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Global moduli for elliptic surfaces with a section

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Global moduli for elliptic surfaces with a section

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Introduction

The aim of this and a following paper is to show the existence of moduli spaces for elliptic surfaces, that is, roughly speaking, of quasiprojective schemes whose geometric points correspond in a natural way to isomorphism classes of elliptic surfaces; the method used is Mumford's geometric invariant theory. In this paper, the case of elliptic surfaces with a section is dealt with, the main tools being Weierstraß equations, and the fact that moduli schemes for curves of a given genus are known to exist. Because of the special role played by genus zero and genus one, a variety of cases have to be considered, and in one of them, the case of rational elliptic surfaces with a section, surfaces with "too singular" fibers have to be excluded before one can get a moduli scheme. Also, because of the use of Weierstraß equations, surfaces in characteristic two or three have to be excluded everywhere.

The results in this paper are part of my thesis, which was written under the direction of Professor D. Mumford; most of the work was done while I was a guest at the Tata Institute of Fundamental Research in Bombay.

during the academic year 1978/79, and at Harvard University in 1980/81; I want to thank both institutions for their hospitality. Particular thanks go to Professor Mumford, who suggested the problem of moduli of elliptic surfaces to me, and supplied many ideas, and to Professor Miller, Karlsruhe, with whom I had many useful discussions in various stages of the preparation of my thesis.

1. Preliminaries

Let k be an algebraically closed field. An elliptic surface over k is an integral projective surface X over k with at most rational double points as singularities, for which there exists an elliptic fibration, that is a k -morphism $p: X \rightarrow C$ onto a nonsingular curve C whose general fiber is a nonsingular curve of genus one, and none of whose fibers contains a rational curve with self-intersection -1 . X is called an elliptic surface with a section, if p admits a section $i: C \rightarrow X$. If, in addition, all fibers of p are irreducible, X is called a Weierstraß surface. It is well known (e.g., [4]) that, for $\text{char } k \neq 2, 3$, a Weierstraß surface has a Weierstraß equation

$$y^2z = x^3 - axz^2 - bz^3$$

in a projective bundle $\mathbb{P}(-2L \oplus -3L \oplus \mathcal{O}_C)$ over C ; here $L = -R^1p_*\mathcal{O}_X$, and $a \in H^0(C, 4L)$, $b \in H^0(C, 6L)$ are sections such that $\Delta = 4a^3 - 27b^2 \neq 0$ and $\min(3 \text{ ord}_P a, 2 \text{ ord}_P b) < 12$ for every geometric point P of C . Similarly, a family of elliptic surfaces over a noetherian scheme S is a diagram $\mathcal{X} \rightarrow \mathcal{C} \rightarrow S$ whose geometric fibers are elliptic fibrations; for a family of elliptic surfaces with a section, I demand that there exists a global section $\mathcal{C} \rightarrow \mathcal{X}$. Combining the construction in [11], theorem 1.1' with [4], one can show that for every family of elliptic surfaces $\mathcal{X} \rightarrow \mathcal{C} \rightarrow S$ with S defined over $\mathbb{Z}[1/6]$, there exists a \mathcal{C} -morphism $\varphi: \mathcal{X} \rightarrow \mathcal{W}$, where $\mathcal{W} \rightarrow \mathcal{C} \rightarrow S$ is a family of Weierstraß surfaces (with a global Weierstraß equation over \mathcal{C}), and φ contracts all components of reducible fibers that are not hit by the section, and leaves everything else unchanged (for details see [14], §1). The idea, in this paper, is to find a scheme parametrizing isomorphism classes of elliptic surfaces with a section. The technical tool to define such a system in a precise way is coarse representability of functors: Let

\mathcal{Sch} = category of connected noetherian schemes over $\mathbb{Z}[1/6]$ and

\mathcal{Sets} = category of sets.

If nothing else is said, a scheme will always be assumed to be defined over $\mathbb{Z}[1/6]$, and an algebraic scheme will be a scheme of finite type over that ring.

Definition: (a) Let $F: \mathcal{S}ch \rightarrow \mathcal{S}ets$ be a (contravariant) functor, and E a scheme over $\mathbb{Z}[1/6]$ with functor of points $h_E: \mathcal{S}ch \rightarrow \mathcal{S}ets; T \rightarrow \text{Hom}(T, E)$. E coarsely represents F , if

- (i) there is a morphism of functors $f: F \rightarrow h_E$
- (ii) $f(K): F(K) \rightarrow h_E(K)$ is a bijection for every algebraically closed field K
- (iii) for every scheme X with a morphism of functors $g: F \rightarrow h_X$, there is a morphism $\varphi: E \rightarrow X$ such that $g = h_\varphi \circ f$.

It is clear that the scheme E , if it exists, is determined uniquely upto isomorphism. The standard method for dealing with the problem of existence is geometric invariant theory ([10]): One starts with a scheme X and a group scheme G acting on X in such a way that the quotient functor

$$h_X/G: \mathcal{S}ch \rightarrow \mathcal{S}ets; T \rightarrow \{G(T)\text{-orbits in } h_X(T)\}$$

is isomorphic to the functor that is to be coarsely represented, and then construct E as a “quotient” of $X \text{ mod } G$:

Definition: Let G be an algebraic group acting on a scheme X . A good quotient of $X \text{ mod } G$ is a G -invariant affine surjective and submersive morphism $f: X \rightarrow Y$, such that $\mathcal{O}_Y = (f_* \mathcal{O}_X)^G$. A geometric quotient of $X \text{ mod } G$ is a good quotient that maps two geometric points of X to the same geometric point of Y iff they lie in the same G -orbit.

