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# Homological characterization of boundary set complements

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## 1. Introduction

H. Torunczyk presents characterizations of both Hilbert cube manifolds [Torunczyk, 1980] and Hilbert space manifolds [Torunczyk, 1981] in terms of geometric general position properties. Since the pseudo-interior $s$ of the Hilbert cube is homeomorphic to the separable Hilbert space [Anderson, 1966], we shall use the term $s$-manifold for separable Hilbert space manifold. [Daverman and Walsh, 1981] refine Torunczyk's Hilbert cube manifold characterization and obtain a characterization in terms of homological general position properties that allows them to characterize those spaces whose product with some finite-dimensional space is a Hilbert cube manifold. The Daverman-Walsh program for homologically characterizing manifolds and their essential factors fails in the setting of $s$-manifolds as shown by examples constructed in [Bestvina *et al.*, 1986]. It is the purpose of this paper to prove that the Daverman-Walsh program succeeds to a great extent in the boundary set setting, which occurs whenever the spaces under consideration have nice ANR local compactifications. In particular, $s$-manifolds are characterized as precisely those complements of $\sigma$-$Z$-sets in locally compact separable ANR's that satisfy the discrete carriers property (homological general positioning) and the discrete 2-cells property (minimal geometric general positioning). This leads to a characterization of those spaces in the boundary set setting whose product with some finite-dimensional space from a particular class of spaces is an $s$-manifold.

After presenting the Torunczyk and Daverman-Walsh characterization theorems in Section 2, the main results of this paper are presented in Section 3. Section 4 presents characterizations of those $\sigma$-$Z$-sets in locally compact separable ANR's whose complements satisfy various geometric general position properties (discrete cells properties) while Section 5 presents such characterizations for those whose complements satisfy various homological general position properties (discrete carriers properties). In Section 6, some 'Hurewicz type' theorems are proved that relate these geometric and homological general position properties to one another. The proofs of the main results are presented in Section 7 while applications are presented in Section 8. Section 9 includes a brief problem list and discussion of the most important unresolved problem that arises from this paper.

*Terminology and notation*

All *spaces* are separable and metric and *maps* are continuous. For a space $X$, $\mathrm{id}_X$ denotes the identity map. If a metric on $X$ is fixed, *dist* denotes distance and *diam* denotes diameter with respect to the fixed metric. The *Hilbert cube* $I^\infty$ is the countable infinite product of intervals $[-1, 1]$ and $s$ is its pseudo-interior. $I^n$ and $S^n$ denote respectively the standard $n$-cell and the standard $n$-sphere. A $I^\infty$-*manifold* (respectively s-*manifold*) is a separable metric space locally modeled on $I^\infty$ (respectively, $s$). An ANR is an absolute neighborhood retract for metric spaces.

If $\mathscr{A}$ is a collection of subsets of a space $X$ and $A$ is a subset of $X$, then $\mathrm{st}(A, \mathscr{A})$ is the union of all elements of $\mathscr{A}$ that hit $A$. $\mathrm{st}\mathscr{A}$ denotes the collection $\{\mathrm{st}(A, \mathscr{A}) \mid A \in \mathscr{A}\}$ and $\mathrm{st}^2\mathscr{A}$ denotes $\mathrm{st}(\mathrm{st}\mathscr{A})$. If $\mathscr{B}$ is another collection of subsets of $X$, then $\mathscr{A}$ *refines* $\mathscr{B}$ if each member of $\mathscr{A}$ is contained in some member of $\mathscr{B}$ and $\mathscr{A}$ *star-refines* $\mathscr{B}$ if $\mathrm{st}\mathscr{A}$ refines $\mathscr{B}$. If $Y$ is a subset of $X$, $\mathscr{A} \cap Y$ denotes the collection $\{A \cap Y \mid A \in \mathscr{A}\}$. Usually $\mathscr{A}$ and $\mathscr{B}$ will denote open covers of $X$. Observe that we do not require $\cup \mathscr{A} = \cup \mathscr{B}$ for $\mathscr{A}$ to refine $\mathscr{B}$.

If $\mathscr{U}$ is an open cover of $Y$ and $f$ and $g$ are maps of $X$ into $Y$, then $f$ is $\mathscr{U}$-*close* to $g$, or a $\mathscr{U}$-*approximation* to $g$, provided $\{f(x), g(x)\}$ is contained in some member of $\mathscr{U}$ for each $x \in X$. If $K$ is an abstract simplicial complex, $K^{(n)}$ denotes its $n$-skeleton while $|K|$ denotes a standard geometric realization of $K$ equipped with the metric topology. Since all complexes in this paper are locally finite, the metric topology coincides with the Whitehead topology. If $\mathscr{A}$ is an open cover of $X$, then $\mathscr{N}(\mathscr{A})$ denotes the abstract nerve of $\mathscr{A}$ whose 0-skeleton is $\mathscr{A}$.

$\check{H}_q$ denotes Čech homology with integer coefficients while $H_q$ denotes either singular or simplicial homology with integer coefficients. The context should make the meaning clear ($H_q(K)$ denotes singular theory if $K$ is a space and simplicial theory if $K$ is a complex). $\tilde{H}_q$ denotes the corresponding reduced homology. $i_*$ usually denotes an appropriate inclusion-induced homomorphism on homology. If $f: X \to Y$ is a map, $f_{\#}$ denotes the induced homomorphism on the singular chain complex and $f_*$ denotes the induced homomorphism on homology. If $c = \sum n_i f_i$ is a singular chain in $X$, then $|c|$ denotes the *support* of $c$, that is, $|c| = \cup \mathrm{im}\, f_i$. A singular chain $c$ in $X$ is *carried* by a subset $A$ of $X$ if $|c| \subset A$. $N$ denotes the positive integers and $\infty + 1$ and $\infty - 1$ both mean $\infty$.

## 2. Discrete properties and Torunczyk's characterization theorems

A collection $\mathscr{D}$ of subsets of a space $X$ is *discrete* in $X$ provided every point in $X$ has a neighborhood that meets at most one member of $\mathscr{D}$. A space $X$ satisfies the *discrete n-cells property* for a given $n \in N \cup \{0, \infty\}$ if for each map $f: \oplus_{i=1}^\infty I_i^n \to X$ of the countable free union of $n$-cells into $X$ ($\infty$-cell $=$

Hilbert cube) and each open cover $\mathcal{U}$ of $X$ there exists a $\mathcal{U}$-approximation $g: \oplus_{i=1}^{\infty} I_i^n \to X$ to $f$ for which $\{g(I_i^n)\}_{i=1}^{\infty}$ is discrete in $X$. [Torunczyk, 1981] has obtained the following characterization of $s$-manifolds in terms of the discrete $\infty$-cells property, usually referred to as the *discrete approximation property*.

### 2.1. Torunczyk's s-manifold characterization theorem [Torunczyk, 1981]

*A topologically complete separable* ANR $X$ *is an s-manifold if and only if* $X$ *satisfies the discrete approximation property.*

Previous to Torunczyk's $s$-manifold characterization, [Torunczyk, 1980] obtained a characterization of $I^{\infty}$-manifolds in the same spirit as 2.1.

### 2.2. Torunczyk's $I^{\infty}$-manifold characterization theorem [Torunczyk 1980]

*A locally compact separable* ANR $X$ *is a* $I^{\infty}$*-manifold if and only if* $X$ *satisfies the disjoint cells property.*

Recall that a space $X$ satisfies the *disjoint n-cells property* for some $n \in N \cup \{0, \infty\}$ provided every pair of maps of the $n$-cell into $X$ is approximable by a pair of maps whose images are disjoint. A space $X$ satisfies the *disjoint cells property* provided it satisfies the disjoint $n$-cells property for all positive integers $n$, equivalently, provided it satisfies the disjoint $\infty$-cells property (usually called the *disjoint Hilbert cubes property*).

  [Daverman and Walsh, 1981] refine Torunczyk's $I^{\infty}$-manifold characterization and obtain a 'homological characterization' of $I^{\infty}$-manifolds that allows them to characterize *essential* $I^{\infty}$*-manifold factors,* those spaces whose product with some finite-dimensional space is a $I^{\infty}$-manifold. The Daverman-Walsh characterizations are stated after some preliminary definitions. Let $V \subset U$ be open subsets of a space $X$ and $z \in H_q(U, V)$ for some integer $q \geqslant 0$; a compact pair $(C, \partial C) \subset (U, V)$ is said to be a *Čech carrier* for $z$ provided

$$z \in \operatorname{im}\left\{ i_* : \check{H}_q(C, \partial C) \to H_q(U, V) \right\},$$

where $i_*$ is the inclusion-induced homomorphism. A space $X$ is said to have the *disjoint Čech carriers property* provided for open sets $V_1 \subset U_1$ and $V_2 \subset U_2$ and elements $z_1 \in H_{q(1)}(U_1, V_1)$ and $z_2 \in H_{q(2)}(U_2, V_2)$ for integers $q(1)$, $q(2)$ $\geqslant 0$, there are Čech carriers $(C_1, \partial C_1)$ for $z_1$ and $(C_2, \partial C_2)$ for $z_2$ with $C_1 \cap C_2 = \phi$.

### 2.3. Daverman-Walsh homological characterization of $I^{\infty}$-manifolds [Daverman and Walsh, 1981]

*A locally compact separable* ANR $X$ *is a* $I^\infty$-*manifold if and only if* $X$ *satisfies the disjoint Čech carriers property and the disjoint 2-cells property.*

This theorem roughly states that a suitably nice space is a $I^\infty$-manifold provided it looks like a $I^\infty$-manifold homologically and satisfies a minimal amount of geometric general positioning. This reflects the philosophy inherent in the characterization of finite-dimensional manifolds that stems from the work of [Cannon, 1979], [Edwards, 1977], and [Quinn, 1979]. As a corollary of their proof of 2.3., Daverman and Walsh obtain the following characterization of essential $I^\infty$-manifold factors.

2.4. COROLLARY [Daverman and Walsh, 1981]. *A locally compact separable* ANR $X$ *is an essential* $I^\infty$-*manifold factor if and only if* $X$ *satisfies the disjoint Čech carriers property.*

Daverman and Walsh then obtain a candidate for a 'minimal' finite-dimensional factor to multiply an essential factor by in order to obtain a manifold.

2.5. COROLLARY [Daverman and Walsh, 1981]. *A locally compact separable* ANR $X$ *is an essential* $I^\infty$-*manifold factor if and only if* $X \times [0, 1]^2$ *is a* $I^\infty$-*manifold.*

It is unknown whether or not the square $[0, 1]^2$ can be replaced by the unit interval $[0, 1]$ in 2.5., however, the results of [Bowers, 1985a] coupled with those of [Daverman and Walsh, 1981] alow one to replace the square by a compact 1-dimensional AR. In particular, let $T$ be a dendrite ($=$ compact 1-dimensional AR) whose endpoints are dense. The following corollary is a consequence of ([Daverman and Walsh, 1981], Corollary 6.4) and ([Bowers, 1985a], Theorem 2.1).

2.6. COROLLARY. *A locally compact separable* ANR $X$ *is an essential* $I^\infty$-*manifold factor if and only if* $X \times T$ *is a* $I^\infty$-*manifold.*

It was hoped that the Daverman-Walsh program would provide a homological characterization of $s$-manifolds leading to a characterization of *essential s-manifold factors,* those spaces whose product with a finite-dimensional space is an $s$-manifold. Unfortunately, the Daverman-Walsh program fails in the non-locally compact setting of $s$ as shown by examples constructed by [Bestvina *et al.,* 1986]. Further examples and results of [Bestvina, to appear] make clear how incomplete is our understanding of essential $s$-manifold factors and the homological structure that is essential for $s$-manifolds. However, as stated in the introduction, it is the purpose of this paper to prove that the Daverman-Walsh program succeeds to a great extent in the special case known as the *boundary set setting,* which occurs whenever the spaces under consideration have 'nice' ANR local compactifications (see Section 3). We

close this section with the definition of the *s*-manifold version of the disjoint Čech carriers property.

A (singular) *carrier* for an element $z \in H_q(U, V)$ for $V \subset U$ subsets of a space $X$ and integer $q \geqslant 0$ is a pair of compact sets $(C, \partial C) \subset (U, V)$ such that

$$z \in \mathrm{im}\{i_* : H_q(C, \partial C) \to H_q(U, V)\}$$

where $i_*$ is the inclusion-induced homomorphism. For example, if $c = \sum_{i=1}^{k} n_i f_i \in z$ is a singular chain, $C = |c|$ and $\partial C = |\partial c|$ form a carrier for $z$. A space $X$ is said to satisfy the *discrete k-carriers property* provided for every open cover $\mathcal{U}$ of $X$ and sequence $\{z_j \in H_{q(j)}(U_j, V_j)\}_{j=1}^{\infty}$ of homology elements where $q(j) \leqslant k$ and $V_j \subset U_j$ are open in $X$, there exists for each $j$ a carrier

$$(C_j, \partial C_j) \subset (\mathrm{st}(U_j, \mathcal{U}), \mathrm{st}(V_j, \mathcal{U}))$$

for $i_*(z_j)$, where

$$i_* : H_{q(j)}(U_j, V_j) \to H_{q(j)}(\mathrm{st}(U_j, \mathcal{U}), \mathrm{st}(V_j, \mathcal{U}))$$

is the inclusion-induced homomorphism, such that $\{C_j\}_{j=1}^{\infty}$ forms a discrete family in $X$. In words, $X$ satisfies the discrete *k*-carriers property provided every sequence of *k*-dimensional homology elements can be made discrete by cover close moves. If we remove the restriction $q(j) \leqslant k$ on the sequence $\{z_j\}_{j=1}^{\infty}$ of homology elements in the definition of the discrete *k*-carriers property, we arrive at the definition of the *discrete carriers property*.

## 3. The main results

A closed subset $F$ of an ANR space $X$ is said to be a *Z-set* (in $X$) provided for each open cover $\mathcal{U}$ of $X$, there exists a map $f : X \to X - F$ that is $\mathcal{U}$-close to $\mathrm{id}_X$. A countable union of *Z*-sets in called a *σ-Z-set* and in a topologically complete ANR $X$, the identity map on $X$ is approximable with cover close controls by maps whose images miss any given *σ*-*Z*-set in X. A *σ*-*Z*-set $B$ in a $I^{\infty}$-manifold $M$ is called a *boundary set* (in $M$) provided $M - B$ is an *s*-manifold. [Curtis, 1985] extensively has studied boundary sets in the Hilbert cube and has characterized them in terms of their intrinsic local homotopy properties in conjunction with extrinsic properties of their embeddings into $I^{\infty}$. This paper provides a characterization of boundary set complements in terms of the intrinsic homological structure of those complements combined with a minimal amount of geometric general positioning. This allows us to characterize certain essential *s*-manifold factors that arise in this boundary set setting. We shall use the phrase *boundary set setting* to refer to the setting in

which all spaces under consideration arise as complements of $\sigma$-$Z$-sets in locally compact separable ANR's. If $X$ is such a space, that is, if $X = Y - F$ for some $\sigma$-$Z$-set $F$ in a locally compact separable ANR $Y$, we shall say that $X$ has a *nice* ANR *local compactification*. Observe that such a space $X$ is a topologically complete separable ANR [Torunczyk, 1978]. It is worth pointing out that every $s$-manifold has a nice ANR local compactification, however, in order to conclude on the strength of the characterization theorem stated below that a given space $X$ is an $s$-manifold, it must be known a priori that $X$ has a nice ANR local compactification.

The following characterization theorem is the $s$-manifold version of the Daverman-Walsh characterization 2.3. (see [Daverman and Walsh, 1981], Theorem 6.1).

3.1. THEOREM. *A space $X$ is an s-manifold if and only if $X$ has a nice* ANR *local compactification and $X$ satisfies the discrete carriers property and the discrete 2-cells property.*

The Theorem is false if we delete the hypothesis that $X$ have a nice ANR local compactification. [Bestvina *et al.*, 1986] present examples of topologically complete separable ANR's that satisfy the discrete carriers property and the discrete 2-cells property, yet fail to be $s$-manifolds.

Only the results of Section 6 along with Theorem 4.5. are needed for the proof of 3.1. We develop the machinery to prove the next theorem in Sections 4 and 5. $\overline{A}$ denotes a dendrite ($=$ compact 1-dimensional AR) whose endpoints are dense.

