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p-adic L-functions for modular forms

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We construct 'many variabled' $\mathcal{S}$-adic $\boldsymbol{L}$-functions for modular forms over arbitrary number field $\boldsymbol{k}$. We take for our form a weight 2 Hecke eigenform (on $\boldsymbol{GL}(2)$, of level $\Gamma_0(\mathfrak{a})$) and for simplicity assume it is cuspidal at infinity. $\mathcal{S}$ is a finite set of primes away from the level of our form, and (if we want boundedness) is such that for $\mathfrak{p} \in \mathcal{S}$ we can choose a root $\rho_{\mathfrak{p}}$ of the $\mathfrak{p}$'th Euler polynomial that is a $\mathfrak{p}$-unit. The $\mathcal{S}$-adic $\boldsymbol{L}$-function is given by a measure on the Galois group of the maximal unramified-outside-$\mathcal{S}$ abelian extension of $\boldsymbol{k}$; the measure obtained by playing the modular symbol game in an adelic setting. We prove that the $\mathcal{S}$-adic $\boldsymbol{L}$-function interpolates the critical values of the classical zeta function of the twists of our form by finite characters of conductor supported at $\mathcal{S}$, and that it satisfies a similar functional equation. The gist of the p-adic continuation is the proof that a certain module in which our distribution takes its values is finitely generated, and the idea is to give this module a geometric interpretation as periods of a harmonic form against certain cycles. From our modular form we get an $\mathrm{r}_1 + \mathrm{r}_2$ harmonic form on the $2\mathrm{r}_1 + 3\mathrm{r}_2$ dimensional symmetric space

$$\mathbb{X} = \boldsymbol{GL}(2;\ \boldsymbol{k}) \backslash \boldsymbol{GL}(2;\ \boldsymbol{k}_{\mathbf{A}}) / \mathcal{K}_{\mathbf{A}} \cdot \mathcal{Z}_\infty$$

where r_1 (resp. r_2) is the number of real (resp. complex) primes of $\boldsymbol{k}$; $\mathcal{K}_{\mathbf{A}}$ the level groups, $\mathcal{Z}_\infty$ the center at infinity. It turns out that one needs to work with an associated $[\boldsymbol{k};\ \mathbf{Q}] = \mathrm{r}_1 + 2\mathrm{r}_2$ form on the $2 \cdot [\boldsymbol{k}\colon \mathbf{Q}]$ dimensional symmetric space

$$\mathbb{X}^{\mathrm{sgn}} = \boldsymbol{GL}(2;\ \boldsymbol{k}) \backslash \boldsymbol{GL}(2;\ \boldsymbol{k}_{\mathbf{A}}) / \mathcal{K}_{\mathbf{A}} \cdot \mathcal{Z}_\infty^+$$

where $\mathcal{Z}_\infty^+$ consist of the real and totally positive elements in $\mathcal{Z}_\infty$; only in $\mathbb{X}^{\mathrm{sgn}}$ can one define the appropriate cycles for a field $\boldsymbol{k}$ which is not totally real or CM. See [Mazur and Swinnerton-Dyer, 1974] for the origin of all this, where the case $\boldsymbol{k} = \mathbf{Q}$ is treated; [Manin, 1976] for totally real $\boldsymbol{k}$; [Kurcanov, 1980] and [Haran, 1983] for CM fields. In order to keep everything in half their size we assume all the places at infinity of $\boldsymbol{k}$ are complex (the necessary adjustments needed for a field $\boldsymbol{k}$ having both real and complex places are indicated at the end of the paper).

§1

In this section we recall the adelic definition of a modular form and fix our notations following mainly those of [Weil, 1971].

Let k denote our totally complex number field, $[k; \mathbb{Q}] = 2\mathsf{n}$, so that $\mathsf{n} = \mathsf{r}_2 = \mathsf{r}_1 + \mathsf{r}_2$; $\mathcal{O}$ the integers of k, k_v the completion of k at a place v, $\mathcal{O}_v$ the integers of k_v, $k_{\mathbf{A}} = k_{\text{fin}} \times k_\infty$ the adeles, and fix a character $\psi : k_{\mathbf{A}}/k \to \mathbb{C}^*$, $\psi = \bigotimes_v \psi_v$, with $\psi_v(x) = \exp[-2\pi i(x + \bar{x})]$ for $v \mid \infty$. We write $k_\infty^* = k_\infty^{\text{sgn}} \times k_\infty^+$ where $k_\infty^{\text{sgn}} = \prod_{v \mid \infty} k_v^{\text{sgn}}$ is the maximal compact subgroup of the infinite ideles k_∞^*, $k_\infty^+ = \prod_{v \mid \infty} k_v^+$, k_v^+ the positive reals inside k_v^*, and we let $x = \text{sgn}(x)\,|x|$ denote the respective decomposition of $x \in k_\infty^*$. We fix ideles $\mathscr{i}_1 \cdots \mathscr{i}_{\mathsf{h}}$ representing $\mathscr{C}\ell(k)$, the class group of k; ∂ representing the absolute different of k; $\mathscr{a}$ representing the level of our modular form (i.e. the classical $\Gamma_0(\mathscr{a})$); $\mathscr{f}$ representing the conductor of a grossencharacter ω; usually $\mathscr{a}$, $\mathscr{f}$ (and the $\mathscr{i}_i$'s) will be taken relative prime. Let $G = GL(2)/k$ and G_k, G_v, $G_{\mathbf{A}} = G_{\text{fin}} \times G_\infty$ its points with values in k, k_v, $k_{\mathbf{A}}$ respectively; $\mathscr{Z}_k$, $\mathscr{Z}_v$, $\mathscr{Z}_{\mathbf{A}} = \mathscr{Z}_{\text{fin}} \times \mathscr{Z}_\infty$ the centers of the above groups, $\mathscr{Z}_\infty^+$ the real and totally positive elements of $\mathscr{Z}_\infty$; $\mathscr{B} = \{(x, y) \overset{\text{def}}{=} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\} = \mathbb{G}_m \ltimes \mathbb{G}_a$ and $\mathscr{B}_k$, $\mathscr{B}_v$, $\mathscr{B}_{\mathbf{A}} = \mathscr{B}_{\text{fin}} \times \mathscr{B}_\infty$ its rational points, $\mathscr{B}_\infty^+ = \{(x, y) \in \mathscr{B}_\infty$ with $x \in k_\infty^+\}$. We define our level groups by $\mathscr{K}_v = SU(2; k_v)$ for $v \mid \infty$, for $v \nmid \infty$ we set

$$\mathscr{K}_v = \left\{ \begin{pmatrix} x & \partial^{-1} y \\ \mathscr{a}\,\partial z & w \end{pmatrix},\ x, y, z, w \in \mathcal{O}_v,\ \det \in \mathcal{O}_v^* \right\}$$

and we write $\mathscr{K}_{\mathbf{A}} = \mathscr{K}_{\text{fin}} \times \mathscr{K}_\infty$ for the associated adelic group. Let $\mathscr{V} = \bigotimes_{v \mid \infty} \mathscr{V}_v$ the value space of our form, where $\mathscr{V}_v$ is a 3-dimensional complex vector space with basis V_v^1, V_v^0, V_v^{-1}, so $\mathscr{V}$ has basis V^e, $e = \{e_v\}$, $e_v \in \{1, 0, -1\}$. We let $\mathscr{K}_\infty$ act on the right on $\mathscr{V}$ via the symmetric square representation M:

$$M\begin{pmatrix} c & b \\ -\bar{b} & \bar{c} \end{pmatrix} = \bigotimes_{v \mid \infty} \begin{pmatrix} c_v^2 & 2c_v b_v & b_v^2 \\ -c_v \bar{b}_v & |c_v|^2 - |b_v|^2 & \bar{c}_v b_v \\ \bar{b}_v^2 & -2\bar{c}_v \bar{b}_v & \bar{c}_v^2 \end{pmatrix}$$

and we extend this action to all of $\mathscr{K}_{\mathbf{A}} \mathscr{Z}_{\mathbf{A}}$ by trivial $\mathscr{K}_{\text{fin}} \mathscr{Z}_{\mathbf{A}}$ action. We define $W : k_\infty^* \to \mathscr{V}$, $W(x) = \bigotimes_{v \mid \infty} W_v(x_v)$, $W_v(x_v) = \sum_{j = 1, 0, -1} W_{v,j}(x_v) \cdot V_v^j$, with $W_{v,0}(x) = |x|^2 K_0(4\pi |x|)$, $W_{v,\pm 1}(x) = \dfrac{1}{2}\left[\dfrac{1}{i}\text{sgn}(x)\right]^{\pm 1} \cdot |x|^2 K_1(4\pi |x|)$, where K_0, K_1 are Hankel's functions [Magnus and Oberhettinger, 1954].