LEMMA 1: *Let G be an algebraic group acting on a scheme X , and let F be a functor which is isomorphic to h_X/G . If a geometric quotient $f: X \rightarrow Y$ exists, then Y coarsely represents F .*

Proof: Easy. For (iii) use that a geometric quotient is universal among G -invariant morphisms by [10], chap. 0 §2, Rem. 6. □

Thus, apart from the problem of finding a suitable scheme X , the problem of coarse representability of a functor is reduced to investigating the existence of geometric quotients. An elementary criterion for this is

LEMMA 2: *For each action of a finite group on a quasiprojective variety, a geometric quotient exists.*

Proof: [8], §7, theorem and remark. □

A larger class of groups for which good criteria exist are geometrically reductive groups: An algebraic group G is called geometrically reductive, if whenever G acts on a projective space P and leaves a point $x \in P$ invariant, there exists an invariant hypersurface $H \subset P$ not containing x . Examples of geometrically reductive groups include

- finite groups and tori (easy)
- Sl_n and Gl_n (Haboush [3], Formanek/Procesi [2]).

Definition: Let G be an algebraic group acting on a scheme X , and let L be a line bundle on X . The action of G on X is called linear with respect to L , if there exists an action of G on L such that the projection $L \rightarrow G$ is G -equivariant. Then G also acts on the vector spaces $H^0(X, nL)$; let $H^0(X, nL)^G$ be the subspace of invariants. A geometric point $x \in X$ is called

- semistable, if there is a function $f \in H^0(X, nL)^G$ for some $n > 0$ such that $f(x) \neq 0$
- properly stable, if there is a function $f \in H^0(X, nL)^G$ for some $n > 0$ such that $f(x) \neq 0$, and each geometric point y with $f(y) \neq 0$ has a finite stabilizer.

The semistable and properly stable points of X are the geometric points of open subschemes of X ; these subschemes will be denoted $X^{ss}(L)$ and $X_0^s(L)$ respectively.

THEOREM 3: *Let G be a geometrically reductive group acting on the algebraic scheme X linearly with respect to an invertible module L . Then there exists a good quotient of $X^{ss}(L) \text{ mod } G$, and a geometric quotient of $X_0^s(L) \text{ mod } G$.*

Proof: [10], theorem 1.10, [13], theorem 4 and remark 9 on page 271. □

THEOREM 4: *Let G be a geometrically reductive group acting on a projective scheme $X \subset \mathbb{P}^n$ linearly with respect to $\mathcal{O}_X(1)$, and let $\mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ be the canonical projection.*

- (a) *A geometric point $x \in X$ is semistable, iff the following holds for one (any) point $\hat{x} \in \mathbb{A}^{n+1} \setminus \{0\}$ lying above x : $\lim_{t \rightarrow 0} \lambda(t) \hat{x} \neq 0$ for every one parameter subgroup $\lambda: \mathbb{G}_m \rightarrow G$ of G .*
- (b) *x is properly stable, iff $\lim_{t \rightarrow 0} \lambda(t) \hat{x}$ does not exist for any nontrivial one parameter subgroup $\lambda: \mathbb{G}_m \rightarrow G$ of G .*

Proof: [10], theorem 2.1. □

2. Functors related to elliptic surfaces with a section

The functor we are mostly concerned with is of course

$$F_{g,\chi}: \mathcal{Sch} \rightarrow \mathcal{Sets}; T \rightarrow \left\{ \begin{array}{l} \text{all isomorphism classes of smooth} \\ \text{families of elliptic surfaces with a} \\ \text{section fibered over a smooth curve} \\ \text{of genus } g \text{ over } T, \text{ and with } \chi(\mathcal{O}) = \chi \\ \text{in all geometric fibers over } T \end{array} \right\}$$

here “isomorphism” will always be understood in the sense of elliptic fibrations (which only in Kodaira dimension zero can be slightly stronger than mere isomorphism of surfaces). For $\chi = 0, g \neq 0$, we shall see in lemma 9 that $F_{g,0}$ splits in a natural way into a union of subfunctors $F_{g,0}^n, n = 0, 1, 2, 3, 4, 6$, parametrizing elliptic fibrations $f: X \rightarrow C$ for which $R^1f_*\mathcal{O}_X$ has order n in the Jacobian of C . We also need the functor $W_{g,d}: \mathcal{Sch} \rightarrow \mathcal{Sets}$ for which $W_{g,d}(T)$ is the set of all quadruples $(\mathcal{C}, \mathcal{L}, a, \ell)$ where \mathcal{C} is a smooth family of curves of genus g over T, \mathcal{L} an invertible $\mathcal{O}_{\mathcal{C}}$ -module of degree d , and $a \in \Gamma_T(\mathcal{C}, 4\mathcal{L}), \ell \in \Gamma_T(\mathcal{C}, 6\mathcal{L})$ sections for which $4a^3 - 27\ell^2$ does not vanish identically on any geometric fiber of \mathcal{C} . Going to Weierstraß surfaces gives a canonical morphism of functors $F_{g,\chi} \rightarrow W_{g,\chi}$ whose image lies in the subfunctor $\tilde{W}_{g,\chi}$ consisting of those quadruples $(\mathcal{C}, \mathcal{L}, a, \ell)$ for which $\min(3 \text{ ord}_P a, 2 \text{ ord}_P \ell) < 12$ for any geometric point P of any fiber of \mathcal{C} .

LEMMA 5:

(a) If $W_{g,d}$ is coarsely represented by a scheme $E, \tilde{W}_{g,d}$ is coarsely represented by an open subscheme \tilde{E} of E .

(b) If $\tilde{W}_{g,d}$ is coarsely represented by a scheme \tilde{E} , so is $F_{g,d}$.

