ as an R-module. See [14], §2.3 for details (the lack of an identity is no problem up to this stage). $Q(R)$ is called the *complete ring of quotients* of R. The development given by Faith in [10] may not be as accessible at Lambek's, but it does prove the following result without the need for assuming the existence of an identity (Proposition 19.34 and Corollary 19.35):

THEOREM 2.1. *If R is a semi-prime commutative ring then $Q(R)$ is an injective hull of R.*

We also require the following lemma, for which we know no explicit reference.

LEMMA 2.2. *If R is a semi-prime commutative ring and S a self-injective ring with $R \subseteq S \subseteq Q(R)$ then $S = Q(R)$.*

Proof. Since S is self-injective it coincides with $Q(S)$, so that we need only verify that $Q(R) \subseteq Q(S)$. If $T \in Q(R)$ and T has domain D, consider $SD = (sd: s \in S, d \in D)$. This is a ring ideal in S and is ring dense since if $t \in S$ and $t(sd) = 0$ for all $d \in D$ and $s \in S$ then let x be in the domain of t, when we regard t as an element of $Q(R)$, so that $tx \in R$. We then have $(tx)d = 0$ for all $d \in D$ so that $tx = 0$. As x was an arbitrary element of the domain of t, $t = 0$ and SD is ring dense in S. Since the extension $\bar{T}: SD \to S$ of T defined by $\bar{T}(sd) = s(Td)$ (regarding s as an element of $Q(R)$) is clearly S-linear, we have the desired embedding of $Q(R)$ into $Q(S)$.

3. Preliminaries about *f*-algebras

Although we will define *f*-algebras as being Riesz spaces with extra structure, we need to state relatively little about Riesz spaces. We follow the terminology of [17] for all unexplained terminology in this section, except in three respects. We will use the term *lattice ideal* to distinguish the order-theoretic notion of ideal from the ring-theoretic one. We will say that a Riesz space E is *laterally complete* if any disjoint subset of E_+ has a supremum. Laterally complete Archimedean Riesz spaces have the projection property [2], so if they are also uniformly complete then they are Dedekind complete. Also we call a sublattice H of a Riesz space E *strongly order dense* if for each $x \in E_+$ with $x > 0$ there is $h \in H_+$ with $x \geqslant h > 0$. Archimedean Riesz spaces may be thought of as function lattices because of the many representation theorems that exist for them. The most useful of these is probably that due to Bernau [1], Theorem 4:

THEOREM 3.1. *If E is an Archimedean Riesz space then there is a Stonean space S and a Riesz space isomorphism of E onto a strongly order-dense sub-lattice of $C^\infty(S)$ which is order continuous as a mapping into $C^\infty(S)$.*

If E is laterally complete and uniformly complete then this isomorphism has image the whole of $C^\infty(S)$.

An *f-algebra* is a Riesz space, A, which is also an associative algebra and where the two structures are related by the conditions:

i) $x, y \geqslant 0 \Rightarrow xy \geqslant 0$ $(\forall x, y \in A)$

ii) $y \perp z \Rightarrow (xy \perp z$ and $yx \perp z)$ $(\forall x, y, z \in A)$.

This definition places some strong constraints on an Archimedean f-algebra A. It must, for example, be commutative. The ring ideal, N, of all nilpotent elements of an Archimedean f-algebra is also a lattice ideal, and even a band. Furthermore, for any $x \in A$, $x \in N \Leftrightarrow x^2 = 0 \Leftrightarrow xy = 0$ $(\forall y \in A)$, so that A is semi-prime if and only if $N = \{0\}$. Clearly an f-algebra with an identity must be semi-prime. For example, if E is any Archimedean Riesz space, a linear operator T on E is an *orthomorphism* if it is order bounded and leaves all bands invariant. With the natural linear and order structure, *Orth(E)*, the set of all orthomorphisms on E is an f-algebra with identity under composition. We may again think of semi-prime Archimedean f-algebras as function spaces because of another result of Bernau [1], Theorem 13:

THEOREM 3.2. *If A is a semi-prime Archimedean f-algebra then there is a Stonean space S and a mapping of A, onto a subspace of $C^\infty(S)$ which is both a strongly order-dense sub-lattice and a sub-ring, which is both a Riesz isomorphism and a ring isomorphism. As a mapping into $C^\infty(S)$ it is order continuous.*

The situation for those f-algebras which are not semi-prime is slightly more complicated, but not greatly:

THEOREM 3.3. *If A is an Archimedean f-algebra then there is a Stonean space S, a function $\phi \in C^\infty(S)_+$ and a Riesz space isomorphism, $x \rightarrow x^\wedge$, of A onto a strongly order-dense sub-lattice of $C^\infty(S)$ which is order continuous as a mapping onto $C^\infty(S)$ and with $(xy)^\wedge(s) = x^\wedge(s)y^\wedge(s)\phi(s)$ for all $x, y \in A$ and for all $s \in S$ for which the product is defined.*

Proof. Let $x \rightarrow x^\wedge$ be a representation of A in some $C^\infty(S)$, as is guaranteed by Theorem 3.1. We need only establish the existence of ϕ and verify its properties. Fix $x \in A$ and consider the operator L_x: $A \rightarrow A$ defined by $L_x y = xy$. This is a positive linear operator on A with the property that $y \perp z \Rightarrow L_x y \perp z$. It follows from [4], [5] or [19] that there is $\phi_x \in C^\infty(S)$ such that $(L_x y)^\wedge(s) = \phi_x(s)y^\wedge(s)$ for all $y \in A$ and for all $s \in S$ for which the product is defined.

Consider now the mapping $T: x^\wedge \to \phi_x$ of $(x^\wedge : x \in A)$ into $C^\infty(S)$. If $x \in A$ and $f \in C^\infty(S)$ are such that $x^\wedge \perp f$ then $\phi_x \perp f$, for if not there is, because of the strong order-density, $y \in A$ with $0 < y^\wedge \leqslant \phi_x \wedge f$. Now we see that $\phi_x y^\wedge \neq 0$ so that $L_x y \neq 0$. But $x^\wedge \perp y^\wedge$ so that $x \perp y$ and $L_x y = xy = 0$. Thus the map T, which is clearly positive and linear, also has the property that $x^\wedge \perp f \Rightarrow Tx^\wedge \perp f\, (\forall x \in A, \forall f \in C^\infty(S))$. The strong order density of $(x^\wedge : x \in A)$ in $C^\infty(S)$ means that the arguments of [18], Theorem 2.5, remain valid here and there is $\phi \in C^\infty(S)_+$ with $\phi_x(s) = Tx(s) = \phi(s) \cdot x^\wedge(s)$ for all $x \in A$ and for all $s \in S$ for which the product is defined. It follows that if $x, y \in A$ then $(xy)^\wedge(s) = (L_x y)^\wedge(s) = x^\wedge(s) y^\wedge(s) \phi(s)$ whenever the product is defined.

Note that A is semi-prime if and only if $\phi^{-1}(0)$ is nowhere dense. In that case the map $x \to x^\wedge \cdot \phi$ has the properties promised in Theorem 3.2.

Various properties of f-algebras have been considered in the past. One that appears to be of great importance is property $(*)$, introduced by Henriksen in [11], §3. An f-algebra A has property $(*)$ if, whenever $0 \leqslant x$, $y \in A$ satisfy $0 \leqslant x \leqslant y^2$, there is $0 \leqslant z \in A$ with $x = zy$. Uniformly complete f-algebras with an identity have property $(*)$ by [12], Theorem 3.11, but a uniformly complete semi-prime f-algebra need not have ([3], p. 136). However such a decomposition is possible on a large subset of A (the following proof is due to the referee):

THEOREM 3.4. *If A is a uniformly complete semi-prime f-algebra then there is a strongly order-dense lattice ideal and sub-ring, I, of A with property* $(*)$.

