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KUMIKO NISHIOKA

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Conditions for algebraic independence of certain power series of algebraic numbers

KUMIKO NISHIOKA
Department of Mathematics, Nara Women's University, Kita-Uoya Nishimachi, Nara 630, Japan

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In what follows, K denotes an algebraic number field. Let $f(z)=\sum_{k=0}^{\infty} a_k z^{e_k}$ be a power series with nonzero coefficients $a_k \in K$, the convergence radius $R>0$ and increasing nonnegative integers e_k satisfying the condition

$$\lim_{k\to\infty} (e_k+\log M_k+\log A_k)/e_{k+1}=0 \tag{1}$$

where $A_k=\max\{1, \overline{|a_0|},\dots, \overline{|a_k|}\}$ and M_k is positive integer such that $M_k a_0,\dots, M_k a_k$ are algebraic integers. By [Cijsouw and Tijdeman, 1973], the number $f(\alpha)$ is transcendental for any algebraic α with $0<|\alpha|<R$. Moreover by [Bundschuh and Wylegala, 1980], the numbers $f(\alpha_1),\dots, f(\alpha_n)$ are algebraically independent for any algebraic numbers $\alpha_1,\dots,\alpha_n$ with $0<|\alpha_1| < \cdots <|\alpha_n|<R$. In this paper, we shall establish necessary and sufficient conditions for algebraic independence of the values $f(\alpha_1),\dots, f(\alpha_n)$ at algebraic numbers $\alpha_1,\dots,\alpha_n$. *Definition.* We say the algebraic numbers $\alpha_1,\dots,\alpha_s$ are $\{e_k\}$-dependent if there exist a number γ, roots of unity $\zeta_i (1\leqslant i\leqslant s)$ and algebraic numbers $d_1,\dots,d_s$ not all zero such that

$$\alpha_i=\zeta_i\gamma(1\leqslant i\leqslant s),$$

$$\sum_{i=1}^{s} d_i\zeta_i^{e_k}=0$$

for any sufficiently large k.

Then we have the following theorem.

THEOREM 1. *Let $\alpha_1,\dots,\alpha_n$ be algebraic numbers with $0<|\alpha_i|<R(1\leqslant i\leqslant n)$. Then the following three properties are equivalent.*

i) $f^{(l)}(\alpha_i)$ $(1\leqslant i\leqslant n, 0\leqslant l)$ are algebraically dependent over the rationals, where $f^{(l)}(z)$ denotes the lth derivative of $f(z)$.

ii) *There is a non-empty subset* $\{\alpha_{i_1},\ldots,\alpha_{i_s}\}$ *of* $\{\alpha_1,\ldots,\alpha_n\}$ *such that* $\alpha_{i_1},\ldots,\alpha_{i_s}$ *are* $\{e_k\}$*-dependent.*
iii) $1, f(\alpha_1),\ldots,f(\alpha_n)$ *are linearly dependent over the algebraic numbers.*

Example 1. Let $f(z)=\sum_{k=1}^{\infty} z^{k!}$ and $\alpha_1,\ldots,\alpha_n$ be algebraic numbers with $0<|\alpha_i|<1 (1\leqslant i\leqslant n)$. Then $f^{(l)}(\alpha_i)$ $(1\leqslant i\leqslant n, 0\leqslant l)$ are algebraically independent if and only if α_i/α_j is not a root of unity for $i\neq j$. This result was conjectured to be true by Masser.

Example 2. Let $f(z)=\sum_{k=1}^{\infty} z^{k!+k}$ and $\alpha_1,\ldots,\alpha_n$ be algebraic numbers with $0<|\alpha_i|<1$ $(1\leqslant i\leqslant n)$. Then $f^{(l)}(\alpha_i)$ $(1\leqslant i\leqslant n, 0\leqslant l)$ are algebraically independent if and only if $\alpha_i\neq\alpha_j$ for $i\neq j$.