Proof: For (a), it suffices to show that $\tilde{W}_{g,d}$ is an open subfunctor of $W_{g,d}$. If $(\mathcal{C}, \mathcal{L}, a, \ell)$ is an element of $W_{g,d}(T)$, the condition $\text{ord}_P a > 4$ or $\text{ord}_P \ell > 6$ clearly defines a closed subset of \mathcal{C} and thus of T, \mathcal{C} being proper over T . Hence the complement is open, as required. (b) is clear by [4]. \square

In some cases, it will simplify matters if the base curve of a family admits a section. This can always be assumed by

LEMMA 6: Let F be one of the functors $F_{g,\chi}, F_{g,0}^n, W_{g,d}$, and let SF denote the subfunctor consisting of all those objects for which the base curve $\mathcal{C} \rightarrow T$ admits a section. Then F is representable by a scheme E , iff SF is representable by that scheme.

Proof: $\mathcal{C} \rightarrow T$ gets a section after a faithfully flat base extension $T' \rightarrow T$, for example with $T' = \mathcal{C}$, and conversely the induced family over T' determines the one over T by descent theory (SGA I, exp. VIII, Cor. 5.3). Therefore the lemma can be easily shown by checking the three points in the definition of coarse representability one by one. \square

The main result of this paper is

THEOREM 7: *Let (g, χ) be any pair of nonnegative integers. If $(g, \chi) \neq (0, 1)$, the functor $F_{g,\chi}$ is coarsely representable by a quasiprojective scheme $E_{g,\chi}$. For $(g, \chi) = (0, 1)$, the case of rational elliptic surfaces with a section, the subfunctor $F'_{0,1}$ of $F_{0,1}$ parametrizing surfaces without fibers of types I_r^* , II^* , III^* , IV^* is representable by a quasiprojective scheme $E_{0,1}$.*

Because of lemma 5 and the table of singular fibers, this follows from

THEOREM 8: *Let (g, d) be any pair of nonnegative integers. If $g \neq 0$ or $g = d = 0$, the functor $W_{g,d}$ is coarsely representable by a quasiprojective scheme $\bar{E}_{g,d}$. For $g = 0$ and $d \geq 0$, the subfunctor $W'_{0,d}$ of $W_{0,d}$ consisting of those quadrupels $(\mathcal{C}, \mathcal{L}, \alpha, \ell)$ for which $\min(3 \text{ord}_P \alpha, 2 \text{ord}_P \ell) < 6d$ for any geometric point P of \mathcal{C} is coarsely representable by a quasiprojective scheme $\bar{E}'_{0,d}$.*

For the *proof*, several cases have to be considered. The most general one is

3. The case $g \geq 2$ or $g = 1, d \geq 3$

Here it is possible to find an integer r , such that $D = d + (2g - 2)r > 2g$; fix one such r . Then, for any curve C of genus g , and any invertible \mathcal{O}_C -module L of degree d on C , $L + r\omega_C$ is very ample, and thus every basis \mathfrak{B} of $H^0(C, L + r\omega_C)$ defines an embedding of C into \mathbb{P}^n , $n = D - g$, with $\mathcal{O}_C(1) = L + r\omega_C$. Now the one dimensional subschemes of genus g and degree D correspond – in a functorial way – to the points of the Hilbert scheme $\text{Hilb}_{\mathbb{P}^n}^{mD+1-g}$ (FGA, exp. 221, theorem 3.1 and page 17). By [10], prop. 5.1 (disregard condition (ii)), $\text{Hilb}_{\mathbb{P}^n}^{mD+1-g}$ has a subscheme \mathbf{H} whose points correspond to irreducible nonsingular curves embedded by a complete linear system. Thus \mathbf{H} represents the functor

$$Sch \rightarrow Sets; T \rightarrow \left\{ \begin{array}{l} (\mathcal{C}, \mathcal{L}, \mathfrak{B}) | \mathcal{C} \subset \mathbb{P}^n \times T \text{ a smooth family of} \\ \text{curves of genus } g, \mathcal{L} \text{ an invertible module} \\ \text{of degree } d \text{ over } \mathcal{C} \text{ such that} \\ \\ \mathcal{O}_{\mathcal{C}}(1) = \mathcal{L} + r\omega_{\mathcal{C}/T}, \\ \\ \mathfrak{B} \text{ a basis of } \Gamma_T(\mathbb{P}^n \times T, \mathcal{O}_p(1)) \end{array} \right.$$

Definition: A small pair $(\mathcal{C}, \mathcal{L})$ over T is a smooth curve $q: \mathcal{C} \rightarrow T$ of genus g together with an invertible $\mathcal{O}_{\mathcal{C}}$ -module of degree d such that $q_*(\mathcal{L} + r\mathcal{O}_{q|T})$ is a free \mathcal{O}_T -module. A basis of $(\mathcal{C}, \mathcal{L})$ is a basis of that \mathcal{O}_T -module.

Then \mathbf{H} also represents the functor

$$\mathit{Sch} \rightarrow \mathit{Sets}; T \rightarrow \left\{ \begin{array}{l} (\mathcal{C}, \mathcal{L}, \mathfrak{B}), (\mathcal{C}, \mathcal{L}) \text{ a small pair} \\ \text{over } T, \mathfrak{B} \text{ a basis thereof} \end{array} \right\},$$

and there exists a universal small pair (\mathbf{C}, \mathbf{L}) over \mathbf{H} . To bring the section a, b of a quadruple (C, L, a, b) into the picture, recall

LEMMA 9: *Let $f: X \rightarrow Y$ be a proper morphism of locally noetherian schemes and \mathcal{F} a coherent \mathcal{O}_X -module, flat over \mathcal{O}_Y . Then the functor*

$$\mathit{Sch}_Y \rightarrow \mathit{Sets}; T \rightarrow \Gamma_T(X \times_Y T, \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_T)$$

is representable by a scheme which is affine over Y .