3.2. THEOREM. *Let $X = \overline{X} - F$ where $F$ is a dense $\sigma$-$Z$-set in the locally compact separable* ANR $\overline{X}$ *and let $A = \overline{A} - F_0$ where $F_0$ is a dense $\sigma$-compact collection of endpoints of the dendrite $\overline{A}$. The following are equivalent*:
 *i*) *$X$ satisfies the discrete carriers property*;
 *ii*) *$X$ satisfies the discrete n-carriers property for all non-negative integers $n$*;
 *iii*) *$F$ is proximately $1c^{\infty}$ rel $\overline{X}$ (see Section 5)*;
 *iv*) *$X \times A$ is an s-manifold*;
 *v*) *$X \times A^n$ is an s-manifold for all positive integers $n$, where $A^n$ denotes the n-fold product of $A$ with itself*;
 *vi*) *$X \times A^n$ is an s-manifold for some positive integer $n$.*

Again, 3.2. is false without the assumption that $X$ have a nice ANR local compactification. Examples are constructed in [Bestvina *et al.*, 1986] of topologically complete separable ANR's that satisfy the discrete carriers property yet fail to become $s$-manifolds upon multiplication by any finite-dimensional space. [Bestvina, to appear], modulo the construction of certain examples in homotopy theory, produces an example for each integer $n > 1$ of a space $X$ for which $X \times A^n$ is an $s$-manifold while $X \times A^{n-1}$ is not an $s$-manifold.

It is desirable to replace statement vi) of 3.2. by the statement

vi)′ $X \times B$ is an *s*-manifold for some finite-dimensional space $B$.

If 3.2 is true with vi)′ in place of vi), then the Daverman-Walsh program for $I^\infty$-manifolds parallels exactly the program for *s*-manifolds in the boundary set setting. Presently, it is unknown whether or not vi)′ implies i). See Section 9.

## 4. Detecting discrete cells properties

In order to prove the results of Section 3, it is important to be able to detect readily which of the discrete properties the complement of a $\sigma$-$Z$-set in a locally compact separable ANR satisfies. In this Section, we introduce various local homotopy properties that such a $\sigma$-$Z$-set might possess that would guarantee that its complement does satisfy particular discrete cells properties.

**4.1. *Definition.*** For a non-negative integer $n$, a subset $F$ of a space $X$ is *locally n-connected rel X* provided for every $x \in X$ and neighborhood $U$ of $x$ in $X$, there exists a neighborhood $V$ of $x$ in $X$ such that every map $f: S^n \to V \cap F$ is null-homotopic in $U \cap F$. $F$ is LC$^n$ *rel X* provided $F$ is dense in $X$ and is locally *i*-connected rel $X$ for $0 \le i \le n$, and $F$ is LC$^\infty$ *rel X* provided $F$ is dense in $X$ and is LC$^i$ rel $X$ for all $i \ge 0$. We shall say that $F$ is LC$^{-1}$ *rel X* if $F$ is dense in $X$.

The property of being LC$^n$ rel $X$ is not an intrinsic property of the subset $F$ of $X$. Rather, this property combines an intrinsic property of $F$ with a property of the particular embedding of $F$ into $X$. Indeed, a dense subset $F$ is LC$^n$ rel $X$ if and only if $F$ is a LC$^n$ space and $X - F$ is a LCC$^n$ subset of $X$, the former property being an intrinsic property of $F$ and the latter a property of the embedding of $F$ into $X$. For example, $(0, 1) \approx \{e^{2\pi i t} \in S^1 \mid 0 < t < 1\}$ is not locally 0-connected rel $S^1$ while $(0, 1) \approx \{e^{it} \in S^1 \mid 0 < t < 1\}$ is locally 0-connected rel $S^1$.

The next definition is inspired by [Curtis, 1985] and arises naturally in the study of discrete cells properties. See [Curtis, 1985; Bowers, 1985a].

**4.2. *Definition.*** For a non-negative integer $n$, a subset $F$ of a space $X$ is *proximately locally n-connected rel X* provided for every $x \in X$ and neighborhood $U$ of $x$ in $X$, there exists a neighborhood $V$ of $x$ in $X$ such that, for every compactum $S \subset V \cap F$, there exists a compactum $T \subset U \cap F$ containing $S$ such that, for every neighborhood $N(T)$ of $T$ in $X$, there exists a neighborhood $N(S)$ of $S$ in $X$ such that every map $f: S^n \to N(S)$ is null-homotopic in $N(T)$. $F$ is *proximately* LC$^n$ *rel X* provided $F$ is dense in $X$ and is proximately locally *i*-connected rel $X$ for $0 \le i \le n$, and $F$ is *proximately* LC$^\infty$ *rel X* provided $F$ is dense in $X$ and is proximately LC$^i$ rel $X$ for all $i \ge 0$. We shall say that $F$ is *proximately* LC$^{-1}$ *rel X* if $F$ is dense in $X$.

If we require $X$ to be an ANR and replace 'for every $x \in X$' by 'for every $x \in F$' in 4.2., we arrive at Curtis' definitions of *proximately locally n-connected, proximately* $LC^n$, and *proximately* $LC^\infty$. These properties are intrinsic properties of $F$, independent of the particular embedding of $F$ into any ANR. The property of being proximately $LC^n$ rel $X$ holds the same relationship to being proximately $LC^n$ as the property of being $LC^n$ rel $X$ holds to being $LC^n$. Indeed, a dense subset $F$ is proximately $LC^n$ rel $X$ if and only if $F$ is proximately $LC^n$, an intrinsic property of $F$, and $F$ is $(n+1)$-*target dense* embedded in $X$ (maps of $(n+1)$-cells into $X$ can be pushed close to $F$ by small moves). See [Curtis, 1985].

   4.1. provides a sufficient condition on a $\sigma$-$Z$-set $F$ in a locally compact separable ANR $X$ to ensure that $X - F$ satisfies the discrete $n$-cells property.

4.3. PROPOSITION. *Let* $n \in N \cup \{0, \infty\}$ *and let F be a (dense)* $\sigma$-$Z$-*set in the locally compact separable* ANR $X$. *If F is* $LC^{n-1}$ *rel X, then* $X - F$ *satisfies the discrete n-cells property. The reverse implication is false.*

A proof of 4.3. appears in [Bowers, 1985a].
   4.2. provides a characterization of those $\sigma$-$Z$-sets in locally compact separable ANR's whose complements satisfy the discrete $n$-cells property.

4.4. THEOREM. *Let* $n \in N \cup \{0, \infty\}$ *and let F be a (dense)* $\sigma$-$Z$-*set in the locally compact separable* ANR X. *The following statements are equivalent:*
(1.n) $X - F$ *satisfies the discrete n-cells property;*
(2.n) *for each open cover* $\mathcal{U}$ *of* $X - F$, *there exists a closed in X subset* $J \subset F$
      *such that for every map* $f : I^n \to X - F$ *and every neighborhood* $N(J)$ *of J*
      *in X, there exists a map* $f' : I^n \to N(J) - F$ *such that* $f'$ *is* $\mathcal{U}$-*close to f;*
(3.n) $F$ *is proximately* $LC^{n-1}$ *rel X.*

*Proof.* The equivalence ȯf (1.n) and (2.n) as well as the implication (2.n) implies (3.n) is proved in [Bowers, 1985a]. See in particular Theorems 3.2. and 3.6. of [Bowers, 1985a]. It remains to prove that (3.n) implies (2.n). We prove this for $n$ finite and then invoke the main result of [Bowers, 1985b] to prove that (3.$\infty$) implies (1.$\infty$). We actually prove that (3.n) for $n$ finite implies a stronger version of (2.n), namely (2.n)':
(2.n)' *for each open cover* $\mathcal{U}$ *of* $X - F$, *there exists a closed in X subset* $J \subset F$
      *such that for every map* $f : L \to X - F$ *from an arbitrary space L of*
      *dimension at most n and every neighborhood* $N(J)$ *of J in X, there*
      *exists a map* $f' : L \to N(J) - F$ *such that* $f'$ *is* $\mathcal{U}$-*close to f.*
Suppose that $F$ is proximately $LC^{n-1}$ rel $X$ for some non-negative integer $n$. Let $\mathcal{U}$ be an open cover of $X - F$ and let $\tilde{\mathcal{U}}$ be a collection of open subsets of $X$ such that $\mathcal{U} = \tilde{\mathcal{U}} \cap (X - F) = \{U \cap (X - F) \mid U \in \tilde{\mathcal{U}}\}$, and let $Y = \cup \tilde{\mathcal{U}}$. Write $Y = \bigcup_{i=1}^{\infty} Y_i$ where each $Y_i$ is compact and for each $i$, $Y_i \subset \text{Int}_Y Y_{i+1}$. Let $\mathscr{A}$ be a locally finite open cover of $Y$ such that $\text{st}^2 \mathscr{A}$ refines $\tilde{\mathcal{U}}$ and so that if

$A \cap Y_i \neq \phi$ for some $A \in \mathscr{A}$, then $A \subset \operatorname{Int}_Y Y_{i+1}$. Observe that if $A \cap (Y_i - Y_{i-1}) \neq \phi$ for some $A \in \mathscr{A}$, then $A \cap Y_{i-2} = \phi$. Since $F$ is proximately $LC^{n-1}$ rel $X$, we can choose a locally finite refinement $\mathscr{B}_{n-1}$ of $\mathscr{A}$ such that for every $B \in \mathscr{B}_{n-1}$ and every compactum $S \subset B \cap F$, there exists an element $A \in \mathscr{A}$ and a compactum $T \subset A \cap F$ containing $S$ such that, for every neighborhood $N(T)$ of $T$ in $X$, there exists a neighborhood $N(S)$ of $S$ in $X$ such that every map $f : S^{n-1} \to N(S)$ is null-homotopic in $N(T)$. Let $\mathscr{A}_{n-1}$ be a star-refinement of $\mathscr{B}_{n-1}$. Continuing in this manner, we obtain a sequence $\mathscr{A}_0, \mathscr{B}_0, \mathscr{A}_1, \mathscr{B}_1, \ldots, \mathscr{A}_{n-1}, \mathscr{B}_{n-1}$ of open covers of $Y$ such that for each $i = 0, 1, \ldots, n-2$, $\mathscr{A}_i$ star-refines $\mathscr{B}_i$ and for every $B \in \mathscr{B}_i$ and every compactum $S \subset B \cap F$, there exists an element $A \in \mathscr{A}_{i+1}$ containing $B$ and a compactum $T \subset A \cap F$ containing $S$ such that every singular $i$-sphere close enough to $S$ contracts close to $T$. We also may assume that $\mathscr{A}_0$ is a countable and star-finite cover of $Y$.

Let $K = \mathscr{N}(\mathscr{A}_0)^{(n)}$, the $n$-skeleton of the abstract nerve of the cover $\mathscr{A}_0$ whose vertices are the elements of $\mathscr{A}_0$. $K$ is a locally finite abstract simplicial complex of dimension at most $n$. For each $A \in \mathscr{A}_0$, choose a point $[A] \in A \cap F$ and let $J_0$ denote the collection of these points. Let $\mu_0 : |K^{(0)}| \to J_0$ denote the obvious function that assigns for each $A \in \mathscr{A}_0 = K^{(0)}$ the corresponding point $\mu_0(|A|) = [A] \in A \cap F$.

If $\sigma = \langle A_0, A_1 \rangle \in K^{(1)}$, then since $\mathscr{A}_0$ star-refines $\mathscr{B}_0$, there exists $B \in \mathscr{B}_0$ with $[\partial \sigma] = \{[A_0], [A_1]\} \subset B \cap F$. Then there exists an element $A \in \mathscr{A}_1$ containing $B$ and a compactum $[\sigma] \subset A \cap F$ containing $[\partial \sigma]$ such that every 0-sphere close enough to $[\partial \sigma]$ contracts close to $[\sigma]$. Let $J_1 = \cup\{[\sigma] \mid \sigma \in K^{(1)}\}$. Assume now that $J_i = \cup\{[\sigma] \mid \sigma \in K^{(i)}\}$ has been defined for $0 \leqslant i \leqslant m < n$ so that for each $i$-simplex $\sigma$, $[\sigma]$ is a compactum contained in $A \cap F$ for some $A \in \mathscr{A}_i$ and, if $\tau$ is a face of $\sigma$, then $[\tau] \subset [\sigma]$. Let $\sigma$ be an $(m+1)$-simplex in $K$ and observe that $[\partial \sigma] = \cup\{[\tau] \mid \tau \text{ is a proper face of } \sigma\}$ is contained in $\operatorname{st}(A, \mathscr{A}_m)$ for some $A \in \mathscr{A}_m$. Hence, there exists $B \in \mathscr{B}_m$ such that $[\partial \sigma] \subset B \cap F$ and thus there is an element $A' \in \mathscr{A}_{m+1}$ and a compactum $[\sigma] \subset A' \cap F$ containing $[\partial \sigma]$ such that every singular $m$-sphere close enough to $[\partial \sigma]$ contracts close to $[\sigma]$. Let $J_{m+1} = \cup\{[\sigma] \mid \sigma \in K^{(m+1)}\}$. After $n+1$ steps, we obtain $J_n = \cup\{[\sigma] \mid \sigma \in K\}$. For each $\sigma \in K$, $[\sigma]$ is a compactum contained in $A \cap F$ for some $A \in \mathscr{A}$ and, if $\tau$ is a face of $\sigma$, then $[\tau] \subset [\sigma]$. Furthermore, singular spheres of dimension $\dim \partial \sigma$ close enough to $[\partial \sigma]$ contract close to $[\sigma]$, where $[\partial \sigma] = \cup\{[\tau] \mid \tau \text{ is a proper face of } \sigma\}$.

We claim that $J_n$ is closed in $Y$. Let $\sigma \in K$ and observe that if $[\sigma] \cap Y_i \neq \phi$, then $[\sigma] \subset Y_{i+1}$. This follows from our choice of $\mathscr{A}$ and the fact that $[\sigma] \subset A$ for some $A \in \mathscr{A}$. Hence, if $[\sigma] \cap Y_i \neq \phi$, then $[A_0] \in Y_{i+1}$ for each vertex $A_0$ of $\sigma$. Since $\mathscr{A}_0$ is star-finite and $Y_{i+1}$ is compact, $J_0 \cap Y_{i+1}$ is finite. Hence $[\sigma] \cap Y_i \neq \phi$ for at most finitely many $\sigma \in K$ and this $J_n \cap Y_i$ is compact for each $i$ and $J_n$ is closed in $Y$.

Let $J = \operatorname{Cl}_X J_n$ and observe that $J \subset F$. Let $f : L \to X - F$ be a map of a space $L$ of dimension at most $n$ and let $N(J)$ be a neighborhood of $J$ in $X$. Let $\nu : Y \to |\mathscr{N}(\mathscr{A}_0)|$ be a canonical map such that $\nu^{-1}(\operatorname{St}(A)) \subset A$ for each

$A \in \mathscr{A}_0$, where St($A$) denotes the open star of $|A|$ in $|\mathscr{N}(\mathscr{A}_0)|$. Since $L$ is at most $n$-dimensional, we can approximate $v \circ f$ by a map $g : L \to |K| \subset |\mathscr{N}(\mathscr{A}_0)|$ so that if $v(f(x)) \in |\sigma|$ for some $\sigma \in \mathscr{N}(\mathscr{A}_0)$, then $g(x) \in |\sigma^{(n)}|$. Suppose $\mu : |K| \to N(J)$ is an extension of $\mu_0 : |K^{(0)}| \to J_0$ such that for each $\sigma \in K$, $\mu(|\sigma|) \subset A$ where $A$ is an element of $\mathscr{A}$ with $[\sigma] \subset A$. Let $x \in L$ and suppose that $f(x) \in A_0 \in \mathscr{A}_0$ and assume that $v(f(x)) \in |\langle A_0, A_1, \ldots, A_p \rangle|$. This last assumption is possible since $v$ is canonical. Thus $g(x) \in |\tau|$ where $\tau$ is a face of $\langle A_0, \ldots, A_p \rangle$. Choose a vertex $A_i$ of $\tau$ and $A \in \mathscr{A}$ such that $\mu(|\tau|) \cup [\tau] \subset A$ and observe that $A_0 \cap A_i \neq \phi$ and $[A_i] \in A_i \cap A$. Hence, since $\mathscr{A}_0$ star-refines $\mathscr{A}$, $f$ is st$\mathscr{A}$-close to $\mu \circ g$. Since $F \cap N(J)$ is a $\sigma$-Z-set in the ANR $N(J)$, a small move produces a map $f' : L \to N(J) - F$ that is st$^2\mathscr{A}$-close, and hence $\tilde{\mathscr{U}}$-close to $f$. It is clear that $f'$ is $\mathscr{U}$-close to $f$.