Let $F\colon G_{\mathbf{A}} = \mathscr{B}_{\mathbf{A}}\mathscr{Z}_{\mathbf{A}}\mathscr{K}_{\mathbf{A}} \to \mathscr{V}$ denote our modular form, so $F(g\kappa z) = F(g)M(\kappa)$ for $\kappa \in \mathscr{K}_{\mathbf{A}}$, $z \in \mathscr{Z}_{\mathbf{A}}$, and $F\left(g\cdot\begin{pmatrix}0 & -\partial^{-1}\\ \partial\mathscr{a} & 0\end{pmatrix}_{fin}\right) = \epsilon_F \cdot F(g)$, $\epsilon_F = \pm 1$. We assume for simplicity that F is cuspidal at infinity and so has Fourier expansion $F(x, y) = \sum_{\xi\in k^*} C((\xi x))W(\xi x_\infty)\psi(\xi y)$, and we write $L_F(\omega) = \sum_{\mathfrak{b}} C(\mathfrak{b})\cdot\omega(\mathfrak{b})$ for the associated L-function. We assume that F is an eigenform of all the Hecke operators $T_{\mathfrak{p}}$, thus $L_F(\omega)$ has an Euler expansion $L_F(\omega) = \prod_{v\nmid\infty} P_v(\mathbb{N}v^{-1}\cdot\omega(v))^{-1}$ with Euler polynomial $P_v(t) = 1 - \lambda_v t + \mathbb{N}v\cdot t^2 = (1-\rho_v t)\cdot(1-\tilde{\rho}_v t)$ for $v \nmid (\mathscr{a})\infty$. Note that everything is normalized so that the functional equation for finite ω has the form $L_F(\omega) = (-1)^{\mathsf{n}}\cdot\epsilon_F\cdot\omega((\mathscr{a}))\cdot\tau(\omega)^2\cdot L_F(\omega^{-1})$, i.e. the critical value is at "$s = 0$"; here the Gaussian sums are defined by $\tau(\omega) = \prod_{v\nmid\infty}\tau_v(\omega)$, $\tau_v(\omega) = \omega_v(\partial)$ for $v \nmid (\mathscr{f})$, and for

$$v \mid (\mathscr{f})\colon \tau_v(\omega) = |\mathscr{f}|_v^{1/2} \sum_{\eta\in(\mathcal{O}_v/(\mathscr{f}_v))^*} \omega_v^{-1}(\partial^{-1}\mathscr{f}^{-1}\eta)\psi_v(\partial^{-1}\mathscr{f}^{-1}\eta)$$

§2

In this section we define the harmonic form, on the symmetric space $\mathbb{X}$, associated with our modular form, following [Weil, 1971], and introduce the new symmetric space $\mathbb{X}^{\mathrm{sgn}}$.

Let $\mathbb{X} = G_k\backslash G_{\mathbf{A}}/\mathscr{K}_{\mathbf{A}}\cdot\mathscr{Z}_\infty$. Decomposing it into connected components we get $\mathbb{X} = \bigcup_{i=1}^{\mathsf{h}} \mathbb{X}_{\mathfrak{t}_i}$ with $\mathbb{X}_{\mathfrak{t}_i} = \Gamma_{\mathfrak{t}_i}\backslash G_\infty/\mathscr{K}_\infty\mathscr{Z}_\infty$, $\Gamma_{\mathfrak{t}_i} = G_k \cap [(\mathfrak{t}_i, 0)\mathscr{K}_{\mathrm{fin}}(\mathfrak{t}_i^{-1}, 0)\times G_\infty]$. We have coordinates (x, y) on $G_\infty/\mathscr{K}_\infty\mathscr{Z}_\infty$ via the map $\mathscr{B}_\infty^+ \xrightarrow{\sim} G_\infty/\mathscr{K}_\infty\mathscr{Z}_\infty$ and the Riemannian structure is the usual $\mathrm{d}s^2 = \frac{1}{x^2}(\mathrm{d}x^2 + \mathrm{d}y\,\mathrm{d}\bar{y})$, so each $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in G_\infty$ acts as an isometry on $\mathscr{B}_\infty^+$; we denote this action by $\gamma\circ(x, y)$ and define $J(\gamma; (x, y)) = \begin{pmatrix}\mathrm{sgn}(\gamma)\overline{(cy+d)} & \mathrm{sgn}(\gamma)\overline{cx}\\ cx & (cy+d)\end{pmatrix} \in \mathscr{K}_\infty\mathscr{Z}_\infty$ where $\mathrm{sgn}(\gamma) = \mathrm{sgn}(\det(\gamma)) \in K_\infty^{\mathrm{sgn}}$. We have $\gamma\circ(x, y) = \gamma\cdot\begin{pmatrix}x & y\\ 0 & 1\end{pmatrix}\cdot J(\gamma; (x, y))^{-1}$ from which we derive the *automorphy relation*

$$J(\gamma_1\gamma_2; (x, y)) = J(\gamma_1; \gamma_2\circ(x, y))\cdot J(\gamma_2; (x, y)).$$

On $\mathscr{B}_\infty^+$ we define an n-form with values in $\mathscr{V}^*$, the vector space dual to $\mathscr{V}$, by $\beta = \sum_e \beta^e\cdot V_e$, where $\{V_e\}$ is the dual basis of $\{V^e\}$, and $\beta^e = \bigwedge_{v\mid\infty}\beta_v^{e_v}$, $\beta_v^{e_v} = -\frac{\mathrm{d}y_v}{x_v}, \frac{\mathrm{d}x_v}{x_v}, \frac{\mathrm{d}\bar{y}_v}{x_v}$ for $e_v = 1, 0, -1$ respectively.

Claim

$$\beta|_{\gamma}(x, y) = \beta(x, y)^{t}M(J(\gamma; (x, y))), \quad \gamma \in G_{\infty}.$$

Using the automorphy relation and the decomposition $G_{\infty} = \mathscr{B}^{+}_{\infty}\mathscr{K}_{\infty}\mathscr{Z}_{\infty}$ it is sufficient to consider the cases:
i) $\gamma \in \mathscr{Z}_{\infty}$ where $J(\gamma; (x, y)) = \gamma$, $M(J(\gamma; (x, y))) = 1$;
ii) $\gamma \in \mathscr{B}^{+}_{\infty}$ where $J(\gamma; (x, y)) = 1$;
iii) $\gamma \in \mathscr{K}_{\infty}$ and $(x, y) = (0, 1)$ where $J(\gamma; (1, 0)) = \gamma$.
The cases i) and ii) are trivial, and iii) is a straightforward calculation.

Claim

$$F(\gamma \circ (x, y) \cdot (\imath_i, 0)) = F((x, y) \cdot (\imath_i, 0)) \cdot M(J(\gamma; (x, y)))^{-1}, \quad \gamma \in \Gamma_{\imath_i}.$$

Indeed, we have,

$$\begin{aligned} F(\gamma \circ (x, y) \cdot (\imath_i, 0)) &= F\big(\gamma^{-1} \cdot \gamma \circ (x, y) \cdot (\imath_i, 0)\big) \quad \text{by left } G_k\text{-invariance} \\ &= F\big(\gamma_{\infty}^{-1} \cdot \gamma \circ (x, y) \cdot (\imath_i, 0) \cdot \big(\imath_i^{-1}, 0\big) \cdot \gamma_{\text{fin}}^{-1} \cdot (\imath_i, 0)\big) \\ &= F\big(\gamma_{\infty}^{-1} \cdot \gamma \circ (x, y) \cdot (\imath_i, 0)\big) \qquad \text{since } \big(\imath_i^{-1}, 0\big)\gamma_{\text{fin}}^{-1}(\imath_i, 0) \in \mathscr{K}_{\text{fin}} \\ &= F\big((x, y) \cdot J(\gamma; (x, y))^{-1} \cdot (\imath_i, 0)\big) \\ &= F((x, y) \cdot (\imath_i, 0)) \cdot M(J(\gamma; (x, y)))^{-1} \end{aligned}$$

Now let $\Omega_{\imath_i}(x, y) = F(\imath_i x, y) \cdot \beta(x, y)$. Using the above two claims we observe that $\Omega_{\imath_i}$ is $\Gamma_{\imath_i}$- invariant, and can be viewed as a $\mathbb{C}$-valued n-form on $\mathbb{X}_{\imath_i} = \Gamma_{\imath_i} \backslash \mathscr{B}^{+}_{\infty}$. (note that elliptic elements in $\Gamma_{\imath_i}$ give whole geodesics that are singular, and $\mathbb{X}_{\imath_i}$ is not a manifold; strictly speaking we should view $\Omega_{\imath_i}$ as a $\Gamma_{\imath_i}/\Gamma_0$-invariant form on $\Gamma_0 \backslash \mathscr{B}^{+}_{\infty}$, where $\Gamma_0 \subseteq \Gamma_{\imath_i}$ is a subgroup of finite index having no torsion). The properties of Hankel's functions, $xK_0'' + K_0' = xK_0$, $K_1 = -K_0'$, imply that the 1-form

$$\begin{aligned} \sum_{e=1,0,-1} W_{v,e}(x)\psi_v(y)\beta_v^e &= xK_0(4\pi x)\, \mathrm{e}^{-2\pi\mathrm{i}(y+\bar{y})}\, \mathrm{d}x \\ &\quad + \frac{\mathrm{i}}{2} xK_1(4\pi x)\, \mathrm{e}^{-2\pi\mathrm{i}(y+\bar{y})}(\mathrm{d}y + \mathrm{d}\bar{y}) \end{aligned}$$

is closed and $*$-closed. Hence we see that $\Omega_{\mathfrak{t}_i}$ is harmonic, and so we have a cohomology class $[\Omega] \in \mathsf{H}^{\mathsf{n}}(\mathbb{X}, \mathbb{C})$ represented by the n-form Ω with $\Omega \mid \mathbb{X}_{\mathfrak{t}_i} = \Omega_{\mathfrak{t}_i}$.