Proof. Since A is semi-prime it embeds canonically as a ring ideal and strongly order dense sublattice of Orth(A), by Proposition 2.1 of [9], letting e denote the identity in Orth(A), we may define

$$I = \{a \in A : \exists b \in A,\ 0 \leqslant b \leqslant e \quad \text{such that} \quad a = ba\}.$$

Clearly I is a lattice ideal and subring of A. To see that I is strongly order dense n A, fix $0 < a \in A$. As Orth(A) is Archimedean we may choose $n \in \mathbb{N}$ such that $(na - e)^+ > 0$. Now the facts that $na = (na \wedge e) + (na - e)^+$ and that $na \wedge e \in A$ (Theorem 2.5 of [9]) together imply that $(na - e)^+ \in A$. If $c = (a - n^{-1}e)^+$ then $0 \leqslant c \leqslant a$ and $c \in I$. The last fact is seen by considering $b = na \wedge e$ and noting that

$$bc = n^{-1}(na \wedge e)(na - e)^+ = n^{-1}(na - e)^+ = c.$$

To see that I has property $(*)$, considering $x, y \in I$ with $0 \leqslant x \leqslant y^2$. Since Orth($A$) has property $(*)$, there is $z \in$ Orth(A) with $x = yz$ and $0 \leqslant z \leqslant y$.

Choose $b \in A$ with $0 \leqslant b \leqslant e$ and $v = bv$, using the definition of I. Then $x = y(bz)$ and $bz \in A$ as A is a ring ideal in Orth(A). But $0 \leqslant bz \leqslant bv = v \in I$ implies that $bz \in I$, so I does have property (*).

Finally, let us note for future use:

LEMMA 3.5. *A laterally complete von Neumann regular Archimedean f-algebra has an identity.*

Proof. The supremum, a, of a maximal disjoint set of non-negative elements will be a weak order unit for the algebra. Choose b with $aba = a$ so that ab is an identity for a^{dd}, which is the whole of the f-algebra.

4. Weak orthomorphisms

DEFINITION 4.1. If E is an Archimedean Riesz space, a *weak orthomorphism* on E is an order bounded linear mapping $T: D \to E$, where D is a strongly order-dense sublattice of E, with the property that if $x \in D$, $y \in E$ and $x \perp y$ then $Tx \perp y$. The operator T is called an *extended orthomorphism* if D is an order-dense lattice ideal in E. If $D = E$ then T is an orthomorphism.

Extended orthomorphisms have been studied in detail recently in [7], [8], [9], [16] and [18]. The argument given in [6], on page 377, shows that every weak orthomorphism, T, may be written as $T^+ - T^-$ where T^+ and T^- are weak orthomorphisms with the same domain as T and with $T^+x = (Tx)^+$ and $T^-(x) = (Tx)^-$ if x is a non-negative element of the domain of T. Luxemburg and Schep's argument in Theorem 1.3 of [16] remains valid for positive weak orthomorphisms, so that weak orthomorphisms are order continuous. Furthermore the arguments of [19], Lemma 2.4 and Theorem 2.5, suffice to establish:

THEOREM 4.2. *Suppose E is an Archimedean Riesz space and T a weak orthomorphism on E with domain D. Let S be a Stonean space and $x \to x^\wedge$ be a Riesz isomorphism of E onto a strongly order-dense ideal in $C^\infty(S)$. There exists $T^\wedge \in C^\infty(S)$ such that*

$$(Tx)^\wedge(s) = T^\wedge(s)x^\wedge(s)$$

for all $x \in D$ and for all $s \in S$ for which the product is defined. Furthermore $T^{+\wedge}(s) = (T^\wedge)(s)^+$ for all $s \in S$.

Indeed a similar result is valid for any admissible representation of E (see [19], page 227) except that we may now only conclude that T is defined (and finite) on a dense open subset of S.