Proof: The corresponding functor of modules is representable by EGA III₂, theorem 7.7.6 and remark 7.7.9, and by EGA I, prop. 9.4.9, we can associate an affine T -scheme to the representing \mathcal{O}_T -module. □

Applying this to the universal curve $\mathbf{C} \rightarrow \mathbf{H}$ and the modules $n\mathbf{L}$, we get schemes \mathbf{V}_n , affine over \mathbf{H} , such that the fiber product $\mathbf{V}_4 \times_{\mathbf{H}} \mathbf{V}_6$ represents the functor

$$\mathit{Sch} \rightarrow \mathit{Sets}; T \rightarrow \left\{ \begin{array}{l} (\mathcal{C}, \mathcal{L}, \mathfrak{B}, a, \ell), (\mathcal{C}, \mathcal{L}) \text{ a small} \\ \text{pair over } T, \mathfrak{B} \text{ a basis thereof,} \\ a \in \Gamma_T(\mathcal{C}, 4\mathcal{L}), \ell \in \Gamma_T(\mathcal{C}, 6\mathcal{L}) \end{array} \right\}.$$

$\mathbf{V}_4 \times_{\mathbf{H}} \mathbf{V}_6$ has an open subscheme U representing the subfunctor of those quintuples for which $\Delta = 4a^3 - 27b^2$ does not vanish identically in any geometric fiber over T . So, in order to represent the functor $W_{g,d}$, only the basis \mathfrak{B} still has to be eliminated. The group scheme Gl_{n+1} acts on U by changing of the basis, and the quotient functor h_U/Gl_{n+1} is isomorphic to the subfunctor of $W_{g,d}$ consisting of those quadruples $(\mathcal{C}, \mathcal{L}, a, \ell)$ for which $(\mathcal{C}, \mathcal{L})$ is a small pair. Since any pair $(\mathcal{C}, \mathcal{L})$ has an open covering by small pairs, one easily sees that coarse representability of this subfunctor implies coarse representability of $W_{g,d}$, so, by lemma 1, the problem is reduced to showing the existence of a geometric quotient of $U \bmod Gl_{n+1}$. This is done

in two steps: As Gl_{n+1} is generated by Sl_{n+1} and its center \mathbb{G}_m , one can take a quotient mod Sl_{n+1} first, and then a quotient mod \mathbb{G}_m . For the first step, analysis of stability gives the result: Let $\mathcal{O}_H(1)$ be an invertible \mathcal{O}_H -module defined by a projective embedding of \mathbf{H} , and let $f: U \rightarrow \mathbf{H}$ be the projection. By [9], theorem 4.15, every point of \mathbf{H} is properly stable for Sl_{n+1} , and since $f: U \rightarrow \mathbf{H}$ is quasiaffine, so is every point of U with respect to the invertible \mathcal{O}_U -module $f^*\mathcal{O}_H(1)$ by [10], prop. 1.18. Hence there are geometric quotients $\varphi: U \rightarrow Y$, $\psi: \mathbf{H} \rightarrow M \bmod Sl_{n+1}$, and a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & Y \\ f \downarrow & & \downarrow g \\ \mathbf{H} & \xrightarrow{\psi} & M \end{array}$$

By definition of a geometric quotient, $\mathcal{O}_Y = (\varphi_*\mathcal{O}_U)^{Sl_{n+1}}$, and $\mathcal{O}_M = (\psi_*\mathcal{O}_H)^{Sl_{n+1}}$. U being an open subscheme of $\mathbf{V}_4 \times_{\mathbf{H}} \mathbf{V}_6$, \mathcal{O}_U is a graded \mathcal{O}_H -algebra, hence \mathcal{O}_Y is a graded \mathcal{O}_M -algebra, and of course $Y = \text{Spec } \mathcal{O}_Y$. Therefore, in order to get a geometric quotient of $Y \bmod \mathbb{G}_m$, we can simply take the canonical projection

$$Y = \text{Spec } \mathcal{O}_Y \rightarrow \text{Proj}_{\mathcal{O}_M} \mathcal{O}_Y =: \bar{E}_{g,d}.$$

This is well defined, because the condition $\Delta \neq 0$ on U ensures that in no fiber over M the \mathbb{G}_m -action on Y has a fixed point. Thus $\bar{E}_{g,d}$ coarsely represents the functor $W_{g,d}$ for $g \geq 2$ and for $g = 1, d \geq 3$. Note that there are canonical projections $\bar{E}_{g,d} \rightarrow M := M_{g,d} \rightarrow M_g$, where M_g is the moduli scheme for nonsingular curves of genus g .

4. The case $g = d = 1$

Since every elliptic curve is a group, all invertible modules of degree one on it are isomorphic; it suffices therefore to distinguish one geometric point P of C , and to consider only the module $\mathcal{O}_C(P)$. If

$$\mathcal{C} = \{(g_2, g_3, x, y, z) \in \mathbb{A}^2_{\Delta} \times \mathbb{P}^2 \mid y^2z = x^3 - g_2xz^2 - g_3z^3\}$$

$$\mathbb{A}^2_{\Delta} = \mathbb{A}^2 \setminus \{(g_2, g_3) \in \mathbb{A}^2 \mid \Delta = 4g_2^3 - 27g_3^2 = 0\}$$

denotes the Weierstraß family of elliptic curves, the condition $x = z = 0$ cuts out a (Weil-) divisor \mathcal{D} which hits every fiber in its point $(0, 1, 0)$. Let

$\mathcal{L} = \mathcal{O}_\varphi(\mathcal{D})$. Then, for each point $s \in \mathbb{A}_\Delta^2$,

$$H^0(C_s, 4L_s) = k \oplus kx \oplus ky \oplus kx^2 \quad \text{and}$$

$$H^0(C_s, 6L_s) = k \oplus kx \oplus ky \oplus kx^2 \oplus kxy \oplus kx^3;$$

thus the global sections of $4L_s$ and $6L_s$ form a trivial vector bundle $E = \mathbb{A}_\Delta^2 \times \mathbb{A}^4 \times \mathbb{A}^6$ over \mathbb{A}_Δ^2 . E contains an open subscheme U consisting of those quadruples (C, L, a, b) , for which the discriminant does not vanish identically. Now let $(C, L, a, b), (C', L', a', b')$ be quadruples corresponding to two geometric points of U , and let (g_2, g_3) and (g'_2, g'_3) be the coefficients of the Weierstraß equations for C and C' . In order that (C, L, a, b) and (C', L', a', b') be isomorphic, C and C' must be isomorphic, and there must be an isomorphism $L \rightarrow L'$ mapping a to a' and b to b' . Thus there have to exist units λ, μ in the ground field such that

$$g'_2 = \lambda^4 g_2, \quad g'_3 = \lambda^6 g_3, \quad a' = \mu^4 a, \quad b' = \mu^6 b.$$