Letting $\mathscr{H} = \mathscr{B}_\infty^+ \cup \mathbb{P}^1(\boldsymbol{k})$, and taking for neighborhoods of $\eta \in \boldsymbol{k}$ the sets $\{\eta\} \cup \left\{(x, y) \mid \prod_{v \mid \infty} \frac{1}{x_v}\left(|\eta - y|_v^2 + |x|_v^2\right) < r\right\}$ and for ∞ the sets $\{\infty\} \cup \left\{(x, y) \mid \prod_{v \mid \infty} \frac{1}{x_v} < r\right\}$, for all $r > 0$, we see that $\boldsymbol{G}_{\boldsymbol{k}}$ acts continuously on the Hausdorff space $\mathscr{H}$, and we get the compactification $\overline{\mathbb{X}} = \bigcup_{i=1}^{\mathsf{h}} \overline{\mathbb{X}}_{\mathfrak{t}_i}$ of $\mathbb{X}$, where $\overline{\mathbb{X}}_{\mathfrak{t}_i} = \Gamma_{\mathfrak{t}_i} \backslash \mathscr{H}$.

Let $\mathbb{X}^{\mathrm{sgn}} = \boldsymbol{G}_{\mathbf{k}} \backslash \boldsymbol{G}_{\mathbb{A}} / \mathscr{K}_{\mathbb{A}} \cdot \mathscr{Z}_\infty^+ = \bigcup_{i=1}^{\mathsf{h}} \Gamma_{\mathfrak{t}_i} \backslash \mathscr{B}_\infty$, where similarly to above we put coordinates via $\mathscr{B}_\infty \xrightarrow{\sim} \boldsymbol{G}_\infty / \mathscr{K}_\infty \mathscr{Z}_\infty^+$, and note that the canonical projection $\boldsymbol{G}_\infty / \mathscr{K}_\infty \mathscr{Z}_\infty^+ \to \boldsymbol{G}_\infty / \mathscr{K}_\infty \mathscr{Z}_\infty$ is given by $\mathscr{B}_\infty \to \mathscr{B}_\infty^+$, $(x, y) \mapsto (|x|, y)$. On $\mathscr{B}_\infty$ we define an $\mathbb{R}$-valued n-form $\Theta = \bigwedge_{v \mid \infty} \Theta_v$, by $\Theta_v(x, y) = \frac{1}{2\pi i} \mathrm{d} \log(\operatorname{sgn}(x_v))$; this is $\boldsymbol{G}_{\boldsymbol{k}}$-invariant since $\operatorname{sgn}(\gamma \circ (x, y)) = \operatorname{sgn}(\gamma) \cdot \operatorname{sgn}(x)$, and so we have a closed n-form Θ on $\mathbb{X}^{\mathrm{sgn}}$. We denote by $\overline{\mathbb{X}}^{\mathrm{sgn}} = \bigcup_{i=1}^{\mathsf{h}} \Gamma_{\mathfrak{t}_i} \backslash \mathscr{B}_\infty \cup \mathbb{P}^1(\boldsymbol{k})$ the obvious compactification of $\mathbb{X}^{\mathrm{sgn}}$ induced by the Seifert-fibration $\mathbb{X}^{\mathrm{sgn}} \to \mathbb{X}$ (this becomes an actual fibration after passage to subgroups $\Gamma_0 \subseteq \Gamma_{\mathfrak{t}_i}$ having no torsion).

Fixing an infinite place $v \mid \infty$, one can look at the action of $\boldsymbol{G}_v$ on $\mathscr{B}_v$ in the following way. Denote by $\boldsymbol{j}$ the element of the quaternions $\mathbb{H}$, and identify $\mathscr{B}_v$ with $\mathbb{H} \backslash \mathbb{C}$ via $(x, y) \to z = x + yj$. The action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \boldsymbol{G}_v$ on $\mathbb{H} \backslash \mathbb{C}$ becomes the Möbius action $\gamma \circ z = (az + b) \cdot (cz + d)^{-1}$.

§3

In this section we study the periods $\boldsymbol{L}(\mathfrak{t}, \eta)$; these are first introduced as an adelic integral, then after Lemma 1, we transform it to an archimeadian integral, and finally after Lemma 2, we show it is given by an integral of our harmonic form pulled back to $\mathbb{X}^{\mathrm{sgn}}$ against a relative cycle "going from" the cusp at infinity to the cusp '$(\mathfrak{t}, \eta)$'. Besides giving us a geometrical intuition, we can deduce from this interpretation the crucial result that the module generated by these periods is finitely generated.

For $\mathfrak{t} \in \boldsymbol{k}_{\mathbb{A}}^*$, $\eta \in \boldsymbol{k}_{\mathrm{fin}}$, such that $|\eta|_v < |\mathfrak{t}|_v$ for $v \mid (\mathfrak{a})$, we define the '*periods*':

$$\boldsymbol{L}(\mathfrak{t}, \eta) = \frac{1}{[\mathscr{O}^* : \mathscr{E}]} \int_{k_\infty^* \prod_{v \nmid \infty} \mathscr{O}_v^* / \mathscr{E}} \boldsymbol{F}_0(\partial \mathfrak{t} x, -\eta) \, \mathrm{d}^* x$$

where $F_0 : G_{\mathbf{A}} \to \mathbb{C}$ is the $\bigotimes_{v|\infty} V_v^0$-component of F, $\mathscr{E}$ the subgroup of $e \in \mathscr{O}^*$ satisfying $e \equiv 1 \bmod(\imath_v/\eta_v)$, (which holds trivially when $|\eta|_v \leqslant |\imath|_v$, i.e. for almost all v's), and the Haar measure $\mathrm{d}^*x = \bigotimes_v \mathrm{d}^*x_v$ being normalized by $\int_{\mathscr{O}_v^*} \mathrm{d}^*x_v = 1$ for $v \nmid \infty$, and $\mathrm{d}^*x_v = \dfrac{\mathrm{d}\,\mathrm{sgn}(x_v) \wedge \mathrm{d}\,|x|_v}{2\pi \mathrm{i} \cdot x_v}$ for $v \mid \infty$.

LEMMA 1:

0. $L(\imath, \eta)$ *is well defined.*
1. $L(\imath, \eta)$ *depends only on the ideal* $((\imath)) \overset{\text{def}}{=} k \cap (\imath)$, $(\imath) \overset{\text{def}}{=} \prod_{v \nmid \infty} \imath_v \mathscr{O}_v$.
2. $L(\imath, \eta)$ *depends only on the image* $\eta \in k_{\text{fin}}/(\imath)$.
3. $L(\imath, \eta) = L(\imath\xi, \eta\xi)$ *for* $\xi \in k^*$.
4. $L(\imath, \eta) = (-1)^{\mathrm{n}} \epsilon_F L(a\imath_{\mathscr{S}}^2 \imath^{-1}, -\eta^{-1})$ *for* $\eta_v = 0$, $v \notin \mathscr{S}$; *and* $\eta_v \in \mathscr{O}_v^*$, $|\imath|_v < 1$, $v \in \mathscr{S}$.

Proof: 1 is clear; 2 follows since by right $\mathscr{K}_{\text{fin}}$-invariance, for $\mu \in (\imath)$, $F(\partial \imath x, -\eta) = F((\partial \imath x, -\eta)\ (1, -\partial^{-1}\imath^{-1}x\mu)) = F(\partial \imath x, -\eta - \mu)$; 3 follows since by left G_k-invariance, for $\xi \in K^*$, $F(\partial \imath x, -\eta) = F((\xi, 0)(\partial \imath x, -\eta)) = F(\partial \imath \xi x, -\eta\xi)$. As for 0, using 1, 2, 3 it's easily seen that the integrand in the definition of $L(\imath, \eta)$ is $\mathscr{E}$-invariant so integration mod $\mathscr{E}$ is o.k. if it converges. For convergence, we first use 2 and assume $\eta_v = 0$ for v outside a finite set of places $\mathscr{S}$, then using 3 we can assume $\eta_v \in \mathscr{O}_v^*$ and $|\imath|_v < 1$ for $v \in \mathscr{S}$, now

$$(\partial \imath x, -\eta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \partial a \eta^{-1} & -a\imath_{\mathscr{S}}^2 \imath^{-1} x^{-1} \\ \partial a & 0 \end{pmatrix} \cdot$$

$$\begin{pmatrix} \imath & -\partial^{-1} x^{-1} \eta \\ \partial x \eta^{-1} & 0 \end{pmatrix}_{\mathscr{S}} \cdot a^{-1} \imath_{\mathscr{S}}^{-1} \imath x$$