Since weak orthomorphisms seem to be well-behaved, why have they received no previous attention? The reason is that they do not, in general, have an additive structure.

DEFINITION 4.3. If M is a vector space of continuous functions on a topological space X, then we shall say that a function on X is *locally in* M if f is defined on a dense open subset of X and if it coincides, on some neighbourhood of each point of its domain, with some element of L. Write $L_M(X)$ for the Riesz space of equivalence classes of such functions, under the relation of coinciding on a dense open subset of X and with vector and lattice operations defined modulo dense open sets. Provided X has the Baire property $L_M(X)$ will be Archimedean and will be an f-algebra if M is an algebra. See [20] pages 90 and 91 for more details of this construction.

EXAMPLE 4.4. Let E be the space of functions on $I = [0, 1]$ of the form $x \to a + bx + ce^x$, F those of the form $x \to bx$ and G those of the form $x \to ce^x$. $L_F(I)$ is a strongly order dense sublattice of $L_E(I)$ as is $L_G(I)$. Define T_1: $L_F(I) \to L_E(I)$ by $T_1 f(x) = f(x)/x$ and T_2: $L_G(I) \to L_E(I)$ by $T_2 g(x) = g(x)/e^x$, so that both $T_1 f$ and $T_2 g$ must be locally constant. Clearly T_1 and T_2 are weak orthomorphisms and are defined on their largest possible domain, yet $L_F(I) \cap L_G(I) = (0)$, so there is no hope of defining $T_1 + T_2$.

Fortunately, this kind of behaviour is avoided if E is replaced by a semi-prime Archimedean f-algebra.

PROPOSITION 4.5. *Amongst those extensions of weak orthomorphisms defined on an Archimedean f-algebra, which are weak orthomorphisms, there is one which has a largest domain. This domain is both a strongly order-dense sublattice and ring ideal.*

Proof. Let $x \to x^\wedge$ be a representation of the Archimedean f-algebra A in some $C^\infty(S)$. Let T be a weak orthomorphism on E, represented by $T^\wedge$ as in Theorem 4.2. Consider now the set $M = (a \in A\colon \exists b \in A$ with $T^\wedge(s)a^\wedge(s) = b^\wedge(s)$, whenever the product is defined). This set, M, is a linear subspace of A. It is also a sublattice, for

$$\begin{aligned} T^\wedge(s)\,a^{+\wedge}(s) &= (T^+)^\wedge(s)\,a^{+\wedge}(s) - (T^-)^\wedge(s)\,a^{+\wedge}(s) \\ &= [(T^+)^\wedge(s)\,a^\wedge(s)]^+ - [(T^-)^\wedge(s)\,a^\wedge(s)]^+ \\ &= [(T^+a)^+ - (T^-a)^+]^\wedge(s) \end{aligned}$$

whenever all the products are defined. It is also a ring ideal in A, for if $m \in M$ and $a \in A$ then

$$\begin{aligned} T^\wedge(s)(ab)^\wedge(s) &= T^\wedge(s)\,a^\wedge(s)\,b^\wedge(s)\,\phi(s) \\ &= (Ta)^\wedge(s)\,b^\wedge(s)\,\phi(s) \\ &= [(Ta)b]^\wedge(s) \end{aligned}$$

whenever the product is defined. With this notation, the map which takes a to b is clearly a weak orthomorphism. Since any extension of T to a weak orthomorphism will be represnted by $T^\wedge$ it will have domain contained in M. This suffices to establish the claim, since M will be this largest domain.

DEFINITION 4.6. We denote by $\underline{M(T)}$ the largest domain of a weak orthomorphism extension of a weak orthomorphism T.