This suggests that one should look for a geometric quotient of U modulo the torus $T = \mathbb{G}_m \times \mathbb{G}_m$. The action of T on U can be given explicitly: With

$$a = a_0 + a_2x + a_3y + a_4x^2$$

$$b = b_0 + b_2x + b_3y + b_4x^2 + b_5xy + b_6x^3,$$

(C, L, a, b) is given by the 12-tuple

$$(g_2, g_3; a_0, a_2, a_3, a_4; b_0, b_2, b_3, b_4, b_5, b_6),$$

which is mapped to

$$(\lambda^4 g_2, \lambda^6 g_3; \mu^4 a_0, \mu^4 \lambda^2 a_2, \mu^4 \lambda^3 a_3, \mu^4 \lambda^4 a_4; \mu^6 b_0, \mu^6 \lambda^2 b_2, \mu^6 \lambda^3 b_3, \mu^6 \lambda^4 b_4, \mu^6 \lambda^5 b_5, \mu^6 \lambda^6 b_6)$$

by the element (λ, μ) of T . Taking monomials in the twelve variables g_2, g_3, a_i, b_j , one can now easily write down an explicit map which is a geometric quotient of $U \bmod T$. It remains to be shown that the quotient scheme V coarsely represents the functor $W_{1,1}$. By lemma 6, it suffices to consider quadruples $(\mathcal{C}, \mathcal{L}, a, \ell)$ for which $p: \mathcal{C} \rightarrow T$ admits a section. Then \mathcal{C}/T is a family of elliptic curves, and therefore has a Weierstraß normal form ([11],

§3, theorem 1.1'). We may assume that $R^1 p_* \mathcal{O}_\mathcal{C} \cong \mathcal{O}_T$, because every quadruple is covered by quadruples for which this holds. Then \mathcal{C} can be embedded into $\mathbb{P}^2 \times T$ and described there by a Weierstraß equation $y^2 z = x^3 - g_2 x z^2 - g_3 z^3$ with $g_2, g_3 \in H^0(T, \mathcal{O}_T)$. \mathcal{L} is isomorphic, and hence without loss of generality equal, to the line bundle defined by the section $\{(0, 1, 0)\} \times T$. As above, we can find now a global basis for the sections of $4L, 6L$, and $(\mathcal{C}, \mathcal{L}, a, \ell)$ defines a point

$$(g_2, g_3; a_0, a_2, a_3, a_4; b_0, b_2, b_3, b_4, b_5, b_6) \in S \times U,$$

that is an S -rational point of U . Via the geometric quotient $U \rightarrow V$, this point can be extended to an S -rational point of V , and one easily sees that this procedure defines a morphism of functors which has all the properties required for coarse representability.

5. The case $g = 1, d = 2$

Here the starting point is the modular family

$$\mathcal{C} = \{(\lambda, x, y, z) \in \mathbb{A} \times \mathbb{P}^2 \mid y^2 z = x(x - z)(x - \lambda z)\}$$

$$A = \mathbb{A}^1 \setminus \{0, 1\}$$

\mathcal{D} once again denotes the divisor which is cut out by $x = z = 0$, and $\mathcal{L} = \mathcal{O}_\mathcal{C}(2\mathcal{D})$. Since all invertible modules of degree two on an elliptic curve are isomorphic, every quadruple (C, L, a, b) is isomorphic to one of the quadruples $(C_\lambda, L_\lambda, a', b')$ for some $\lambda \in A$. As in the case $d = 1$, the sections of $4\mathcal{L}$ and $6\mathcal{L}$ form a trivial vector bundle $E = A \times \mathbb{A}^8 \times \mathbb{A}^{12}$ over A , and E has an open subset U where $\Delta \neq 0$. As isomorphisms between different quadruples in the family, we still have those which already occurred for $d = 1$, namely

- isomorphisms of the base curves and
- the equivalence given by $(a, b) \rightarrow (\mu^4 a, \mu^6 b)$ for $\mu \in \mathbb{G}_m$.

In addition, we also have those isomorphisms which are induced by interchanging $(0, 1, 0)$ with another 2-division point of C_λ . In the modular family, these 2-division points lie on sections of \mathcal{C}/A , namely on

$$\mathcal{D}_1: x = y = 0, \mathcal{D}_2: x = z, y = 0 \text{ and } \mathcal{D}_3: x = \lambda z, y = 0.$$

Thus one gets three isomorphisms

$$\sigma_i: P \in C_\lambda \text{ is mapped to } P + D_{i,\lambda} - D_\lambda$$

(+ is the addition in the group C_λ), which, together with the identity map, form a noncyclic group of order four modulo the action of \mathbb{G}_m . Since \mathcal{L} is not invariant under the action of the σ_i , the group generated by \mathbb{G}_m and the σ_i is a nontrivial extension H of \mathbb{G}_m by V_4 , the so called Heisenberg group. The isomorphisms $C_\lambda \cong C_{\lambda'}$ can be got rid of by taking the quotient modulo an action of the symmetric group S_3 , so on the whole the group $G = H \times S_3$ acts on U . The existence of a quotient of $U \bmod \mathbb{G}_m$ is clear, and since that quotient is quasiprojective, lemma 2 shows the existence of a further quotient mod $V_4 \times S_3$, hence a geometric quotient of $U \bmod G$ exists. It remains to show that the quotient scheme V coarsely represents the functor $W_{1,2}$. The crucial point is, once again, the existence of a morphism of functors $W_{1,2} \rightarrow h_V$; the rest is routine and will be omitted.