(where $\imath_{\mathscr{S}} = \imath_v$ (resp. 1), $\eta^{-1} = \eta_v^{-1}$ (resp. 0) for $v \in \mathscr{S}$ (resp. $v \notin \mathscr{S}$)) and so

$$\begin{aligned} F_0(\partial \imath x, -\eta) &= F_0 \begin{pmatrix} \partial a \eta^{-1} & -a\imath_{\mathscr{S}}^2 \imath^{-1} x^{-1} \\ \partial a & 0 \end{pmatrix} \\ &= F_0 \left(\begin{pmatrix} \partial a \imath_{\mathscr{S}}^2 \imath^{-1} x^{-1} & \eta^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\partial^{-1} \\ \partial a & 0 \end{pmatrix} \right) \\ &= (-1)^{\mathrm{n}} \epsilon_F \cdot F_0 \left(\partial a \imath_{\mathscr{S}}^2 \imath^{-1} x^{-1}, \eta^{-1} \right). \end{aligned}$$

By using the fact that F is cuspidal at infinity and trivial estimates on Hankel's K_0, we get $|F_0(\partial \imath x, -\eta)| = O(|x|^\sigma)$ for all $\sigma \in \mathbb{R}$ as $|x| \to \infty$, and from the above formula also when $|x| \to 0$; this proves convergence (that is, our condition, $|\eta|_v < |\imath|_v$ for $v|(\mathfrak{a})$, imply that the cusp $(\imath, \eta)$ is congruent to the cusp at infinity). Integrating the above formula over $k_\infty^* \prod_{v \nmid \infty} \mathcal{O}_v^*/\mathscr{E}$ we obtain 4.

Note that by part 3 of the lemma we can translate any $L(\imath, \eta)$ into some $L(\partial^{-1}\imath_i, \eta')$, and then using part 2 we can assume $\eta' = \alpha_{\text{fin}}$ for some $\alpha \in k^*$, finally using left G_k-invariance we obtain the *archimedean integral expression*:

$$L(\imath, \eta) = L\left(\partial^{-1}\imath_i, \alpha_{\text{fin}}\right) = \frac{1}{[\mathcal{O}^* : \mathscr{E}]} \int_{k_\infty^*/\mathscr{E}} F_0(\imath_i x, \alpha_\infty) \, \mathrm{d}^* x.$$

We shall now define our relative cycles. Let $\mathfrak{z}(\imath_i, \alpha): k_\infty^* \to \mathbb{X}_{\imath_i}^{\text{sgn}}$, $\mathfrak{z}(\imath_i, \alpha)(x) =$ image of (x, α_∞) in $\mathbb{X}_{\imath_i}^{\text{sgn}}$. Note that for $e \in \mathscr{E}$, $(1-e)\alpha \in (\imath_i \, \partial^{-1})$, hence $(e, (1-e)\alpha) \in \Gamma_{\imath_i}$, and we get $\mathfrak{z}(\imath_i, \alpha)(ex) =$ image of $(e, (1-e)\alpha) \circ (x, \alpha) = \mathfrak{z}(\imath_i, \alpha)(x)$, so we can view $\mathfrak{z}(\imath_i, \alpha)$ as a smooth map $k_\infty^*/\mathscr{E} \to \mathbb{X}_{\imath_i}^{\text{sgn}}$. Moreover, let $\mathbb{D}(\mathscr{E})$ denote the obvious compactification of $k_\infty^*/\mathscr{E} \approx (0, \infty) \times (\mathbb{R}/\mathbb{Z})^{2n-1}$ obtained by adding $0 \times (\mathbb{R}/\mathbb{Z})^{2n-1}$ and $\infty \times (\mathbb{R}/\mathbb{Z})^{2n-1}$, so that $\mathbb{D}(\mathscr{E}) \approx [0, \infty] \times (\mathbb{R}/\mathbb{Z})^{2n-1}$. Setting $\mathfrak{z}(\imath_i, \alpha)[0 \times (\mathbb{R}/\mathbb{Z})^{2n-1}] = \alpha$, $\mathfrak{z}(\imath_i, \alpha)[\infty \times (\mathbb{R}/\mathbb{Z})^{2n-1}] = \infty$, we get a continuous $2n$ relative cycle, $\mathfrak{z}(\imath_i, \alpha): \mathbb{D}(\mathscr{E}) \to \overline{\mathbb{X}}_{\imath_i}^{\text{sgn}}$, with $\partial\mathfrak{z}(\imath_i, \alpha)$ supported on $\{\alpha, \infty\} \subseteq \partial\overline{\mathbb{X}}_{\imath_i}^{\text{sgn}}$.

LEMMA 2: $[\mathcal{O}^* : \mathscr{E}] L(\imath, \eta) = \int_{\mathfrak{z}(\imath_i, \alpha)} \Omega_{\imath_i}^{\text{sgn}} \wedge \Theta$ *where* $\Omega_{\imath_i}^{\text{sgn}}$ *is the pull-back of* $\Omega_{\imath_i}$ *along* $\pi: \mathbb{X}_{\imath_i}^{\text{sgn}} \to \mathbb{X}_{\imath_i}$.

Proof: We have:

$$\int_{\mathfrak{z}(\imath_i, \alpha)} \Omega_{r_i}^{\text{sgn}} \wedge \Theta$$

$$= \int_{k_\infty^*/\mathscr{E}} \left[F(\imath_i |x|, \alpha_\infty) \cdot \left(\mathfrak{z}(\imath_i, \alpha)^* \pi^* \beta\right)\right] \wedge \bigwedge_{v|\infty} \left[\frac{1}{2\pi i} \mathrm{d} \log(\mathrm{sgn}(x_v))\right]$$

but since all the "y-components" of $\mathfrak{z}(\imath_i, \alpha)$ are constant, $y_v = \alpha_v$, the above simplify to

$$\int_{k_\infty^*/\mathscr{E}} F_0(\imath_i x, \alpha_\infty) \bigwedge_{v|\infty} \frac{\mathrm{d}|x|_v}{|x|_v} \wedge \bigwedge_{v|\infty} \frac{\mathrm{d}\, \mathrm{sgn}(x_v)}{2\pi i \cdot \mathrm{sgn}(x_v)} = [\mathcal{O}^* : \mathscr{E}] L(\imath, \eta)$$

by the above archimedean integral expression.

COROLLARY: *The* $\mathbb{Z}$*-module* $\mathscr{L}^0 \subseteq \mathbb{C}$ *generated by all the numbers* $\{[\mathcal{O}^* : \mathscr{E}]\mathscr{L}(\imath, \eta)\}$, $\imath \in k_{\mathbb{A}}^*$, $\eta \in k_{\text{fin}}$, $|\eta|_v < |\imath|_v$ *for* $v|(\mathfrak{a})$, *is finitely generated.*

Proof: The forms $\Omega^{\mathrm{sgn}}_{\imath_i} \wedge \Theta$ are closed and so the integral in Lemma 2 depends only on the homology class of $\mathfrak{z}(\imath_i, \alpha)$ in $\mathsf{H}_{[k:\mathbb{Q}]}(\overline{\mathbb{X}}^{\mathrm{sgn}}, \partial\overline{\mathbb{X}}^{\mathrm{sgn}}; \mathbb{Z})$.

§4

In this section, following Manin's and Kurcanov's generalization [Manin, 1976; Kurcanov, 1980] of the basic idea of [Birch, 1971], we prove 'Birch's Lemma' expressing the critical values of the $\boldsymbol{L}$-functions as linear combinations of our periods.

Let ω denote now a *finite* grossencharacter primitive of conductor $(\not{f})$, and set $\boldsymbol{F}_0^{\omega}(x) = \sum_{\xi \in \boldsymbol{k}^*} \boldsymbol{C}((\xi x))\omega((\xi x)) \cdot \boldsymbol{W}_0(\xi x_\infty)$, where $\boldsymbol{W}_0(x)$ is the $\bigotimes_{v|\infty} V_v^0$-component of $\boldsymbol{W}(x)$. An easy calculation gives

$$\Gamma(s) \cdot \boldsymbol{L}_{\boldsymbol{F}}(\omega_s) = \int_{\boldsymbol{k}^*_{\mathbb{A}}/\boldsymbol{k}^*} \boldsymbol{F}_0^{\omega}(x)\,|x|^{s}_{\mathbb{A}}\; \mathrm{d}^*x$$

where $\omega_s(x) = \omega(x) \cdot |x|^s_{\mathbb{A}}$, $\Gamma(s) = (4\pi)^{-2n}(2\pi)^{-2ns}\Gamma(s+1)^{2n}$, and $\mathrm{Re}(s)$ is large. Decomposing the above integral into ideal classes, we get

$$\Gamma(s)\boldsymbol{L}_{\boldsymbol{F}}(\omega_s) = \sum_{i=1}^{\mathsf{h}} |\imath_i|^s_{\mathbb{A}} \cdot \frac{1}{[\mathcal{O}^* : \mathscr{E}]} \int_{\boldsymbol{k}^*_\infty/\mathscr{E}} \boldsymbol{F}_0^{\omega}(\imath_i\, x_\infty)\,|x|^s_\infty\; \mathrm{d}^*x_\infty.$$