PROPOSITION 4.7. *If M and N are strongly order dense sublattices and ring ideals in a semi-prime f-algebra A, then the sublattice of A generated by the products $\{m \cdot n: m \in M, n \in N\}$ is also a strongly order dense sublattice and ring ideal in A.*

Proof. In order to show that it is order dense, consider the representation $b \to b^\wedge$, of A, in $C^\infty(S)$, that is given by Theorem 3.2. If $0 < a \in A$ then choose $m \in M$ with $0 < m < a$, since M is order dense in A. Now let $b \in A$ with $0 < b < m$ and $b^\wedge(s) \leqslant 1$ for all $s \in S$. This is possible by the strong order density of $(a^\wedge : a \in A)$ in $C^\infty(S)$. Now let $n \in N$ with $0 < n < b$. Since $n^\wedge(s) \leqslant 1$ for all $s \in S$, $0 < mn < m < a$, so that the strong order density is established.

To see that it is a ring ideal it suffices to consider $a \in A_+$ and a sum $\Sigma_{i=1}^k m_i n_i$. Note that $a(\Sigma_{i=1}^k m_i n_i) = \Sigma_{i=1}^k (am_i)n_i$ and hence $a \cdot (\Sigma_{i=1}^k m_i n_i)^+ = [a \cdot (\Sigma_{i=1}^k m_i n_i)]^+$ lie in it.

Let us denote by $\underline{\mathrm{Orth}^w(A)}$ the set of all weak orthomorphisms on A which have maximal domain. We may define an addition on $\mathrm{Orth}^w(A)$ by defining $(T_1 + T_2)(m) = T_1(m) + T_2(m)$ for m in the vector sublattice of A generated by the products $M(T_1) \cdot M(T_2)$, which is clearly contained in $M(T_1) \cap M(T_2)$, and then extending $T_1 + T_2$ to its largest domain. Consideration of $T + (-T)$ shows that the largest domain of $T_1 + T_2$ may be strictly larger than $M(T_1) \cap M(T_2)$. Ordering $\mathrm{Orth}^w(A)$ by $T_1 \geqslant T_2 \Leftrightarrow T_1(M) \geqslant T_2(m)$ for all $m \in M(T_1) \cap M(T_2)$ makes $\mathrm{Orth}^w(A)$ into an

Archimedean Riesz space. Notice that if $a \in A$ and $m \in M(T)$ then $T(am) = aT(m)$, so that if $T_1, T_2 \in \mathrm{Orth}^w(A)$ then the composition $T_1 T_2$ is defined on $M(T_1)$. $M(T_2)$, for if $m_1 \in M(T_1)$ and $m_2 \in M(T_2)$ then $T_2(m_1 m_2) = m_1(T_2 m_2) \in M(T_1)$. Thus we may extend $T_1 T_2$ to an element of $\mathrm{Orth}^w(A)$. It is routine to verify that $\mathrm{Orth}^w(A)$, as thus defined, is an Archimedean f-algebra with identity. We shall denote by $\mathrm{Orth}^\infty(A)$ the subspace of $\mathrm{Orth}^w(A)$ consisting of those operators T for which $M(T)$ contains an order-dense lattice ideal. The f-algebra $\mathrm{Orth}^w(A)$ is in fact rather a special one:

THEOREM 4.8. *If A is a semi-prime Archimedean f-algebra then* $\mathrm{Orth}^w(A)$ *is a laterally complete, von Neumann regular f-algebra.*

Proof. The argument of Theorem 2.3 of [7] shows that $\mathrm{Orth}^w(A)$ is laterally complete. If $T \in \mathrm{Orth}^w(A)$ then $T^+[M(T)]$ and $T^-(M(T)]$ are easily seen to be sub-lattices of A using Theorems 3.2 and 4.1 to represent A and T. With such a representation, suppose $a \in (T^+[M(T)])^{dd}$ and choose a non-empty open set K on which $a^\wedge$ is bounded below by $\varepsilon > 0$ and $T^\wedge$ is bounded above by n and below by $(n - 1)$ (say). Using the strong order density of $(m^\wedge : m \in M)$ in $C^\infty(S)$ to find $0 \neq b \in M(T)$ with b supported by K and $|b^\wedge(S)| < \varepsilon/n$ for $s \in K$, we see that $0 < Tb < a$. It follows that $T[M(T) \cap T[M(T)]^{dd}] \oplus T[M(T)]^d$ is a strongly order-dense sub-lattice of A. On it we may define S by

$$S(Tm) = m \quad (m \in M(T) \cap T[M(T)]^{dd})$$

$$Sn = 0 \quad (n \in T[M(T)]^d).$$

Clearly S is a weak orthomorphism and extends to an element $\bar{S}$ of $\mathrm{Orth}^w(A)$. We now have $T\bar{S}T = T$ as required.