It suffices to prove the existence of the extension $\mu$ of $\mu_0$ with the properties described in the preceding paragraph. This is done by extending $\mu_0$ to the higher dimensional skeleta of $K$ through one skeleton at a time. If $\sigma$ is a 1-simplex of $K$, then $\mu_0 | |\partial\sigma|$ extends to a map $\mu_1 | |\sigma|$ so that $\mu_1(|\sigma|) \subset N([\sigma])$, where $N([\sigma])$ is any prechosen neighborhood of $[\sigma]$. Hence, we may extend $\mu_0$ to a map $\mu_1 : |K^{(1)}| \to N(J)$ so that if $\sigma$ is a 2-simplex of $K$, then $\mu_1(|\partial\sigma|)$ lies so close to $[\partial\sigma]$ that the restriction of $\mu_1$ extends to a map $\mu_2 | |\sigma|$ so that $\mu_2(|\sigma|) \subset N([\sigma])$ for any prechosen neighborhood $N([\sigma])$ of $[\sigma]$. We may assume that at the $i^{\text{th}}$ step, $\mu_i : |K^{(i)}| \to N(J)$ has been constructed so that for each $(i+1)$-simplex $\sigma$ in $K$, $\mu_i(|\partial\sigma|)$ lies so close to $[\partial\sigma]$ that $\mu_i$ extends to a map $\mu_{i+1} : |K^{(i+1)}| \to N(J)$ so that $\mu_{i+1}(|\sigma|) \subset N([\sigma])$ for any prechosen neighborhood $N([\sigma])$ of $[\sigma]$. After $n$ steps, if $N([\sigma]) \subset A \cap N(J)$ for some element $A$ of $\mathscr{A}$ that contains $[\sigma]$ for a simplex $\sigma$ of $K$, we obtain a map $\mu = \mu_n : |K| \to N(J)$ such that $\mu(|\sigma|) \subset A$ where $A$ is an element of $\mathscr{A}$ with $[\sigma] \subset A$. The reader should observe that in order to carry out the construction of $\mu$ described above, one must first choose the neighborhoods of $[\sigma]$ for the principal simplices $\sigma$ from $K$ and, working backwards one dimension at a time choose $N([\sigma])$ for smaller dimensional simplices. One then constructs $\mu_1$ after $N([\alpha])$ has been chosen for all $\sigma \in K$. This completes the proof that (3.$n$) for $n$ finite implies (2.$n$)$'$, and hence (2.$n$).

If (3.$\infty$) holds, then (3.$n$) holds for all $n$, hence (2.$n$) and finally (1.$n$) holds for all $n$. Thus $X - F$ satisfies the discrete $n$-cells property for each non-negative integer $n$. In general this is not enough to guarantee that $X - F$ satisfies the discrete approximation property as examples of [Bestvina et al., 1986] illustrate; however in the boundary set setting, this suffices. An application of the following theorem completes the proof of 4.4.

4.5. THEOREM [Bowers, 1985b]. *Let $F$ be a dense $\sigma$-Z-set in the locally compact separable ANR $X$. Then $X - F$ satisfies the discrete approximation property if and only if $X - F$ satisfies the discrete n-cells property for each non-negative integer n.*

Our first application of 4.4. appears below. $\bar{A}$ denotes a dendrite whose endpoints are dense.

**4.6. PROPOSITION.** *Let $X = \overline{X} - F$ where $F$ is a dense $\sigma$-Z-set in the locally compact separable ANR $\overline{X}$ and let $A = \overline{A} - F_0$ where $F_0$ is a dense $\sigma$-compact collection of endpoints of the dendrite $\overline{A}$. If for some $n \in N \cup \{0\}$ $X$ satisfies the discrete n-cells property, then $X \times A$ satisfies the discrete $(n + 1)$-cells property.*

*Proof.* Since $X \times A = \overline{X} \times \overline{A} - F_*$ where $F_*$ is the $\sigma$-Z-set $(F \times \overline{A}) \cup (\overline{X} \times F_0)$ in $\overline{X} \times \overline{A}$, it suffices to prove that $F_*$ is proximately $LC^n$ rel $\overline{X} \times \overline{A}$. First observe that since $X$ satisfies the discrete $n$-cells property, Theorem 4.4. guarantees that $F$ is proximately $LC^{n-1}$ rel $\overline{X}$. Let $x \in Y' \times B'$ for arbitrary relatively compact open sets $Y'$ of $\overline{X}$ and $B'$ of $\overline{A}$ and choose open sets $Y$ of $\overline{X}$ and $B$ of $\overline{A}$ so that $x \in Y \times B \subset Y' \times B'$ and $Y$ is contractible in $Y'$ and $B$ is contractible. Let $S \subset (Y \times B) \cap F_*$ be compact and let $\alpha : c * S \to Y' \times B'$ be an extension of the inclusion-induced map $\alpha_0 : \{0\} \times S \to S \subset Y' \times B'$ where $c * S$ denotes the cone $[0, 1] \times S / \{1\} \times S$. Since $S$ is contained in the $\sigma$-Z-set $F_*$, we may assume that $\alpha(c * S - \{0\} \times S) \cap S = \phi$. Let $p_1 : \overline{X} \times \overline{A} \to \overline{X}$ and $p_2 : \overline{X} \times \overline{A} \to \overline{A}$ be the projection mappings and choose collections of connected product open subsets of $Y' \times B'$, say $\mathcal{V}_0, \mathcal{U}_0, \ldots, \mathcal{V}_{n-1}, \mathcal{U}_{n-1}, \mathcal{V}, \mathcal{U}$, such that the following hold:

i) for each $U \in \mathcal{U}$, $\mathrm{Cl}_{\overline{X} \times \overline{A}} U \subset Y' \times B'$ and $\mathrm{dist}(S, U) > \mathrm{diam}\, U$ (we assume that we have fixed some metric on $\overline{X} \times \overline{A}$);

ii) $\mathcal{V}_0$ covers $\alpha(c * S) - S$;

iii) for each $i$, $\mathcal{V}_i$ refines $\mathcal{U}_i$ and $\mathcal{U}_i$ star-refines $\mathcal{V}_{i+1}$, and $\mathcal{U}_{n-1}$ star-refines $\mathcal{V}$ while $\mathcal{V}$ refines $\mathcal{U}$;

iv) for each $i$ and for each $V \in \mathcal{V}_i$, there exists $U \in \mathcal{U}_i$ such that $V \subset U$ and, for every compactum $S' \subset p_1(V) \cap F$, there exists a compactum $T' \subset p_1(U) \cap F$ containing $S'$ such that, given any neighborhood $N(T')$ of $T'$ in $\overline{X}$, there exists a neighborhood $N(S')$ of $S'$ in $\overline{X}$ such that any map $S^i \to N(S')$ is null-homotopic in $N(T')$);

v) for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subset U$ and $p_1(V)$ contracts in $p_1(U)$ to a point.

This choice is possible since $\overline{X}$ is an ANR and $F$ is proximately $LC^{n-1}$ rel $\overline{X}$. Choose these collections in reverse order, starting with $\mathcal{U}$ and ending with $\mathcal{V}_0$.

Choose a countable open star-finite star-refinement $\mathcal{W}$ of $\mathcal{V}_0$ that covers $\alpha(c * S) - S$ such that $W \cap \alpha(c * S) \neq \phi$ for each $W \in \mathcal{W}$, and let $K = \mathcal{N}(\mathcal{W})^{(n+1)}$, the $(n + 1)$-skeleton of the abstract nerve of the collection $\mathcal{W}$ whose vertices are the elements of $\mathcal{W}$. For each $W \in \mathcal{W}$, choose a point $[W] \in p_1(W) \cap F$. As in the proof that $(3.n)$ implies $(2.n)$ in Theorem 4.4., construct a collection $M = \{[\sigma] \mid \sigma \in K^{(n)}\}$ such that each $[\sigma]$ for $\sigma \in K^{(n)}$ is a compactum contained in $p_1(U) \cap F$ for some $U \in \mathcal{U}_{n-1}$ and, if $\tau$ is a face of $\sigma$, then $[\tau] \subset [\sigma]$. Furthermore, if $[\partial \sigma] = \cup \{[\tau] \mid \tau \text{ is a proper face of } \sigma \}$, then every singular sphere of dimension $\dim \partial \sigma$ close enough to $[\partial \sigma]$ contracts close to $[\sigma]$. In addition, the step-by-step construction of $M$ through the skeleta of $K$ using ii) and iii) allows us to construct $M$ so that for each simplex $\sigma = \langle W_0, \ldots, W_p \rangle$ of $K$ there is an element $V_\sigma$ of $\mathcal{V}$ such that $W_0 \cup \ldots \cup W_p \subset$

$V_\sigma$ and $[\partial\sigma] \subset p_1(V_\sigma) \cap F$ if $p = n + 1$ and $[\sigma] \subset p_1(V_\sigma) \cap F$ if $p < n + 1$. Using (v), for each $\sigma$ there exists $U_\sigma \in \mathcal{U}$ such that $V_\sigma \subset U_\sigma$ and $p_1(V_\sigma)$ is contractible in $p_1(U_\sigma)$. Let $b_\sigma \in p_2(U_\sigma) \cap F_0$, which exists since $F_0$ is dense in $\bar{A}$ and $p_2(U_\sigma)$ is open in $\bar{A}$.

Let $\sigma$ be an arbitrary principal simplex in $K$. Define $\{\sigma\} \subset \mathrm{Cl}_{\bar{X} \times \bar{A}} U_\sigma$ as follows:

$$\{\sigma\} = \begin{cases} [\partial\sigma] \times \mathrm{Cl}_{\bar{A}}(p_2(U_\sigma)) \cup \mathrm{Cl}_{\bar{X}} p_1(U_\sigma) \times \{b_\sigma\} & \text{if} \quad \sigma \notin K^{(n)} \\ [\sigma] \times \mathrm{Cl}_{\bar{A}}(p_2(U_\sigma)) & \text{if} \quad \sigma \in K^{(n)}. \end{cases}$$

It is easy to see that $\{\sigma\}$ is a compact subset of $(Y' \times B') \cap F_*$. Let

$$T = S \cup \left[ \cup \{\{\sigma\} \mid \sigma \text{ is a principal simplex of } K\} \right].$$

It is clear that $T \subset (Y' \times B') \cap F_*$ and we claim that $T$ is compact. It suffices to prove that $T$ is closed in $\bar{X} \times \bar{A}$ (recall that $Y' \times B'$ is relatively compact). It is straightforward to prove that if $x_i \in \{\sigma_i\}$ for distinct principal simplices $\sigma_i$ and if $x_i \to x$, then $x \in S$.

We must show that singular $n$-spheres close enough to $S$ contract close to $T$. Let $N(T)$ be an arbitrary neighborhood of $T$ in $\bar{X} \times \bar{A}$ and choose an open cover $\mathcal{D}$ of $N(T)$ such that $\mathcal{D}$-close maps are homotopic in $N(T)$. Let $N_1(S)$ be an open neighborhood of $S$ whose closure is contained in $N(T)$ such that if $U \in \mathcal{U}$ and $U \cap N_1(S) \neq \phi$, then $\mathrm{st}(U, \mathcal{U})$ is contained in some element of $\mathcal{D}$. This is possible by i). Observe that $\alpha(c * S)$ is contained in the open subset $Q = [\cup \mathcal{W}] \cup N_1(S)$ of $\bar{X} \times \bar{A}$. Hence, since $Q$ is an ANR and $S$ contracts in $Q$ to a point, there exists a neighborhood $N(S)$ of $S$ in $N_1(S)$ that contracts in $Q$ to a point. Let $h: S^n \to N(S)$ be a map and let $H: B^{n+1} \to Q$ be an extension of $h$ to the $(n + 1)$-ball $B^{n+1}$. Let $E = H^{-1}(Q - N_1(S))$ and $E' = H^{-1}(\mathrm{Cl}_{\bar{X} \times \bar{A}} N_1(S))$ and observe that $E$ and $E'$ are closed subsets of $B^{n+1}$ for which $B^{n+1} = E \cup E'$, $S^n \subset E'$, and $S^n \cap E = \phi$. Suppose that $H \mid E$ is $\mathrm{st}\mathcal{U}$-close to a map $G: E \to N(T)$. Then $H \mid E \cap E'$ is $\mathcal{D}$-close and hence homotopic in $N(T)$ to $G \mid E \cap E'$. The homotopy extension theorem [Hu, 1965] guarantees that $G$ may be extended on $B^{n+1}$ to a map still called $G$ for which $G = H = h$ on $S^n$ and whose image is contained in $N(T)$. Thus, it suffices to show that $H \mid E$ is $\mathrm{st}\mathcal{U}$-close to a map $G: E \to N(T)$. We show in fact that if $f: Z \to \cap \mathcal{W}$ is any map of a space $Z$ of dimension at most $n + 1$, then $f$ is $\mathrm{st}\mathcal{U}$-close to a map $f': Z \to N(T)$. Let $\nu: \cup \mathcal{W} \to |\mathcal{N}(\mathcal{W})|$ be a canonical map such that $\nu^{-1}(\mathrm{St}(W)) \subset W$ for each $W \in \mathcal{W}$. Since $Z$ is at most $(n + 1)$-dimensional, we can approximate $\nu \circ f$ by a map $g: Z \to |K| \subset |N(\mathcal{W})|$ so that if $\nu(f(x)) \in |\sigma|$ for $x \in Z$ and $\sigma \in \mathcal{N}(\mathcal{W})$, then $g(x) \in |\sigma^{(n+1)}|$. Suppose $\mu: |K| \to N(T)$ is a map such that $\mu(|\sigma|) \subset U_\sigma$ for each principal simplex $\sigma$ of $K$. If $x \in Z$ and $f(x) \in W \in \mathcal{W}$, then $\nu(f(x)) \in |\langle W, W_0, \ldots, W_p \rangle|$ for some principal simplex $\langle W, W_0, \ldots, W_p \rangle$ in $\mathcal{N}(\mathcal{W})$.

There is a principal simplex $\tau$ of $K$ such that $\tau$ is a face of $\langle W, W_0, \ldots, W_p \rangle$ and $g(x) \in |\tau|$. Without loss of generality, $W_0$ is a vertex of $\tau$, hence $W_0 \subset V_\tau \subset U_\tau$. We have the following: $W \cap W_0 \neq \phi$, $W_0 \subset U_\tau$, $f(x) \in W$, $\mu(g(x)) \in \mu(|\tau|) \subset U_\tau$. This implies that $\{f(x), \mu(g(x))\} \subset \mathrm{st}(U_\tau, \mathscr{U})$, hence $f$ is st$\mathscr{U}$-close to $f' = \mu \circ g$.

It remains only to prove the existence of $\mu$. We begin by constructing a map $m: |K^{(n)}| \to \overline{X}$ so that, if $\sigma$ is a principal simplex of $K$ of dimension at most $n$, then $m(|\sigma|)$ is so close to $[\sigma]$ that

$$m(|\sigma|) \times p_2(U_\sigma) \subset N(T) \cap U_\sigma \tag{1}$$

and, if $\sigma$ is a simplex of $K$ of dimension $n+1$, then $m(|\partial\sigma|)$ is so close to $[\partial\sigma]$ that

$$m(|\partial\sigma|) \subset p_1(V_\sigma) \quad \text{and} \quad m(|\partial\sigma|) \times p_2(U_\sigma) \subset N(T) \cap U_\sigma. \tag{2}$$

The construction of $m$ is done inductively as in the construction of $\mu$ in Theorem 4.4. Choose for each principal simplex $\tau$ of $K^{(n)}$, a neighborhood $N([\tau])$ of $[\tau]$ so that, if $\tau$ is also a principal simplex of $K$, then

$$N([\tau]) \times p_2(U_\tau) \subset N(T) \cap U_\tau$$

and, if $\tau$ is a face of an $(n+1)$-simplex of $K$, then

$$N([\tau]) \subset p_1(V_\sigma) \quad \text{and} \quad N([\tau]) \times p_2(U_\sigma) \subset N(T) \cap U_\sigma$$

for each $(n+1)$-simplex $\sigma$ having $\tau$ as a face. For each such $\tau$, choose neighborhood $N([\partial\tau])$ of $[\partial\tau]$ so that singular spheres of dimension dim $\partial\tau$ in $N([\partial\tau])$ are contractible in $N([\tau])$. Assume that $m$ has been defined on $|J|$, where $J$ is the subcomplex of $K^{(n)}$ consisting of those simplices of $K^{(n)}$ that are not principal, so that for each $\tau$ as above, $m(|\partial\tau|) \subset N([\partial\tau])$. Then we may extend $m$ to $|K^{(n)}|$ so that $m(|\tau|) \subset N([\tau])$ for each such $\tau$. The existence of $m$ on $|J|$ follows as in Theorem 4.4. by induction, noting that we may begin the inductive procedure by defining $m$ on $K^{(0)}$ by $m(|W|) = [W]$ for each $W \in \mathscr{W} = K^{(0)}$ (recall that $[W]$ is a prechosen element of $p_1(W) \cap F$).