Let $(\mathcal{C}, \mathcal{L}, a, \ell) \in W_{1,2}(S)$ be a quadruple whose base scheme S is so small that the sheaf $\mathcal{M} = R^1 p_* \mathcal{O}_{\mathcal{C}}$ is trivial. Then the sections of \mathcal{M} define a morphism $f: \mathcal{C} \rightarrow \mathbb{P}(p_* \mathcal{M}) \cong \mathbb{P}^1 \times S$. f is a two-sheeted covering and has four ramification points in each geometric fiber over S . Suppose, these ramification points lie on four sections $\varrho_i: S \rightarrow \mathbb{P}^1 \times S$. Then an easy calculation shows that there is an automorphism of $\mathbb{P}^1 \times S$ mapping $\varrho_1, \varrho_2, \varrho_3$ to the sections

$$S \rightarrow \{0\} \times S, S \rightarrow \{1\} \times S \text{ and } S \rightarrow \{\infty\} \times S$$

respectively, i.e., \mathcal{C}/S has a Legendre normal form, and the existence of a morphism $S \rightarrow V$ can be shown by the same method as in the case $d = 1$.

If the ramification locus of \mathcal{C} over $\mathbb{P}^1 \times S$ cannot be split into four sections, it still can be described by a quaternary form, and this form can be viewed as an S -valued point of \mathbb{P}^4 . \mathbb{P}^4 is a geometric quotient of $(\mathbb{P}^1)^4 \bmod S_4$; since we have excluded characteristics two and three, S_4 is a linearly reductive group, so by [10], theorem 1.1 (iii), this quotient is universal, i.e., also the base extension $S' = S \times_{\mathbb{P}^4} (\mathbb{P}^1)^4 \rightarrow S$ is a geometric quotient. For the corresponding base extension $(\mathcal{C}', \mathcal{L}', a', \ell')$ of $(\mathcal{C}, \mathcal{L}, a, \ell)$, the branch locus in $\mathbb{P}^1 \times S'$ obviously splits into four sections, hence there is a morphism $S' \rightarrow V$. This morphism is S_4 -invariant, because it factorizes via $W_{1,2}$, therefore the morphism $S' \times V$ has to factor via the geometric quotient $S' \rightarrow S$, thus giving the desired morphism $S \rightarrow V$. This completes the case $g = 1, d = 2$.

6. The case $g = 0, d \geq 1$

Here we consider elliptic surfaces with a section over \mathbb{P}^1 ; this case has been dealt with extensively by R. Miranda in his papers [5], [6] and [7]; my line

of approach is almost identical to that in [6], and at the same time a specialization of the arguments in §2 above, so I can be rather short.

Since $\mathcal{O}_p(d)$ is the only invertible module of degree d on \mathbb{P}^1 , quadruples (\mathbb{P}^1, L, a, b) with $\deg L = d$ are parametrized by an open subset U of

$$X_d = \mathbb{A}(H^0(\mathbb{P}^1, \mathcal{O}_p(4d)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_p(6d))) \cong \mathbb{A}^{10d+2}.$$

Two such quadruples are isomorphic, *iff* they can be connected by a linear transformation $\sigma \in Gl(H^0(\mathbb{P}^1, \mathcal{O}_p(1))) \cong Gl_2$; as in §2, it follows that a geometric quotient of $U \text{ mod } Gl_2$, if it exists, coarsely represents $W_{0,d}$. In order to show the existence of a geometric quotient, it is useful once again to consider Gl_2 as being generated by its subgroups \mathbb{G}_m and Sl_2 , and to form the quotient in two steps: The ring

$$R = S(H^0(\mathbb{P}^1, \mathcal{O}_p(4d)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_p(6d))),$$

where $S(\cdot \cdot \cdot)$ denotes the symmetric algebra of a module, becomes a graded algebra by assigning degree $4d$ resp. $6d$ to every element of $H^0(\mathbb{P}^1, \mathcal{O}_p(4d))$ resp. $H^0(\mathbb{P}^1, \mathcal{O}_p(6d))$, and clearly the canonical morphism from U to its image V in $\text{Proj } R$ is a geometric quotient mod \mathbb{G}_m . Sl_2 acts on V and – compatibly – on U ; this defines a linearization of the action on V with respect to a certain very ample sheaf. Now let $\lambda: \mathbb{G}_m \rightarrow Sl_2$ be a one-parameter subgroup of Sl_2 . With respect to a suitable basis of $H^0(\mathbb{P}^1, \mathcal{O}_p(1))$, λ has the form

$$t \rightarrow \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix},$$

so the point $(a, b) \in V$ with

$$a = \sum_{i=0}^{4d} a_i x^i y^{4d-i}, \quad b = \sum_{i=0}^{6d} b_i x^i y^{6d-i}$$

is mapped to

$$\lambda(t)(a, b) = \left(\sum_{i=0}^{4d} a_i t^{4d-2i} x^i y^{4d-i}, \sum_{i=0}^{6d} b_i t^{6d-2i} x^i y^{6d-i} \right),$$

and the specialization $\lim_{t \rightarrow 0} \lambda(t)(a, b)$ exists in X_d , *iff* a_{2d+1}, \dots, a_{4d} and b_{3d+1}, \dots, b_{6d} vanish; it is zero, if in addition $a_{2d} = b_{3d} = 0$. Fixing a basis

of $H^0(\mathbb{P}^1, \mathcal{O}_p(1))$ and letting λ vary, one thus gets from theorem 4 that (a, b) is properly stable, iff for every point $P \in \mathbb{P}^1$, $\min(3 \text{ord}_p a, 2 \text{ord}_p b) < 6d$; for semistability, also equality is permitted. This proves the theorem for the case $g = 0$.