EXAMPLE 4.9. If E consists of the polynomials on $I = [0, 1]$ and F consists of the rational functions on I, then $\mathrm{Orth}^w(L_E(I))$ may be identified with $L_F(I)$ whilst $\mathrm{Orth}\,(L_E(I))$ may be identified with $I_E(I)$ itself.

In the uniformly complete case, we have:

THEOREM 4.10. *If A is a uniformly complete semi-prime Archimedean f-algebra then* $\mathrm{Orth}^w(A) = \mathrm{Orth}^\infty(A)$.

Proof. Let I be a strongly order-dense lattice ideal and sub-ring in A such that if $0 \leqslant x, y \in I$ with $0 \leqslant x \leqslant y^2$ then there is $0 \leqslant z \in A$ with $x = zy$, as is guaranteed by Theorem 3.4. If $T \in \mathrm{Orth}^w(A)$, consider the set $J_+ = (x \in I\colon \exists y \in M(T) \text{ with } 0 \leqslant x \leqslant y^2)$. This is closed under multiplication by non-negative scalars and is closed under addition as $0 \leqslant x \leqslant y^2$ and

$0 \leqslant x' \leqslant y'^2 \Rightarrow 0 \leqslant x + x' \leqslant y^2 + y'^2 \leqslant (y + y')^2$. Hence $J = J_+ - J_-$ is a lattice ideal in A, which is order dense since it contains $I \cap M(T) \cdot M(T)$, for $m, n \in M(T)_+$ implies that $m \cdot n \leqslant (m + n)^2$. In fact $J \subseteq M(T)$, for our hypothesis on I guarantees that if $x \in J$ and $0 \leqslant x \leqslant y^2$ with $y \in M(T)$ then there is $z \in A$ with $x = yz$, and we need now only recall that $M(T)$ is a ring ideal to see that $x \in M(T)$. Thus $M(T)$ contains the order-dense lattice ideal J and hence $T \in \mathrm{Orth}^\infty(A)$.

5. Self-injectivity and injective hulls of *f*-algebras

Let us first present our characterization of self-injective *f*-algebras.

THEOREM 5.1. *An Archimedean f-algebra is self-injective if and only if it is laterally complete and von Neumann regular.*

Proof. Suppose first that A is a laterally complete, von Neumann regular *f*-algebra. Let M be an A-module, M_0 a submodule of M and $T_0: M_0 \to A$ be A-linear. It is routine to write M as a direct sum of modules on one of which the identity of A acts as the identity operator and on the other of which all products are zero. M_0 is decomposed in the same manner. The extension to the second summand needs only be additive so is easily seen to exist. We may thus concentrate on the case that the identity of A acts as the identity operator on M. A Zorn's lemma argument shows the existence of a maximal A-linear extension of T_0 to $T_1: M_1 \to A$, where M_1 is a submodule of M. Suppose that $x \in M \backslash M_1$, we show how to construct an A-linear extension of T_1 to the submodule $M_2 = (ax + m: a \in A, m \in M_1)$ of M. In particular, $x \in M_2$ and therefore $M_2 \neq M_1$. This contradicts the maximality of M_1 so that in fact $M_1 = M$ and we already have the desired extension.