We now use $m$ to construct $\mu$. Define $\mu_0: |K^{(0)}| \to \overline{X} \times \overline{A}$ as follows: if $W \in \mathscr{W} = K^{(0)}$, choose $b(W) \in p_2(W)$ and let $\mu_0(|W|) = (m(|W|), b(W)) \in W$. For $i < n$, suppose that $\mu_i: |K^{(i)}| \to \overline{X} \times \overline{A}$ has been defined so that for each $\sigma \in K^{(i)}$, $p_1(\mu_i(x)) = m(x)$ for $x \in |\sigma|$ and $\mu_i(|\sigma|) \subset m(|\sigma|) \times b(\sigma)$ where $b(\sigma)$ is the smallest subcontinuum of $\overline{A}$ that contains $\{b(W) | W$ is a vertex of $\sigma\}$. Let $\sigma$ be an $(i+1)$-simplex of $K$ and let $b(\sigma)$ be the smallest subcontinuum of $\overline{A}$ that contains $\{b(W) | W$ is a vertex of $\sigma\}$. Let $\theta: b(\sigma) \times [0,1] \to b(\sigma)$ be a contraction of $b(\sigma)$ and write $|\sigma| = |\partial\sigma| \times [0,1] / |\partial\sigma| \times \{1\}$. Define $\mu_{i+1}$ on $\sigma$ via

$$\mu_{i+1}([s, t]) = \left(m([s, t]), \theta(p_2(\mu_i(s)), t)\right) \text{ for } (s, t) \in |\partial\sigma| \times [0,1].$$

Observe that $\mu_{i+1}$ is well-defined on $|\sigma|$ and extends $\mu_i|\partial\sigma$. Hence, we obtain $\mu_{i+1}: |K^{(i+1)}| \to \overline{X} \times \overline{A}$ such that $p_1(\mu_{i+1}(x)) = m(x)$ for $x \in |\sigma|$ and $\mu_{i+1}(|\sigma|) \subset m(|\sigma|) \times b(\sigma)$ for each simplex $\sigma \in K^{(i+1)}$. Inductively, we assume that $\mu_n$ exists. Let $\sigma$ be an $(n+1)$-simplex of $K$ and let $b(\sigma)$ be the smallest subcontinuum of $\overline{A}$ that contains $\{b(W) \mid W$ is a vertex of $\sigma\} \cup \{b_\sigma\}$ (recall that $b_\sigma \in p_2(U_\sigma) \cap F_0$). Let $\theta: b(\sigma) \times [0, 1] \to b(\sigma)$ be a contraction of $b(\sigma)$ to the endpoint $b_\sigma$ of $\overline{A}$ and let $\Psi: p_1(V_\sigma) \times [1, 2] \to p_1(U_\sigma)$ be a homotopy from the inclusion $\Psi_1$ to a constant map $\Psi_2$. Write $|\sigma| = |\partial\sigma| \times [0, 2]/|\partial\sigma| \times \{2\}$ and define $\mu$ on $|\sigma|$ via

$$\mu([s, t]) = \begin{cases} \big(m(s), \theta\big(p_2(\mu_n(s)), t\big)\big) & 0 \leqslant t \leqslant 1 \\ \big(\Psi(m(s), t), b_\sigma\big) & 1 \leqslant t \leqslant 2 \end{cases}$$

for $(s, t) \in |\partial\sigma| \times [0, 2]$. Hence, we obtain a well-defined map $\mu: |K| \to \overline{X} \times \overline{A}$ that extends $\mu_n$. Let $\sigma$ be a simplex of $K$ of dimension $n+1$ and observe that $\mu(|\sigma|)$ is contained in $[m(|\partial\sigma|) \times b(\sigma)] \cup [p_1(U_\sigma) \times \{b_\sigma\}]$. By our choice of $b(W)$ for $W \in \mathcal{W}$, since $U_\sigma \supset W_0 \cup \ldots \cup W_{n+1}$ if $\sigma = \langle W_0, \ldots, W_{n+1}\rangle$ and $U_\sigma$ is connected, we have that $b(\sigma) \subset p_2(U_\sigma)$. By (2), $m(|\partial\sigma|) \subset p_1(V_\sigma)$ and therefore, $\mu(|\sigma|) \subset p_1(U_\sigma) \times p_2(U_\sigma) = U_\sigma$. Also, since by (2), $m(|\partial\sigma|) \times p_2(U_\sigma) \subset N(T)$ and $p_1(U_\sigma) \times \{b_\sigma\} \subset \{\sigma\} \subset T$, $\mu(|\sigma|) \subset N(T)$. More easily, (1) shows that if $\sigma$ is a principal simplex of $K$ of dimension at most $n$, then $\mu(|\sigma|) \subset m(|\sigma|) \times b(\sigma) \subset N(T) \cap U_\sigma$. Thus $\mu: |K| \to N(T)$ and $\mu(|\sigma|) \subset U_\sigma$ for each principal simplex $\sigma$ of $K$. This completes the proof of 4.6.

## 5. Detecting discrete carriers properties

In this Section we focus our attention on the homological versions of the discrete properties, namely, the various discrete carriers properties defined in Section 2. Our approach for detecting discrete carriers properties parallels our approach for detecting discrete cells properties that is the content of the previous Section. First, observe that an ANR satisfies the discrete 1-cells property if and only if it satisfies the discrete 1-carriers property. As a corollary of Proposition 4.6. and this observation, we obtain:

5.1. COROLLARY. *Let $X$ and $A$ be as in Proposition 4.6. If $X$ satisfies the discrete 1-carriers property, then $X \times A$ satisfies the discrete 2-cells property.*

Next we define the homological versions of the proximate $LC^n$ rel $X$ property.

5.2. *Definition.* For a non-negative integer $n$, a subset $F$ of a space $X$ is *proximately locally homologically $n$-connected rel $X$* provided for every $x \in X$ and neighborhood $U$ of $x$ in $X$, there exists a neighborhood $V$ of $x$ in $X$ such

that, for every compactum $S \subset V \cap F$ there exists a compactum $T \subset U \cap F$ containing $S$ such that, for every neighborhood $N(T)$ of $T$ in $X$, there exists a neighborhood $N(S)$ of $S$ in $X$ such that the inclusion-induced homomorphism $\tilde{H}_n(N(S)) \to \tilde{H}_n(N(T))$ is the zero homomorphism. $F$ is *proximately* $1c^n$ *rel* $X$ provided $F$ is dense in $X$ and is proximately locally homologically *i*-connected rel $X$ for $0 \leqslant i \leqslant n$, and $F$ is *proximately* $1c^\infty$ *rel* $X$ provided $F$ is dense in $X$ and is proximately $1c^i$ rel $X$ for all $i \geqslant 0$. We shall say that $F$ is *proximately* $1c^{-1}$ *rel* $X$ if $F$ is dense in $X$.

The following theorem is the homological version of Theorem 4.4. and characterizes those $\sigma$-$Z$-sets in locally compact separable ANR's whose complements satisfy various discrete carriers properties.

**5.3. THEOREM.** *Let* $n \in N \cup \{0\}$ *and let* $F$ *be a (dense)* $\sigma$-$Z$-set *in the locally compact separable* ANR $X$. *The following statements are equivalent*:
[1.$n$]  $X - F$ *satisfies the discrete* $n$-*carriers property*;
[2.$n$]  *for each open cover* $\mathcal{U}$ *of* $X - F$, *there exists a closed in* $X$ *subset* $J \subset F$ *such that for every* $z \in H_q(U, V)$ *where* $q \leqslant n$ *and* $V \subset U$ *are open in* $X - F$ *and for every neighborhood* $N(J)$ *of* $J$ *in* $X$, *there exists a carrier* $(C, \partial C)$ *for* $i_*(z)$, *where*

$$i_* : H_q(U, V) \to H_q(\mathrm{st}(U, \mathcal{U}), \mathrm{st}(V, \mathcal{U}))$$

*is inclusion-induced, such that* $C \subset N(J)$ *(observe then that* $C \subset N(J) - F$*)*;
[3.$n$]  $F$ *is proximately* $1c^{n-1}$ *rel* $X$.

In the proof that [3.$n$] implies [2.$n$] in Theorem 5.3. and in the proofs of later results in this paper, we need the homological version of the fact that in an ANR $X$, every open cover $\mathcal{U}$ admits an open refinement $\mathcal{V}$ covering $X$ such that $\mathcal{V}$-close maps into $X$ are $\mathcal{U}$-homotopic. This often is stated less precisely as 'close maps into an ANR are homotopic via a small homotopy'. Theorem 5.3. is proved after Lemma 5.4., which asserts that 'close chain maps into an ANR (more generally, an $1c^n$-space) $X$ are chain homotopic via a small chain homotopy'. In Lemma 5.4. $\mathcal{C}(K)$ denotes the oriented chain complex of the simplicial complex $K$ and $\mathcal{S}(X)$ denotes the singular chain complex of the space $X$.

**5.4. LEMMA.** *Let* $X$ *be an* $1c^n$-*space. For each open cover* $\mathcal{U}$ *of* $X$, *there exists an open refinement* $\mathcal{V}$ *of* $\mathcal{U}$ *covering* $X$ *with the following property*:
    *Let* $K$ *be a simplicial complex of dimension at most* $n$ *and let* $\alpha, \beta : \mathcal{C}(K) \to \mathcal{S}(X)$ *be chain maps such that i) for each vertex* $v$ *of* $K$, $\alpha(v)$ *and* $\beta(v)$ *are singular* $0$-*simplices in* $X$ *(i.e., singleton maps) and ii) for each* $\sigma \in K$, $|\alpha(\sigma)| \cup |\beta(\sigma)| \subset V$ *for some* $V \in \mathcal{V}$ *and for each vertex* $v$ *of* $\sigma$, *the singleton* $|\alpha(v)|$ *is contained in* $|\alpha(\sigma)|$. *Then there is a chain homotopy* $D : \alpha \simeq \beta$ *such*

*that for each* $\sigma \in K$, $|D(\sigma)| \subset U$ *for some* $U \in \mathcal{U}$ *with* $|\alpha(v)| \in U$ *for each vertex* $v$ *of* $\sigma$.

*Proof.* Choose open covers

$$\mathcal{U} = \mathcal{U}_n, \quad \mathcal{V}_n, \quad \mathcal{U}_{n-1}, \quad \mathcal{V}_{n-1}, \ldots, \quad \mathcal{U}_0, \quad \mathcal{V}_0 = \mathcal{V}$$

of $X$ such that $\mathcal{U}_i$ star-refines $\mathcal{V}_{i+1}$ for $i = 0, \ldots, n-1$, and for each $V \in \mathcal{V}_i$ there exists $U \in \mathcal{U}_i$ containing $V$ such that the inclusion $V \subset U$ induces the zero homomorphism $\tilde{H}_i(V) \to \tilde{H}_i(U)$ for $i = 0, \ldots, n$. Let $K$ be a simplicial complex of dimension at most $n$ and let $\alpha$ and $\beta$ be chain maps from the simplicial chain complex $\mathscr{C}(K) = \{C_p(K), \partial\}$ of $K$ to the singluar chain complex $\mathscr{S}(X) = \{S_p(X), \partial\}$ of $X$ that satisfy i) and ii). First we construct $D$ on $C_0(K)$. For a vertex $v$ of $K$, there is an element $V$ of $\mathcal{V}$ containing both the singletons $|\alpha(v)|$ and $|\beta(v)|$. Let $U \in \mathcal{U}_0$ satisfy $V \subset U$ and $\tilde{H}_0(V) \to \tilde{H}_0(U)$ is zero. By i), $\alpha(v) - \beta(v)$ is a singular reduced 0-cycle in $V$ and hence represents the zero element of $\tilde{H}_0(U)$. Thus there is a singular 1-chain $Dv$ in $X$ carried by $U$ such that $\partial Dv = \alpha(v) - \beta(v)$. Define $Dv$ for each vertex $v$ of $K$ and extend linearly on $C_0(K)$ and observe that $Dv$ is carried by an element of $\mathcal{U}_0$ that contains the singleton $|\alpha(v)|$. For $i = 0, \ldots, k < n$ assume that $D$ has been defined on $C_i(K)$ so that

$$\partial Dc = \alpha(c) - \beta(c) - D\, \partial c \tag{3}$$

for $c \in C_i(K)$ and, for each oriented $i$ simplex $\sigma$ of $K$, $D\sigma$ is carried by an element of $\mathcal{U}_i$ that contains the singleton $|\alpha(v)|$ for all vertices $v$ of $\sigma$. Let $\sigma$ be an oriented $(k+1)$-simplex of $K$ with oriented boundary $\partial \sigma = \sum n_j \sigma_j$ for oriented $k$-faces $\sigma_j$. Let $V \in \mathcal{V}$ be such that $V$ carries $\alpha(\sigma)$ and $\beta(\sigma)$ and for each $j$, choose $U_j$ in $\mathcal{U}_k$ that carries $D\sigma_j$ and contains the singletons $|\alpha(v)|$ for all vertices $v$ of $\sigma_j$. Then $V \cap U_j \neq \phi$ for all $j$ since each contains $|\alpha(v)|$ for any vertex $v$ of $\sigma_j$ (recall ii)). Hence $D\, \partial \sigma = \sum n_j D\sigma_j$ is carried by $\mathrm{st}(V, \mathcal{U}_k)$ as are the chains $\alpha(\sigma)$ and $\beta(\sigma)$. Since $\mathcal{V}$ refines $\mathcal{U}_k$ and $\mathcal{U}_k$ star-refines $\mathcal{V}_{k+1}$, the singular $(k+1)$-chain $\gamma = \alpha(\sigma) - \beta(\sigma) - D\, \partial\sigma$ is carried by an element $V'$ of $\mathcal{V}_{k+1}$ that contains the singleton $|\alpha(v)|$ for all vertices $v$ of $\sigma$. (3) shows that $\gamma$ is a cycle:

$$\partial\gamma = \partial(\alpha(\sigma) - \beta(\sigma) - D\, \partial\sigma) = \partial\alpha(\sigma) - \partial\beta(\sigma) - \partial D\, \partial\sigma$$

$$= \partial\alpha(\sigma) - \partial\beta(\sigma) - (\alpha(\partial\sigma) - \beta(\partial\sigma) - D\, \partial\, \partial\sigma)$$

$$= \partial\alpha(\sigma) - \alpha\partial(\sigma) - \partial\beta(\sigma) + \beta\partial(\sigma) = 0$$

Hence, there is an element $U$ of $\mathcal{U}_{k+1}$ containing $V'$ and a $(k+2)$-chain $D\sigma$ in $X$ carried by $U$ with $\partial D\sigma = \gamma$. Extend $D$ linearly on $C_{k+1}(K)$ and observe that

$$\partial Dc = \alpha(c) - \beta(c) - D\, \partial c$$

for all $c \in C_{k+1}(K)$. Inductively, we may assume that $D$ is defined on $C_n(K)$. For $k > n$, define $D$ to be zero on $C_k(K)$. Then $D$ is a chain homotopy from $\alpha$ to $\beta$ that satisfies the desired properties.

