

COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 62, n° 2 (1987), p. 187-213

http://www.numdam.org/item?id=CM_1987__62_2_187_0

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Global moduli for polarized elliptic surfaces

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Received 30 May 1986; accepted 7 October 1986

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Introduction

This paper is a continuation of [23], where I showed the existence of moduli schemes for elliptic surfaces with a section; now this result is applied to show the existence of moduli schemes for polarized elliptic surfaces not necessarily admitting a section. The idea is to show that the moduli functor to be represented is finite over a coarsely representable moduli functor, which is an extension of the moduli functor for elliptic surfaces with a section, and to show – using ideas of Seshadri – that one has coarse representability in such a situation. Since I need finiteness in the algebrogeometric sense (including properness), it turns out that even if one is only interested in nonsingular surfaces, one still has to consider surfaces with rational double points, too, and in some constructions even worse singularities have to be

admitted. A sufficiently general definition of elliptic surfaces is therefore given in §1.1, and it is shown that these surfaces still behave more or less in the usual way. The rest of §1 deals with constructions that are needed for the existence of the moduli scheme, which is shown in §2. The ground field is an arbitrary algebraically closed field, but because of the use of Weierstraß equations, characteristics two and three have to be excluded, and because of the use of Ogg–Šafarevič theory for the finiteness part, I can deal in characteristic $p > 0$ only with those surfaces which admit a polarization whose degree when restricted to a fiber is not divisible by p .

Work on this paper was begun while I was a guest of the Sonderforschungsbereich Theoretische Mathematik in Bonn, and completed here in Mannheim. I want to thank Professor Mumford, who suggested the problem of moduli of elliptic surfaces to me, and supplied many ideas, and Professor Popp for his help with the Galois theoretic approach to the finiteness condition.

§1. Families of elliptic surfaces

1.1. Basic properties of elliptic surfaces

Definition: An elliptic fibration is a morphism $f: X \rightarrow C$ from an integral projective surface X over an algebraically closed field k onto a nonsingular curve C over k , such that

- (i) all fibers of f are connected
- (ii) the general fiber of f is a nonsingular curve of genus one
- (iii) no fiber of f contains a one-dimensional singularity of X
- (iv) if ω is the dualizing sheaf on X , and ω^\vee its dual, then

$$(\omega \cdot \omega) := \chi(\mathcal{O}_X) - 2\chi(\omega^\vee) + \chi(\omega^\vee \otimes \omega^\vee) = 0.$$

X is called an elliptic surface, if it admits an elliptic fibration $f: X \rightarrow C$. A fiber F of f is called tame, if $h^0(F, \mathcal{O}_F) = 1$, and wild otherwise. f and X are called tame, if there are no wild fibers.

For nonsingular elliptic surfaces, (iv) means that the canonical class on X has self-intersection 0, and this is easily seen to be equivalent to the fact that no fiber of f contains an exceptional curve of the first kind, that is a nonsingular rational curve with self-intersection -1 . We shall see in a moment, that such curves are still impossible in the singular case, and that in fact almost everything can be generalized from nonsingular elliptic surfaces to elliptic surfaces as they are defined here. The reason for this is,

that an elliptic surface is locally a complete intersection, as one easily sees from the fact that all fibers of f have arithmetic genus one, and therefore the dualizing sheaf ω of X is invertible, so that we can speak of canonical divisors as in the nonsingular case. Also, every elliptic surface is normal, because conditions (ii) and (iii) ensure that X is nonsingular in codimension one. Therefore we also have an intersection pairing on X , constructed by Mumford in [14], II b, which has all the usual properties, except that the intersection numbers can be rational numbers instead of integers. Using these two tools, we can proceed as in the nonsingular case; in particular we have the fundamental canonical bundle formula: The sheaf $R^1 f_* \mathcal{O}_X$ splits into a direct sum of an invertible sheaf, which I shall call $-L$, and a torsion sheaf T which is supported over the base points of the wild fibers, and in terms of these sheaves we have

THEOREM 1.1: *Let $f: X \rightarrow C$ be an elliptic fibration with multiple fibers $m_i G_i$. Then*

- (a) $\omega = f^*(L \otimes \omega_C) \otimes \mathcal{O}_X(\sum a_i G_i)$ with $0 \leq a_i < m_i$,
and $\text{deg } L = \chi(\mathcal{O}_X) + \text{length } T$.
- (b) If $m_i G_i$ is a tame multiple fiber, then $a_i = m_i - 1$.
- (c) All wild fibers are multiple, and their multiplicities are divisible by the characteristic of the ground field. In particular, all elliptic surfaces in characteristic zero are tame.
- (d) No fiber of f contains an exceptional curve of the first kind.

The *proof* can be taken almost literally from [6], theorem 2; by [1], prop. 2.3 + 4, the formula $\omega \otimes \mathcal{O}_F \cong \mathcal{O}_F$ still holds for every elliptic fiber F of f , and throughout the proof, an intersection pairing with rational values suffices. □

LEMMA 1.2: *Let X be an elliptic surface with elliptic fibration $f: X \rightarrow C$, and let $p: Y \rightarrow X$ be a minimal resolution of the singularities of X . The following are equivalent:*

- (i) X has at most rational double points as singularities
- (ii) All singularities of X are rational
- (iii) $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)$
- (iv) There are no exceptional curves of the first kind in the fibers of $p \circ f$, that is, Y is a (minimal) elliptic surface.

Proof: (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii) is a consequence of Leray's spectral sequence, which gives $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) - \chi(R^1 p_* \mathcal{O}_Y)$, and $R^1 p_* \mathcal{O}_Y = 0$ by definition of a rational singularity.

(iii) \Rightarrow (iv) By theorem 1.1(d), there are no exceptional curves of the first kind in the fibers of f ; thus an exceptional curve E of the first kind in a fiber of $p \circ f$ must be the proper transform of a component Z of a fiber of f , and Z must contain a singular point of X , which does not lie on any other component of that fiber. Therefore Z is a singular rational curve, whose total transform still has genus one. Since E does not affect the genus of the total transform, the resolution cycle of P also has genus one, and that is impossible, as Artin has shown in [5], theorem 2.3.

(iv) \Rightarrow (i) Let P be a singular point of X . Then $p: Y \rightarrow X$ is a minimal resolution of P , hence $f^{-1}(P)$ is a subcycle of a reducible fiber of $p \circ f$ with at least one curve missing, namely the proper transform of $f^{-1}(f(P))$. Comparing the table of singular fibers of an elliptic surfaces in [12], §6 or [16] with the table of resolutions of rational double points in [5] or [7], one finds that any such cycle is a resolution of a rational double point. \square

COROLLARY: In each birational equivalence class of elliptic surfaces, there are only finitely many isomorphism classes of surfaces with at most rational singularities. If, moreover, the minimal model of the equivalence class has only irreducible fibers, then every singular surface in that class has an irrational singularity.

Proof: Because of condition (iv) in the lemma, rational singularities can only arise as contractions of components of reducible fibers, and there are only finitely many combinations of such components. \square

1.2. The Jacobi–Weierstraß fibration associated to a family of elliptic surfaces

Let X be a nonsingular elliptic surface with elliptic fibration $f: X \rightarrow C$. Then, if $K = k(C)$ denotes the function field of C , the general fiber X_K of f is a curve of genus one over K . The Jacobian J_K of X_K is therefore an elliptic curve over K , and looking at its function field over k , we can find a nonsingular surface J over k , which carries an elliptic fibration $j: J \rightarrow C$. J is called the Jacobian surfaces, and j the Jacobian fibration associated to X resp. f . j admits a section, namely the closure of the zero divisor on X_K , and therefore by [10] is birationally equivalent to a surface given by a Weierstraß equation $y^2z = x^3 - axz^2 - bz^3$ in a projective bundle $\mathbb{P}(-2L \oplus -3L \oplus \mathcal{O}_C)$ over C . In the sequel, I want to show that for an arbitrary family of elliptic surfaces, one can define these Weierstraß surfaces in such a way, that they form a flat family, too: If $f: X \rightarrow C$ is a tame elliptic fibration, the relative Picard functor $\mathcal{P}ic_{X/C}^0$ is representable by an algebraic

space, as was shown by Raynaud in [19], theorem 8.2.1. According to Anantharaman [3], theorem 4.B, this algebraic space is in fact a scheme, which is formally smooth, because $R^2f_*\mathcal{O}_X = 0$. Therefore J is a compactification of that scheme. As a consequence we get

LEMMA 1.3: Let $f: X \rightarrow C$ be a tame elliptic fibration, and $j: J \rightarrow C$ the corresponding Jacobian fibration. Then $R^1f_*\mathcal{O}_X \cong R^1j_*\mathcal{O}_J$.

Proof: $R^1f_*\mathcal{O}_X$ is the sheaf of tangent vectors to $\mathcal{P}ic_{X/C}^0$ along the zero section, and hence that of J over C . As $\mathcal{P}ic_{J/C}^0$ is isomorphic to an open subset of J , the tangent sheaves along the zero section are also isomorphic, which is just the claim. □

COROLLARY: $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_J)$.