If $a \in A$ there is $b \in A$ with $aba = a$. The element ab of A is an idempotent and it acts on A, via multiplication, as the band projection onto the band generated by a. We denote this by $P(a)$. Note that $P(a) \leqslant P(b)$ if and only if $a \in b^{dd}$. Consider the set $\mathrm{S} = (a \in A: ax \in M_1 \text{ and } a^{dd} = T_1(ax)^{dd})$ and let N be a maximal disjoint subset of S. If $n \in \mathrm{N}$ then we may, by the von Neumann regularity of A, find $m(n) \in A$ with $n^2 m(n) = n$. If we now define $c(n) = T_1(nx)m(n)$ then

$$c(n)n = T_1(nx)m(n)n = T_1(nx)$$

since $m(n)n$ is the identity on $n^{dd} = T_1(nx)^{dd}$. Note, furthermore, that $c(n)^{dd} = n^{dd}$. Let

$$s = \bigvee_{n \in N} c(n)^+ - \bigvee_{n \in N} c(n)^-$$

which exists by the lateral completeness of A, and note that $P(n)s = c(n)$ for each $n \in N$.

Define $T_2(ax + m) = as + T_1 m (a \in A, m \in M_1)$, so that (provided it is well-defined) T_2 will certainly be an A-linear extension of T_1, and will provide the desired contradiction. To see that T_2 is well-defined, suppose that $ax \in M$ for some $a \in A$. If $n \in N$ then

$$\begin{aligned} P(n)T_1(ax) &= T_1(P(n)ax) \\ &= T_1(m(n)nax) \\ &= am(n)T_1(nx) \\ &= ac(n) \\ &= aP(n)s \\ &= P(n)as. \end{aligned}$$

On the other hand, if $k \perp n$ for all $n \in N$ then $k \perp s$, so that $k \perp as$. We also have $T_1(ax) \perp k$ for, if not, let $k' = |T_1(ax)| \wedge |k|$ so that

$$\begin{aligned} P(k') &= P(T_1(ax))P(k) \\ &\leqslant P(T_1(ax)) \\ &= P(T_1(P(a)ax)) \\ &= P(P(a)T_1(ax)) \\ &\leqslant P(a). \end{aligned}$$

Thus we have:

$$\begin{aligned} P(T_1(k'ax)) &= P(T_1(k'P(a)x)) \\ &= P(k'P(a)T_1(ax)) \\ &= P(k')P(a)P(T_1(ax)) \\ &= P(k')P(a) \\ &= P(k'a). \end{aligned}$$

Thus we could extend N by adjoining $k'a$, contradicting the maximality of N. Thus we see that $T_1(ax) = as$, and the proof of this implication is complete.

Now suppose that A is self-injective. Endow $A \times \mathbb{R}$ with its usual addition derived from those in A and in $\mathbb{R}$, and define

$$a(b, \lambda) = (ab + \lambda a, 0) \quad (\forall a, b \in A, \forall \lambda \in \mathbb{R})$$

which makes $A \times \mathbb{R}$ into an A-module. The A-linear map defined on the submodule $A \times (0)$ by $T_0(\langle a, 0\rangle) = a$ must extend to an A-linear map of $A \times \mathbb{R}$ into A. Since, if $a \in A$,

$$aT(\langle 0, 1\rangle) = T(a\langle 0, 1\rangle) = T(\langle a, 0\rangle) = a,$$

$T(\langle 0, 1\rangle)$ is an identity for A. Let us denote this identity by e.

Now let B be a band in A. The map from $B \oplus B^d$ into A taking $b + c$ to b (for $b \in B$, $c \in B^d$) is A-linear so extends to $\bar{T}$ on the whole of A. If $b \in B$ and $c \in B^d$ then

$$(b + c)\bar{T}e = \bar{T}((b + c)e) = T(b + c) = b$$

so that $\bar{T}e$ acts, by multiplication, as the identity on B and as the zero operator on B^d. It follows that $\bar{T}e$ is the band projection of A onto B, so that B is a projection band in A. We thus see that A has the projection property.