*Proof of 5.3.*

[2.n] implies [1.n]. (For this implication, $X$ does not need to be locally compact.) Let $\mathcal{U}$ be an open cover of $X - F$ and let $\{z_i\}_{i=1}^{\infty}$ be a sequence of homology elements where $z_i \in H_{q(i)}(U_i, V_i)$ for open sets $V_i \subset U_i$ in $X - F$ and $q(i) \leq n$. Let $J$ be as hypothesized in [2.n] for the cover $\mathcal{U}$ of $X - F$ and write $J = \cap_{i=1}^{\infty} N_i(J)$ for a decreasing sequence of neighborhoods $N_i(J)$ of $J$ in $X$. Let $(C_1, \partial C_1)$ be a carrier for $z_1$. According to [2.n], there is a carrier $(C_2, \partial C_2)$ for $i_*(z_2)$ with $C_2 \subset N_1(J) - C_1$. Inductively, assume that a carrier $(C_i, \partial C_i)$ for $i_*(z_i)$ has been chosen for $1 \leq i \leq k$ so that for each $i \geq 2$, $C_i \subset N_{i-1}(J)$ and so that $\{C_i\}_{i=1}^{k}$ is pairwise disjoint. According to [2.n], there is a carrier $(C_{k+1}, \partial C_{k+1})$ for $i_*(z_{k+1})$ with $C_{k+1} \subset N_k(J) - \cup_{i=1}^{k} C_i$. Hence, we obtain a sequence $\{(C_i, \partial C_i)\}_{i=1}^{\infty}$ of carriers and since for each $i$, $C_i \subset N_{i-1}(J)$ and $\{C_i\}_{i=1}^{\infty}$ is pairwise disjoint, $\{C_i\}_{i=1}^{\infty}$ forms a discrete family in $X - F$.

[1.n] implies [2.n]. Let $\mathcal{U}$ be an open cover of $X - F$ and let $\tilde{\mathcal{U}}$ be a collection of open subsets of $X$ such that $\mathcal{U} = \tilde{\mathcal{U}} \cap (X - F) = \{U \cap (X - F) \mid U \in \tilde{U}\}$, and let $Y = \cup \tilde{\mathcal{U}}$. Without loss of generality, we may assume that $\tilde{\mathcal{U}}$ is locally finite and, since $Y$ is locally compact, we may assume further that $\tilde{\mathcal{U}}$ consists of relatively compact open subsets of $Y$. Let $\mathcal{V}$ consist of all finite unions of elements from a countable basis for $X - F$ for which each basis element has compact closure in $Y$.

Let $\sum = \{\sigma_1, \sigma_2, \dots\}$ be an ordering of the elements of

$$\overline{\sum} = \left\{ (U, V, z, q) \mid U, V \in \mathcal{V}; \; V \subset U; \; z \in H_q(U, V); \; q \leq n \right\}$$

where each element of $\overline{\sum}$ appears infinitely often in the list $\sum$ and write $\sigma_i = (U_i, V_i, z_i, q(i))$. Since $X - F$ satisfies the discrete $n$-carriers property, there are carriers $(C_i, \partial C_i)$ for $\alpha_*(z_i)$ where

$$\alpha_* : H_{q(i)}(U_i, V_i) \to H_{q(i)}\big(\mathrm{st}(U_i, \mathcal{U}), \mathrm{st}(V_i, \mathcal{U})\big)$$

is inclusion-induced such that $\{C_i\}_{i=1}^{\infty}$ is discrete in $X - F$. Let $J$ be the limit points in $X$ of $\{C_i\}_{i=1}^{\infty}$, that is, $x \in X$ is in $J$ if and only if every neighborhood of $x$ in $X$ meets infinitely many of the sets $C_i$. Since $\{C_i\}_{i=1}^{\infty}$ is discrete in $X - F$, $J \subset F$ and easily $J$ is closed in $X$.

Let $z \in H_q(U, V)$ for open sets $V \subset U$ in $X - F$ and $q \leq n$, and let $N(J)$ be a neighborhood of $J$ in $X$. Choose $V' \subset U'$ with $U', V' \in \mathcal{V}$ such that $(C, \partial C) \subset (U', V') \subset (U, V)$ where $(C, \partial C)$ is a given carrier for $z$ and let $z' \in H_q(U', V')$ be an element whose image in $H_q(U, V)$ is $z$ under the inclusion-induced homomorphism and for which $(C, \partial C)$ is a carrier. Let

$\{\sigma_{j(1)}, \sigma_{j(2)}, \ldots\}$ be the list of elements of $\Sigma$ for which $\sigma_{j(k)} = (U', V', z', q)$ for all $k$ and observe that $(C_{j(k)}, \partial C_{j(k)})$ is a carrier for $i_*(z)$ for all $k$ where $i_*$ is induced by the inclusion $(U, V) \subset (\text{st}(U, \mathcal{U}), \text{st}(V, \mathcal{U}))$. Now $C_{j(k)} \subset \text{st}(U', \mathcal{U}) \subset \text{Cl}_Y(\text{st}(U', \mathcal{U})) = D$ for all $k$. It follows from the facts that $U' \in \mathcal{V}$, each element of $\mathcal{V}$ has compact closure in $Y$, and $\mathcal{U} = \tilde{\mathcal{U}} \cap (X - F)$ for the locally finite cover $\tilde{\mathcal{U}}$ of $Y$ by relatively compact open sets, that $D$ is compact. It now easily follows that there exists some $k$ for which $C_{j(k)} \subset N(J)$, for otherwise, there are limit points of $\{C_i\}_{i=1}^\infty$ in $D$ not contained in $J$.

[2.n] implies [3.n]. First note that since $F$ is a $\sigma$-$Z$-set in the ANR $X$, the inclusion $U - F \subset U$ is a weak homotopy equivalence for any given open set $U$ of $X$, hence this inclusion induces isomorphisms $\tilde{H}_q(U - F) \cong \tilde{H}_q(U)$ for all $q \geqslant 0$. Let $x \in X$ and let $U$ be an open neighborhood of $x$ in $X$. Choose relatively compact open sets $V$, $P$, and $Q$ in $X$ so that

$$x \in V \subset \text{Cl}_X V \subset P \subset \text{Cl}_X P \subset Q \subset \text{Cl}_X Q \subset U$$

and so that the inclusion induced homomorphism $\tilde{H}_q(V) \to \tilde{H}_q(P)$ is zero for all $q \geqslant 0$. Let $S \subset V \cap F$ be compact. Choose an open cover $\mathcal{W}$ of $X - F$ by sets open in $X - F$ such that $\text{st}(P, \mathcal{W}) \subset Q$ and for each $W \in \mathcal{W}$, $\text{diam}[\text{Cl}_X W] < \text{dist}(\text{Cl}_X W, S)$ where we assume that some metric for $X$ has been fixed. Let $J$ be as promised in [2.n] for the cover $\mathcal{W}$ and without loss of generality assume that $S \subset J$. Let $T = J \cap (\text{Cl}_X Q)$, a compact subset of $U \cap F$, and let $N(T)$ be a neighborhood of $T$ in $X$. Choose any neighborhood $N(S)$ of $S$ in $X$ with $N(S) \subset V$ and $\text{st}(N(S), \mathcal{W}) \subset N(T)$, which is possible by our choice of $\mathcal{W}$.

Let $0 \leqslant q < n$ be an integer. Consider the commutative diagram below where all homomorphisms are inclusion induced (Fig. 1).

$$\tilde{H}_q(N(S)) \to \tilde{H}_q(N(T))$$

$$\uparrow \quad \uparrow$$

$$\tilde{H}_q(N(S) - F) \underset{\alpha}{\to} \tilde{H}_q(N(T) - F)$$

Fig. 1.

By our first observation in the preceding paragraph, the vertical arrows are isomorphisms, hence, $\tilde{H}_q(N(S)) \to \tilde{H}_q(N(T))$ is zero if and only if $\alpha$ is zero. Similarly, since $\tilde{H}_q(N(S)) \to \tilde{H}_q(V) \to \tilde{H}_q(P)$ is zero, we have that $\tilde{H}_q(N(S) - F) \overset{\beta}{\to} \tilde{H}_q(P - F)$ is zero. Let $z \in \tilde{H}_q(N(S) - F)$. Since $\beta(z) = 0$, there is an element $z_1 \in H_{q+1}(P - F, N(S) - F)$ such that $\partial_*^1(z_1) = z$ where $\partial_*^1: H_{q+1}(P - F, N(S) - F) \to \tilde{H}_q(N(S) - F)$ is the connecting homomorphism in the long exact sequence in reduced homology of the obvious pair. Let

$$\gamma: H_{q+1}(P - F, N(S) - F) \to H_{q+1}(\text{st}(P - F, \mathcal{W}), \text{st}(N(S) - F, \mathcal{W}))$$

be inclusion-induced. According to [2.n], since $q + 1 \leqslant n$, there is a carrier $(C, \partial C)$ for $\gamma(z_1)$ such that

$$C \subset \underbrace{N(T) \cup (X - \mathrm{Cl}_X Q)}_{\text{open neighborhood of } J \text{ in } X}$$

Since $C \subset \mathrm{st}(P, \mathscr{W}) \subset Q$, $C \subset N(T)$ and thus $C \subset N(T) - F$. Let $z_2 \in H_{q+1}(C, \partial C)$ so that $\delta(z_2) = \gamma(z_1)$ where $\delta$ is induced by the inclusion $(C, \partial C) \subset (\mathrm{st}(P - F, \mathscr{W}), \mathrm{st}(N(S) - F, \mathscr{W}))$. Consider the following diagram (Fig. 2).
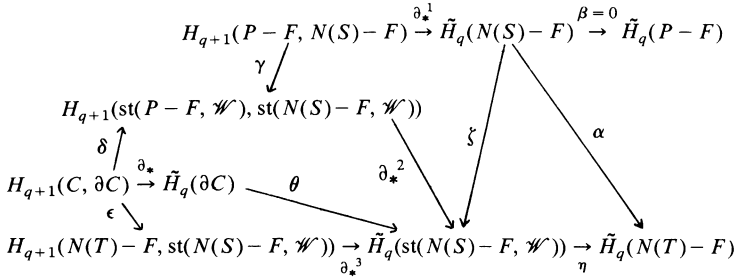
$$H_{q+1}(P - F, N(S) - F) \xrightarrow{\partial_*^1} \tilde{H}_q(N(S) - F) \xrightarrow{\beta = 0} \tilde{H}_q(P - F)$$

Fig. 2.

The two horizontal rows are exact at the middle terms, $\partial_*$, $\partial_*^1$, $\partial_*^2$, $\partial_*^3$ are connecting homomorphisms, and all other homomorphisms are inclusion-induced. All rectangles and triangles in the diagram commute. An easy diagram chase now shows that $\alpha(z) = 0$:

$$\alpha(z) = \alpha\, \partial_*^1(z_1) = \eta \zeta\, \partial_*^1(z_1) = \eta\, \partial_*^2 \gamma(z_1) = \eta\, \partial_*^2 \delta(z_2)$$

$$= \eta \theta\, \partial_*(z_2) = \eta\, \partial_*^3 \epsilon(z_2) = 0$$

since $\eta\, \partial_*^3 = 0$. Hence $\alpha = 0$ and we have the conclusion that $\tilde{H}_q(N(S)) \to \tilde{H}_q(N(T))$ is zero for $0 \leqslant q < n$, and thus $F$ is proximately $1c^{n-1}$ rel $X$.

[3.n] implies [2.n]. Let $\mathscr{U}$ be an open cover of $X - F$ and $\tilde{\mathscr{U}}$ a collection of open subsets of $X$ such that $\tilde{\mathscr{U}} \cap (X - F) = \mathscr{U}$. Let $Y = \cup \tilde{\mathscr{U}}$ and write $Y$ as in 4.4 as an increasing union of compacta: $Y = \bigcup_{i=1}^{\infty} Y_i, \; Y_i \subset \mathrm{Int}_Y Y_{i+1}$.

Let $\mathscr{W}$ be an open star-refinement of $\tilde{\mathscr{U}}$ that covers $Y$ and apply Lemma 5.4. to the space $Y$ and cover $\mathscr{W}$ to obtain an open cover $\mathscr{V}$ of $Y$ refining $\mathscr{W}$ that satisfies the conclusion of 5.4. Choose open covers

$$\mathscr{A} = \mathscr{A}_n, \quad \mathscr{B}_{n-1}, \quad \mathscr{A}_{n-1}, \ldots, \quad \mathscr{B}_0, \quad \mathscr{A}_0$$

of $Y$ that satisfy the following properties:
i) $\mathscr{A}$ is locally finite and $\mathrm{st}^2 \mathscr{A}$ refines $\mathscr{V}$ and, if $A \cap Y_i \neq \phi$ for some $A \in \mathscr{A}$, then $A \subset \mathrm{Int}_Y Y_{i+1}$;

ii) for $i = 0, \ldots, n-1$, $\mathscr{A}_i$ star-refines $\mathscr{B}_i$ and, for every $B \in \mathscr{B}_i$ and every compactum $S \subset B \cap F$, there exists an element $A$ of $\mathscr{A}_{i+1}$ containing $B$ and a compactum $T \subset A \cap F$ containing $S$ such that every singular reduced $i$-cycle close enough to $S$ is null-homologous close to $T$ (that is, given a neighborhood $N(T)$, there exists a neighborhood $N(S)$ such that $\tilde{H}_i(N(S)) \to \tilde{H}_i(N(T))$ is zero);

iii) $\mathscr{A}_0$ is countable and star-finite and there are maps $Y \overset{\nu}{\to} |\mathscr{N}(\mathscr{A}_0)| \overset{\mu}{\to} Y$ such that $\nu$ is canonical, $\mu(|A|) \in A$ for each $A \in \mathscr{A}_0 = \mathscr{N}(\mathscr{A}_0)^{(0)}$, $\mu \circ \nu$ is $\mathscr{A}$-homotopic to $\mathrm{id}_Y$, and $\mu$ is an $\mathscr{A}$-realization (that is, for $\sigma \in \mathscr{N}(\mathscr{A}_0)$, $\mu(|\sigma|) \subset A$ for some $A \in \mathscr{A}$).

ii) is possible since $F$ is proximately $1c^{n-1}$ rel $X$ and iii) is possible since $Y$ is an ANR.

Let $K = \mathscr{N}(\mathscr{A}_0)^{(n)}$ and for each $A \in K^{(0)} = \mathscr{A}_0$, let $[A]$ be a point of $A \cap F$. As in 4.4., by proceeding through one skeleton at a time, we may inductively construct a collection $M = \{[\sigma] \mid \sigma \in K\}$ such that each $[\sigma]$ for $\sigma \in K$ is a compactum contained in $A \cap F$ for some $A \in \mathscr{A}$, and, if $\tau$ is a face of $\sigma$, then $[\tau] \subset [\sigma]$. Furthermore, if $[\partial \sigma] = \cup\{[\tau] \mid \tau$ is a proper face of $\sigma\}$, then every singular (reduced) cycle of dimension $\dim \partial \sigma$ close enough to $[\partial \sigma]$ is null-homologous close to $[\sigma]$. Exactly as in 4.4., $\cup M$ is a closed subset of $Y$, hence, its closure in $X$, denoted by $J$, is contained in $F$.

Let $N(J)$ be an arbitrary neighborhood of $J$ in $X$. In the following discussion, we need to deal with oriented simplices of $K$. For notational convenience, we make the following assumption: assume that the set of vertices of $K$, that is, $\mathscr{A}_0$, has been linearly ordered and whenever we write a simplex $\sigma$ of $K$ as $\sigma = \langle A_0, \ldots, A_p \rangle$, we shall assume that $A_{i-1}$ precedes $A_i$ for $i = 1, \ldots, p$ in the linear order on $\mathscr{A}_0$. We shall use the single letter $\sigma$ to denote either a simplex of $K$ or an oriented simplex of $K$ induced from the linear order on $\mathscr{A}_0$. The context should make the meaning clear.