Proof of Theorem 1. Obviously the property ii) implies the property iii) and the property iii) implies the property i). We prove the property i) implies the property ii). Suppose the property i) is satisfied. We may assume $f^{(l)}(\alpha_i)$ $(1\leqslant i\leqslant n, 0\leqslant l\leqslant L)$ are algebraically dependent and $f^{(l)}(\alpha_i)$ are algebraically independent for any subset of $n-1$ numbers α_i. Changing the indices of the α's, we may suppose

$$|\alpha_{11}|=\cdots=|\alpha_{1s_1}|=|\alpha_{21}|=\cdots=|\alpha_{2s_2}|=\cdots$$

$$=|\alpha_{t1}|=\cdots=|\alpha_{ts_t}|>|\alpha_{t+1,1}|\geqslant\cdots\geqslant|\alpha_{t+1,s_{t+1}}|,$$

$$s_1+s_2+\cdots+s_{t+1}=n,$$

$$\alpha_{iq}/\alpha_{i1}(1\leqslant i\leqslant t, 1\leqslant q\leqslant s_i) \text{ are roots of unity,}$$

$$\alpha_{i1}/\alpha_{j1}(1\leqslant i<j\leqslant t) \text{ are not roots of unity.}$$

Define U and $U_m\in\mathbb{C}^{n(L+1)}$ by

$$U=\left(\alpha_{iq}^l f^{(l)}(\alpha_{iq})\right)_{1\leqslant i\leqslant t+1,1\leqslant q\leqslant s_i,0\leqslant l\leqslant L},$$

$$U_m=\left(\sum_{k=0}^{m-1} P_l(e_k)a_k\alpha_{iq}^{e_k}\right)_{1\leqslant i\leqslant t+1,1\leqslant q\leqslant s_i,0\leqslant l\leqslant L},$$

where $P_0(X)=1$, $P_l(X)=X(X-1)\ldots(X-l+1)$. Then $\lim_{m\to\infty}U_m=U$ and there is a nonzero polynomial $F\in\overline{\mathbb{Q}}[\{y_{iq}^{(l)}\}_{1\leqslant i\leqslant t+1,1\leqslant q\leqslant s_i,0\leqslant l\leqslant L}]$ such that $F(U)=0$. We may assume F has the least total degree among such polynomials and algebraic integer coefficients. By the assumption on the minimality of n, for any i and q $(1\leqslant i\leqslant t+1, 1\leqslant q\leqslant s_i)$ there exists an integer l $(0\leqslant l\leqslant L)$

such that $\partial F/\partial y_{iq}^{(l)} \neq 0$. Since the total degree of $\partial F/\partial y_{iq}^{(l)}$ is less than the total degree of F, $\partial F/\partial y_{iq}^{(l)}(U) \neq 0$. By Taylor expansion, we have

$$-F(U_m) = F(U) - F(U_m) = \sum_{|J| \geqslant 1} J!^{-1}\partial^{|J|}F/\partial y^J(U_m)(U-U_m)^J,$$

where $J = (j_{iq}^{(l)})_{1 \leqslant i \leqslant t+1, 1 \leqslant q \leqslant s_i, 0 \leqslant l \leqslant L}$ with $j_{iq}^{(l)}$ being nonnegative integers and $|J|$, $J!$, $\partial^{|J|}/\partial y^J$ and $(U-U_m)^J$ are defined in the usual way. There is a positive number $\theta < 1$ such that $\mathrm{e}_m^L |a_m| |\alpha_{11}|^{\mathrm{e}_m} = \mathrm{O}(\theta^{\mathrm{e}_m})$. (In what follows, the constants implicit in the symbol O and positive constants c_1, c_2, ... depend only on K, $f(z)$, $\alpha_1, \ldots, \alpha_n$ and F.) Then we have

$$\sum_{k=m}^{\infty} P_l(\mathrm{e}_k) a_k \alpha_{iq}^{\mathrm{e}_k} = P_l(\mathrm{e}_m) a_m \alpha_{iq}^{\mathrm{e}_m} + \mathrm{O}(\theta^{\mathrm{e}_{m+1}}).$$