If $c \in A$, let p denote that element of A which acts, by multiplication, as the band projection onto c^{dd}. From the submodule $N_0 = (ac\colon a \in A)$ of A and define T_0 on it by $T_0(ac) = ap$. This is A-linear so extends A-linearly to T defined on the whole of A. Since $T(c) = p$ and $pc = c$ we have

$$c^2T(p) = cT(cp) = cT(c) = cp = c$$

and thus A is von Neumann regular.

Finally, let $(x_i)_{i\in I}$ be a disjoint family in A_+. Let M be the A-module consisting of all mappings of I into A with $f(i) \in Ax_i$ for each $i \in I$, and let M_0 be the submodule consisting of those f with $f(i) = 0$ for all but finitely many i. The map $f \to \Sigma_{i\in I} f(i)$ of M_0 into A is A-linear, so extends to $T\colon M \to A$. Consider the element $g \in M$ with $g(i) = x_i$ for all $i \in I$, noting that A has an identity. For each $i \in I$,

$$P(x_i)Tg = T(P(x_i)g) = T(g_i) = x_i$$

where $g_i(j) = x_i$ if $i = j$ and $g_i(j) = 0$ if $i \neq j$. It follows that if B is the band generated by the x_i's and P_B is the band projection of B onto A, then $P_B(Tg)$ is the supremum in A of the family $(x_i)_{i \in I}$, showing that A is laterally complete and terminating the proof.

COROLLARY 5.2. *If A is a semi-prime Archimedean f-algebra then* $\mathrm{Orth}^w(A)$ *may be identified with $Q(A)$ and is an injective hull of A.*

Proof. By Theorem 4.2 and Proposition 4.5, we have embeddings of $A \subseteq \mathrm{Orth}^w(A) \subseteq Q(A)$. $\mathrm{Orth}^w(A)$ is von Neumann regular and laterally complete (Theorem 4.8) so is self-injective (Theorem 5.1). Lemma 2.2 now shows that $\mathrm{Orth}^w(A) = Q(A)$ and Theorem 2.1 completes the proof.

COROLLARY 5.3. *If A is a uniformly complete semi-prime Archimedean f-algebra then* $\mathrm{Orth}^\infty(A)$ *may be identified with $Q(A)$ and is an injective hull of A.*

This has been proved by de Pagter [18] in the case that A also has an identity.

These results are of interest because they enable us to describe concretely the complete ring of quotients of some function algebras via a representation of $\mathrm{Orth}^w(A)$.

EXAMPLE 5.4. If Σ is a locally compact Hausdorff space and $C_0(\Sigma)$ the algebra of all continuous real-valued functions on Σ that vanish at infinity, then $Q(C_0(\Sigma))$ may be identified with the algebra of all real-valued functions defined on dense open subsets of Σ, identifying functions that coincide on dense open subsets of Σ, with the algebra operation defined pointwise on the appropriate intersection of domains.

This is because the note after Theorem 4.1 implies that every fraction on $C_0(\Sigma)$ is of this form, and it is routine to verify that any such function has, as its maximal domain, a ring dense ring ideal in $C_0(\Sigma)$.

As a counterpoint to our positive results for semi-prime Archimedean f-algebras let us, finally, note:

PROPOSITION 5.5. *If an Archimedean f-algebra, A, has an injective hull that can be given an Archimedean f-algebra structure extending that of A and compatible with its A-module structure, then A is semi-prime.*

Proof. If E is an injective hull of A with this structure then the argument, used in the proof of Theorem 5.1 to show that a self-injective f-algebra has an identity, shows that there is $e \in E$ with $ea = a$ for all $a \in A$. Now if $0 \leqslant a \in A$ and $a^2 = 0$ we have

$$0 \leqslant (n^{-1}e - na)^2 = n^{-2}e - 2a + n^2a^2$$
$$= n^{-2}e - 2a$$

for all $n \in \mathbb{N}$. As this implies that $0 \nleqslant 2a \leqslant n^{-2}e$, we see that $a = 0$ and A is semi-prime.

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