For each $\sigma = \langle A_0, A_1 \rangle \in K^{(1)}$, $(\partial \sigma)\hat{} = [A_1] - [A_0]$ is a singular reduced $0$-cycle in $[\partial \sigma]$, hence there is a singular $1$-chain $\hat{\sigma}$ in $N([\sigma])$, where $N([\sigma])$ is any prechosen neighborhood of $[\sigma]$ in $Y$, such that $\partial \hat{\sigma} = [A_1] - [A_0] = (\partial \sigma)\hat{}$. If $\tau = \sum n_i \sigma_i$ is an oriented $p$-chain in $K$ for some $p$ and if singular $p$-chains $\hat{\sigma}_i$ have been defined, define $\hat{\tau}$ or $(\sum n_i \sigma_i)\hat{}$ to be the singular $p$-chain $\sum n_i \hat{\sigma}_i$. Inductively, assume that for each oriented simplex $\sigma$ of $K^{(n-1)}$, we have chosen a singular chain $\hat{\sigma}$ of dimension $\dim \sigma$ in some prechosen neighborhood $N([\sigma])$ of $[\sigma]$ in $Y$ such that $\partial \hat{\sigma} = (\partial \sigma)\hat{}$. Let $\sigma$ be an oriented $n$-simplex of $K$ and, letting $\partial \sigma = \sum n_i \sigma_i$ where $\sigma_i$ are oriented $(n-1)$-faces of $\sigma$, we have

$$\partial(\partial \sigma)\hat{} = \partial(\sum n_i \sigma_i)\hat{} = \partial(\sum n_i \hat{\sigma}_i) = \sum n_i \, \partial \hat{\sigma}_i = \sum n_i (\partial \sigma_i)\hat{}$$
$$= (\sum n_i \, \partial \sigma_i)\hat{} = (\partial \sum n_i \sigma_i)\hat{} = (\partial \partial \sigma)\hat{} = (0)\hat{} = 0.$$

Hence, $(\partial \sigma)\hat{}$ is a singular $(n-1)$-cycle and we may assume that $(\partial \sigma)\hat{}$ is carried so close to $[\partial \sigma]$ that there is a singular $n$-chain $\hat{\sigma}$ in some prechosen

neighborhood $N([\sigma])$ of $[\sigma]$ such that $\partial \hat{\sigma} = (\partial \sigma)\hat{\,}$. We may assume that the neighborhoods $N([\sigma])$ have been chosen so close to $[\sigma]$ that, for each $\sigma$ in $K$, $\hat{\sigma}$ is carried by $A \cap N(J)$ where $A$ is an element of $\mathscr{A}$ with $[\sigma] \subset A$. Notice that if $\tau$ is an oriented $p$-chain in $K$, then $\hat{\tau}$ is a singular $p$-chain in $N(J) \cap Y$ and $\partial \hat{\tau} = (\partial \tau)\hat{\,}$.

For $z \in H_q(U, V)$ where $q \leqslant n$ and $V \subset U$ are open in $X - F$, suppose that $z$ is represented by a singular $q$-chain $c$ in $U$ with boundary $\partial c$ in $V$. Let $L$ (respectively, $L_0$) be the full-subcomplex of $\mathscr{N}(\mathscr{A}_0)$ whose vertices are those $A$ in $\mathscr{A}_0$ that meet $|c|$ (respectively, that meet $|\partial c|$). Let $\tilde{\nu}$ denote the restriction of $\nu$ to $|c|$ and $\tilde{\mu}$ the restriction of $\mu$ to $L$ and observe that $\tilde{\nu}$ and $\tilde{\mu}$ are maps of pairs in the following diagram:

$$\left( |c|, |\partial c| \right) \xrightarrow{\tilde{\nu}} \left( |L|, |L_0| \right) \xrightarrow{\tilde{\mu}} \left( \mathrm{st}^2(|c|, \mathscr{A}), \mathrm{st}^2(|\partial c|, \mathscr{A}) \right).$$

This follows from iii). Let $\eta$ be the chain map from oriented chains on $L$ to singular chains on $|L|$ defined on oriented simplices of $L$ by: $\eta(\langle A_0, \ldots, A_p \rangle)$ is the singular $p$-simplex that maps the standard $p$-simplex $\Delta_p$ onto $|\langle A_0, \ldots, A_p \rangle|$ linearly taking the $i$th vertex of $\Delta_p$ to $|A_i|$. $\eta$ induces an isomorphism $\eta_* : H_q(L, L_0) \to H_q(|L|, |L_0|)$ and there is a chain homotopy inverse $\lambda$ for $\eta$ [Munkres, 1984].

Let $\sigma$ be a simplex in $L^{(n)} \subset K$. Let $A_1$ be a vertex of $\sigma$ and recall that $[A_1] \in A_1 \cap [\sigma]$ and $A_1 \cap |c| \neq \phi$. By construction, there is an element $A$ of $\mathscr{A}$ such that $[\sigma] \subset A$ and $\hat{\sigma}$ is carried by $A$. Hence, $[A_1] \in A_1 \cap A$ and therefore $\hat{\sigma}$ is carried by $\mathrm{st}^2(|c|, \mathscr{A})$. Similarly, if $\sigma$ is a simplex in $L_0^{(n)} \subset K$, then $\hat{\sigma}$ is carried by $\mathrm{st}^2(|\partial c|, \mathscr{A})$. It now follows that for any oriented $q$-chain $\tau$ in $L$ with $\partial \tau$ carried by $L_0$, $\hat{\tau}$ is a singular $q$-chain in $\mathrm{st}^2(|c|, \mathscr{A})$ whose boundary $\partial \hat{\tau} = (\partial \tau)\hat{\,}$ is carried by $\mathrm{st}^2(|\partial c|, \mathscr{A})$.

Consider the following singular $q$-chains in $\mathrm{st}^2(|c|, \mathscr{A})$ whose boundaries are carried by $Q = \mathrm{st}^2(|\partial c|, \mathscr{A})$:

$$c, \quad c_1 = \tilde{\mu}_\# \tilde{\nu}_\#(c), \quad c_2 = \tilde{\mu}_\# \eta \lambda \tilde{\nu}_\#(c), \quad c_3 = \left( \lambda \tilde{\nu}_\#(c) \right)\hat{\,}.$$

Observe that $(\lambda \tilde{\nu}_\#(c))\hat{\,}$ makes sense since $\lambda \tilde{\nu}_\#(c)$ is an oriented $q$-chain of $K$ since $q \leqslant n$. Let

$$i'_* : H_q(U, V) \to H_q\left( \mathrm{st}(U, \tilde{\mathscr{U}}), \mathrm{st}(V, \tilde{\mathscr{U}}) \right)$$

be inclusion-induced. We claim that all four chains above represent $i'_*(z)$ If so, then $(|c_3|, |\partial c_3|)$ is a carrier for $i'_*(z)$ with $|c_3| \subset N(J)$. Since $F$ is $\sigma$-$Z$-set in the ANR $X$, there is a carrier $(C, \partial C)$ for $i_*(z)$ so close to $(|c_3|, |\partial c_3|)$ that $C \subset N(J)$ and the proof is complete. It is clear that $c$ represents the element $i'_*(z)$. Since $\mu \circ \nu$ is $\mathscr{A}$-homotopic to $\mathrm{id}_Y$, $c$ is homologous mod $Q$ to $c_1$ and $c_1$ represents $i'_*(z)$. Since $\lambda$ is a chain homotopy inverse to $\eta$, $c_1$ is homologous mod $Q$ to $c_2$ and $c_2$ represents

$i'_*(z)$. To finish, it suffices to show that if $\tau$ is an oriented $q$-chain in $L$ whose boundary is carried by $L_0$, then $\tilde{\mu}_{\#}\eta(\tau)$ and $\hat{\tau}$ represent the same element of $H_q(\mathrm{st}(U, \tilde{\mathcal{U}}), \mathrm{st}(V, \tilde{\mathcal{U}}))$.

Let $\mathscr{C}$ (respectively, $\mathscr{C}_0$) denote the oriented chain complex of $L^{(n)}$ (respectively, $L_0^{(n)}$) and $\mathscr{S}$ (respectively, $\mathscr{S}_0$) denote the singular chain complex of $\mathrm{st}(U, \tilde{\mathcal{U}})$ (respectively, $\mathrm{st}(V, \tilde{\mathcal{U}})$). Let $\mathscr{S}(Y)$ denote the singular chain complex of $Y$. $\tilde{\mu}_{\#}\eta$ is a chain map from $\mathscr{C}$ to $\mathscr{S}(Y)$ and the previous paragraphs, noting in particular that $\partial\hat{\tau} = (\partial\tau)\hat{\;}$ for an oriented chain $\tau$ of $K$, show that the restriction of $\hat{\;}$ (still denoted $\hat{\;}$) to $\mathscr{C}$ is a chain map from $\mathscr{C}$ to $\mathscr{S}\subset\mathscr{S}(Y)$ (we define $\hat{A}$ for a 0-simplex $A$ to be the singular 0-simplex $[A]$). We wish to apply Lemma 5.4. to $\alpha = \tilde{\mu}_{\#}\eta$, $\beta = \hat{\;}: \mathscr{C}\to\mathscr{S}(Y)$: first, for $\sigma\in L^{(n)}$, it is straightforward to show that there exists $A\in\mathscr{A}_0$ such that $\tilde{\mu}_{\#}\eta(\sigma)$ and $\hat{\sigma}$ are both carried by $\mathrm{st}(A, \mathscr{A})$, hence according to i), by some element of $\mathscr{V}$. Furthermore, for each vertex $A$ of $L$, $\tilde{\mu}_{\#}\eta(A)$ is the singular 0-simplex $\tilde{\mu}(|A|)$ and $\hat{A}$ is the singular 0-simplex $[A]$ and, if $A$ is a vertex of $\sigma\in L^{(n)}$, then $\tilde{\mu}(|A|)\in \mu(|\sigma|) = |\tilde{\mu}_{\#}\eta(\sigma)|$. Lemma 5.4. guarantees the existence of a chain homotopy $D: \tilde{\mu}_{\#}\eta\simeq\hat{\;}$ such that for each $\sigma\in L^{(n)}$, $|D(\sigma)|\subset W$ for some $W\in\mathscr{W}$ with $\tilde{\mu}(|A|)\in W$ for each vertex $A$ of $\sigma$. From the definitions of $L$ and $L_0$ and the fact that $\tilde{\mu}(|A|)\in A$ (by iii)), $\mathscr{A}_0$ refines $\mathscr{W}$, and $\mathscr{W}$ star-refines $\mathscr{U}$, the reader may observe that for any simplex $\sigma$ of $L^{(n)}$ (respectively, $L_0^{(n)}$), $D\sigma$ is carried by $\mathrm{st}(U, \tilde{\mathcal{U}})$ (respectively, $\mathrm{st}(V, \tilde{\mathcal{U}})$). Hence, $D$ is a chain homotopy of the chain maps $\tilde{\mu}_{\#}\eta$ and $\hat{\;}$ from $\mathscr{C}$ to $\mathscr{S}$ that restricts to a chain homotopy of the restrictions of $\tilde{\mu}_{\#}\eta$ and $\hat{\;}$ to $\mathscr{C}_0$ and $\mathscr{S}_0$. Therefore, $D$ induces a chain homotopy of $\tilde{\mu}_{\#}\eta$ and $\hat{\;}$ on the corresponding complexes of pairs and thus $\tilde{\mu}_{\#}\eta$ and $\hat{\;}$ induce the same homomorphisms on homology. In particular for $q\leqslant n$, $\tilde{\mu}_{\#}\eta(\tau)$ and $\hat{\tau}$ represent the same element of $H_q(\mathrm{st}(U, \tilde{\mathcal{U}}), \mathrm{st}(V, \tilde{\mathcal{U}}))$ for an oriented $q$-chain $\tau$ in $L$ whose boundary is carried by $L_0$. This completes the proof of 5.3.

The proof of the following proposition uses 5.3. and is similar to the proof of 4.6. Again, $\bar{A}$ denotes a dendrite whose endpoints are dense.

5.5. PROPOSITION. *Let $X = \bar{X} - F$ where $F$ is a dense $\sigma$-Z-set in the locally compact separable ANR $\bar{X}$ and let $A = \bar{A} - F_0$ where $F_0$ is a dense $\sigma$-compact collection of endpoints of the dendrite $\bar{A}$. If for some $n\in N\cup\{0\}$ $X$ satisfies the discrete $n$-carriers property, then $X\times A$ satisfies the discrete $(n+1)$-carriers property.*

*Proof.* Since $X\times A = \bar{X}\times\bar{A} - F_*$ where $F_*$ is the $\sigma$-Z-set $(F\times\bar{A})\cup(\bar{X}\times F_0)$ in $\bar{X}\times\bar{A}$, it suffices to prove that $F_*$ is proximately $1c^n$ rel $\bar{X}\times\bar{A}$. Theorem 5.3. guarantees that $F$ is proximately $1c^{n-1}$ rel $\bar{X}$. The proof proceeds exactly as the proof of 4.6. except the homological version of iv) is used in place of iv). Continuing, one constructs a collection $M = \{[\sigma]\,|\,\sigma\in K^{(n)}\}$ such that each $[\sigma]$ for $\sigma\in K^{(n)}$ is a compactum contained in $p_1(U)\cap F$ for some $U\in\mathscr{U}_{n-1}$ and, if $\tau$ is a face of $\sigma$, then $[\tau]\subset[\sigma]$. Furthermore, if $[\partial\sigma] = \cup\{[\tau]\,|\,\tau$ is a

proper face of $\sigma$}, then every singular (reduced) cycle of dimension dim $\partial\sigma$ carried close enough to $[\partial\sigma]$ bounds close to $[\sigma]$. Continue and define $\{\sigma\}$ and the compact subset $T$ as in 4.6. We must show that for every neighborhood $N(T)$ of $T$ in $\overline{X}\times\overline{A}$, there exists a neighborhood $N(S)$ of $S$ in $\overline{X}\times\overline{A}$ such that the inclusion-induced homomorphism $\tilde{H}_n(N(S))\to\tilde{H}_n(N(T))$ is zero. The proof of this is similar to the proof that singular $n$-spheres in $N(S)$ are contractible in $N(T)$ in 4.6., except that homology is used in place of homotopy where pertinent. Instead of using the maps $\nu: \cup\mathscr{W}\to|\mathscr{N}(\mathscr{W})|$ and $\mu: |K|\to N(T)$ in 4.6., one uses the chain map $\nu_{\#}$ induced from $\nu$, the chain equivalences $\eta$ and $\lambda$, and the chain map $\hat{}$ in place of $\mu$ as in the proof that $[3.n]$ implies $[2.n]$ in 5.3. The details of this argument are left to the reader.

## 6. Hurewicz theorems for discrete properties

In this section we state and prove some Hurewicz-type theorems for discrete properties. In general, these theorems state that $n$-dimensional homotopy data (for instance discrete $n$-cells or the proximate $LC^{n-1}$ property) together with $(n+1)$-dimensional homology data (discrete $(n+1)$-carriers or proximate $1c^n$ property) combine to give $(n+1)$-dimensional homotopy data.

6.1. LEMMA. *If $X$ is an ANR that satisfies the discrete 2-cells property and the discrete n-carriers property for each non-negative integer $n$, then compact subsets of $X$ are Z-sets in $X$.*

*Proof.* Let $D$ be a compact subset of $X$ and $U$ an open subset of $X$. For any element $z\in H_q(U, U-D)$ for a non-negative integer $q$, let $z_i=z$ for all non-negative integers $i$ and use the fact that $X$ satisfies the discrete carriers property to obtain carriers $(C_i, \partial C_i)$ for $z_i$ such that $\{C_i\}_{i=1}^{\infty}$ is discrete. Since $D$ is compact, all but finitely many $C_i$ miss $D$ and hence there is a carrier $(C, \partial C)$ for $z$ such that $C\cap D=\phi$. This implies that $z=0$ and thus $H_q(U, U-D)=0$. A similar argument using the discrete 2-cells property shows that $D$ is a 1-LCC subset of $X$. The Hurewicz theorem then implies that $D$ is a $Z$-set in $X$ (see [Daverman and Walsh, 1981], Proposition 4.2).

A subset $F$ of an ANR $X$ is a $Z_n$-set for some integer $n\geqslant 0$ provided each map of the $n$-cell $I^n$ into $X$ is approximible by maps whose images miss the subset $F$.

Given a collection $\{f_\alpha\}$ of maps of spaces $Y_\alpha$ into a space $X$ and an open cover $\mathscr{U}$ of $X$, we say that the collection $\{f_\alpha\}$ is $\mathscr{U}$-small provided $\{f_\alpha(Y_\alpha)\}$ refines $\mathscr{U}$.