By the fundamental inequality: for any algebraic $\alpha \neq 0$, $\log|\alpha| \geqslant -[\mathbb{Q}(\alpha):\mathbb{Q}]\{\log \overline{|\alpha|} + \log(\text{den } \alpha)\}$, we have $\log|a_m| \geqslant -[K:\mathbb{Q}]\{\log M_m + \log A_m\}$, Therefore by (1), $\theta^{\mathrm{e}_{m+1}} = \mathrm{O}(|a_m| |\alpha_{11}|^{\mathrm{e}_m}\theta^{\mathrm{e}_m})$ and

$$\begin{aligned} -F(U_m) &= \sum_{i=1}^{t} \sum_{l=0}^{L} \sum_{q=1}^{s_i} \partial F/\partial y_{iq}^{(l)}(U_m) P_l(\mathrm{e}_m) a_m \alpha_{iq}^{\mathrm{e}_m} \\ &\quad + \mathrm{O}\big(\mathrm{e}_m^L |a_m| |\alpha_{t+1,1}|^{\mathrm{e}_m}\big) + \mathrm{O}\big(\mathrm{e}_m^L |a_m| |\alpha_{11}|^{\mathrm{e}_m}\theta^{\mathrm{e}_m}\big) \\ &= \mathrm{O}(\theta^{\mathrm{e}_m}). \end{aligned} \tag{2}$$

Let g be the total degree of F and d be a positive integer with $\alpha_i d$ $(1 \leqslant i \leqslant n)$ being algebraic integers. Then $\overline{|F(U_m)|} = \mathrm{O}((A_{m-1}c_1^{\mathrm{e}_{m-1}})^g)$ and $(M_{m-1} d^{\mathrm{e}_{m-1}})^g F(U_m)$ is an algebraic integer. Hence by (1), (2) and the fundamental inequality, we have $F(U_m) = 0$ for sufficiently large m. By (2),

$$\sum_{i=1}^{t} \sum_{l=0}^{L} P_l(\mathrm{e}_m) \sum_{q=1}^{s_i} \partial F/\partial y_{iq}^{(l)}(U_m) \alpha_{iq}^{\mathrm{e}_m} = \mathrm{O}(A^{\mathrm{e}_m}), \tag{3}$$

where A is a number with $\max(|\alpha_{t+1,1}|, |\alpha_{11}|\theta) < A < |\alpha_{11}|$. For each i, $l (1 \leqslant i \leqslant t,\ 0 \leqslant l \leqslant L)$, let $Q_i^{(l)}$ be a maximal subset of $\{1, 2, \ldots, s_i\}$ such that $\partial F/\partial y_{iq}^{(l)}$ $(q \in Q_i^{(l)})$ are linearly independent over the algebraic numbers. Then we have

$$\sum_{q=1}^{s_i} \partial F/\partial y_{iq}^{(l)}(U_m)\alpha_{iq}^{\mathrm{e}_m} = \sum_{q \in Q_i^{(l)}} \partial F/\partial y_{iq}^{(l)}(U_m) \sum_{p=1}^{s_i} \mathrm{d}_{iqp}^{(l)} \alpha_{ip}^{\mathrm{e}_m}, \tag{4}$$

where $\overline{\mathbb{Q}}^{s_i} \ni (\mathrm{d}_{iq1}^{(l)}, \ldots, \mathrm{d}_{iqs_i}^{(l)}) \neq (0, \ldots, 0)$. By assumption, for any i there exists l $(0 \leqslant l \leqslant L)$ with $Q_i^{(l)}$ being non-empty. Let $\alpha_{iq} = \zeta_{iq}\gamma_i$ $(1 \leqslant q \leqslant s_i)$, $\zeta_{iq}^N = 1$

$(1 \leqslant i \leqslant t,\ 1 \leqslant q \leqslant s_i)$, and take a triplet (i, l, q) with $1 \leqslant i \leqslant t$, $0 \leqslant l \leqslant L$ and $q \in Q_i^{(l)}$. We assert that $\sum_{p=1}^{s_i} \mathrm{d}_{iqp}^{(l)} \zeta_{ip}^{\mathrm{e}_k} = 0$ for any sufficiently large k. This implies the property ii). On the contrary, suppose that there is an integer a $(0 \leqslant a < N)$ such that $\mathrm{e}_m \equiv a (\mathrm{mod}\ N)$ for infinitely many m and $\sum_{p=1}^{s_i} \mathrm{d}_{iqp}^{(l)} \zeta_{ip}^{a} \neq 0$. Define