Proof: Clear via Leray’s spectral sequence. □

Now, let $f: X \rightarrow C$ again be a tame elliptic fibration, $j: J \rightarrow C$ its Jacobian fibration, and $g: W \rightarrow C$ the associated Weierstraß fibration. By construction, J and W become isomorphic when restricted to $U = \{t \in C \mid X_t \text{ is irreducible}\}$, hence, if $X' = f^{-1}(U)$, W is the closure of $\mathcal{P}ic_{X'/U}^0$ in the projective bundle $\mathbb{P}(-2L \oplus -3L \oplus \mathcal{O}_C)$, where $-L = R^1f_*\mathcal{O}_X$. This construction can easily be generalized to arbitrary, not necessarily smooth families of tame elliptic surfaces: Let $f: \mathcal{X} \rightarrow \mathcal{C}$ be the elliptic fibration,

$$U = \{t \in \mathcal{C} \mid X_t \text{ is a simple elliptic curve}\},$$

and $\mathcal{X}' = f^{-1}(U)$. Then $\mathcal{P}ic_{\mathcal{X}'/U}^0$ is representable by a family $j: \mathcal{J}' \rightarrow U$ of elliptic curves ([2], theorem 3.1), and \mathcal{J}' can be embedded into $\mathbb{P}(2R^1j_*\mathcal{O}_{\mathcal{J}'} \oplus 3R^1j_*\mathcal{O}_{\mathcal{J}'} \oplus \mathcal{O}_U)$. Since $R^1j_*\mathcal{O}_{\mathcal{J}'}$ is isomorphic to $R^1f_*\mathcal{O}_{\mathcal{X}'}$, this bundle can be extended to the bundle $\mathbb{P} = \mathbb{P}(-2\mathcal{L} \oplus -3\mathcal{L} \oplus \mathcal{O}_{\mathcal{C}})$ over \mathcal{C} with $-\mathcal{L} = R^1f_*\mathcal{O}_{\mathcal{X}'}$, and the closure \mathcal{W} of \mathcal{J}' in \mathbb{P} is a family of Weierstraß surfaces, with the projection map g to \mathcal{C} as elliptic fibration.

Definition: $g: \mathcal{W} \rightarrow \mathcal{C}$ is called the Jacobi-Weierstraß fibration associated to $f: \mathcal{X} \rightarrow \mathcal{C}$ (or simply \mathcal{X}), and $\mathcal{W}/\mathcal{C}/S$ is called the family of Jacobi-Weierstraß surfaces associated to $\mathcal{X}/\mathcal{C}/S$.

LEMMA 1.4: Let $f: X \rightarrow C$ be an elliptic fibration, and $g: W \rightarrow C$ the corresponding Jacobi-Weierstraß fibration.

- (a) $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_W)$
- (b) *The following are equivalent:*
 - (i) *X has at most rational double points as singularities*
 - (ii) *W has at most rational double points as singularities*
 - (iii) *W is a minimal Weierstraß surface in the sense of Tate ([24], §3), i.e. if a, b are the coefficients of its Weierstraß equation, then $\min(3\text{ord}_p a, 2\text{ord}_p b) < 12$ for every geometric point $P \in C$.*

Proof: (a) is clear, for, by construction, $R^1 f_* \mathcal{O}_X \cong R^1 g_* \mathcal{O}_W$.

(b) Let Y be the minimal resolution of X .

(i) \Rightarrow (ii) If X has at most rational double points as singularities, Y is a (minimal) elliptic surface by lemma 1.2, and it has the same Jacobi–Weierstraß fibration as X . But the Jacobi–Weierstraß fibration of Y is the Weierstraß surface of the nonsingular Jacobian surface to Y , and therefore it has at most rational double points as singularities by [10].

(ii) \Rightarrow (i) By [10], there is only one Weierstraß surface with at most rational double points as singularities in each birational equivalence class of elliptic surfaces; therefore, W is the Jacobi–Weierstraß fibration associated to the nonsingular minimal model \tilde{Y} of X , so $\chi(\mathcal{O}_W) = \chi(\mathcal{O}_Y)$. But Y and \tilde{Y} are birationally equivalent nonsingular surfaces, therefore $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_{\tilde{Y}})$, and hence $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_W) = \chi(\mathcal{O}_X)$. This implies, by lemma 1.2, that X has at most rational double points as singularities.

(ii) \Leftrightarrow (iii) was shown by Kas in [10]. □

1.3. Families of elliptic surfaces with at most rational double points as singularities

Throughout this section, S is a connected noetherian scheme none of whose residue fields has characteristic two or three. A family of surfaces over S is a flat projective morphism $\pi: \mathcal{X} \rightarrow S$ whose geometric fibers are surfaces with at most rational double points as singularities; a polarization on \mathcal{X}/S is a line bundle on \mathcal{X} , relatively ample for π .

LEMMA 1.5: *Let $\pi: \mathcal{X} \rightarrow S$ be a family of surfaces, and assume that one of the geometric fibers of \mathcal{X} is an elliptic surface. Then one of the following holds:*

- (a) *All geometric fibers of π are honest elliptic surfaces, i.e. have Kodaira dimension one*
- (b) *All geometric fibers of π are ruled surfaces over an elliptic base curve*
- (c) *All geometric fibers of π are rational surfaces*
- (d) *All geometric fibers of π are K3 surfaces*
- (e) *All geometric fibers of π are Enriques surfaces*

- (f) All geometric fibers of π are abelian surfaces
- (g) All geometric fibers of π are hyperelliptic surfaces.

Note that in cases (c) and (d), not every surface in the family has to be elliptic.

Proof: Since all fibers of π have only rational double points as singularities, π is a Gorenstein morphism, and the relative dualizing sheaf $\omega_{\mathcal{X}/S}$ exists. Its self-intersection number is defined via Euler characteristics, and therefore invariant under flat deformations, hence $(\omega \cdot \omega) = 0$ for every surface in the family. In particular, no fiber of π can be a minimal surface of general type. Suppose that the family contains a nonminimal surface X , and let E be one of its exceptional curves. The obstruction to deform E to an exceptional curve on a neighbouring surface lies in $H^1(E, \mathcal{N}_{E/X})$, where $\mathcal{O}_{E/X} = \mathcal{O}_X(-E)|_E$ is the normal sheaf of E in X . Since $E \cong \mathbb{P}^1$, and $E^2 = -1$, $\mathcal{N}_{E/X} = \mathcal{O}_E(1)$, hence the obstruction vanishes, and we get an exceptional curve in the general fiber of π . Now consider that general surface X ! Like the plurigenera, the Kodaira dimension is upper semi-continuous, so X cannot be of general type. If it is an honest elliptic surface, it contains no exceptional curve, because no elliptic surface contains an exceptional curve in its fibers by theorem 1.1 (d), and a transversal curve $E \cong \mathbb{P}^1$ with $E^2 = -1$ would have intersection multiplicity -1 with the canonical class, which is absurd in Kodaira dimension $\kappa \neq -1$. Thus X cannot degenerate into a surface of general type, and we are in case (a).

Now suppose that X has Kodaira dimension zero. Then $\omega_{\mathcal{X}/S}^{\otimes 12} = 0$, hence all fibers of π have Kodaira dimension zero, and we are in one of the cases (d), (e), (f), or (g), depending on the values of the deformation invariant numbers χ and $q = \text{irregularity of } \mathcal{X}$.

The only remaining case is that X has Kodaira dimension -1 . This is equivalent to saying that X contains a curve D whose intersection number with the canonical divisor is negative. Letting D degenerate to a special fiber, we see that each fiber must have Kodaira dimension -1 , and we are in case (b) for $q = 1$, and (c) for $q = 0$. □

LEMMA 1.6: *Let $\pi: \mathcal{X} \rightarrow S$ be a family of algebraic surfaces with at most rational double points as singularities, such that all fibers of \mathcal{X} are*

- honest elliptic surfaces or
- rational elliptic surfaces or
- ruled elliptic surfaces or
- hyperelliptic surfaces.

Then there exists a smooth family $\gamma: \mathcal{C} \rightarrow S$, and an S -morphism $f: \mathcal{X} \rightarrow \mathcal{C}$ that induces the elliptic fibration on each surface in the family. In the hyperelliptic case, one can always get the elliptic fibration with genus $\mathcal{C} = 1$; the other one can be obtained too, provided that there exists a relatively very ample line bundle \mathcal{P} on \mathcal{X} .

Proof: First consider the case of honest elliptic surfaces. Here, the elliptic fibration is given by any m -canonical map with sufficiently large m ; in fact Katsura and Ueno [11] have shown that $m \geq 14$ suffices. By EGA III₂, theorem 7.7.6 and remark 7.7.9, the sections of the fibers of $\omega_{\mathcal{X}/S}^{\otimes m}$ over S form a coherent \mathcal{O}_S -module \mathcal{F}_m , so we get a map $p: \mathcal{X} \rightarrow \mathbb{P}(\mathcal{F}_m)$ to the projective bundle associated to \mathcal{F}_m for any $m \geq 14$, and its image is the base curve of the elliptic fibration of \mathcal{X} . For rational or ruled elliptic surfaces, the same argument applies, if we replace $\omega_{\mathcal{X}/S}$ by its dual, but there is no universal bound on m in that case, as one easily sees by looking at a rational surface with one or a ruled surface with two multiple elliptic fibers. So we must be content to find a bound depending on the given family, and that is easy enough, because S is noetherian, and the subschemes Z_m of S , over which m suffices, cover S .