An application of finite Fourier inversion gives for $\xi \in \boldsymbol{k}^*$:

$$\omega((\xi)) = \tau(\omega)\,|\not{f}|^{1/2}_{\mathbb{A}} \cdot \sum_{\eta \in (\mathcal{O}/(\not{f}))^*} \omega(\partial^{-1}\not{f}^{-1}\eta)\,\psi(-\partial^{-1}\not{f}^{-1}\eta\xi)$$

where in any multiplicative context (e.g. in $\omega(\ldots)$) we view η as an idele equal to 1 outside $(\not{f})$, and in any additive content (e.g. in $\psi(\ldots)$) we view η as an adele equal to 0 outside $(\not{f})$. Using this we get:

$$\boldsymbol{F}_0^{\omega}(\imath_i x) = \tau(\omega)\,|\not{f}|^{1/2}_{\mathbb{A}} \cdot \sum_{\eta \in (\mathcal{O}/(\not{f}))^*} \omega(\imath_i\partial^{-1}\not{f}^{-1}\eta)\boldsymbol{F}_0(\imath_i x,\, -\partial^{-1}\not{f}^{-1}\eta)$$

substituting this in the above and evaluating at $s = 0$, we obtain:

$$\boldsymbol{L}_{\boldsymbol{F}}(\omega) = \tau(\omega)\,|\not{f}|^{1/2}_{\mathbb{A}}(4\pi)^{2n} \sum_{i=1}^{\mathsf{h}} \sum_{\eta \in (\mathcal{O}/(\not{f}))^*} \omega(\imath_i\, \partial^{-1}\not{f}^{-1}\eta)\frac{1}{[\mathcal{O}^* : \mathscr{E}]}$$

$$\times \int_{\boldsymbol{k}^*_\infty/\mathscr{E}} \boldsymbol{F}_0(\imath_i x_\infty,\, -\partial^{-1}\not{f}^{-1}\eta)\; \mathrm{d}^*x_\infty.$$

Letting $\xi \in \boldsymbol{k}^*$ be such that $|\xi|_v = |\partial\not{f}|_v^{-1}$ for $v\,|(\not{f})$, putting $\eta\xi\,\partial\not{f}$ for η, and $x_\infty \cdot \xi_\infty$ for x_∞, then using left $\boldsymbol{G}_{\boldsymbol{k}}$-invariance to multiply the argument of

F_0 by $(\xi^{-1}, 0)$, and finally substituting $\imath_i(\xi\partial)^{-1}_{(\mathscr{f})}\,\partial(\xi)$ for $\imath_i$, we have the following:

Lemma Birch [Birch, 1971]: *For finite character ω, primitive of conductor $(\mathscr{f})$,*

$$L_F(\omega) = \tau(\omega)\cdot|\mathscr{f}|_{\mathbf{A}}^{1/2}(4\pi)^{2n}\sum_{i=1}^{h}\sum_{\eta\in(\mathscr{O}/(\mathscr{f}))^*}\omega(\imath_i\eta)L(\imath_i\mathscr{f}, \eta).$$

§5

In this section, following Manin's adelization [Manin, 1976] of [Mazur and Swinnerton-Dyer, 1974], we construct distributions $\mu_\imath$ by specifying its values on open sets to be a certain linear combination of our periods. The additivity of $\mu_\imath$ follows from the Hecke Relations among the periods.

Fix $\mathscr{S}$, a finite set of primes of $\boldsymbol{k}$ away from $(\mathscr{a})\infty$. Denote by $\mathscr{L}^0_{\mathscr{S}}$ the $\mathbb{Z}[\rho_v^{-1};\ v\in\mathscr{S}]$-module generated by $[\mathscr{O}^*:\mathscr{E}_{\mathfrak{d},\eta}]\cdot L(\imath\mathfrak{d}, \eta)$ with $\imath$ prime-to-$\mathscr{S}$, $\mathfrak{d}$ supported-on-$\mathscr{S}$, $\eta\in\boldsymbol{k}_{\mathscr{S}}$, and recall that $\mathscr{E}_{\mathfrak{d},\eta}=\{e\in\mathscr{O}^*\mid |(e-1)\eta|_v\leqslant|\mathfrak{d}|_v$ for $v\in\mathscr{S}\}$, and that ρ_v is *one of the two* roots of the v'th Euler polynomial; we also set $\rho_{\mathfrak{d}}=\prod_{v\in\mathscr{S}}\rho_v^{\mathrm{ord}_v\mathfrak{d}}$. Whenever $\eta\in\boldsymbol{k}_{\mathscr{S}}/(\mathfrak{d})$ is given by the context as $\eta\in\boldsymbol{k}_{\mathscr{S}}/(\mathfrak{b}\mathfrak{d})$, (e.g. when $\mathfrak{b}^{-1}$ is integral), we can define a formal operator $\mathscr{R}_{\mathfrak{b}}L(\imath\mathfrak{d}, \eta)=L(\imath\mathfrak{b}\mathfrak{d}, \eta)$; these are only formal conveniences and whenever we have an expression involving $\mathscr{R}_{\mathfrak{b}}$'s and $L(\imath\mathfrak{d}, \eta)$'s we first apply the $\mathscr{R}_{\mathfrak{b}}$'s and only thereafter can evaluate the periods. We define the operators $\mathscr{U}_{\mathfrak{p}}$ for $\mathfrak{p}\in\mathscr{S}$ by $\mathscr{U}_{\mathfrak{p}}L(\imath, \eta)=\sum_{u\in\mathscr{O}/\mathfrak{p}}L(\imath\mathfrak{p}, \eta+u)$, and extend this to all the $L(\imath\mathfrak{d}, \eta)$'s by using Lemma 1.3.

Lemma Hecke: *When acting on $L(\imath, \eta)$, $\imath$ prime-to-$\mathfrak{p}$, we have the following relations*:

1. $(\rho_{\mathfrak{p}}+\tilde{\rho}_{\mathfrak{p}})=\mathscr{R}_{\mathfrak{p}^{-1}}+\mathscr{U}_{\mathfrak{p}}$
2. $\rho_{\mathfrak{p}}\cdot\tilde{\rho}_{\mathfrak{p}}=\mathscr{R}_{\mathfrak{p}^{-1}}\cdot\mathscr{U}_{\mathfrak{p}}$.

Proof: 2. is clear since $\rho_{\mathfrak{p}}\cdot\tilde{\rho}_{\mathfrak{p}}=\mathbb{N}\mathfrak{p}$ and $\mathscr{R}_{\mathfrak{p}^{-1}}\mathscr{U}_{\mathfrak{p}}$ also equal $\mathbb{N}\mathfrak{p}$ since for all u: $L(\imath, \eta+u)=L(\imath, \eta)$ by Lemma 1.2. For 1 we use the fact that F is a Hecke eigenform with eigenvalue $\rho_{\mathfrak{p}}+\tilde{\rho}_{\mathfrak{p}}$, and the fact that $T_{\mathfrak{p}}=\mathscr{R}_{\mathfrak{p}^{-1}}+\mathscr{U}_{\mathfrak{p}}$ when acting on any $L(\imath, \eta)$ with $\imath$ prime-to-$\mathfrak{p}$.

We define a $\mathbb{Q}\otimes\mathscr{L}^0_{\mathscr{S}}$-valued distribution $\mu_\imath$ on $\mathscr{O}^*_{\mathscr{S}}$ by giving its values on 'elementary sets' as follows. We write $\mathscr{S}=\mathscr{S}_0\cup\mathscr{S}_1$, and denote by $\mathfrak{p}$'s (resp. $\mathfrak{q}$'s) the primes in $\mathscr{S}_1$ (resp. $\mathscr{S}_0$); we let $\mathfrak{f}=\prod\mathfrak{p}^{e_{\mathfrak{p}}}$ with $e_{\mathfrak{p}}>0$, and let $\eta\in\mathscr{O}^*_{\mathscr{S}_1}$ be extended to $\eta\in\mathscr{O}_{\mathscr{S}}$ by decreeing that $\eta_{\mathfrak{q}}=0$; we set $\eta+\mathfrak{f}^*\overset{\text{def}}{=}\mathscr{O}^*_{\mathscr{S}_0}\times\prod_{\mathfrak{p}}(\eta+\mathfrak{p}^{e_{\mathfrak{p}}})\subseteq\mathscr{O}^*_{\mathscr{S}}$. Every open set is a finite union of such elementary open sets.

Definition:

$$\mu_{\imath}(\eta+\mathfrak{f}^{*})=\prod_{\mathfrak{q}}\left(1-\rho_{\mathfrak{q}}^{-1}\mathscr{R}_{\mathfrak{q}}\right)\left(1-\rho_{\mathfrak{q}}^{-1}\mathscr{R}_{\mathfrak{q}^{-1}}\right)\prod_{\mathfrak{p}}\left(1-\rho_{\mathfrak{p}}^{-1}\mathscr{R}_{\mathfrak{p}^{-1}}\right)$$

$$\cdot\rho_{\mathfrak{f}}^{-1}\mathscr{R}_{\mathfrak{f}}\boldsymbol{L}(\imath,\eta).$$

Note that this depends only on the image of η in $\mathcal{O}_{\mathscr{S}_1}^{*}/(1+\mathfrak{f})$ by Lemma 1.2.

