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DIANE MEUSER

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On the degree of a local zeta function

DIANE MEUSER
Department of Mathematics, Harvard University, Cambridge, MA 02138, USA; present address: Dept. of Mathematics, Boston University, 111 Cunningham Street, Boston, MA 02215, USA

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Let K be a finite algebraic extension of Q, R the ring of integers of K and $\{v\}$ the set of finite places of K. For $v \in \{v\}$ let $|\ |_v$ be the non-archimedian absolute value on K and K_v the completion of K with respect to this absolute value. Let R_v be the ring of integers of K_v, P_v the unique maximal ideal of R_v and $k_v = R_v/P_v$. Then k_v is a finite field and we let $q_v = \text{card } k_v$. Let $f(x) = f(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree m. Then for any v we can consider

$$Z(t) = \int_{R_v^{(n)}} |f(x)|_v^s |\mathrm{d}x|_v$$

where $s \in C$, $\mathrm{Re}(s) > 0$ and $t = q_v^{-s}$. This has been shown to be a rational function of t by Igusa in [Igusa, 1977]. Writing $Z(t) = P(t)/Q(t)$ we define $\deg Z(t) = \deg P(t) - \deg Q(t)$. Igusa has conjectured in [Igusa, 1984], p. 1027, and [Igusa, 1986], that for almost all v, i.e. except for a finite number of v, one has $\deg Z(t) = -m$. In this paper Igusa gives many examples where f satisfies the additional property that it is the single invariant polynomial for a connected irreducible simple linear algebraic group.

In this paper we show this conjecture is true if f is non-degenerate with respect to its Newton Polyhedron. This establishes the conjecture for "generic" homogeneous polynomials in a sense to be described below.

§1. The Newton polyhedron of f and its associated toroidal modification

We first recall some of the terminology and basic properties of the Newton polyhedron of an arbitrary polynomial. Other references for this include [Danilov, 1978; Kouchnirenko, 1976; Lichtin, 1981; Varchenko, 1977].

Let $f \in K[x_1, \ldots, x_n]$. We write $f = \sum_{I \in N^n} a_I x^I$, where $I = (i_1, \ldots, i_n)$ and $x^I = x_1^{i_1} \cdots x_n^{i_n}$. Let $\mathrm{Supp}(f) = \{I \in N^n \mid a_I \neq 0\}$. Let $S(f)$ denote the convex hull of $\cup_{I \in \mathrm{Supp}(f)}(I + \boldsymbol{R}_+^n)$. Let $\Gamma_+(f)$ be the union of all faces of $S(f)$. Let $\Gamma(f)$ be the union of compact faces only. $\Gamma_+(f)$ is called the Newton polyhedron of f and $\Gamma(f)$ is called the Newton diagram. We will denote a fixed Newton polyhedron and diagram by Γ_+ and Γ respectively. Given a Newton polyhedron Γ_+ and its associated Newton diagram Γ we define $\Omega_{\Gamma_+} = \{g \in K[x_1, \ldots, x_n] \mid \Gamma_+(g) = \Gamma_+\}$. If $g \in \Omega_{\Gamma_+}$, and γ is a face of Γ, we define g_γ to be $\sum_{I \in \gamma} b_I x^I$ if $g = \sum_{I \in \gamma} b_I x^I + \sum_{I \in \gamma} b_I x^I$. Then we define non-degeneracy as in [Kouchnirenko, 1976].

Definition: f is non-degenerate with respect to its Newton polyhedron if for any face γ of $\Gamma_+(f)$ the functions $(x_i \cdot \partial f/\partial x_i)$, have no common zero in $(\bar{K} - \{0\})^n$, where $\bar{K}$ denotes the algebraic closure of K.

Fix m and n. Identify homogeneous polynomials of degree m in n variables with $\boldsymbol{P}_K^N$, where $N = \binom{m+n-1}{m} - 1$. For Γ_+ a fixed Newton polyhedron $X_{\Gamma_+} = \{f \mid \Gamma_+(f) = \Gamma_+\}$ is a Zariski subset of $\boldsymbol{P}_K^N$. Let

$$Y_{\Gamma_+} = \{f \mid f \text{ is non-degenerate with respect to } \Gamma_+\}.$$

Then in a completely analogous manner to the proof of Theorem 6.1 in [Kouchnirenko, 1976] we have the following result which shows the non-degeneracy condition is generic.

PROPOSITION 1: *Y_{Γ_+} is a Zariski open, dense subset of X_{Γ_+}.*
Let K be a finite algebraic extension of Q, $\{v\}$ the finite places of K, and K_v, R_v, P_v and k_v as defined in the introduction. Let $U_v = R_v - P_v$ be the units of R_v. We first recall some definitions concerning the reduction of varieties modulo P_v.