For hyperelliptic surfaces, one of the elliptic fibrations is given by the Albanese morphism $X \rightarrow (\text{Pic}^0 X)^\vee$, which can be globalized in our case: $\text{Pic}_{\mathcal{X}/S}^0$ is representable by [2], theorem 3.1, and it is in fact an abelian scheme; for it is proper over S and has reduced fibers, because a hyperelliptic surface has no wild fibers in characteristics different from two or three. Thus the dual abelian scheme exists by [15], Cor. 6.8, and the Poincaré bundle on $\mathcal{X} \times_S \text{Pic}_{\mathcal{X}/S}^0$ defines a morphism $\mathcal{X} \rightarrow (\text{Pic}_{\mathcal{X}/S}^0)^\vee$, inducing the elliptic fibration on each surface in the family. If there exists a relatively very ample line bundle \mathcal{P} on \mathcal{X} , also the second elliptic fibration on \mathcal{X} can be constructed by the same argument as in [6], theorem 3: Let $\mathcal{F} \rightarrow S$ be the fiber over the zero section of $(\text{Pic}_{\mathcal{X}/S}^0)^\vee$, and choose $a, b \in \mathbb{N}$ such that $\mathcal{L} = \mathcal{P}^{\otimes a} \otimes \mathcal{O}_{\mathcal{X}}(-b\mathcal{F})$ has self-intersection zero in each geometric fiber. Then, for each geometric fiber, it is shown there, that some n -fold of \mathcal{L} gives the second elliptic fibration, and since in my definition of a family the base is always assumed to be noetherian, one can find a global n . □

1.4. Multiple fibers and the relative canonical sheaf of a family

In this section, I want to focus on multiple fibers and their behaviour under deformations, and show that a globalization of the canonical bundle formula (theorem 1.1) to families of elliptic surfaces exists.

LEMMA 1.7: Let $\mathcal{X}/\mathcal{C}/S$ be a family of elliptic surfaces, and \mathcal{P} a polarization on \mathcal{X} .

- (a) The rational number $\text{length } T + \sum a_i/m_i$ is the same for every surface in the family
- (b) Let e' be the l.c.m. of all multiplicities of multiple fibers occurring in surfaces in the family, and let e be the degree of \mathcal{P} restricted to a simple elliptic fiber of one of these surfaces. Then $e' | e$, and the m -th plurigenus P_m is invariant for every m divisible by e' .

Proof: (a) \mathcal{P} and $\omega_{\mathcal{X}/S}$ being flat over S , their intersection number is the same in every fiber over S , and by the formula for the canonical bundle it is equal to

$$e(2g - 2 + \chi(\mathcal{O}_{\mathcal{X}}) + \text{length } T + \sum a_i/m_i).$$

Since e , g , and $\chi(\mathcal{O}_{\mathcal{X}})$ are constant, so is $\text{length } T + \sum a_i/m_i$.

(b) e' divides e , because the degree of \mathcal{P} restricted to the reduced curve G of a multiple fiber mG is equal to e/m , which must therefore be an integral number. The m -th plurigenus is

$$P_m = m(2g - 2 + \chi(\mathcal{O}_{\mathcal{X}}) + \text{length } T) + \Sigma[ma_i/m] + 1 - g,$$

where $[\dots]$ denotes the Gauß bracket, hence (a) implies (b). □

LEMMA 1.8: With notations as above, assume that S is irreducible, and suppose that the surface X_s over $s \in S$ has a multiple fiber mG_s (with G_s not necessarily reduced). If the normal sheaf \mathcal{N}_{G_s/X_s} on G_s is nontrivial, then there exists a unique flat family $\mathcal{G} \rightarrow S$ of curves, a closed S -immersion $\mathcal{G} \rightarrow \mathcal{X}$, and a section $\sigma: S \rightarrow \mathcal{C}$, such that G_s is the fiber of G over s , and for every $t \in S$, X_t has a multiple fiber mG_t with base point $\sigma(t)$.

Proof: Let $\mathcal{O} = \mathcal{O}_{S,s}$ be the local ring of s in S , and m its maximal ideal. Since \mathcal{N}_{G_s/X_s} is a nontrivial invertible sheaf of degree zero on G_s , $H^0(G_s, \mathcal{N}_{G_s/X_s}) = H^1(G_s, \mathcal{N}_{G_s/X_s}) = 0$, and this implies by SGA 3, exp. III, prop. 4.5 and remark 4.10, that G_s can be extended successively to a curve in $\mathcal{X} \times_s \text{Spec } \mathcal{O}/m^r$ for any r . These extensions define an \mathcal{O} -rational point in $\text{Hilb}_{\mathcal{X}/S}$, and thus a curve G in $\mathcal{X} \times_s \mathcal{O}$ extending G_s . This defines a flat family \mathcal{G}_U of curves over an open neighbourhood U of s , and since $\text{deg } \mathcal{P}|_{(\mathcal{G}_U)}$ must be the same for every $t \in U$, $m(\mathcal{G}_U)_t$ is a multiple fiber in X_t for every $t \in U$. Let \mathcal{G} be the closure of \mathcal{G}_U in \mathcal{X} . This is still a flat family of curves,

for otherwise some fiber of \mathcal{G} were a surface, i.e. the family \mathcal{X}/S would have to contain a multiple surface, which is excluded in the definition. Thus mG_t is a multiple fiber of X_t for any $t \in S$, and $f(\mathcal{G})$ is a closed subset of \mathcal{C} which is mapped bijectively onto S by the canonical projection $\mathcal{C} \rightarrow S$. In fact it is even mapped there isomorphically, because $f(\mathcal{G})$ supports the relative Cartier divisor $f_*(m\mathcal{G})$. Thus there exists a section $\sigma: S \rightarrow \mathcal{C}$ with image $f(\mathcal{G})$, and everything is proved. \square

LEMMA 1.9: *With notations as above, suppose that all surfaces X_s are tame. Then there exist integers n, m_1, \dots, m_n , sections $\sigma_1, \dots, \sigma_n: S \rightarrow \mathcal{C}$, and flat families $\mathcal{G}_1 \rightarrow \sigma_1(S), \dots, \mathcal{G}_n \rightarrow \sigma_n(S)$ of reduced curves in \mathcal{X} , such that each surface X_s has exactly n multiple fibers, namely $m_1G_{1,s}, \dots, m_nG_{n,s}$. Furthermore,*

$$\omega_{\mathcal{X}/S} = f^*(\omega_{\mathcal{C}/S} \otimes \mathcal{L}) \otimes \mathcal{O}_{\mathcal{X}}(\Sigma(m_i - 1)\mathcal{G}_i),$$

where $\mathcal{L} = \mathcal{H}om_{\mathcal{O}_S}(R^1f_*\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{C}})$.

Proof: Start with any surface X_s ; let m_1G_1, \dots, m_nG_n be its multiple fibers, and apply the preceding lemma to each of these. This gives curves \mathcal{G}_i and sections σ_i , for which we have to show that $\sigma_i(t) \neq \sigma_j(t)$ for every geometric point $t \in S$ and all $i \neq j$. Suppose that $\sigma_i(t) = \sigma_j(t)$ for some t , and let mG be the fiber of X_t over that point. Since it is tame, lemma 1.8 can be applied and yields a curve $\mathcal{G} \rightarrow S$, such that each surface X_u has a multiple fiber mG_u . Over the point t , $mG_t = m_iG_{i,t} = m_jG_{j,t}$, hence $G_{i,t} = (m/m_i)G_t$ and $G_{j,t} = (m/m_j)G_t$. The normal sheafs of $G_{i,t}$ and $G_{j,t}$ have orders m_i, m_j respectively, so lemma 1.8 can be applied once again, and shows that $\mathcal{G}_i = (m/m_i)\mathcal{G}$ and $\mathcal{G}_j = (m/m_j)\mathcal{G}$. Looking at the point s , we see that this can only happen for $i = j$, hence every surface X_t has the n distinct multiple fibers $m_1G_{1,t}, \dots, m_nG_{n,t}$. For $t = s$, only these multiple fibers occur, and since all surfaces are tame, and $\Sigma(m_i - 1)/m_i$ is a constant by lemma 1.7, no surface X_t can have more multiple fibers. Also, it is clear by lemma 1.8 that each fiber $m_iG_{i,t}$ has exact multiplicity m_i . So only the formula for $\omega_{\mathcal{X}/S}$ remains to be shown, but is clear from the fact that both sides are flat over S and coincide in each geometric fiber. \square

Lemma 1.9 becomes wrong for families containing a wild surface, because then several tame fibers can come together to form a wild fiber, see [11]. Over \mathbb{C} , however, the lemma even holds for families of compact complex surfaces, as Iitaka has shown in [9], prop. 10.