LEMMA 3. *$\mu_{\imath}$ is indeed a distribution*:

$$\mu_{\imath}\left(\bigcup_{j=1}^{N}\mathfrak{U}_j\right)=\sum_{j=1}^{N}\mu_{\imath}(\mathfrak{U}_j)\ \textit{for disjoint open sets}\ \mathfrak{U}_j\subseteq\mathcal{O}_{\mathscr{S}}^{*}.$$

Proof: It's enough to check that

$$\sum_{\substack{u_{\mathfrak{q}}\bmod\mathfrak{q}\\ u_{\mathfrak{q}}\neq 0}}\mu_{\imath}\left(\eta+\sum_{\mathfrak{q}}u_{\mathfrak{q}}+\left(\prod\mathfrak{q}\right)\mathfrak{f}^{*}\right)=\mu_{\imath}(\eta+\mathfrak{f}^{*})\tag{1}$$

and to check that for $\mathfrak{f}$ divisible by all $\mathfrak{p}\in\mathscr{S}$, $\eta\in\mathcal{O}_{\mathscr{S}}^{*}$, and any $\mathfrak{p}_0\in\mathscr{S}$

$$\sum_{\substack{\eta'\bmod\mathfrak{f}\mathfrak{p}_0\\ \eta'\equiv\eta\bmod\mathfrak{f}}}\mu_{\imath}(\eta'+\mathfrak{p}_0\mathfrak{f}^{*})=\mu_{\imath}(\eta+\mathfrak{f}^{*}).\tag{2}$$

Letting $(-1)^{\mathfrak{d}}$ denote the Möbius function we have the additive expression

$$\mu_{\imath}(\eta+\mathfrak{f}^{*})=\rho_{\mathfrak{f}}^{-1}\sum_{\mathfrak{d}\mid\mathfrak{f}}(-1)^{\mathfrak{d}}\rho_{\mathfrak{d}}^{-1}\mathscr{R}_{\mathfrak{f}\mathfrak{d}^{-1}}\boldsymbol{L}(\imath,\eta)$$

whenever $\mathfrak{f}$ is divisible by all places in $\mathscr{S}$. Using this expression for the left hand side of (1), then grouping terms back into a multiplicative form, we obtain

$$\rho_{\mathfrak{f}\pi\mathfrak{q}}^{-1}\prod_{\mathfrak{q}}\left(\mathscr{R}_{\mathfrak{q}}-\rho_{\mathfrak{q}}^{-1}\right)\left(\mathscr{R}_{\mathfrak{q}^{-1}}\mathscr{U}_{\mathfrak{q}}-1\right)\prod_{\mathfrak{p}}\left(1-\rho_{\mathfrak{p}}^{-1}\mathscr{R}_{\mathfrak{p}^{-1}}\right)\cdot\mathscr{R}_{\mathfrak{f}}\boldsymbol{L}(\imath,\eta)$$

and (1) follows upon invoking the Hecke Lemma to put

$$\rho_{\mathfrak{q}}^{-1}\left(\mathscr{R}_{\mathfrak{q}}-\rho_{\mathfrak{q}}^{-1}\right)\left(\mathscr{R}_{\mathfrak{q}^{-1}}\mathscr{U}_{\mathfrak{q}}-1\right)=\left(1-\rho_{\mathfrak{q}}^{-1}\mathscr{R}_{\mathfrak{q}^{-1}}\right)\left(1-\rho_{\mathfrak{q}}^{-1}\mathscr{R}_{\mathfrak{q}}\right).$$

For (2) we choose $\xi\in\boldsymbol{k}^{*}$, such that $(\xi)_{\mathscr{S}}=\mathfrak{f}$, and writing $\eta'=\eta+\xi u$, with

$u \in \mathcal{O}_{\mathfrak{p}_0}$ running through a complete set of representatives of $\mathcal{O}/\mathfrak{p}_0$, we use again the additive expression for the left hand side of (2), and we obtain

$$\rho_{\mathfrak{f}\mathfrak{p}_0}^{-1} \sum_{u \bmod \mathfrak{p}_0} \sum_{\mathfrak{d} \mid \mathfrak{f}\mathfrak{p}_0} (-1)^{\mathfrak{d}} \rho_{\mathfrak{d}}^{-1} \mathscr{R}_{\mathfrak{f}\mathfrak{p}_0\mathfrak{d}^{-1}} \boldsymbol{L}(\mathfrak{z}, \eta + u\xi).$$

Writing $\sum_{\mathfrak{d} \mid \mathfrak{f}\mathfrak{p}_0}$ as $\sum_{\substack{\mathfrak{d} \mid \mathfrak{f} \\ \mathfrak{p}_0 \nmid \mathfrak{d}}} + \sum_{\substack{\mathfrak{d} \mid \mathfrak{f}\mathfrak{p}_0 \\ \mathfrak{p}_0 \mid \mathfrak{d}}}$, and substituting $\mathfrak{d}\mathfrak{p}_0$ for $\mathfrak{d}$ in the second sum, then using Lemma 1.3. to divide by ξ, we get

$$\rho_{\mathfrak{f}}^{-1} \sum_{\substack{\mathfrak{d} \mid \mathfrak{f} \\ \mathfrak{p}_0 \nmid \mathfrak{d}}} (-1)^{\mathfrak{d}} \rho_{\mathfrak{d}}^{-1} \sum_{u \bmod \mathfrak{p}_0} \Big[\rho_{\mathfrak{p}_0}^{-1} \boldsymbol{L}(\mathfrak{z}\xi^{-1}\mathfrak{f}\mathfrak{p}_0\mathfrak{d}^{-1}, \eta\xi^{-1} + u)$$

$$- \rho_{\mathfrak{p}_0}^{-2} \boldsymbol{L}(\mathfrak{z}\xi^{-1}\mathfrak{f}\mathfrak{d}^{-1}, \eta\xi^{-1} + u)\Big].$$

Using Hecke Lemma 1 and 2 for the first and second terms inside the brackets respectively, then using Lemma 1.3. to multiply back by ξ, we get the additive expression for $\mu_{\mathfrak{z}}(\eta + \mathfrak{f}^*)$ upon canceling terms inside the brackets.

Note that by Lemma 1.1. and 1.3., we have for $e \in \mathcal{O}^*$, $\boldsymbol{L}(\mathfrak{z}, e\eta) = \boldsymbol{L}(\mathfrak{z}, \eta)$, hence $\mu_{\mathfrak{z}}(e \cdot \mathfrak{U}) = \mu_{\mathfrak{z}}(\mathfrak{U})$ for all $\mathfrak{U} \subseteq \mathcal{O}_{\mathscr{S}}^*$, and we view $\mu_{\mathfrak{z}}$ as a distribution on $\mathcal{O}_{\mathscr{S}}^*/\bar{\mathcal{O}}^*$, where $\bar{\mathcal{O}}^*$ denotes the closure of $\mathcal{O}^*$ in $\mathcal{O}_{\mathscr{S}}^*$. As such, $\mu_{\mathfrak{z}}$ takes its values in $\mathscr{L}_{\mathscr{S}}^0$; indeed, if $\mathfrak{U} \subseteq \mathcal{O}_{\mathscr{S}}^*$ is stable under multiplication by $\mathcal{O}^*$, it can be written as a disjoint union $\mathfrak{U} = \bigcup_{e \in \mathcal{O}^*/\mathscr{E}_{\mathfrak{f},\eta}} (e\eta + \mathfrak{f}^*)$, and hence $\mu_{\mathfrak{z}}(\mathfrak{U}) = \sum_{e \in \mathcal{O}^*/\mathscr{E}_{\mathfrak{f},\eta}} \mu_{\mathfrak{z}}(e\eta + \mathfrak{f}^*) = [\mathcal{O}^* : \mathscr{E}_{\mathfrak{f},\eta}] \cdot \mu_{\mathfrak{z}}(\eta + \mathfrak{f}^*) \in \mathscr{L}_{\mathscr{S}}^0$. Using the corollary to Lemma 2, we have:

THEOREM 1. *$\mu_{\mathfrak{z}}$ is a distribution on $\mathcal{O}_{\mathscr{S}}^*/\bar{\mathcal{O}}^*$ with values in the finitely generated $\mathbb{Z}[\rho_{\mathfrak{p}}^{-1};\ \mathfrak{p} \in \mathscr{S}]$-module $\mathscr{L}_{\mathscr{S}}^0$.*

§6

In this section we average the distributions $\mu_{\mathfrak{z}}$ over all ideal classes, and use class field theory to get a measure on the Galois group. The 'Mellin-transform' of this measure is the $\mathscr{S}$-adic $\boldsymbol{L}$-function. We prove the interpolation property relating the $\mathscr{S}$-adic $\boldsymbol{L}$-function to its classical counterpart, and the functional equation.