6.2. LEMMA. *Let $X$ be a separable ANR and suppose that for some integer $n>1$, compact subsets and $Z_{n+1}$-sets and $X$ satisfies the discrete n-cells property and the discrete $(n+1)$-carriers property. Then for each open cover $\mathscr{U}$ of $X$, there exists*

*an open cover $\mathcal{W}$ of X refining $\mathcal{U}$ such that if $\{f_i : I^{n+1} \to X\}_{i=1}^{\infty}$ is a $\mathcal{W}$-small family of maps for which $\{f_i(\partial I^{n+1})\}_{i=1}^{\infty}$ forms a discrete family, then there exists a $\mathcal{U}$-small family $\{g_i : I^{n+1} \to X\}_{i=1}^{\infty}$ of maps that satisfies $g_i = f_i$ on $\partial I^{n+1}$ for all i and for which $\{g_i(I^{n+1})\}_{i=1}^{\infty}$ forms a discrete family.*

*Proof.* Let $\mathcal{U}'$ be a refinement of $\mathcal{U}$ such that each element of $\mathcal{U}'$ is contractible in some element of $\mathcal{U}$ and let $\mathcal{W}$ be a locally finite star-refinement of $\mathcal{U}'$. We find it convenient to use the standard $(n+1)$-simplex $\Delta_{n+1}$ rather than the $(n+1)$-cell $I^{n+1}$ in the proof. For $i = 1, 2, \ldots$, let $f_i : \Delta_{n+1} \to X$ be a map such that $f_i(\Delta_{n+1}) \subset W_i$ for some $W_i \in \mathcal{W}$ and such that $\{f_i(\partial \Delta_{n+1})\}_{i=1}^{\infty}$ forms a discrete family. Choose open sets $V_i$ and $V_i'$ in X such that

$$f_i(\partial \Delta_{n+1}) \subset V_i \subset \mathrm{Cl}_X V_i \subset V_i' \subset W_i$$

and such that $\{\mathrm{Cl}_X V_i'\}_{i=1}^{\infty}$ is discrete. Let $\mathcal{V}$ be a refinement of $\mathcal{W}$ by open sets such that $\mathrm{st}(\mathrm{Cl}_X V_i, \mathcal{V}) \subset V_i'$ for all i.

For each i, $f_i$ represents an element $z_i = [f_i]$ in $H_{n+1}(W_i, V_i)$. Since X satisfies the discrete $(n+1)$-carriers property, there exist carriers $(C_i, \partial C_i) \subset (\mathrm{st}(W_i, \mathcal{V}), \mathrm{st}(V_i, \mathcal{V}))$ for $i_*(z_i)$, where $i_*$ is induced by the inclusion $(W_i, V_i) \subset (\mathrm{st}(W_i, \mathcal{V}), \mathrm{st}(V_i, \mathcal{V}))$, such that $\{C_i\}_{i=1}^{\infty}$ forms a discrete family. Since $\mathrm{st}(W_i, \mathcal{V}) \subset \mathrm{st}(W_i, \mathcal{W}) \subset U_i'$ for some $U_i' \in \mathcal{U}'$ and $\mathrm{st}(V_i, \mathcal{V}) \subset V_i'$, $(C_i, \partial C_i)$ is also a carrier for $i'_*(z_i)$ where $i'_*$ is induced by the inclusion $(W_i, V_i) \subset (U_i', V_i')$. Let $z_i^* \in H_{n+1}(C_i, \partial C_i)$ be an element whose image in $H_{n+1}(U_i', V_i')$ is $i'_*(z_i)$ and let $b_i$ be a relative $(n+1)$-chain in $(C_i, \partial C_i)$ representing $z_i^*$. Since $[b_i] = [f_i]$ in $H_{n+1}(U_i', V_i')$, $f_i$ is homologous mod $V_i'$ to $b_i$ and there exists an $(n+1)$-chain $a_i$ in $V_i'$ such that $f_i - b_i - a_i = \partial d_i$ for some $(n+2)$-chain $d_i$ in $U_i'$. Let $c_i = b_i + a_i = f_i - \partial d_i$ so that $\partial c_i = \partial f_i$. Then $c_i$ can be represented by a map $\Psi_i : (N_i, \partial N_i) \to (U_i', V_i')$ where $N_i$ is a compact $(n+1)$-dimensional cell-complex with subcomplex $\partial N_i = \partial \Delta_{n+1}$ and such that $\Psi_i | \partial N_i = f_i | \partial \Delta_{n+1}$ and $\Psi_i(N_i) = |c_i| = |b_i + a_i| \subset C_i \cup |a_i|$. Observe that the families $\{C_i\}_{i=1}^{\infty}$ and $\{|a_i|\}_{i=1}^{\infty}$ are discrete in X $(|a_i| \subset \mathrm{Cl}_X V_i')$. Since each $|a_i|$ is compact and hence a $Z_{n+1}$-set in X, we may assume that $\{C_i \cup |a_i|\}_{i=1}^{\infty}$ is pairwise disjoint, and hence also discrete. This is possible since $C_i$ is the support of a singular chain of dimension $n+1$. Thus, the collection $\{\Psi_i(N_i)\}_{i=1}^{\infty}$ is discrete in X.

Let $M_i$ be the union of $N_i$ and the cone on its $(n-1)$-skeleton. Using the fact that $U_i'$ contracts in $U_i$ for some $U_i \in \mathcal{U}$, we can extend $\Psi_i$ to a map $\phi_i : M_i \to U_i$. Since X satisfies the discrete $n$-cells property, $\{\phi_i(N_i)\}_{i=1}^{\infty}$ is discrete, $\phi_i(N_i)$ is compact and thus a $Z_{n+1}$-set and $M_i - N_i$ is $n$-dimensional, we may assume that $\{\phi_i(M_i)\}_{i=1}^{\infty}$ also is discrete in X. Note that $M_i$ is $(n-1)$-connected and that $\partial i_{n+1}$ is null-homologous in $N_i$, hence in $M_i$, where $i_{n+1} : \Delta_{n+1} \to \Delta_{n+1}$ is the identity. Therefore, by the classical Hurewicz theorem, $i_{n+1} | \partial \Delta_{n+1}$ is null-homotopic in $M_i$, hence $f_i | \partial \Delta_{n+1} = \phi_i \circ i_{n+1} | \partial \Delta_{n+1}$ is null-homotopic in $\phi_i(M_i)$. Since $\{\phi_i\}_{i=1}^{\infty}$ is a $\mathcal{U}$-small family of maps whose images form a discrete family, we are done.

6.3. PROPOSITION. *Let $X$ be a separable* ANR *in which compact subsets are $Z_{n+1}$-sets where $n > 1$ is an integer. If $X$ satisfies the discrete $n$-cells property and the discrete $(n + 1)$-carriers property, then $X$ satisfies the discrete $(n + 1)$-cells property.*

*Proof.* Let $\mathscr{U}$ be open cover of $X$ and $\{ f_i : I^{n+1} \to X \}_{i=1}^{\infty}$ a sequence of maps. Let $\mathscr{W}$ be an open refinement of $\mathscr{U}$ as promised in 6.2. and choose an open refinement $\mathscr{V}$ of $\mathscr{W}$ covering $X$ such that elements of $\mathscr{V}$ are contractible in elements of $\mathscr{W}$. For each $i$, choose a finite triangulation $T_i$ of $I^{n+1}$ so fine that for each $\sigma \in T_i$, $f_i(|\sigma|)$ is contained in some $V \in \mathscr{V}$. Since $X$ satisfies the discrete $n$-cells property, we may assume that $\{ f_i(|T_i^{(n)}|) \}_{i=1}^{\infty}$ forms a discrete collection of compacta in $X$. For each $i$, choose a closed neighborhood $N_i$ of $f_i(|T_i^{(n)}|) \}$ such that $\{ N_i \}_{i=1}^{\infty}$ is discrete. For each $(n + 1)$-simplex $\sigma$ of $T_i$, let $c(\sigma) = |\partial\sigma| \times [0, 1]$ be a collar in $|\sigma|$ on $|\partial\sigma|$ and write $|\sigma| = c(\sigma) \cup \tilde{\sigma}$ where $c(\sigma) \cap \tilde{\sigma} = |\partial\sigma| \times \{1\}$. Assume that $c(\sigma)$ is chosen so that $f_i(c(\sigma)) \subset \text{Int}_X N_i$. Use the fact that $X$ is an ANR that satisfies the discrete $n$-cells property to obtain maps $g_{i,\sigma} : |\partial\sigma| \times \{1\} \to \text{Int}_X N_i$, where $\sigma$ is an $(n + 1)$-simplex in $T_i$, such that

$$\left\{ g_{i,\sigma}(|\partial\sigma| \times \{1\}) \mid \text{all } i \text{ and } \sigma \right\}$$

is discrete and each $g_{i,\sigma}$ is homotopic to $f_i \mid \mid \partial\sigma \mid \times \{0\}$ via a homotopy whose image is contained in $N_i \cap V$ for some $V \in \mathscr{V}$. Use these homotopies to extend $f_i \mid |T_i^{(n)}|$ to maps $g_i$ defined on $|T_i^{(n)}| \cup [\cup \{ c(\sigma) \mid \sigma \text{ is an } (n + 1)\text{-simplex in } T_i \}]$ such that the image of $g_i$ is contained in $N_i$, each $g_i(c(\sigma))$ is contained in some element $V$ of $\mathscr{V}$, $g_i \mid \mid \partial\sigma \mid \times \{1\} = g_{i,\sigma}$, and $g_i = f_i$ on $|T_i^{(n)}|$. Use the fact that each $V$ in $\mathscr{V}$ is contractible in some $W$ in $\mathscr{W}$ to extend $g_i \mid \mid \partial\sigma \mid \times \{1\}$ to $\tilde{\sigma}$ and then use 6.2. to obtain extensions of the $g_i$ to maps still called $g_i$ that are defined on $|T_i^{(n+1)}| = I^{n+1}$ such that

$$\left\{ g_i(\tilde{\sigma}) \mid \text{all } i \text{ and } \sigma \right\}$$

is discrete and refines $\mathscr{U}$. For each $i$, let

$$C_i = g_i \left( |T_i^{(n)}| \cup [\cup \{ c(\sigma) \mid \sigma \text{ is an } (n + 1)\text{-simplex of } T_i \}] \right),$$

$$D_i = \cup \{ g_i(\tilde{\sigma}) \mid \sigma \text{ is an } (n + 1)\text{-simplex of } T_i \}.$$

Since $C_i \subset N_i$, $\{ C_i \}_{i=1}^{\infty}$ is discrete in $X$ and obviously $\{ D_i \}_{i=1}^{\infty}$ is discrete in $X$. Since $C_i$ is compact for each $i$ and therefore a $Z_{n+1}$-set, we may assume that $\{ C_i \cup D_i \}_{i=1}^{\infty}$ forms a discrete family (via small adjustments of the $(g_i \mid \tilde{\sigma})$'s). Observe that $g_i$ is $\mathscr{U}$-close to $f_i$ and $\{ g_i(I^{n+1}) \}_{i=1}^{\infty}$ is discrete since $g_i(I^{n+1}) = C_i \cup D_i$.

The reader should observe that it is not necessary to restrict ourselves to the boundary set setting in order to prove the results 6.2. and 6.3. Similarly, the following holds in the general case.

6.4. THEOREM. *If $X$ is a topologically complete separable* ANR *that satisfies the discrete 2-cells property and the discrete n-carriers property for each non-negative integer n, then $X$ satisfies the discrete n-cells property for each non-negative integer n.*

*Proof.* Apply 6.1. and 6.3.

If we restrict ourselves to the boundary setting, we obtain the following result.

6.5. COROLLARY. *Let $F$ be a dense $\sigma$-Z-set in the locally compact separable* ANR $X$. *If for some $n > 0$, $F$ is proximately $LC^n$ rel $X$ and proximately $1c^{n+1}$ rel $X$, then $F$ is proximately $LC^{n+1}$ rel $X$.*

*Proof.* The hypotheses imply that compact subsets of $X - F$ are $Z_{n+1}$-sets. Apply 4.4., 5.3., 6.3., then 4.4. again. We note, however, that this could be proved directly, similarly to the proofs of 6.2. and 6.3.


## 7. Proofs of the main results

*Proof of 3.1.* If $X$ is an $s$-manifold, then $X$ satisfies the discrete approximation property and easily $X$ satisfies the discrete carriers property. For the converse, if $X$ satisfies the discrete carriers property and the discrete 2-cells property, then 6.4. implies that $X$ satisfies the discrete $n$-cells property for each non-negative integer $n$. Then 4.5. implies that $X$ satisfies the discrete approximation property and Torunczyk's $s$-manifold characterization theorem (2.1) guarantees that $X$ is an $s$-manifold.

*Proof of 3.2.* i) implies ii) is a trivial consequence of the definitions; ii) is equivalent to iii) follows from 5.3.; ii) implies iv) since 5.1. and 5.5. combine to imply that $X \times A$ satisfies the discrete 2-cells property and the discrete $n$-carriers property for all $n$, hence 6.4. and 4.5. apply to show that $X \times A$ satisfies the discrete approximation property; iv) implies v) follows from 2.1., 4.6. and 4.5.; v) implies vi) is trivial. It remains to prove that vi) implies ii) and ii) proves i).

vi) implies ii) follows from the following result.

7.1. THEOREM. *Let $X = \overline{X} - F$ and $A = \overline{A} - F_0$ be as in Theorem 3.2. and let n be a non-negative integer. Then the following are equivalent*:
 *i)* $F$ *is proximately $1c^{n-1}$ rel $\overline{X}$*;
 *ii)* $X$ *satisfies the discrete n-carriers property*;
 *iii)* $(\overline{X} \times F_0) \cup (F \times \overline{A})$ *is proximately $1c^n$ rel $\overline{X} \times \overline{A}$*;
 *iv)* $X \times A$ *satisfies the discrete $(n + 1)$-carriers property.*

*Proof.* The equivalence of i) and ii) and of iii) and iv) follows from Theorem 5.3.; ii) implies iv) follows from Proposition 5.5. It remains only to prove that iii) implies i). The proof is almost exactly the same as the proof of ([Bowers, 1985c], Proposition 3.4) except that one uses open neighborhoods rather than $\epsilon$'s and $\delta$'s. The necessary changes are straightforward.

We now prove that ii) implies i). ([Bowers, 1985b], Lemma 3.2) guarantees that $X$ satisfies the following property:

> For every open cover $\mathscr{U}$ of $X$, there exists a countable open cover $\mathscr{V} = \{V_i\}_{i=1}^{\infty}$ of $X$, a locally finite countable complex $K$ with maps $\nu : X \to |K|$ and $\mu : |K| \to X$, and positive integers $n(1), n(2), \ldots$ (4) such that $\mu \circ \nu$ is $\mathscr{U}$-close to $\mathrm{id}_X$ and, if $\sigma$ is a simplex of $K$ of dimension greater than $n(i)$, then $\nu(|\sigma|) \cap V_i = \phi$ for $i = 1, 2, \ldots$.

Assume that $X$ satisfies the discrete $n$-carriers property for all non-negative integers $n$ and, given an open cover $\mathscr{U}$ of $X$, choose an open cover $\tilde{\mathscr{W}}$ of $X$ such that $\tilde{\mathscr{W}}$-close maps into $X$ are $\mathscr{U}$-homotopic. Choose an open cover $\mathscr{W}$ of $X$ such that $\mathrm{st}^2 \mathscr{W}$ refines $\tilde{\mathscr{W}}$ and let $\mathscr{V}$, $K$, $\mu$, $\nu$, $n(1)$, $n(2)$, $\ldots$ be as promised in (4) for the cover $\mathscr{W}$ of $X$. Without loss of generality, we may assume that $\mathscr{V}$ is star-finite. Let $\{z_i\}$ be a sequence of homology elements with $z_i \in H_{q(i)}(A_i, B_i)$ for open sets $A_i \supset B_i$ and non-negative integers $q(i)$. The idea of making '$\{z_i\}$' discrete is first to push each $z_i$ up to $K$ via $\nu_\#$ and represent each as a simplicial chain. Then push each such chain back to $X$ via $\mu_\#$ and use the discrete $n$-carriers property to make '$\{z_i \mid q(i) = n\}$' discrete for each $n$. By property (4), this new collection of carriers for $\{z_i\}$ will be locally finite and the disjoint carriers property, which follows from the discrete carriers property, can be used to make these carriers pairwise disjoint while preserving the local finiteness. Details of this argument follow.