§2. Moduli for polarized elliptic surfaces

2.1. A Hilbert scheme for elliptic surfaces

We would like to have a scheme whose geometric points correspond biuniquely to isomorphism classes of elliptic surfaces. Unfortunately such a scheme cannot exist except for special cases like elliptic surfaces with a section or ruled elliptic surfaces [4], because in general, compact complex surfaces form families the general member of which is not algebraic, and the algebraic surfaces in these families lie in a countably infinite number of sub-families, as Kodaira has shown in [12], §11. In order to get a moduli scheme, we shall therefore consider pairs (X, P) consisting of an elliptic surface X , and an equivalence class of projective embeddings of X , given by a very ample \mathcal{O}_X -module P . I shall assume that X has at most rational double points as singularities, and exclude the case that X is an abelian surface, because moduli of abelian surfaces are well understood, and here, this case would involve several extra arguments. In this section, the existence of Hilbert schemes for the pairs (X, P) will be shown. We fix the following invariants of a pair:

- the Euler–Poincaré characteristic $\chi = \chi(\mathcal{O}_X)$
- the genus g of the base curve C of the elliptic fibration: if $\kappa(X) \neq 0$, C and hence g is uniquely determined by X , if $\kappa(X) = 0$, X is a K3 or Enriques surface, in which case $g = 0$ for all elliptic fibrations, or X is a hyperelliptic surface, in which case we can either set $g = 0$ or $g = 1$ for *all* hyperelliptic surfaces. Recall that the remaining case of abelian surfaces is not considered here.
- the number n and the multiplicities m_1, \dots, m_n of the multiple fibers; from the classification of surfaces it follows that these numbers are determined uniquely by X and g even if there are several elliptic fibrations
- the degree $d = P^2$ of the polarization
- the degree $e = PF$ of P when restricted to a fiber F of X over a curve of genus g .

The sequence $\mathfrak{I} = (g, \chi; n, m_1, \dots, m_n, d, e)$ will be called the type of the pair (X, P) . We shall always use the abbreviation

$$v = v(\mathfrak{I}) = \sum_{i=0}^m (m_i - 1)/m_i,$$

and call $\kappa(\mathfrak{I}) = \text{sign}(2g - 2 + \chi + v)$ the Kodaira dimension of \mathfrak{I} . Because of difficulties in positive characteristics (see §2.3), I shall always

assume that e is invertible, so from now on, \mathcal{Sch} will denote the category of connected noetherian schemes over $\mathbb{Z}[1/6e]$.

The first problem in proving the existence of a Hilbert scheme for elliptic surfaces is to show that the notion of an elliptic surface behaves well under deformations.

The difficulty here is that an elliptic surface as defined in §1.1 need not be smooth, but only integral and without one-dimensional singularities. Therefore we cannot just use the Jacobian criterion for smoothness, which is an open condition, but have to work a bit more. The first step is

LEMMA 2.1: *Let X be a projective surface (i.e. a two-dimensional subscheme of some \mathbb{P}^N), and assume that there exists a flat morphism $f: X \rightarrow C$ onto a nonsingular curve C , such that the fibers of f are connected and have at most isolated singularities, and f is smooth over a non-empty subset of C . Then X is integral.*

Proof: The subset U over which f is smooth is open; since it is non-empty by assumption, it must therefore be dense. Let $X' = f^{-1}(U)$. The fibers of f over U are connected and smooth, hence irreducible. Since f is flat and projective over a one-dimensional base, the valuative criterion for the Hilbert scheme shows that X must be the closure of X' in \mathbb{P}^N , so X is irreducible too. Similarly, X is reduced, hence integral, because X_{red} and X coincide over U . □

THEOREM 2.2: *For every type \mathfrak{I} with $\kappa(\mathfrak{I}) \neq 0$, there exists a quasiprojective Hilbert scheme for pairs (X, P) of type \mathfrak{I} , that is a scheme \mathcal{H} representing the functor*

$$\mathcal{Sch} \rightarrow \mathcal{Sets}; T \rightarrow \left\{ \begin{array}{l} \mathcal{X} \subset \mathbb{P}_T^M \mid (X, \mathcal{O}_X(1)) \text{ is a pair of type } \mathfrak{I} \\ \text{with } h^1(X, \mathcal{O}_X(1)) = h^2(X, \mathcal{O}_X(1)) = 0 \\ \text{for every geometric fiber } X \text{ of } \mathcal{X} \end{array} \right\}$$

with $M = \chi(X, P) = (d/2) + (e/2)(2g - 2 + \chi + v)m + \chi$.

Proof: Every pair (X, P) of type \mathfrak{I} has Hilbert polynomial

$$Q(m) = \chi(X, mP) = \frac{d}{2} m^2 + \frac{e}{2} (2g - 2 + \chi + v)m + \chi,$$

and by FGA, exp. 221, theorem 3.1, there exists a projective scheme \mathcal{H}_0 representing the functor

$$\mathcal{Sch} \rightarrow \mathcal{Sets}; T \rightarrow \left\{ \begin{array}{l} \mathcal{X} \subset \mathbb{P}_T^M \mid (X, \mathcal{O}_X(1)) \text{ has Hilbert polynomial} \\ Q(m) \text{ for every geometric fiber} \\ X \text{ of } \mathcal{X} \end{array} \right\},$$

and a universal surface $p: \mathcal{X}_\mathcal{O} \rightarrow \mathcal{H}_\mathcal{O}$ in $\mathbb{P}_{\mathcal{H}_\mathcal{O}}^M$. By [8], ex. 9.7, p has Gorenstein fibers over a subscheme U of $\mathcal{H}_\mathcal{O}$, iff $p^! \mathcal{O}_U$ is a complex consisting of a single invertible module. Since the condition that a coherent module be zero resp. invertible is constructible, the condition that p should be a Gorenstein morphism defines a subscheme \mathcal{H}_1 of \mathcal{H}_0 with a universal surface \mathcal{X}_1 , and on \mathcal{X}_1 there is an invertible relative dualizing sheaf $\omega_1 = \omega_{\mathcal{X}_1/\mathcal{H}_1}$. Its self-intersection number is defined in terms of Euler–Poincaré characteristics, hence $(\omega_1 \cdot \omega_1) = 0$ defines a subscheme \mathcal{H}_2 of \mathcal{H}_1 with a universal surface \mathcal{X}_2 and a dualizing sheaf ω_2 . In order to introduce an elliptic fibration, recall that on an elliptic surface of Kodaira dimension different from zero the fibration is always given by any m -canonical map with sufficiently large ($\kappa = +1$) resp. small ($\kappa = -1$) m ; in fact we can take any m such that

$$m(2g - 2 + \chi) + \sum_{i=0}^n [m(m_i - 1)/m_i] > 2g - 2.$$

Fix one such m and let \mathcal{H}_3 be the subscheme of \mathcal{H}_2 over which the fiber of $\omega_2^{\otimes m}$ is an invertible module whose sections form a vector space of the expected dimension

$$L = m(2g - 2 + \chi) + \sum_{i=0}^n [(m_i - 1)/m_i] + 1 - g,$$

and let $\mathcal{E} = p_*(\omega_3^{\otimes m})$, where $p: \mathcal{X}_3 \rightarrow \mathcal{H}_3$ is the universal surface. In order to show that \mathcal{H}_3 has a subscheme \mathcal{H} over which all geometric fibers of \mathcal{X}_3 are elliptic surfaces, it clearly suffices to show this for every sufficiently small open subset of \mathcal{H}_3 , so let U be an open subset over which \mathcal{E} is free. Then a basis of \mathcal{E} defines a morphism $f: \mathcal{X}_{3|U} \rightarrow U \times \mathbb{P}^{L-1}$; let $q: \mathcal{C} \rightarrow U$ be its schemetheoretic image. The condition that q be a smooth family of curves of genus g , and that all fibers of f be one-dimensional, defines a subscheme U' of U , over which we have a fibration $f': \mathcal{X}' \rightarrow \mathcal{C}'$ with a smooth family $q': \mathcal{C}' \rightarrow U'$ of curves of genus g . Let Z be the singular locus of f' in \mathcal{X}' . Then \mathcal{C}' has a subscheme \mathcal{C}'' which is defined by the following conditions: The geometric fibers of Z over \mathcal{C}'' are finite, the geometric fibers of f' are

connected (i.e. the finite part of the Stein factorization is an isomorphism), and f' is flat over \mathcal{C}'' with geometric fibers of genus one. Let $U'' = U' \setminus q'(\mathcal{C}' \setminus \mathcal{C}'')$; then the fibration $f'': \mathcal{X}'' \rightarrow \mathcal{C}''$ is flat with connected fibers of arithmetic genus one having at most isolated singularities. We still have to impose the condition that the general fiber of an elliptic surface has to be smooth, so let \mathcal{V} be the open subscheme of \mathcal{C}'' over which f'' is smooth, and define $\tilde{\mathcal{H}} \cap U$ as the image of \mathcal{V} in U'' . This gives us a Hilbert scheme $\tilde{\mathcal{H}}$ for elliptic surfaces together with a universal surface $\tilde{\mathcal{X}}$ and a canonical fibration $\tilde{f}: \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{C}}$. To this we can associate, by §1.2, a Jacobi-Weierstraß fibration $\tilde{j}: \tilde{\mathcal{W}} \rightarrow \tilde{\mathcal{C}}$, and a geometric fiber of $\tilde{\mathcal{X}}$ has at most rational double points as singularities, *iff* the corresponding fiber of $\tilde{\mathcal{W}}$ has at most rational double points as singularities (lemma 1.4). But the condition that $\tilde{\mathcal{W}}$ should have only rational double points can be expressed – by the same lemma – in terms of the vanishing orders of the coefficients of the Weierstraß equation and thus defines a subscheme of $\tilde{\mathcal{H}}$ (cf. [23], lemma 5). Now apply lemmas 1.5 and 1.9, which show that a suitable union \mathcal{H} of connected components of $\tilde{\mathcal{H}}$ is a Hilbert scheme for pairs of type \mathfrak{I} . □

In Kodaira dimension zero, there is only one case with a canonical elliptic fibration, namely that of hyperelliptic surfaces, that is $\mathfrak{I} = (1, 0; 0; d, e)$ for the fibration over an elliptic curve, and certain types of the form $(0, 0; . . .)$ if we consider the other one. Here we have

LEMMA 2.3: There exists a Hilbert scheme \mathcal{H} for smooth hyperelliptic surfaces X together with a polarization P with $P^2 = d, PF = e$. \mathcal{H} splits into connected components parametrizing the different classes of hyperelliptic surfaces in the list of Bagnera–De Franchis (see [6], p. 33).