Let $\boldsymbol{k}(1)$ denote the Hilbert class field of $\boldsymbol{k}$, and let $\boldsymbol{k}(\mathscr{S})$ denote the maximal abelian extension of $\boldsymbol{k}$ unramified outside $\mathscr{S}$. By means of the Artin symbol we have identifications:

$$
\begin{array}{rcl}
 & & * \\
 & & \downarrow \\
\mathcal{O}_{\mathcal{S}}^*/\bar{\mathcal{O}}^* \cong k^* \prod_{v \nmid \infty} \mathcal{O}_v^* \cdot k_\infty^* \Big/ \overline{k^* \prod_{v \nmid \mathcal{S}^\infty} \mathcal{O}_v^* \cdot k_\infty^*} & \xrightarrow{\sim} & \mathrm{Gal}(k(\mathcal{S})/k(1)) \\
 & & \downarrow \\
k_{\mathbb{A}}^* \Big/ \overline{k^* \prod_{v \nmid \mathcal{S}^\infty} \mathcal{O}_v^* \cdot k_\infty^*} & \xrightarrow{\sim} & \mathrm{Gal}(k(\mathcal{S})/k) \\
 & & \downarrow \\
\mathcal{C}\ell(k) \cong k_{\mathbb{A}}^* \Big/ k^* \prod_{v \nmid \infty} \mathcal{O}_v^* \cdot k_\infty^* & \xrightarrow{\sim} & \mathrm{Gal}(k(1)/k) \\
 & & \downarrow \\
 & & *
\end{array}
$$

We define a distribution on $\mathcal{G}_{\mathcal{S}} = \mathrm{Gal}(k(\mathcal{S})/k)$ by $\mu_F = \sum_{i=1}^{h} \delta_{\mathfrak{t}_i} * \mu_{\mathfrak{t}_i}$; that is for a locally constant function g on $\mathcal{G}_{\mathcal{S}}$, we have

$$\int_{\mathcal{G}_{\mathcal{S}}} g \, d\mu_F = \sum_{i=1}^{h} \int_{\mathcal{O}_{\mathcal{S}}^*/\overline{\mathcal{O}^*}} g(\mathfrak{t}_i \eta)\, d\mu_{\mathfrak{t}_i}(\eta).$$

The distribution μ_F is determined by its values on finite characters ω, we let $\mathcal{L}_{\mathcal{S}}^0[\omega]$ denote the $\mathbb{Z}[\omega]$-module generated by $\mathcal{L}_{\mathcal{S}}^0$, where $\mathbb{Z}[\omega] \subseteq \mathbb{C}$ denotes the subring generating by the values of ω.

THEOREM 2. *For a finite character $\omega : \mathcal{G}_{\mathcal{S}} \to \mathbb{Z}[\omega]$, primitive of conductor $\mathfrak{f}$, we have inside $\mathcal{L}_{\mathcal{S}}^0[\omega]$:*

$$\int_{\mathcal{G}_{\mathcal{S}}} \omega \, d\mu_F = \left(\tau(\omega)(4\pi)^{2n}\rho_{\mathfrak{f}}\right)^{-1} \mathbb{N}\mathfrak{f}^{1/2} \prod_{\mathfrak{q} \in \mathcal{S}} \left(1 - \rho_{\mathfrak{q}}^{-1}\omega(\mathfrak{q})\right)\left(1 - \rho_{\mathfrak{q}}^{-1}\omega^{-1}(\mathfrak{q})\right)$$

$$\cdot L_F(\omega).$$

Proof: Using an additive expression for our measure we have

$$\int_{\mathcal{G}_{\mathcal{S}}} \omega \, d\mu_F = \sum_{i=1}^{h} \sum_{\eta \in (\mathcal{O}/\mathfrak{f})^*} \omega(\mathfrak{t}_i \eta) \rho_{\mathfrak{f}}^{-1} \sum_{\mathfrak{d} \mid \pi\mathfrak{q}\pi\mathfrak{p}} (-1)^{\mathfrak{d}} \rho_{\mathfrak{d}}^{-1} \mathcal{R}_{\mathfrak{d}^{-1}}$$

$$\times \sum_{\mathfrak{d}' \mid \pi\mathfrak{q}} (-1)^{\mathfrak{d}'} \rho_{\mathfrak{d}'}^{-1} \mathcal{R}_{\mathfrak{d}'} \cdot \mathcal{R}_{\mathfrak{f}} L(\mathfrak{t}_i, \eta)$$

By invoking Lemma 1.2. we see that we may assume $(\mathfrak{d}, \mathfrak{f}) = 1$ and take the summation only over $\mathfrak{d} \,|\, \prod \mathfrak{q}$, then substituting $\imath_i \mathfrak{d}' \mathfrak{d}^{-1}$ for $\imath_i$, we get

$$\rho_{\mathfrak{f}}^{-1} \sum_{\mathfrak{d} \,|\, \pi \mathfrak{q}} (-1)^{\mathfrak{d}} \rho_{\mathfrak{d}}^{-1} \omega(\mathfrak{d}) \sum_{\mathfrak{d}' \,|\, \pi \mathfrak{q}} (-1)^{\mathfrak{d}'} \omega^{-1}(\mathfrak{d}')$$

$$\cdot \sum_{i=1}^{\mathsf{h}} \sum_{\eta \in (\mathcal{O}/\mathfrak{f})^*} \omega(\imath_i \eta) \boldsymbol{L}(\imath_i \mathfrak{f}, \eta)$$

and the expression in the theorem follows from Birch Lemma upon transforming the additive $\sum_{\mathfrak{d} \,|\, \pi \mathfrak{q}} \cdots \sum_{\mathfrak{d}' \,|\, \pi \mathfrak{q}} \cdots$ into the Euler product $\prod_{\mathfrak{q}} (\ldots)(\ldots)$.

Assume that the $\rho_{\mathfrak{p}}$'s, $\mathfrak{p} \in \mathcal{S}$, can be chosen to be $\mathfrak{p}$-units (hence $\mathcal{S}$-units). Let $\mathcal{L}_{\mathcal{S}} = \mathbf{Z}_{\mathcal{S}} \otimes \mathcal{L}_{\mathcal{S}}^0$ denote the $\mathcal{S}$-adic completion of $\mathcal{L}_{\mathcal{S}}^0$; where $\mathbf{Z}_{\mathcal{S}} = \prod_p \mathbf{Z}_p$ the product taken over all rational primes p such that there exists a prime $\mathfrak{p} \in \mathcal{S}$ above p. $\mathcal{L}_{\mathcal{S}}$ is a finitely generated $\mathbf{Z}_{\mathcal{S}}$-module; and so if $\tilde{\mathcal{O}}$ is any $\mathcal{S}$-adically complete and separated flat $\mathbf{Z}_{\mathcal{S}}$-algebra, we can associate to every *continuous* function $g : \mathcal{G}_{\mathcal{S}} \to \tilde{\mathcal{O}}$ the well define integral of g with respect to μ_F,

$$\int_{\mathcal{G}_{\mathcal{S}}} g \, d\mu_F \in \tilde{\mathcal{O}} \underset{\mathbf{Z}_{\mathcal{S}}}{\otimes} \mathcal{L}_{\mathcal{S}}.$$

In particular, for any continuous $\mathcal{S}$-adic character $\omega : \mathcal{G}_{\mathcal{S}} \to \tilde{\mathcal{O}}^*$, we can define the $\mathcal{S}$-adic L-function:

$$\boldsymbol{L}_{F,\mathcal{S}}(\omega) = \int_{\mathcal{G}_{\mathcal{S}}} \omega \, d\mu_F \in \tilde{\mathcal{O}} \underset{\mathbf{Z}_{\mathcal{S}}}{\otimes} \mathcal{L}_{\mathcal{S}}.$$

Theorem 2 gives the precise sense in which the $\boldsymbol{L}_{F,\mathcal{S}}$ interpolates the classical $\boldsymbol{L}_F$.

THEOREM 3. *We have the functional equation*

$$\boldsymbol{L}_{F,\mathcal{S}}(\omega) = (-1)^{\mathsf{n}} \epsilon_F \omega(\mathscr{a}) \cdot \boldsymbol{L}_{F,\mathcal{S}}(\omega^{-1}).$$

Proof: By Lemma 1.4.

$$\boldsymbol{L}(\imath \mathfrak{f}, \eta) = (-1)^{\mathsf{n}} \epsilon_F \boldsymbol{L}(\mathscr{a} \imath^{-1} \mathfrak{f}, -\eta^{-1}).$$

This implies a functional equation for our measures

$$\mu_{\imath}(\eta) = (-1)^{\mathsf{n}} \epsilon_F \mu_{\imath^{-1}\mathscr{a}}(-\eta^{-1})$$

from which the functional equation for $\boldsymbol{L}_{F,\mathcal{S}}(\omega)$ follows immediately.

§7

We end this paper with a few remarks.