$$D_{iq}^{(l)} = \sum_{p=1}^{s_i} \mathrm{d}_{iqp}^{(l)} \zeta_{ip}^{a}$$

for each i, l, q with $1 \leqslant i \leqslant t$, $0 \leqslant l \leqslant L$, $q \in Q_i^{(l)}$, and

$$B = \left\{ (i, l, q) \mid D_{iq}^{(l)} \neq 0 \right\}.$$

Then B is not empty. Let D be a positive integer with $DD_{iq}^{(l)}$ $((i, l, q) \in B)$ being algebraic integers, and define

$$E_{iq}^{(l)}(m) = DM_{m-1}^{g} P_l(\mathrm{e}_m)\ \partial F / \partial y_{iq}^{(l)}(U_m) D_{iq}^{(l)}$$

for each $(i, l, q) \in B$. Since $\lim_{m \to \infty} U_m = U$, there is a positive constant M such that $E_{iq}^{(l)}(m) \neq 0$ for any $(i, l, q) \in B$ and any $m > M$. By (3) and (4), if $\mathrm{e}_m \equiv a (\mathrm{mod}\ N)$, then

$$\sum_{(i, l, q) \in B} E_{iq}^{(l)}(m) \gamma_i^{\mathrm{e}_m} = \mathrm{O}(A^{\mathrm{e}_m} M_{m-1}^{g}). \tag{5}$$

We may assume $\alpha_1, \ldots, \alpha_n$, $\gamma_1, \ldots, \gamma_t$, $D_{iq}^{(l)}$ and the coefficients of F are in K, by extending K if necessary.

Before proceeding, we must here explain Evertse's theorem, which will play an important role to prove our theorem. By a prime on K we mean an equivalence class of non-trivial valuations on K. We denote by S_K the set of all primes on K by S_∞ the set of all infinite primes on K. For every prime v on K lying above a prime p on $\mathbb{Q}$, we choose a valuation $\| \cdot \|_v$ such that

$$\| \alpha \|_v = | \alpha |_p^{[K_v : \mathbb{Q}_p]} \quad \text{for all } \alpha \in \mathbb{Q}.$$

Then we have the product formula:

$$\prod_{v \in S_K} \| \alpha \|_v = 1 \ \text{ for all } \alpha \in K, \quad \alpha \neq 0.$$

For $X = (x_0 : x_1 : \ldots : x_n) \in P^n(K)$, Put

$$H_K(X) = H(X) = \prod_{v \in S_K} \max(\|x_0\|_v, \|x_1\|_v, \ldots, \|x_n\|_v).$$

By the product formula, this height is well-defined. Put

$$h_K(\alpha) = h(\alpha) = H(1:\alpha) \quad \text{for } \alpha \in K.$$

Then we have the following fundamental inequality:

$$-\log h(\alpha) \leqslant \sum_{v \in S} \log \|\alpha\|_v \leqslant \log h(\alpha) \quad \text{for } \alpha \in K, \quad \alpha \neq 0,$$

where S is any set of primes on K. Let S be a finite set of primes on K, enclosing S_∞, and c, d be constants with $c > 0$, $d \geqslant 0$. A projective point $X \in P^n(K)$ is called (c, d, S)-admissible if its homogeneous coodinates $x_0, x_1, \ldots, x_n$ can be chosen such that
i) all x_k are S-integers, i.e. $\|x_k\|_v \leqslant 1$ if $v \notin S$, and
ii) $\prod_{v \in S} \prod_{k=0}^{n} \|x_k\|_v \leqslant cH(x)^d$.
The following theorem is due to [Evertse, 1984]: let c, d be constants with $c > 0$, $0 \leqslant d < 1$ and let n be a positive integer. Then there are only finitely many (c, d, S)-admissible projective points $X = (x_0 : x_1 : \ldots : x_n) \in P^n(K)$ satisfying

$$x_0 + x_1 + \cdots + x_n = 0$$

but

$$x_{i_0} + x_{i_1} + \cdots + x_{i_s} \neq 0$$

for each proper, non-empty subset $\{i_0, i_1, \ldots, i_s\}$ of $\{0, 1, \ldots, n\}$.