(Note that a hyperelliptic surface, like any elliptic surface with $\chi = 0$, cannot have rational double points, because all its fibers are elliptic curves.)

Proof: Again we start with the Hilbert scheme of all surfaces with the right Hilbert polynomial. It has an open subscheme U over which the universal surface \mathcal{X} is smooth. Applying lemma 1.5 to each connected component of U , we get a subscheme V parametrizing hyperelliptic surfaces. By lemma 1.6, the universal surface over V has an elliptic fibration over a curve of genus zero, and the multiple fibers of that fibration separate the different classes of hyperelliptic surfaces. Therefore the results follows from lemma 1.9. □

Since I have excluded abelian surfaces, the only remaining cases are Enriques and K3 surfaces, i.e. $\mathfrak{I} = (0, 1; 2, 2, 2, d, e)$ and $\mathfrak{I} = (0, 2; 0; d, e)$. In both

cases the elliptic fibration need not be unique, and there is no obvious way to distinguish one particular fibration. Therefore, in order to apply the method used in this paper, I have to consider elliptic fibrations on Enriques resp. K3 surfaces instead of the surfaces themselves. The right “Hilbert scheme” for these fibrations is given in

LEMMA 2.4: *If \mathfrak{T} is a type for Enriques or elliptic K3 surfaces, then there exists a quasiprojective scheme \mathcal{G} representing the functor*

$$Sch \rightarrow Sets; T \rightarrow \left\{ \begin{array}{l} f: \mathcal{C} \rightarrow \mathcal{C} \mid \mathcal{X} \subset \mathbb{P}_T^M, (X, \mathcal{O}_X(1)) \text{ is of type } \mathfrak{T} \\ \text{with } h^1(X, \mathcal{O}_X(1)) = h^2(X, \mathcal{O}_X(1)) = 0 \\ \text{for every geometric fiber } X \text{ of } \mathcal{X}, \\ \text{and } f: \mathcal{X} \rightarrow \mathcal{C} \text{ is an elliptic fibration} \\ \text{over a curve } \mathcal{C} \rightarrow T \text{ of genus zero} \end{array} \right\}$$

Proof: The starting point still is the Hilbert scheme \mathcal{H} of all surfaces with the given Hilbert polynomial, together with its universal surface $\mathcal{X} \rightarrow \mathcal{H}$. By FGA, exp. 221, 4c, the functor $\mathcal{H}om_{\mathcal{X}}(\mathcal{X}, \mathcal{H} \times \mathbb{P}^1)$ is representable by a quasiprojective scheme $\tilde{\mathcal{G}}$, over which we have a universal surface $y \rightarrow \tilde{\mathcal{G}} \times \mathbb{P}^1$. Now the methods used in the proof of theorem 2.2 yield \mathcal{G} as a subscheme of $\tilde{\mathcal{G}}$. □

2.2. A lemma from geometric invariant theory

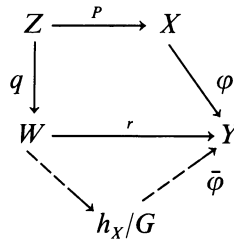
In order to get from Hilbert schemes to moduli schemes, we have to divide out by the projective group; this will be done using geometric invariant theory. I do not want, however, to go the usual way via the numerical criterion for stability, but rather use the fact, that there exist moduli schemes for elliptic surfaces with a section (see [23]), and relate these to the present moduli problem via the Jacobi-Weierstraß fibrations constructed in §1.2. We shall see in the next section, that – in an appropriate setting – these fibrations determine an elliptic surface up to finitely many possibilities. In this paragraph, I want to show how geometric invariant theory can deal with such a situation:

Definition: A morphism $\varphi: F \rightarrow G$ between contravariant functors $F, G: Sch \rightarrow Sets$ is called proper, if for every discrete valuation ring A and every $y \in G(A)$ with general fiber $y' \in G(K)$, $K = \text{Quot } A$, every $x' \in F(K)$ with $\varphi(K)(x') = y'$ has a unique continuation $x \in F(A)$ with $\varphi(A)(x) = y$.

φ is called finite, if in addition $\varphi(k)$ has finite fibers for every algebraically closed field k .

LEMMA 2.5: *Let G be a connected algebraic group over an algebraically closed field k , acting properly on an algebraic k -scheme X , and suppose that there is a k -morphism $\varphi: X \rightarrow Y$ to some quasiprojektive k -scheme Y , such that the induced morphism of functors $\bar{\varphi}: h_X/G \rightarrow h_Y$ is finite. Then φ is affine, and there exists a quasiprojective geometric quotient X/G , finite over Y .*

Proof: As Seshadri has shown in [22], Cor. 6.1, there is a normal scheme Z with a G -action, and with a finite G -equivariant morphism $p: Z \rightarrow X$, such that a geometric quotient $q: Z \rightarrow W$ exists, and W is a separated scheme. Then $\varphi \circ p: Z \rightarrow Y$ is a G -invariant morphism, hence it must factorize over q and a morphism $r: W \rightarrow Y$; since both $\bar{\varphi}: h_X/G \rightarrow Y$ and $p: Z \rightarrow X$ are proper with finite geometric fibers, one easily checks that the same holds for r , i.e. r is a finite morphism and hence affine. $q: Z \rightarrow W$ is affine, because it is a geometric quotient, so $r \circ q = \varphi \circ p$ is affine, and since p is finite, Chevalley’s theorem (EGA II, Th^m 6.7.1) implies that φ is affine.



Now let M be an ample line bundle on Y , and $L = \varphi^*M$. Y is covered by affine open subsets Y_s , $s \in H^0(Y, dM)$, $d \in \mathbb{N}$, hence X is covered by the affines $\varphi^{-1}(Y_s) = X_{\varphi^*s}$, $\varphi^*s \in H^0(X, dL)^G$. G acting properly, this implies that every point of X is properly stable, and thus the existence of a geometric quotient X/G . □

2.3. A morphism with finite fibers

In order to apply the lemma just proved, we must find a suitable scheme Y . This will be constructed as a product of moduli schemes whose existence we already know: Fix a type $\mathfrak{X} = (g, \chi; n, m_1, \dots, m_n; d, e)$, and let \mathcal{H} be the corresponding Hilbert scheme. We shall first assume that g is positive. Since

\mathcal{H} comes together with a universal elliptic fibration $f: \mathcal{X} \rightarrow \mathcal{C}$ with a smooth curve \mathcal{C} of genus g , it becomes a scheme over the moduli scheme M_g of nonsingular curves of genus g . Also, by §1.2, f has a Jacobi–Weierstraß fibration; hence we get an M_g -morphism $\mathcal{H} \rightarrow E_{g,\chi}$ to the moduli scheme for minimal Weierstraß surfaces with invariants g, χ , which exists by [23]. This morphism does not yet have finite numbers of orbits as its fibers, but Ogg–Šafarevič theory gives us quite a good picture of these fibers: If we define the period of an elliptic fibration $f: X \rightarrow C$ as the minimal degree over C of a transversal curve on X , we have

THEOREM 2.6: *Let $j: J \rightarrow C$ be a Jacobian fibration, P_1, \dots, P_n points of C , and m, m_1, \dots, m_n positive integers not divisible by $p = \text{char } k$. Then there exist elliptic fibrations $f: X \rightarrow C$ with Jacobian fibration j and period m , whose multiple fibers are precisely the fibers $f^{-1}(P_i)$ with multiplicities m_i , if and only if*

- (i) *None of the fibers $j^{-1}(P_i)$ is additive*
- (ii) *m_i divides m for all i*
- (iii) *If $J \cong C \times E$ is a product, then there exist elements $\alpha_1, \dots, \alpha_n$ in $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$, such that α_i has order m_i , and the sum of the α_i is zero.*

The number of birational equivalence classes of such surfaces is finite and depends only on

- *the numbers m, m_1, \dots, m_n*
- *the second Betti number and the Picard number of J*
- *the question whether $j^{-1}(P_i)$ is elliptic or multiplicative*
- *if j is locally constant, a certain finite abelian group, whose order is at most four in characteristic zero.*

Proofs can be found in [17] or [20]; see also [18]. The assumption that p must not divide m is essential; as Vvedenskii shows in his paper [26], the finiteness result is “almost always” wrong for $p|m$, and in his papers [25] and others he gives examples, that X can exist then, even though (i) is not satisfied. In our case, since m is obviously a divisor of e , and p does not divide e , the theorem tells us that there are at most finitely many isomorphism classes of surfaces X with (X, P) of type \mathfrak{Z} , if we fix j and the base points of the singular fibers. By lemma 1.9, $\mathcal{C} \rightarrow \mathcal{H}$ admits n sections $\sigma_1, \dots, \sigma_n$, corresponding to the base points of the multiple fibers, so we get an M_g -morphism from \mathcal{H} to the moduli scheme $M_{g,n}$ of curves with n distinguished points. For $g \geq 2$, this suffices, because then the automorphism group of the base curve is finite, and it also suffices for $g = 1$ and $\chi = 0$, because then every automorphism of the base curve extends to an automorphism of the surface, so we have