Especially with respect to the case $d = 1$, which leads to rational elliptic surfaces, and where the result above does not even show the existence of a coarse moduli scheme for smooth surfaces, it would be interesting to know, whether theorems 7 and 8 can be improved for $g = 0$. In [7], Miranda conjectures that a rational elliptic surface with a singular fiber of type I_m^* , II^* , III^* or IV^* is unstable in any projective embedding; he has proofs for types II^* , III^* and IV^* . In [5] he considers the elliptic pencil in \mathbb{P}^2 defined by a rational elliptic surface with a section, and shows that essentially the same also holds for this pencil. So there does not seem to be a method by which one can get a bigger quasiprojective moduli scheme for rational elliptic surfaces with a section.

7. The case $d = 0$

The last two open cases for theorems 7 and 8 are $g = 0$ or 1 and $d = 0$, the cases $d = 0$ and $g \geq 2$ being included in §2. Nevertheless I want to investigate the cases $d = 0$ here for arbitrary values of g in order to get some more detailed results than mere existence of a moduli scheme. A first observation is

LEMMA 9: *Let $(\mathcal{C}, \mathcal{L}, a, \ell) \in W_{g,0}(S)$ be a quadruple over a connected scheme S . Then there exists an integer $n \in \{1, 2, 3, 4, 6\}$ such that L_i has order n in the Jacobian of C_i for every geometric point t of S .*

Proof: Each L_i has an order $n_i \in \{1, 2, 3, 4, 6\}$, because either $4L_i$ or $6L_i$ has to have a non-trivial section; let n be the smallest n_i . By semicontinuity, $Z = \{t \in S \mid mL_t \cong \mathcal{O}_{C_t}\}$ is closed; for Z is the subset of S over which $h^0(C_t, mL_t) = 1$. But also $S \setminus Z$ is closed: After a finite base change $S' \rightarrow S$, $\mathcal{C}' = \mathcal{C} \times_S S'$, one can assume that $\mathcal{P}ic^0(\mathcal{C}'/S')$ has a level $12/n$ structure, and that there are invertible modules $\mathcal{L}_1, \dots, \mathcal{L}_r, r = (12/n)^{2g} - 1$, on \mathcal{C} which induce all nontrivial invertible modules of order $12/m$ on the fibers of \mathcal{C}'/S' . The subsets

$$Z'_i = \{t \in S' \mid mL_t \cong L_{i,t}\} = \{t \in S' \mid mL_t - L_{i,t} \cong \mathcal{O}_{C_t}\}$$

of S' are closed, and so is their union Z' and its image under the (finite) projection $S' \rightarrow S$. But this image is the complement of Z , because for each

$s' \in S'$ with image $s \in S$, $L_s \cong L'_{s'}$. Thus the connected space is the union of two disjoint closed subsets Z and Z' ; since $Z \neq \emptyset$, Z' must be empty, and $Z = S$, i.e., $n_t = n$ for all t . □

As an immediate corollary we get that a scheme $\bar{E}_{g,0}$ coarsely representing $W_{g,0}$ for $g \geq 1$ splits into a disjoint union of subschemes $\bar{E}_{g,0}^n$, $n = 1, 2, 3, 4, 6$, representing the subfunctors $W_{g,0}^n$ of those quadrupels for which the order of the module L in the Jacobian of the curve is n . For $n = 1$, the module is trivial, and one easily shows

LEMMA 10: *Let M_g be the coarse moduli scheme for non-singular curves of genus g , $g \geq 0$. Then $E_{g,0}^1 = M_g \times \mathbb{A}^1$ coarsely represents $W_{g,0}^1$.*

Proof: The coordinate in \mathbb{A}^1 is of course $j = 12^3 \cdot 4a^3/\Delta$, $\Delta = 4a^3 - 27b^2$. □

This lemma completes the proof of theorems 7 and 8 for the case $g = d = 0$, because there are no nontrivial modules of degree zero on a projective line. For positive g , we still have to consider the other values of n ; the first case is

LEMMA 11: *The affine plane $\mathbb{A}^2 = E_{1,0}^2$ coarsely represents the functor $W_{1,0}^2$. For arbitrary $g \geq 1$, there is a canonical finite morphism $E_{g,0}^2 \rightarrow E_{g,0}^1 \cong M_g \times \mathbb{A}^1$ of degree at most $2^{2g} - 1$ which is an isomorphism for $g = 1$. $E_{g,0}^2$ is irreducible and has dimension $3g - 2$ for $g \geq 2$, and 2 for $g = 1$.*

Proof: Let $(\mathcal{C}, \mathcal{L}, a, \ell) \in W_{g,0}^2(S)$ be a quadruple whose base curve admits a section. By [10], prop. 6.9 and its proof, there is an embedding $\mathcal{C} \rightarrow \mathcal{J} = \mathcal{P}ic^0(\mathcal{C}/S)$, and a canonical isomorphism $\mathcal{J} \rightarrow \hat{\mathcal{J}}$, where $\hat{\mathcal{J}}$ is the dual abelian scheme of \mathcal{J} . This gives an embedding $\mathcal{C} \rightarrow \hat{\mathcal{J}}$. Since $n = 2$, the invertible $\mathcal{O}_{\mathcal{C}}$ -module \mathcal{L} is an element of order two in \mathcal{J} . Factoring by the subgroup generated by \mathcal{L} gives a quotient $\mathcal{J} \rightarrow \mathcal{Q}$ whose dual map $\hat{\mathcal{Q}} \rightarrow \hat{\mathcal{J}}$ is an etale twofold covering. Then also the induced map $\mathcal{C}' = \mathcal{C} \times_{\hat{\mathcal{J}}} \hat{\mathcal{Q}} \rightarrow \mathcal{C}$ is etale and of degree two, and the inverse image of \mathcal{L} on \mathcal{C}' is trivial. Thus the induced quadruple with base curve \mathcal{C}' is an element of $W_{g,0}^1$, and this defines a canonical morphism $W_{g,0}^2 \rightarrow W_{g,0}^1$. For $g = 1$, this morphism is an isomorphism onto the subfunctor of $W_{1,0}^1$ consisting of those quadruples for which the base curve admits a section which is a point of order two of the abelian scheme; since \mathbb{A}^2 also coarsely represents this subfunctor, it coarsely represents $W_{1,0}^2$, too. For arbitrary $g = 1$, we only get a morphism $E_{g,0}^2 \rightarrow E_{g,0}^1$, which must be finite of degree at most 2^{2g-1} , because there are at most 2^{2g-1} nontrivial invertible modules of order two on a curve of genus g .