Remark 1: Let $\boldsymbol{E} = \mathbb{Q}(\rho_v) \subseteq \mathbb{C}$ denote the subfield generated by all the ρ_v's, $v \nmid \infty$. Assume $\boldsymbol{F}$ is a form such that $\boldsymbol{E} \cdot \mathscr{L}^0_{\mathscr{S}} \approx \boldsymbol{E} \cdot \ell$ is a one dimensional $\boldsymbol{E}$-vector space. Take for $\mathscr{S}$ a set of finite primes away from (a) and containing all the p-placed of $\boldsymbol{k}$, p a 'good' rational prime (i.e., such that we can find ρ_v's which are p-units for $v \mid p$). Let $\tilde{\boldsymbol{E}}$ denote the field generated over $\boldsymbol{E}$ by all roots of unity of order dividing $\mathbb{N}v - 1$, or some power of $\mathbb{N}v$, for all $v \in \mathscr{S}$. Choose a place $\mathfrak{P}$ of $\tilde{\boldsymbol{E}}$ above p and let $\tilde{\boldsymbol{E}}_{\mathfrak{P}}$ denote the completion of $\tilde{\boldsymbol{E}}$ at $\mathfrak{P}$. $\mathscr{G}_{\mathscr{S}}$ is the Galois group of the maximal $\mathscr{S}$-ramified abelian extension of $\boldsymbol{k}$, and for each continuouis character $\omega : \mathscr{G}_{\mathscr{S}} \to \tilde{\boldsymbol{E}}^*_{\mathfrak{P}}$, we obtain for the value of *$\mathfrak{P}$-adic $\boldsymbol{L}$-function* at ω: $\boldsymbol{L}_{F,\mathscr{S},\mathfrak{P}}(\omega) \in \tilde{\boldsymbol{E}}_{\mathfrak{P}} \cdot \ell$. (this is the '$\mathfrak{P}$-component' of the above $\boldsymbol{L}_{F,\mathscr{S}}(\omega) \in \mathbb{Z}_{\mathscr{S}} \otimes \tilde{\boldsymbol{E}} \cdot \ell$).

Remark 2: If the $\rho_{\mathfrak{p}}$'s were not $\mathscr{S}$-adic units the μ_F defined above would still be a distribution but would not be bounded. Nevertheless, it would have a 'moderate growth' [i.e. $\rho_{\mathfrak{f}} \mu_{\imath_i}$ (image of $(\eta + \mathfrak{f}^*) \bmod \bar{\mathcal{O}}^*$) takes values in a finitely generated $\mathbb{Z}_{\mathscr{S}}$-module, and $\rho_{\mathfrak{p}}$ is such that at worst $|\rho_{\mathfrak{p}}|_p = |\mathbb{N}\mathfrak{p}|_p^{1/2}$] and hence analytic functions (e.g. $\mathscr{S}$-adic characters) could be integrated against it. But continuous function could not be integrated and our $\mathscr{S}$-adic $\boldsymbol{L}$-functions might have infinitely many zeros, cf. [Visik, 1976].

Remark 3: The presence of real places v slightly complicates the situation, since for finite grossencharacter ω, ω_v need not be trivial, $\omega_v(-1) = \pm 1$, and so now we have to keep track of the 'directions' at the real places. We shall indicate the needed modifications in the order of their appearance above. We take $\imath_i$ representing the *wide* class group $\boldsymbol{k}^* \backslash \boldsymbol{k}^*_{\mathbb{A}} / \prod_{v \nmid \infty} \mathcal{O}_v^* \boldsymbol{k}^0_\infty$, $\boldsymbol{k}^0_\infty =$ the connected component of $\boldsymbol{k}^*_\infty$; $\mathscr{K}_v = \boldsymbol{O}(2; \boldsymbol{k}_v)$; $\mathscr{V}_v$ a 2-dimensional complex vector space with basis V_v^1, V_v^{-1}, on which $\mathscr{K}_v$ acts on the right via the representation $M_v\begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} = \begin{pmatrix} e^{-2i\vartheta} & 0 \\ 0 & e^{2i\vartheta} \end{pmatrix}$, $M_v\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $W_v : \boldsymbol{k}_v^* \to \mathscr{V}_v$ is given by $W_v(x) = |x| \exp(2\pi |x|) V_v^{\operatorname{sgn} x}$; $\beta_v^1 = \frac{1}{x}(\mathrm{d}y + \mathrm{i}\mathrm{d}x)$, $\beta_v^{-1} = \frac{1}{x}(-\mathrm{d}y + \mathrm{i}\mathrm{d}x)$, so that the v-component of our form is: $W_v(x)\psi_v(y)\beta_v = \exp[2\pi(|x| - \mathrm{i}y)] \cdot (\mathrm{d}y + \mathrm{i}\operatorname{sgn}(x)\,\mathrm{d}x)$. Note that since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathscr{K}_v$ there is no difference between $\mathbb{X}$ and $\mathbb{X}^{\operatorname{sgn}}$ from the point of view of a real place v, that is: $\mathscr{B}_v^+ \xrightarrow{\sim} G_v/\mathscr{K}_v\mathscr{Z}_v^+ \xrightarrow{\sim} G_v/\mathscr{K}_v\mathscr{Z}_v$, and $\mathbb{X}^{\operatorname{sgn}}_{\imath_i}$ depends only on the ideal class of $(\imath_i)$, but $\Omega_{\imath_i}$ will depend also on $\operatorname{sgn}(\imath_i)_\infty$; $\Theta = \bigwedge_v \Theta_v$, the product is taken only over the complex v's. Now fix a direction $d = \{d_v \mid v \text{ real}\}$, $d_v = \pm 1$. The definition of the periods is altered by replacing $\boldsymbol{k}^*_\infty$ by $\boldsymbol{k}^0_\infty$, V_v^0 by $V_v^1 + d_v V_v^{-1}$, and requiring the units in $\mathscr{E}$ to be *positive* in all the *real* places v; lemma 1.1.: $L(\mathfrak{s}, \eta)$ depends also on the sign of $\mathfrak{s}_v$, $L(\mathfrak{s}(-1)_v, \eta) = d_v \cdot L(\mathfrak{s}, \eta)$ (and of course also on our choice of d); Lemma 1.4.: $(-1)^n$ is replaced by $(-1)^{r_1+r_2}$. In the definition of the cycles replace again k_∞^* by k_∞^0, so that $\mathbb{I}(\mathscr{E}) \approx [0, \infty] \times (\mathbb{R}/\mathbb{Z})^{[k:\mathbb{Q}]-1}$, and note again that $\mathfrak{z}(\mathfrak{s}_i, \alpha)$ depends only on the ideal class of $(\mathfrak{s}_i)$; lemma 2 and its corollary remain unchanged. The proof of Birch lemma needs the obvious modification of keeping track of the directions, but its statement remains true for all finite characters ω satisfying $\omega_v(-1) = d_v$ (where we sum over the wide ideal class representatives $\mathfrak{s}_i$'s, and replace $(4\pi)^{2n}$ by $(4\pi)^{[k:\mathbb{Q}]}$). From this point onwards everything remains the same if only we replace 'class-group' by '*wide class group*', k_∞^* by k_∞^0, and we obtain a distribution μ_F on $\mathscr{G}_{\mathscr{S}}$, such that for finite characters $\omega : \mathscr{G}_{\mathscr{S}} \to \mathbb{Z}[\omega]$, $L_{F,\mathscr{S}}(\omega) = \int_{\mathscr{G}_{\mathscr{S}}} \omega \, d\mu_F$ interpolates the classical $L_F(\omega)$ in the sense of Theorem 2 (replacing $(4\pi)^{2n}$ by $(4\pi)^{[k:\mathbb{Q}]}$) and satisfies the functional equation $L_{F,\mathscr{S}}(\omega) = \omega_\infty(-1)(-1)^{r_1+r_2} \epsilon_F \omega(\mathfrak{a}) L_{F,\mathscr{S}}(\omega^{-1})$.

Remark 4: Having started with a modular form corresponding to a harmonic form on $\mathbb{X}$ we pulled it back to $\mathbb{X}^{\mathrm{sgn}}$ in order to construct the p-adic L-functions. Thus, from the 'p-adic point of view', it seems more natural to *start* with a modular form corresponding to a harmonic form on $\mathbb{X}^{\mathrm{sgn}} = \bigcup_{i=1}^{h} \Gamma_{\mathfrak{s}_i} \backslash \prod_{v \text{ complex}} (\mathbb{H} \setminus \mathbb{C}) \prod_{v \text{ real}} (\mathbb{C} \setminus \mathbb{R})$. Such forms when written adelically take values in $\bigotimes_{v|\infty} \mathscr{V}_v$. For v complex $\mathscr{V}_v$ is the complexification of $\mathscr{T}_j(\mathbb{H})$, the tangent space to the quaternions at j. Note that under the action of the maximal compact subgroup $SU(2; \mathbb{C})$, $\mathscr{T}_j(\mathbb{H})$ splits as a direct sum of two irreducible representations, one 3-dimensional and the other 1-dimensional. In particular, $\mathbb{H} \setminus \mathbb{C}$ has no complex structure, invariant under the G_v action, and hence $\mathbb{X}^{\mathrm{sgn}}$ has no natural complex structure. It seems interesting to inquire what further structure $\mathbb{X}^{\mathrm{sgn}}$ possess (besides the Riemannian structure), and what kind of moduli interpretation $\mathbb{X}^{\mathrm{sgn}}$ admits.

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