LEMMA 2.7: *If $g \geq 2$, or $g = 1$ and $\chi = 0$, then there is a canonical M_g -morphism $\varphi: \mathcal{H} \rightarrow Y' := E_{g,\chi} \times_{M_g} M_{g,n}$, such that the image of the Hilbert point of a pair (X, P) determines the isomorphism class of X upto finitely many possibilities. \square*

In order to take care of the polarization, we have to push it down to the base curve and consider its moduli there: For any pair (X, P) of type \mathfrak{T} , with elliptic fibration $f: X \rightarrow C$, $R^1 f_* P = 0$, because the degree e of P on each fiber is at least equal to the multiplicity of that fiber. Therefore $f_* P$ is a vector bundle of rank e , and

$$\chi(f_* P) = \chi(P) - \chi(R^1 f_* P) = h^0(X, P) = M + 1,$$

hence $\wedge^e fP$ is a line bundle of degree $M - g$ on C . If $\eta \in NS(X)$ is the algebraic equivalence class of P , we thus get a morphism $\varphi: Pic^n X \rightarrow Pic^{M-g} C$, $P \mapsto \wedge^e fP$. This morphism is surjective, because for any $L \in Pic^0 C$,

$$\wedge^e f_*(P \otimes f^*L) = \wedge^e f_* P \otimes eL,$$

and since it is equivalent to a morphism of abelian varieties, it has finite fibers if $q = g$, so that in this case $\wedge^e f_* P$ determines P in its algebraic equivalence class upto finitely many possibilities. The only case with $q \neq g$ occurs, if the Jacobian surface of X is a product $C \times E$, then $q = g + 1$, and the continuous part of $\ker \varphi$ is isogeneous to E (cf. FGA exp. 232, prop. 2.1), so we have a canonical morphism $\psi: Pic^n X \rightarrow Pic^e E$, such that $\varphi(P)$ and $\psi(P)$ determine P upto finitely many possibilities in $Pic^n X$.

Let's globalize this construction! We know that there exists a moduli scheme $L_{g,M-g}$ for nonsingular curves of genus g together with a line bundle of degree $M - g$, and if $f: \mathcal{X} \rightarrow \mathcal{C}$ is the universal elliptic fibration over the Hilbert scheme \mathcal{H} for pairs of type \mathfrak{T} , the pair $(\mathcal{C}, \wedge^e f_* \mathcal{O}_{\mathcal{X}}(1))$ defines an M_g -morphism $\mathcal{H} \rightarrow L_{g,M-g}$.

If we are in a case where the Jacobian surface is a product, let again $f: \mathcal{X} \rightarrow \mathcal{C}$ denote the universal elliptic fibration over the corresponding Hilbert scheme \mathcal{H} . (This is not the Hilbert scheme of all pairs of type \mathfrak{T} , but only a union of connected components thereof, because the type does not contain the order of $R^1 f_* \mathcal{O}_{\mathcal{X}}$ in the cases where this line bundle has degree zero.) Then $R^1 f_* \mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{C}}$, and the direct image of $\mathcal{O}_{\mathcal{C}}$ on \mathcal{H} is $\mathcal{O}_{\mathcal{H}}$, hence the coefficients of the Weierstraß equation of the Jacobian fibration also define an elliptic curve $\mathcal{E} \rightarrow \mathcal{H}$ together with a canonical section $0: \mathcal{H} \rightarrow \mathcal{E}$. Then

both $e \cdot 0$ and the image of $\mathcal{O}_x(1)$ are sections of $\mathcal{P}ic^e \mathcal{C}|\mathcal{H}$, and thus define a morphism from \mathcal{H} to $M_{1,2}$, the moduli scheme for elliptic curves with two distinguished points. Putting everything together, we get

THEOREM 2.8: *Let \mathfrak{T} be a type with $g \geq 2$ or $g = 1, \chi = 0$, and let \mathcal{H} be the corresponding Hilbert scheme (resp. for $\chi = 0$ the part of it corresponding to a given order of $R^1 f_* \mathcal{O}_X$). Then there exists an M_g -morphism $\mathcal{H} \rightarrow Y' \times_{M_g} L_{g,M-g}$ whose geometric fibers are finite unions of orbits, unless we are in a case where the Jacobian surface is a product. In these cases, there is a morphism $\mathcal{H} \rightarrow (Y' \times_{M_g} L_{g,M-g}) \times M_{1,2}$ with finite unions of orbits as its fibers.*

Proof: So far we only know that the polarization is determined up to finitely many possibilities in its algebraic equivalence class; we have to show that there are only finitely many possibilities at all. But since $NS(X)$ is a discrete group, the geometric fibers of the above morphism must be discrete unions of orbits, and as we are dealing with algebraic schemes, these geometric fibers are again algebraic, hence finite unions of orbits. \square

This completes the cases with $g \geq 2$ or $g = 1$ and $\chi = 0$; the next problems are those with $g = 1, \chi \geq 1$. Here the problem is that the automorphism group of the surface is finite, whereas that of the base curve is one-dimensional, so that the base curve has to be rigidified. But if $g = 1$, the elliptic fibration is given by the Albanese map, and thus the base curve has a canonical section. Again, let $f: \mathcal{X} \rightarrow \mathcal{C}$ be the universal elliptic fibration over \mathcal{H} , let σ_0 be the canonical section, and $\sigma_1, \dots, \sigma_n$ the sections given by the base points of the multiple fibers. The polarization $\mathcal{P} = \mathcal{O}_x(1)$ can be used to define yet another section of \mathcal{C} : $\wedge^e \mathcal{P}$ is a line bundle of degree $M - 1$ on \mathcal{C} , hence $\wedge^e \mathcal{P} \otimes \mathcal{O}_{\mathcal{C}}((g - M - 1)\sigma_0(\mathcal{H}))$ is a line bundle of degree one, and under the canonical isomorphism $\mathcal{C} \rightarrow \mathcal{P}ic^1 \mathcal{C}|\mathcal{H}$ defines a section $\tau: \mathcal{H} \rightarrow \mathcal{C}$. The same arguments used above show

THEOREM 2.9: *Let \mathfrak{T} be a type with $g = 1$ and $\chi \geq 1$, and let \mathcal{H} be the corresponding Hilbert scheme. Then there exists an M_1 -morphism $\mathcal{H} \rightarrow Y = E_{1,\chi} \times_{M_1} M_{1,n+2}$, given by the Jacobian fibration and the tuple $(\mathcal{C}, \sigma_0, \dots, \sigma_n, \tau)$, such that all geometric fibers are finite unions of orbits. \square*

Next we consider the case $g = 0, \chi \geq 1$. Since there is no coarse moduli scheme for all minimal Weierstraß surfaces with $g = 0, \chi = 1$, I won't use the moduli schemes $E_{0,\chi}$, but will use a slightly more direct construction: If $X \rightarrow \mathbb{P}^1$ is an elliptic surface over $\mathbb{P}^1 = \mathbb{P}(V)$, let $a \in S^{4\chi} V, b \in S^{6\chi} V$ be the

coefficients of the Weierstraß equation of its Jacobian surface, and let $(P_1, \dots, P_n) \in (\mathbb{P}^1)^n$ be the base points of the multiple fibers. Then we have an action of PGl_1 on $\mathbb{P}(S^{4\chi}V \oplus S^{6\chi}V) \times (\mathbb{P}^1)^n$. Let W be the sub-scheme of those points $((a, b), P_1, \dots, P_n)$, for which $\min(3\text{ord}_p a, 2\text{ord}_p b) < 12$ for all $P \in \mathbb{P}^1$, and for which all P_i are distinct; obviously every elliptic surface with at most rational double points as singularities gives rise to a geometric point of W . The same calculation as in [23], §6, gives

LEMMA 2.10: (a) *If $\chi > 1$, or $m > 3$, then every geometric point of W is properly stable.*

(b) *If $\chi = 1$ and $m \leq 2$, then a geometric point of X is properly stable, iff $\min(3\text{ord}_p a, 2\text{ord}_p b) < 6$ for all $P \in \mathbb{P}^1$.*

Let Y be the geometric quotient of W , resp. the subset of W defined by (b), modulo the action of PGl_1 . Since $\text{Pic}^0 X = 0$ for an elliptic surface with $g = 0, \chi \geq 1$, there are no problems about the polarization, hence the usual argument via Ogg–Šafarevič theory gives

THEOREM 2.11: (a) *Let \mathfrak{C} be a type with $g = 0$ and $\chi > 1$ or $m > 3$, and let \mathcal{H} be the corresponding Hilbert scheme. Then there is a morphism $\mathcal{H} \rightarrow Y$ whose geometric fibers are finite unions of orbits.*

(b) *Let \mathfrak{C} be a type with $g = 0, \chi = 1$, and $m \leq 3$, and let $\mathcal{H} \xrightarrow{\sim} \tilde{\mathcal{H}}$ be the open subscheme of the corresponding Hilbert scheme over which the surface has no fibers of types I_m^*, II^*, III^* or IV^* . Then there is a morphism $\tilde{\mathcal{H}} \rightarrow Y$ whose geometric fibers are finite unions of orbits.*