Let S be a finite set of primes on K which includes S_∞ and all prime divisors of $\alpha_1, \ldots, \alpha_n$. Then $E_{iq}^{(l)}(m)\gamma_i^{e_m}((i, l, q) \in B,\ m > M)$ are nonzero S-integers and

$$h\left(E_{iq}^{(l)}(m)\right) \leqslant c_2^{\log e_m + e_{m-1} + \log M_{m-1} + \log A_{m-1}}. \tag{6}$$

PROPOSITION 1. *Let* $(i_j, l_j, q_j) \in B (j = 1, 2)$, $i_1 \neq i_2$ *and* $m_1 > m_2 > M$. *If* m_1 *is sufficiently large*, *then*

$$\left(E_{i_1 q_1}^{(l_1)}(m_1)\gamma_{i_1}^{e_{m_1}} : E_{i_2 q_2}^{(l_2)}(m_1)\gamma_{i_2}^{e_{m_1}}\right) \neq \left(E_{i_1 q_1}^{(l_1)}(m_2)\gamma_{i_1}^{e_{m_2}} : E_{i_2 q_2}^{(l_2)}(m_2)\gamma_{i_2}^{e_{m_2}}\right).$$

Proof. Suppose the proposition is not true. Then

$$\left(\gamma_{i_1}/\gamma_{i_2}\right)^{e_{m_1} - e_{m_2}} = E_{i_2 q_2}^{(l_2)}(m_1) E_{i_1 q_1}^{(l_1)}(m_2) E_{i_1 q_1}^{(l_1)}(m_1)^{-1} E_{i_2 q_2}^{(l_2)}(m_2)^{-1}$$

hence

$$h\left(\gamma_{i_1}/\gamma_{i_2}\right)^{e_{m_1}-e_{m_2}} \leqslant c_2^{4(\log e_{m_1}+e_{m_1-1}+\log M_{m_1-1}+\log A_{m_1-1})}$$

for infinitely many m_1. Since $\gamma_{i_1}/\gamma_{i_2}$ is not a root of unity and so $h(\gamma_{i_1}/\gamma_{i_2}) > 1$, this contradicts the equality (1).

PROPOSITION 2. *Let B_0 be any non-empty subset of B. If m is sufficiently large, then*

$$\sum_{(i,\,l,\,q)\in B_0} E_{iq}^{(l)}(m)\gamma_i^{e_m} \neq 0.$$

Proof. We prove the proposition by induction on the cardinal number $|B_0|$ of B_0. If $|B_0| = 1$, then the proposition is true. Let $|B_0| \geqslant 2$ and suppose that

$$\sum_{(i,\,l,\,q)\in B_0} E_{iq}^{(l)}(m)\gamma_i^{e_m} = 0 \tag{7}$$

for infinitely many m. First we consider the case where i_0 is the only integer such that $(i_0, l, q) \in B_0$ for some q and l. Define

$$l_0 = \max\{l \mid (i_0, l, q) \in B_0 \text{ for some } q\}.$$

Then

$$P_{l_0}(e_m) \sum_{(i_0,\,l_0,\,q)\in B_0} \partial F/\partial y_{i_0 q}^{(l_0)}(U_m) D_{i_0 q}^{(l_0)}$$

$$= -\sum_{\substack{(i_0,\,l,\,q)\in B_0\\ 0\leqslant l<l_0}} P_l(e_m)\, \partial F/\partial y_{i_0 q}^{(l)}(U_m) D_{i_0 q}^{(l)}$$

for infinitely many m. When we divide both sides by $P_{l_0}(e_m)$ and let m tend to infinitely, we have

$$\sum_{(i_0,\,l_0,\,q)\in B_0} \partial F/\partial y_{i_0 q}^{(l_0)}(U) D_{i_0 q}^{(l_0)} = 0.$$

Since the total degrees of $\partial F/\partial y_{i_0 q}^{(l_0)}(q \in Q_{i_0}^{(l_0)})$ are less than the total degree of F,

$$\sum_{(i_0,\,l_0,\,q)\in B_0} \partial F/\partial y_{i_0 q}^{(l_0)} D_{i_0 q}^{(l_0)} = 0.$$