For the proof of irreducibility, recall that for sufficiently large integers N , which are invertible in the ground ring, there is a fine moduli scheme M for curves of genus g with a level- N -structure, i.e., a scheme representing the functor

$$G: Sch \rightarrow Sets; T \rightarrow \left\{ \begin{array}{l} (\mathcal{C}; \mathcal{L}_1, \dots, \mathcal{L}_{2g}) | \mathcal{C}/T \text{ a curve of} \\ \text{genus } g, \mathcal{L}_1, \dots, \mathcal{L}_{2g} \text{ a basis of} \\ \text{the group of } N\text{-division points} \\ \text{in } Pic^0(\mathcal{C}/T) \end{array} \right\}$$

([10], theorem 7.9), and by [1], theorem 5.15, M is irreducible. Now let $N = 2N'$ be even. Since $Pic^0(\mathcal{C}/T)$ is a divisible group, the morphism of functors

$$G \rightarrow M_{g,0}; (\mathcal{C}; \mathcal{L}_1, \dots, \mathcal{L}_{2g}) \rightarrow (\mathcal{C}, \mathcal{L}_1^{N'})$$

with

$$M_{g,0}: Sch \rightarrow Sets; T \rightarrow \left\{ \begin{array}{l} (\mathcal{C}, \mathcal{L}) | \mathcal{C}/T \text{ a curve of genus } g, \\ \mathcal{L} \in Pic^0(\mathcal{C}/T) \end{array} \right\}$$

is surjective onto the subfunctor of those pairs for which \mathcal{L} has exactly order two. This subfunctor is coarsely represented by the image Z of the projection $p: E_{g,0}^2 \rightarrow M_{g,0}$, hence Z is irreducible. But then also $E_{g,0}^2$ is irreducible, because all fibers of p are isomorphic to \mathbb{A}^1 , i.e., they are irreducible and equidimensional, which suffices by [12], I §6, theorem 8. □

For $n > 2$, the j -invariant is determined by n : For $n = 3, 6, j \equiv 0$, and for $n = 4, j \equiv 12^3$. Therefore the quadruple (C, L, a, b) is already determined – upto isomorphism – by the pair (C, L) , and in analogy to lemma 11 one gets

LEMMA 12: *The affine line $\mathbb{A}^1 = E_{1,0}^3 = E_{1,0}^4 = E_{1,0}^6$ coarsely represents the functors $W_{1,0}^3, W_{1,0}^4$, and $W_{1,0}^6$. For arbitrary $g \geq 1$, there is a canonical finite morphism $E_{g,0}^n \rightarrow M_g$ whose degree is at most*

$$3^{2g} - 1 \quad \text{for } n = 3$$

$$4^{2g} - 2^{2g} \quad \text{for } n = 4$$

$$6^{2g} - 3^{2g} \quad \text{for } n = 6. \quad \square$$

8. A first result about irreducibility

Strictly speaking, this is the second result, because we already know that the schemes $E_{g,0}^n$ are irreducible. In this section, the problem will be approached from the other end, i.e., for large values of d :

LEMMA 13: For $d > (g - 1)/2$, $\bar{E}_{g,d}$ is irreducible of dimension $10d + 2g - 2$, unless $g = d = 0$, when the dimension is one.

Proof: By construction, there always is a morphism $f: \bar{E}_{g,d} \rightarrow M_{g,d} =$ coarse moduli scheme of pairs (C, L) , where C is a curve of genus g , and L an invertible \mathcal{O}_C -module of degree d , and of course we also have a canonical projection $h: M_{g,d} \rightarrow M_g =$ coarse moduli scheme of curves of genus g . M_g is irreducible by [1], §3, and has dimension $3g - 3$ for $g = 2$, 1 for $g = 1$, and 0 for $g = 0$. For $g \geq 2$, the fibers of h are abelian varieties of dimension g , thus all fibers of h are irreducible of the same dimension, and thus $M_{g,d}$ is irreducible of dimension $4g - 3$ by [12], I §6, theorem 8. If $g = 1$, $M_{g,d} = M_g$, hence the dimension of $M_{g,d}$ is again $4g - 3$. For $g = 0$, $M_{g,d}$ is of course just a point. Since $d > (g - 1)/2$, both $4L$ and $6L$ are non-special for any invertible module of degree d , hence $h^0(C, 4L) = 4d + 1 - g$ and $h^0(C, 6L) = 6d + 1 - g$. The fibers of f are quotients of an open subset of $H^0(C, 4L) \oplus H^0(C, 6L) \bmod \mathbb{G}_m$ if $g \geq 2$, modulo some other one-dimensional group for $g = 1$, and modulo Gl_2 for $g = 0$. Therefore all fibers are irreducible and have dimension $10d + 1 - 2g$ for $g \geq 1$, $10d - 2$ for $g = 0$, $d > 0$, and 1 for $g = d = 0$. Adding up, we get the result. \square

For smaller values of d , the problem is more difficult and must be attacked by local deformation theory. This will be done in a subsequent paper, and can also be found in [14].

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