Finally, suppose that $g = \chi = 0$. Then the Jacobian fibration of X is a product $\mathbb{P}^1 \times E$, $\text{Pic}^0 X = \text{Pic}^0 E$, and the moduli of the Jacobian surface are those of E . Therefore the morphism $\mathcal{H} \rightarrow M_{1,2}$ constructed above for theorem 2.8 takes care of both the polarization and the Jacobian fibration, so that we are only left with the base points of the multiple fibers. If there are at most three of these, we can forget them, because then all m -tuples of distinct points are isomorphic. For $m > 3$, each m -tuple of distinct points of \mathbb{P}^1 defines a stable point of $(\mathbb{P}^1)^m$ with respect to the action of PGl_1 , hence there is a coarse moduli scheme Y' for these, and we get

THEOREM 2.12: (a) *Let \mathfrak{Z} be a type with $g = \chi = 0$ and $m \leq 3$, and let \mathcal{H} be the corresponding Hilbert scheme. Then there exists a morphism $\mathcal{H} \rightarrow Y := M_{1,2}$ whose geometric fibers are finite unions of orbits.*

(b) *Let \mathfrak{Z} be a type with $g = \chi = 0$ and $m > 3$, and let \mathcal{H} be the corresponding Hilbert scheme. Then there exists a morphism $\mathcal{H} \rightarrow Y := M_{1,2} \times Y'$ whose geometric fibers are finite unions of orbits. \square*

2.4. Verification of properness

We have to show that the morphism from the modular functor to h_Y , the functor of points of the scheme Y constructed in the previous section, is proper. Obviously the morphism taking care of the polarization poses no problem, because it is equivalent to a morphism of abelian schemes and thus proper. So, what we really need is

THEOREM 2.13: *Let R be a discrete valuation ring with quotient field K and algebraically closed residue field k , $J_R \rightarrow C_R \rightarrow \text{Spec } R$ a smooth family of elliptic surfaces with a section, $\sigma_1, \dots, \sigma_n: \text{Spec } R \rightarrow C_R$ sections of $C_R \rightarrow \text{Spec } R$, no two of which coincide in the special fiber, $X_K \rightarrow C_K \rightarrow \text{Spec } K$ a family of elliptic surfaces with multiple fibers over $\sigma_1(\text{Spec } K), \dots, \sigma_n(\text{Spec } K)$, and P_K a very ample line bundle on X_K , whose degree on the fibers of $X_K \rightarrow C_K$ is relatively prime to $\text{char } k$. Assume that the general fiber $J_K \rightarrow C_K$ of $J_R \rightarrow C_R$ is the Jacobian fibration of $X_K \rightarrow C_K$. Then there exists a finite extension K'/K , such that $P_{K'} \rightarrow X_{K'} \rightarrow C_{K'} \rightarrow \text{Spec } K'$ can be extended to a family $P_{R'} \rightarrow X_{R'} \rightarrow C_{R'} \rightarrow \text{Spec } R'$ over the normalization R' of R in K' , with multiple fibers over the points $\sigma_i(\text{Spec } R')$. $P_{R'}$ is relatively ample for $X_{R'} \rightarrow \text{Spec } R'$, and the special fiber X_k of $X_{R'}$ has at most rational double points as singularities.*

Proof: $X_{R'}$ will be constructed in several steps:

Step I: There exists an extension K'/K , a Galois covering $\pi: D_{K'} \rightarrow C_{K'}$, whose degree is not divisible by $\text{char } k$, and a cohomology class (u_g) in $H^1(G, J_{K'}(D_{K'}))$, where $G = \text{Gal}(D_{K'}/C_{K'})$, such that the following holds: If $j: J \rightarrow D_{K'}$ is a nonsingular model of $J \times_{C_{K'}} D_{K'}$, then $X_{K'} = J/G$ with respect to the G -action $x \rightarrow g(x) + u_g(j(x))$, where $x \rightarrow g(x)$ is an extension of the G -action on $D_{K'}$ to J . □

Proof: If such a covering exists over the algebraic closure of K , it can be defined already over a finite extension of K ; therefore we can assume without loss of generality that K is algebraically closed. But then the result is classical, and can be found for example in [21], VII, §5; in fact we can take a hyperplane section corresponding to $P_{K'}$ as $D_{K'}$, so that the degree of $D_{K'}$ over $C_{K'}$ is relatively prime to $\text{char } k$. □

Step II: A first candidate for $X_{R'}$: Let $D_{K'} \rightarrow C_{K'}$ be the Galois covering from step I; extending K' once more, if necessary, we can assume that $D_{K'} \rightarrow C_{K'}$

extends to a Galois covering $D_{R'} \rightarrow C_{R'}$ with a stable curve $D_{R'}$. Since this was the last extension of K , I shall now simplify notations and simply write K, R instead of K', R' . The nonsingular model $\tilde{j}: \tilde{J}_K \rightarrow D_K$ of $J_K \times_{C_K} D_K$ can be obtained by blowing up a certain ideal \mathcal{I} , and G acts on \tilde{J}_K in two ways:

- (1) $x \rightarrow g(x)$ with $\tilde{J}_K/G = J_K$
- (2) $x \rightarrow g(x) + u_g(j(x))$ with $\tilde{J}_K/G = X_K$.

Define \tilde{J}_R as the blow up of $J_R \times_{C_R} D_R$ along the ideal generated by \mathcal{I} ; note that \tilde{J}_R will usually have singularities. Since the sections u_g can be extended to sections of $\tilde{J}_R \rightarrow D_R$, both G -actions extend to \tilde{J}_R ; define the candidate Y_R as \tilde{J}_R/G with respect to the second one.

Step III: Let z be a point of \tilde{J}_R , and d its image in D_R . The stabilizer of z with respect to the first G -action is $\text{Stab}_1 z = \text{Stab } d$, the one with respect to the second G -action is

$$\text{Stab}_2 z = \{g \in \text{Stab } d \mid u_g(d) = 0\},$$

hence the image of z in Y_R lies in a $(\text{Stab } d : \text{Stab}_2 z)$ – fold fiber.

Proof: $g \in \text{Stab } z$, iff $z = g(z) + u_g(d)$. Now $z \rightarrow g(z)$ permutes the fibers of \tilde{j} , whereas addition of $u_g(d)$ does not leave the given fiber, therefore $z = g(z)$ and $u_g(d) = 0$. The rest is clear. □

Step IV: The special fiber Y_k of Y_R is \tilde{J}_k/G .

Proof: In characteristic zero this is clear, because then every finite group is linearly reductive, and thus its quotients are universal. In positive characteristics, we only get a purely inseparable morphism $f: \tilde{J}_k/G \rightarrow X_k$. This morphism is generically 1–1, because G acts faithfully on D_k , and hence a fortiori on \tilde{J}_k . Now, locally only the stabilizers of G act, and these are linearly reductive, because their orders divide $\text{deg } D_K/C_K$, which is prime to $\text{char } k$. So, locally f is an isomorphism, and thus also globally, because it has degree one. □

Step V: If $y \in Y_k$ lies on a non-multiple fiber, then it is a regular point of Y_k .

Proof: Let z be an inverse image of y in \tilde{J}_k ; then $\mathcal{O}_y \cong \mathcal{O}_z^{\text{Stab}_2 z}$. But since y lies on a non-multiple fiber, $\text{Stab}_1 z = \text{Stab}_2 z$, hence $\mathcal{O}_y \cong \mathcal{O}_z^{\text{Stab}_1 z}$, and this is a regular local ring, because J_k is nonsingular. □

Step VI: If $y \in Y_k$ lies on a multiple fiber, then it is either regular, or – in finitely many cases – a rational double point of type A_r .

Proof: Again let z be an inverse image of y in \tilde{J}_k , let w be the image of z in J_k , and c the common image of y and w in C_k . Then $\mathcal{O}_y = \mathcal{O}_z^{\text{Stab}_2 z}$, $\mathcal{O}_w = \mathcal{O}_z^{\text{Stab}_1 z} = \mathcal{O}_z^{\mathbb{Z}/m\mathbb{Z}}$. Now \mathcal{O}_w is an algebra over the discrete valuation ring \mathcal{O}_c , and if t is a prime of \mathcal{O}_c , \mathcal{O}_y is either the ring $\mathcal{O}' = \mathcal{O}_w[\sqrt[m]{t}]$, or – geometrically – a resolution of it. If w is a regular point of the fiber of J_k over c , then \mathcal{O}' is again a regular local ring, and thus y is a regular point. Otherwise, w is one of the finitely many nodes of that fiber (which is either a nodal curve, or a Néron polygone), hence the completion of \mathcal{O}_w is $k[[u, v, t]]/(uv - t)$, and that of \mathcal{O}' is $k[[u, v, t]]/(uv - t^m)$, so that \mathcal{O}' is the local ring of a rational double point of type A_r , which may or may not be (partially) resolved when going from $J_R \times_{C_R} D_R$ to \tilde{J}_R . \square

Step VII: Let P_R be the extension of P_K on $X_K = Y_K \rightarrow Y_R$, and let X_R be the image of Y_R under the map given by the sections of P_R . Then $P_R \rightarrow X_R \rightarrow C_R \rightarrow \text{Spec } R$ satisfies the theorem.