This contradicts that $\partial F/\partial y_{i_0 q}^{(l_0)}(q \in Q_{i_0}^{(l_0)})$ are linearly independent over the algebraic numbers. Next we suppose there exist two triplets $(i_j, l_j, q_j) \in B_0$ $(j=1, 2)$ with $i_1 \neq i_2$. Let ϵ be any positive number <1. By Proposition 1 and the induction hypothesis on $|B_0|$, applying Evertse's theorem to the equality (7) as $c=1$ and $d=1-\epsilon$, we have

$$\prod_{v \in S} \prod_{(i, l, q) \in B_0} \| E_{iq}^{(l)}(m) \gamma_i^{e_m} \|_v > H\big(\dots : E_{iq}^{(l)}(m) \gamma_i^{e_m} : \dots \big)^{1-\epsilon}$$

for infinitely many m. By the facts that $\prod_{v \in S} \| \gamma_i \|_v = 1$ and there exists a prime v on K such that $\| \gamma_{i_1}/\gamma_{i_2} \|_v > 1$, we have

$$c_3^{\log e_m + e_{m-1} + \log M_{m-1} + \log A_{m-1}} > \| E_{i_1 q_1}^{(l_1)}(m) / E_{i_2 q_2}^{(l_2)}(m) \|_v^{1-\epsilon} \| \gamma_{i_1}/\gamma_{i_2} \|_v^{e_m(1-\epsilon)}.$$

This is a contradiction.

Now we can complete the proof of Theorem 1. For infinitely many m with $e_m \equiv a (\mathrm{mod}\ N)$, we have, by (5),

$$\sum_{(i, l, q) \in B} E_{iq}^{(l)}(m) \gamma_i^{e_m} + \delta_m = 0, \quad \delta_m = \mathrm{O}\big(A^{e_m} M_{m-1}^{g} \big) \tag{8}$$

where $E_{iq}^{(l)}(m) \gamma_i^{e_m}$ and δ_m are S-integers. First we consider the case where i_0 is the only integer such that $(i_0, l, q) \in B$ for some l and q. Define

$$l_0 = \max\{ l \mid (i_0, l, q) \in B \text{ for some } q \}.$$

Then

$$P_{l_0}(e_m) \sum_{(i_0, l_0, q) \in B} \partial F/\partial y_{i_0 q}^{(l_0)}(U_m) D_{i_0 q}^{(l_0)} \gamma_{i_0}^{e_m}$$

$$= - \sum_{\substack{(i_0, l, q) \in B \\ 0 \leq l < l_0}} P_l(e_m)\ \partial F/\partial y_{i_0 q}^{(l)}(U_m) D_{i_0 q}^{(l)} \gamma_{i_0}^{e_m} + \mathrm{O}(A^{e_m}).$$

When we divide the both sides by $P_{l_0}(e_m) \gamma_{i_0}^{e_m}$ and make m tend to infinity, we have

$$\sum_{(i_0, l_0, q) \in B} \partial F/\partial y_{i_0 q}^{(l_0)}(U) D_{i_0 q}^{(l_0)} = 0,$$

since $A < |\alpha_{11}| = |\gamma_{i_0}|$. This is a contradiction. Next we suppose that there are two triplets $(i_j, l_j, q_j) \in B$ $(j=1, 2)$ with $i_1 \neq i_2$. Let ϵ be any positive number <1. We may assume K is not a real field and $|\cdot|^2 = \|\cdot\|_{v_0}$ for some infinite prime v_0 on K. By Proposition 1 and Proposition 2, applying the

Evertse's theorem to the equality (8), we have

$$\left(\prod_{v\in S}\prod_{(i,l,q)\in B}\| E_{iq}^{(l)}(m)\|_v\right)\prod_{v\in S}\|\delta_m\|_v$$

$$> H\left(\ldots : E_{iq}^{(l)}(m)\gamma_i^{e_m} : \ldots : \delta_m\right)^{1-\epsilon} \tag{9}$$

for sufficiently large m with $e_m \equiv a(\text{mod } N)$. By (6) and (8), the left hand side of the equality (9) is not greater than

$$c_4^{\log e_m + e_{m-1} + \log M_{m-1} + \log A_{m-1}}\left(\prod_{\substack{v\in S\\ v\neq v_0}}\max_{(i,l,q)\in B}\|\gamma_i\|_v\right)^{e_m} A^{2e_m}.$$