As in Theorem 5.3., [3.$n$] implies [2.$n$], letting $c_i$ be a singular chain representing $z_i$, we may obtain singular chains $\bar{c}_i = \mu_\# \eta \lambda \nu_\#(c_i)$ that represent $i'_*(z_i)$. Here $i'_*$ is induced by the conclusion $(A_i, B_i) \subset (\mathrm{st}(A_i, \mathscr{U}), \mathrm{st}(B_i, \mathscr{U}))$ and $\eta$ is a chain map from oriented chains on an appropriate subcomplex of $K$ to singular chains with chain homotopy inverse $\lambda$. If for some $i$, $q(i) > n(i)$, then $|\bar{c}_i| \cap V_i = \phi$. This follows from (4) and the fact that $\eta \lambda \nu_\#(c_i)$ is supported on a union of $q(i)$-simplices of $K$. Use the fact that $X$ satisfies the discrete $n$-carriers property for all $n$ to obtain carriers $\{(C_i, \partial C_i)\}$ for $\{i_*(z_i)\}$, where $i_*$ is induced by the inclusion $(A_i, B_i) \subset (\mathrm{st}^2(A_i, \mathscr{U}), \mathrm{st}^2(B_i, \mathscr{U}))$, for which $\{C_i \mid q(i) = n\}$ is discrete for each $n$ and each $C_i$ is contained in $\mathrm{st}^2(|\bar{c}_i|, \mathscr{V})$. Observe that if $C_i \cap V_k \neq \phi$, then there are integers $k'$ and $k''$ such that $V_k \cap V_{k'} \neq \phi$, $V_{k'} \cap V_{k''} \neq \phi$, and $V_{k''} \cap |\bar{c}_i| \neq \phi$ and hence $q(i) \leqslant n(k'')$. In general, if $C_i \cap V_k \neq \phi$, then $q(i) \leqslant m(k)$ where $m(k) = \max\{n(j) \mid V_j \subset \mathrm{st}^2(V_k, \mathscr{V})\}$, which exists and is finite since $\mathscr{V}$ is star-finite. This implies that $\{C_i\}$ is locally finite since $\{C_i \mid q(i) = n\}$ is discrete for each $n$. Now using the fact that $X$ satisfies the disjoint carriers property, we can adjust the collection $\{C_i\}$ to obtain a collection $\{(D_i, \partial D_i)\}$ of carriers for

$\{i_*(z_i)\}$ such that $\{D_i\}$ is both pairwise disjoint and locally finite, hence discrete. This completes the proof of 3.2.

## 8. Applications

The homological analog of a $Z$-set in an ANR $X$ is a closed subset $D$ that has *infinite codimension* in $X$, that is, $H_q(U, U - D) = 0$ for all open subsets $U$ and all integers $q \geqslant 0$. It is known that if $X$ has a nice ANR local compactification, then $X - D$ is an $s$-manifold for a $Z$-set $D$ in $X$ if and only if $X$ is an $s$-manifold [Bestvina *et al.*, 1986; Bowers, 1986]. In this section we prove the homological analog of this result.

In [Bestvina *et al.*, 1986] the usual notion of $Z$-set is refined to give the correct and useful notion of homotopic negligibility in the setting of non-locally compact ANR's. A closed subset $D$ of a separable ANR $X$ is a *strong-Z-set* if, for each open cover $\mathcal{U}$ of $X$, there is an open set $V$ containing $D$ and a map $f : X \to (X - V)$ $\mathcal{U}$-close to the identity. Equivalently, $D$ is a strong-$Z$-set in $X$ if $D$ is a $Z$-set and $X$ satisfies the discrete approximation property *at* $D$, that is, for each open cover $\mathcal{U}$ of $X$ and map $f : \oplus_{i=1}^{\infty} I_i^{\infty} \to X$, there exists a $\mathcal{U}$-approximation $g$ to $f$ such that $\{g(I_i^{\infty})\}_{i=1}^{\infty}$ is a discrete collection *at* $D$, meaning that each point of $D$ has a neighborhood meeting at most one element of $\{g(I_i^{\infty})\}_{i=1}^{\infty}$. In general, as examples of [Bestvina *et al.*, 1986] illustrate, the concepts of $Z$-set and strong-$Z$-set do not coincide, the latter being a strictly stronger concept. However, if $X$ has a nice ANR local compactification (in particular, if $X$ is locally compact), every $Z$-set is a strong-$Z$-set [Bowers, 1986]. We need the homological analog of this result in order to prove the main results of this section. This is the content of the next lemma. In general, a closed subset $D$ of an arbitrary separable ANR $X$ that satisfies the conclusion of the lemma might be said to have *strong infinite codimension* in $X$.

8.1. LEMMA. *Suppose the closed subset $D$ of $X$ has infinite codimension in $X$, where $X = \bar{X} - F$ for a $\sigma$-$Z$-set $F$ in the locally compact separable ANR $\bar{X}$. Then for each non-negative integer $n$ and open cover $\mathcal{U}$ of $X$ and sequence $\{z_i \in H_{q(i)}(U_i, V_i)\}_{i=1}^{\infty}$ of homology elements in $X$ where $V_i \subset U_i$ are open and $q(i) \leqslant n$, there exists carriers $(C_i, \partial C_i)$ for $i_*(z_i)$, where*

$$i_* : H_{q(i)}(U_i, V_i) \to H_{q(i)}(\mathrm{st}(U_i, \mathcal{U}), \mathrm{st}(V_i, \mathcal{U}))$$

*is inclusion-induced, such that $\{C_i\}_{i=1}^{\infty}$ is discrete at $D$.*

We delay the proof of 8.1. until the end of this section.

8.2. COROLLARY. *Let $X$ and $D$ be as in Lemma 8.1. If $n$ is a non-negative integer and $X - D$ satisfies the discrete $n$-carriers property, then $X$ satisfies the discrete $n$-carriers property.*

The proof of 8.2. is an easy application of 8.1.

**8.3. THEOREM.** *Let $X = \overline{X} - F$ for a $\sigma$-Z-set $F$ in the locally compact separable ANR $\overline{X}$ and $A = \overline{A} - F_0$ for a dense $\sigma$-compact collection $F_0$ of endpoints of the dendrite $\overline{A}$. If $D$ is a closed subset of $X$ with infinite codimension in $X$, then $(X - D) \times A$ is an s-manifold if and only if $X \times A$ is an s-manifold.*

*Proof.* The 'if' implication is obvious. For the 'only if' implication, since $X - D$ has the nice ANR local compactification $\overline{X} - \text{Cl}_{\overline{X}} D$ and since $(X - D) \times A$ is an $s$-manifold, 3.2. applies and therefore $X - D$ satisfies the discrete carriers property. By 8.2., $X$ satisfies the discrete $n$-carriers property for each non-negative integer $n$ and hence by 3.2., $X \times A$ is an $s$-manifold.

**8.4. COROLLARY.** *Let $X$ and $A$ be as in Theorem 8.3. If $D$ is a closed subset of $X$ with infinite codimension in $X$ and if $X - D$ is an s-manifold, then $X \times A$ is an s-manifold.*

Compare 8.4. with ([Bowers, 1986], Corollary 1, Section 4) where 8.4 is proved in the special case that $D$ is compact. There, this special case of 8.4. is used to show that various examples of fake $s$-manifolds that arise as images of fine homotopy equivalences defined on $s$ stabilize to $s$-manifolds upon multiplication by $A$.

The remainder of this section is devoted to the proof of 8.1.

*Proof of 8.1.* The proof is similar to the proof that $[3.n]$ implies $[2.n]$ in Theorem 5.3. Given an open cover $\mathcal{U}$ of $X$, as usual let $\tilde{\mathcal{U}}$ be an open cover of an open subset $Y$ of $\overline{X}$ containing $X$ so that $\tilde{\mathcal{U}} = \mathcal{U} \cap X$ and and write $Y$ as an increasing union of compacta: $Y = \bigcup_{i=1}^{\infty} Y_i$. Choose open covers

$$\mathcal{W}, \quad \mathcal{V}, \quad \mathcal{A} = \mathcal{A}_n, \quad \mathcal{B}_{n-1}, \quad \mathcal{A}_{n-1}, \ldots, \quad \mathcal{B}_0, \quad \mathcal{A}_0$$

of $Y$ that satisfy the following properties:
 i) $\mathcal{V}$ is as promised in 5.4. for the space $Y$ and the star refinement $\mathcal{W}$ of $\tilde{\mathcal{U}}$ that covers $Y$;
 ii) $\mathcal{A}$ is locally finite and $\text{st}^2 \mathcal{A}$ refines $\mathcal{W}$ and, if $A \cap Y_i \neq \phi$ for some $A \in \mathcal{A}$, then $A \subset \text{Int}_Y Y_{i+1}$;
 iii) for $i = 0, \ldots, n-1$, $\mathcal{A}_i$ star-refines $\mathcal{B}_i$ and each element of $\mathcal{B}_i$ is contractible in some element of $\mathcal{A}_{i+1}$;
 iv) $\mathcal{A}_0$ is countable and star-finite and there are maps $Y \xrightarrow{\nu} |\mathcal{N}(\mathcal{A}_0)| \xrightarrow{\mu} Y$ so that $\mu(|A|) \in A$ for each $A \in \mathcal{A}_0 = \mathcal{N}(\mathcal{A}_0)^{(0)}$, $\mu \circ \nu$ is $\mathcal{A}$-homotopic to $\text{id}_Y$, and $\mu$ is an $\mathcal{A}$-realization.
Let $K = \mathcal{N}(\mathcal{A}_0)^{(n)}$ and assume that the vertices of $K$ are linearly ordered and, as in the proof of 5.3., whenever we write $\langle A_0, \ldots, A_p \rangle$, we shall assume that $A_i$ precedes $A_{i+1}$ in the linear order on $K^{(0)}$ and we shall use $\sigma$ to denote not

only a simplex in $K$, but also the oriented simplex induced by the linear ordering on $K^{(0)}$. Using iii) and the facts that $D$ has infinite codimension and $F$ is a $\sigma$-$Z$-set, construct a chain map

$$\hat{\ } : \mathscr{C}(K) \to \mathscr{S}(X)$$

where $\mathscr{C}(K)$ (respectively, $\mathscr{S}(X)$) denotes the complex of oriented (respectively, singular) chains in $K$ (respectively, $X$), so that for each oriented simplex $\sigma = \langle A_0, \ldots, A_p \rangle$ of $K$, $\hat{\sigma}$ is singular $p$-chain in $X$ carried by an element $A$ in $\mathscr{A}$ for which $A \cap A_i \neq \phi$ for $i = 0, \ldots, p$ and $|\hat{\sigma}| \cap D = \phi$. Using ii) and the fact that $\mathscr{A}_0$ is star-finite, it is easy to show that each point in $X$ has a neighborhood that meets at most finitely many $|\hat{\sigma}|$ for $\sigma \in K$, hence $\cup_{\sigma \in K} |\hat{\sigma}|$ is a closed subset of $X$ that misses $D$.

As in 5.3., let $\eta : \mathscr{C}(K) \to \mathscr{S}(|K|)$ be the natural chain equivalence from oriented chains in $K$ to singular chains in $|K|$ with chain homotopy inverse $\lambda$. Using i), iv) and 5.4. and the fact that $F$ is a $\sigma$-$Z$-set, one can obtain a chain homotopy $G$ of $\mu_{\#}\eta$ to $\hat{\ }$ such that $G(\sigma)$ is carried by some $W$ in $\mathscr{W}$ for each $\sigma$ in $K$. If $c$ is any singular $p$-chain in $X$ with $p \leqslant n$, then the chains $c$, $\mu_{\#}\nu_{\#}(c)$, $\mu_{\#}\eta\lambda\nu_{\#}(c)$, and $(\lambda\nu_{\#}(c))\hat{\ }$ are all carried by $\mathrm{st}^2(|c|, \mathscr{A})$ and their boundaries are carried by $\mathrm{st}^2(|\partial c|, \mathscr{A})$. Using iv) and the chain homotopy $G$, one can show that all four chains represent the same element of $H_p(\mathrm{st}(|c|, \mathscr{U}), \mathrm{st}(|\partial c|, \mathscr{U}))$, hence

$$\left( |(\lambda\nu_{\#}(c))\hat{\ }|, |\partial(\lambda\nu_{\#}(c))\hat{\ }| \right)$$

is a carrier for such an element.

Now if $z_i \in H_{q(i)}(U_i, V_i)$ as hypothesized, then, letting the singular chain $c_t$ represent $z_i$,

$$\left( C_i = |(\lambda\nu_{\#}(c_i))\hat{\ }|, \partial C_i = |\partial(\lambda\nu_{\#}(c_i))\hat{\ }| \right)$$

is a carrier for $i_*(z_i)$. Our claim is that $\{C_i\}_{i=1}^{\infty}$ is discrete at $D$. This follows easily since each $C_i$ is contained in the closed subset $\bigcup_{\sigma \in K} |\hat{\sigma}|$ of $X$ that misses $D$.

# 9. Questions

The main unresolved question in this paper is the following question.

## 9.1. Question

Can statement vi) of Theorem 3.2. be replaced by the statement: $X \times B$ is an $s$-manifold for some finite-dimensional separable metric space $B$?

Question 9.1. also should be asked in the boundary set setting for $B$.

*9.2. Question*

Can statement vi) of Theorem 3.2. be replaced by the statement: $X \times B$ is an $s$-manifold for some finite-dimensional space $B$, where $B$ has a nice ANR local compactification?

The answer to the following more general question is presently unknown.

*9.3. Question*

Let $X$ and $B$ be separable metric ANR's with $B$ finite-dimensional. If $X \times B$ satisfies the discrete carriers property, must $X$ satisfy the discrete carriers property?

We point out that 9.3. has an affirmative answer in case $X$ and $B$ are locally compact and the discrete carriers property is replaced by the disjoint carriers property. See [Daverman and Walsh, 1981].

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**References**

Anderson, R.D.: Hilbert space is homeomorphic to the countable infinite product of lines. *Bull. Amer. Math. Soc.* 72 (1966) 515–519.

Bestvina, M.: Stabilizing fake Hilbert spaces. *Top. and Its Appl.,* to appear.

Bestvina, M., Bowers, P., Mogilski, J. and Walsh, J.: Characterization of Hilbert space manifolds revisited. *Top. and Its Appl.,* 24 (1986) 53–69.

Bowers, P.L.: *Applications of general position properties of dendrites to Hilbert space topology.* Ph.D. dissertation, University of Tennessee (1983).

Bowers, P.L.: General position properties satisfied by finite products of dendrites. *Trans. Amer. Math. Soc.* 288 (1985a) 739–753.

Bowers, P.L.: Discrete cells properties in the boundary set setting. *Proc. Amer. Math. Soc.* 93 (1985b) 735–740.

Bowers P.L.: Fake boundary sets in the Hilbert cube. *Proc. Amer. Math. Soc.* 93 (1985c) 121–127.

Bowers, P.L.: Nonshrinkable 'cell-like' decompositions of s. *Pac. J. Math.* 124 (1986) 257–273.

Cannon, J.W.: Shrinking cell-like decompositions of manifolds, codimension three. *Ann. of Math.* (2) 110 (1979) 83–112.

Curtis, D.W.: Boundary sets in the Hilbert cube. *Top. and Its Appl.* 20 (1985) 201–221.

Daverman, R. and Walsh, J.: Čech homology characterizations of infinite dimensional manifolds. *Amer. J. Math.* 103 (1981) 411–435.

Edwards, R.D.: Approximating certain cell-like maps by homeomorphisms. See Notices *Amer. Math. Soc.* 24 (1977) A649, #751–G5.

Hu, S.T.: *Theory of Retracts.* Wayne State Univ. Press, Detroit, MI (1965).

Munkres, J.R.: *Elements of Algebraic Topology.* Addison-Wesley Pub. Co., Menlo Park, CA (1984).

Quinn, F.: Ends of maps and applications. *Ann. of Math.* (2) 110 (1979) 275–331.

Torunczyk, H.: Absolute retracts as factors of normed linear spaces. *Fund. Math.* 86 (1974) 53–67.

Torunczyk, H.: Concerning locally homotopy negligible sets and characterization of $l_2$-manifolds. *Fund. Math.* 101 (1978) 93–110.

Torunczyk, H.: On CE-images of the Hilbert cube and characterization of $Q$-manifolds. *Fund. Math.* 106 (1980) 31–40.

Torunczyk, H.: Characterizing Hilbert space topology. *Fund. Math.* 111 (1981) 247–262.