Proof: We have to show that X_k is an elliptic surface with at most rational double points as singularities. Since $\text{Spec } R$ is one-dimensional, $X_R \rightarrow \text{Spec } R$ is flat, hence the specializations of the elliptic fibers of X_K still have self-intersection zero, so that the morphism $Y_R \rightarrow C_R$ factors over a fibration $X_R \rightarrow C_R$, hence we get a fibration $X_k \rightarrow C_k$ over a nonsingular curve. By flatness, its fibers are connected and have arithmetic genus one. Also, since P_K is very ample on the generic fibers of $X_K \rightarrow C_K$, P_k is very ample on the generic fibers of $Y_k \rightarrow C_k$, hence their images in X_k remain elliptic curves, and the morphism $Y_k \rightarrow X_k$ is birational. Therefore the resolution Z_k of Y_k is also a resolution of X_k , and the resolution map $Z_k \rightarrow X_k$ is a C_k -morphism. If X_k were not normal, that is if some fiber of X_k would contain a one-dimensional surface singularity H , the inverse image of H in the normalization of X_k would have to be disconnected. But this normalization is dominated by Z_k , so the inverse image of H would correspond to some non-intersecting curves in a singular fiber of Z_k . Looking at the table of singular fibers, one sees that identification of two non-intersecting curves always generated a cycle, and thus increases the genus of the fiber, which is not possible in our case. Therefore X_k is normal, and since $(\omega \cdot \omega) = 0$ by flatness, it is an elliptic surface. Finally we have

$$\chi(\mathcal{O}_{X_k}) = \chi(\mathcal{O}_{X_k}) = \chi(\mathcal{O}_{Y_k}) = \chi(\mathcal{O}_{Z_k}),$$

therefore X_k has at most rational double points as singularities by lemma 1.2. The multiple fibers are no problem because of lemma 1.9, and the σ_i extend uniquely to C_R by properness. Step IV, finally, shows that J_k is the Jacobian fibration of X_k , and this completes the proof of the theorem. \square

2.5. Moduli schemes for polarized elliptic surfaces

So far we have been dealing with pairs (X, P) of surfaces and very ample line bundles, so let's consider moduli schemes for these first:

Definition: Let \mathfrak{T} be a type not corresponding to Enriques or $K3$ surfaces, i.e. $\mathfrak{T} \neq (0, 1; 2, 2, 2; d, e)$ and $\mathfrak{T} \neq (0, 2; 0; d, e)$, and let \mathcal{Sch} be the category of connected noetherian schemes over $\mathbb{Z}[1/6e]$. The moduli functor $P_{\mathfrak{C}}$ for pairs of type \mathfrak{C} is the functor from \mathcal{Sch} to \mathcal{Sets} with $P_{\mathfrak{T}}(T) =$ all isomorphism classes of pairs $(\mathcal{X}, \mathcal{P})$ over T , where \mathcal{X} is a family of elliptic surfaces with at most rational double points as singularities over T , and \mathcal{P} a relatively very ample line bundle on \mathcal{X} , such that each geometric fiber (X, P) is a pair of type \mathfrak{T} with $h^1(X, P) = h^2(X, P) = 0$. For the types of Enriques and $K3$ surfaces, let $P_{\mathfrak{T}}(T)$ consist of pairs $(\mathcal{X} \rightarrow \mathcal{C}, \mathcal{P})$, where $\mathcal{X} \rightarrow \mathcal{C}$ is a family of elliptic fibrations over T , and everything else as above.

The results in sections 2.2–2.4 show

THEOREM 2.14: *$P_{\mathfrak{T}}$ is coarsely representable by a quasiprojective scheme $M_{\mathfrak{C}}$, unless \mathfrak{T} is a type with $g = 0, \chi = 1$, and $n \leq 3$. In this latter case, only the subfunctor of surfaces without singular fibers of types I_m^*, II^*, III^*, IV^* is coarsely representable.* \square

$M_{\mathfrak{T}}$ is a moduli scheme for elliptic surfaces together with a very ample line bundle; in order to get a moduli scheme for elliptic surfaces, we still have to eliminate the line bundle. Unfortunately, however, we cannot expect that an algebraic moduli scheme for all elliptic surfaces with invariants $g, \chi, n, m_1, \dots, m_n$ exists, because the Tate–Šafarevič group of the Jacobian fibration has elements of arbitrarily high order. (There might exist a moduli scheme, if one also fixes that order.) The next best thing after eliminating the line bundle, is to replace it by an equivalence class of line bundles:

Definition: A polarized elliptic surface of type $\mathfrak{T} = (g, \chi; n, m_1, \dots, m_n, d, e)$ is a pair (X, η) resp. $(X \rightarrow C, \eta)$, consisting of an elliptic surface X with invariants $g, \chi, n, m_1, \dots, m_n$, and a numerical equivalence class η of line bundles on X , which contains an ample line bundle, such that $\eta^2 = d$ and $\eta \cdot f = e$, where f is the numerical class of a fiber. η is called an (inhomogeneous) polarization.

Remark: A homogeneous polarization is the ray of all positive rational multiples of an inhomogeneous polarization; in characteristic zero, Lefschetz's theorem shows, that each homogeneous polarization contains a unique deformation invariant inhomogeneous polarization generating it additively, so in characteristic zero moduli of homogeneously polarized and inhomogeneously polarized surfaces are the same.

With this notation we can now define the final moduli functor $G_{\mathfrak{T}}: \mathcal{Sch} \rightarrow \mathcal{Sets}$ by $G_{\mathfrak{T}}(T) =$ set of all isomorphism classes of families of polarized surfaces (resp. polarized surfaces together with an elliptic fibration in the case of Enriques and K3 surfaces) of type \mathfrak{T} over T .

Then the main result of this paper is

THEOREM 2.15: $G_{\mathfrak{T}}$ is coarsely representable by a quasiprojective scheme $E_{\mathfrak{T}}$, unless \mathfrak{T} is a type with $g = 0, \chi = 1$, and $n \leq 3$. In this latter case, only the subfunctor of surfaces without fibers of types I_m^*, II^*, III^*, IV^* is coarsely representable.

Proof: As Matsusaka and Mumford have shown in [13], theorem 4, there exists a constant N_0 depending only on \mathfrak{T} , such that for every line bundle L whose Hilbert polynomial is that of η , the k -th power is very ample for $k \geq N_0$. This implies in particular, that polarized surfaces of a given type \mathfrak{T} form a bounded family, hence we can also assume that $h^1(X, kL) = h^2(X, kL) = 0$ for $k \geq N_0$, and choose k such that multiplication by k cancels the torsion part of the Néron–Severi group of each polarized elliptic surface of type \mathfrak{T} . Therefore every such surface is represented by pairs (X, P) in the moduli scheme $M_{\mathfrak{T}^*}$, where $\mathfrak{T}^* = (g, \chi; n, m_1, \dots, m_n, k^2d, ke)$ is the type of pairs (X, P) with P in the class $k \cdot \eta$. Over the Hilbert scheme \mathcal{H} for this type, the Picard functor of the universal family is representable, hence also the subscheme which is the image of multiplication by k , and the condition that P should define a point of this subscheme is satisfied over a subscheme U of \mathcal{H} . Let V be the geometric quotient of U modulo the projective group. Then V is a coarse moduli scheme for pairs (X, P) of type \mathfrak{T}^* with P divisible by k in $Pic X$. For $g = 0, \chi \geq 1$, this is our moduli scheme $E_{\mathfrak{T}}$, because then $Pic^0 X = 0$, and $Pic^c X$ is annihilated by multiplication with k , so that every class η corresponds to only one P . For $g \geq 2$, or $g = 1$ and $\chi = 0$, consider the morphism $\mathcal{H} \rightarrow Y'$ from lemma 2.7. It restricts to a morphism $U \rightarrow Y''$, which in turn factors over a morphism $V \rightarrow Y''$. If (X, P) is a point of V , its image point determines X upto finitely many possibilities, but does not say anything about P , hence the fibers of this morphism are finite unions of algebraic equivalence classes of line bundles, hence finite unions of connected components of $Pic X$. Let $V \rightarrow E_{\mathfrak{T}} \rightarrow Y''$

be the Stein factorization, that is, $V \rightarrow E_{\bar{x}}$ has connected fibers, which must be algebraic equivalence classes of divisors on X , and $E_{\bar{x}} \rightarrow Y''$ has finite fibers. Since multiplication by k cancels the torsion part of $NS(X)$, the fibers of $V \rightarrow E_{\bar{x}}$ in fact even belong to different numerical equivalence classes, and thus $E_{\bar{x}}$ is really the moduli scheme we want. For $g = 1$ and $\chi \geq 1$, the argument is similar, only that now we use the morphism from theorem 2.9 without its last component, i.e. the morphism $\mathcal{H} \rightarrow E_{1,\chi} \times_{M_1} M_{1,n+1}$ given by the Jacobian fibration and the tuple $(\mathcal{C}, \sigma_0, \dots, \sigma_n)$. For $g = \chi = 0$ finally, we can use the morphism from theorem 2.12 with $M_{1,2}$ replaced by $M_1 = \mathbb{A}^1$, the moduli scheme for elliptic curves and hence also for Jacobian fibrations with $g = \chi = 0$. \square

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