On the other hand, taking a triplet $(i_0, l_0, q_0)\in B$, we have

$$H\left(\ldots : E_{iq}^{(l)}(m)\gamma_i^{e_m} : \ldots : \delta_m\right)$$

$$\geqslant H\left(\ldots : E_{iq}^{(l)}(m)\gamma_i^{e_m} : \ldots\right)$$

$$= H\left(\ldots : E_{iq}^{(l)}(m)E_{i_0q_0}^{(l_0)}(m)^{-1}(\gamma_i/\gamma_{i_0})^{e_m} : \ldots\right)$$

$$\geqslant \prod_{v\in S}\max_{(i,l,q)\in B}\| E_{iq}^{(l)}(m)E_{i_0q_0}^{(l_0)}(m)^{-1}\|_v\|\gamma_i/\gamma_{i_0}\|_v^{e_m}$$

$$\geqslant c_5^{\log e_m + e_{m-1} + \log M_{m-1} + \log A_{m-1}}\left(\prod_{v\in S}\max_{(i,l,q)\in B}\|\gamma_i/\gamma_{i_0}\|_v\right)^{e_m}.$$

Since

$\prod_{v\in S}\max_{(i,l,q)\in B}\|\gamma_i/\gamma_{i_0}\|_v = \prod_{v\in S_K}\max_{(i,l,q)\in B}\|\gamma_i/\gamma_{i_0}\|_v = \prod_{v\in S_K}\max_{(i,l,q)\in B}\|\gamma_i\|_v =$ $\prod_{v\in S}\max_{(i,l,q)\in B}\|\gamma_i\|_v$, the right hand side of the equality (9) is not less than

$$c_5^{\log e_m + e_{m-1} + \log M_{m-1} + \log A_{m-1}}\left(\prod_{v\in S}\max_{(i,l,q)\in B}\|\gamma_i\|_v\right)^{(1-\epsilon)e_m}.$$

Then we have

$$c_6^{\log e_m + e_{m-1} + \log M_{m-1} + \log A_{m-1}} A^{2e_m}$$

$$\geqslant |\alpha_{11}|^{2(1-\epsilon)e_m}\left(\prod_{\substack{v\in S\\ v\neq v_0}} \max_{(i,l,q)\in B} \|\gamma_i\|_v\right)^{-\epsilon e_m},$$

for infinitely many m. As m tends to infinity, we have

$$2\log A \geqslant 2(1-\epsilon)\log|\alpha_{11}| - \epsilon\log\left(\prod_{\substack{v\in S\\ v\neq v_0}} \max_{(i,l,q)\in B} \|\gamma_i\|_v\right).$$

Since ϵ is any positive number <1, this implies $\log A \geqslant \log|\alpha_{11}|$. This contradicts the fact $A < |\alpha_{11}|$. This completes the proof of Theorem 1.

In [Nishioka, to appear in J. Number Theory], the p-adic analogue of Example 1 is proved. Here we have the following theorem. Let p be a prime number and denote by C_p the p-adic completion of $\overline{\mathbf{Q}}_p$ with respect to the valuation $|\cdot|_p$. Let the convergence radius R_p of $f(z)$ in C_p be positive. For an algebraic number α with $0 < |\alpha|_p < R_p$ we denote by $f(\alpha)_p$ the value of $f(z)$ at $z = \alpha$ in C_p.

THEOREM 2. *Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers with $0 < |\alpha_i|_p < R_p (1 \leqslant i \leqslant n)$. Then the following three properties are equivalent.*

i) $f(\alpha_1)_p, \ldots, f(\alpha_n)_p$ are algebraically dependent over the rationals.

ii) There is a non-empty subset $\{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$ of $\{\alpha_1, \ldots, \alpha_n\}$ such that $\alpha_{i_1}, \ldots, \alpha_{i_s}$ are $\{e_k\}$-dependent.

iii) $1, f(\alpha_1)_p, \ldots, f(\alpha_n)_p$ are linearly dependent over the algebraic numbers.

Theorem 2 is proved in the same way as Theorem 1, but ignoring the derivatives of $f(z)$ (i.e. $L = 0$).

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