For $g \in R[x_1, \ldots, x_n]$, v a finite place of K, let $\bar{g}_v$ denote the polynomial in $k_v[x_1, \ldots, x_n]$ obtained by reducing the coefficients of g modulo P_v. We shall abbreviate this to $\bar{g}$ when v is understood and use the same notation when g is a constant in R. Let V be an algebraic set defined over K, i.e., $V = \{x \in \bar{K}^n \mid f_i(x) = 0, 1 \leqslant i \leqslant r\}$, where $f_i(x) \in K[x_1, \ldots, x_n]$.

Let $I(V)$ be the ideal of V, i.e., $I(V) = \{f \in \bar{K}[x_1, \ldots, x_n] \mid f(x) = 0 \ \forall x \in V\}$. Then we define the reduction of V modulo P_v, denoted $\bar{V}_v$ by

$$\bar{V}_v = \left\{x \in \bar{k}_v^n \mid \bar{f}_v(x) = 0 \ \forall f \in I(V) \cap R_v[x_1, \ldots, x_n]\right\}.$$

If $f \in R[x_1, \ldots, x_n]$ then for any finite place v of K we can consider the non-degeneracy of $\bar{f}_v$. We have:

PROPOSITION 2: *Let $f \in R[x_1, \ldots, x_n]$ be non-degenerate with respect to its Newton polyhedron. Then for almost all v*
a) $\Gamma + (\bar{f}_v) = \Gamma_+(f)$
b) *$\bar{f}_v$ is non-degenerate with respect to its Newton polyhedron.*

Proof. Let $S = \{v \mid \text{all coefficients of } f \text{ are in } U_v\}$. Then for $v \in S$, $\Gamma_+(\bar{f}_v) = \Gamma_+(f)$ and a) follows since almost all v are in S.

Let τ be a face of $\Gamma(f)$, and write $f_\tau = f_{\tau,1}, \dots, f_{\tau,t}$ where each $f_{\tau,l}$ is absolutely irreducible. Let $V_{\tau,t}$ be the variety defined by $f_{\tau,l}=0$, Y_i the hyperplane defined by $x_i = 0$, and $Y = \cup_{i=1}^n Y_i$. The condition that f is non-degenerate is equivalent to the condition that for any face τ of $\Gamma(f)$, and $V_{\tau,l}$ as above, the singular points of each $V_{\tau,i}$ are contained in Y and for any $i, j, i \neq j$ we have $V_{\tau,i} \cap V_{\tau,j} \subset Y$.

Let L be a finite extension of K such that the coefficients of $f_{\tau,i}$ for any τ, i are in L. To each place of v of K let v' be any place of L dividing v. As a straightforward consequence of Hilbert's Nullstellensatz, for any τ, i, j we have $(\overline{V}_{\tau,i})_{v'} \cap (\overline{V}_{\tau,j})_{v'} \subset \overline{Y}_{v'}$ for all v, v'. As a consequence of Proposition 30 in [Shimura, 1955], $(\overline{V}_{\tau,i})_{v'}$ is absolutely irreducible and its singularities are contained in $\overline{Y}_{v'}$ for almost all places v' of L. Let $\tilde{S}$ be the set of $v \in S$ satisfying the above property for all τ, i and all $v' \mid v$. Then almost every place of K is in $\tilde{S}$ and $\bar{f}_v$ is non-degenerate for all $v \in \tilde{S}$. Q.E.D.

We next describe a toroidal modification of K_v^n that we shall use to prove the conjecture for homogeneous f that are non-degenerate with respect to their Newton polyhedron. The modification we use is not the one utilized in [Lichtin, 1981] or [Lichtin and Meuser, 1985], which gives a nonsingular variety Y_v and a morphism $h: Y_v \to K_v^n$ such that $f \circ h = 0$ is a divisor with normal crossings, but a weaker modification that has also been used by Denef in [Denef, not yet published].

Let $(\boldsymbol{R}_+^n)^* = \boldsymbol{R}_+^n - \boldsymbol{0}$. Let $a^1, \dots, a^l$ be vectors in $\boldsymbol{R}_+^n$ and $\sigma = \{\alpha_1 a^1 + \cdots + \alpha_l a^l \mid \alpha_i \in \boldsymbol{R}_+, 1 \leqslant i \leqslant l\}$. σ is called a closed cone which we denote by $\langle a^1, \dots, a^l \rangle$. $\check{\sigma} = \{\alpha_1 a^1 + \cdots + \alpha_l \alpha^l \mid \alpha_i > 0, 1 \leqslant i \leqslant l\}$ is called an open cone. The dimension of any cone is the dimension of the smallest vector subspace of $\boldsymbol{R}^n$ containing it. σ, or $\check{\sigma}$, is called a simplicial cone if $a^1, \dots, a^l$ are linearly independent over $\boldsymbol{R}$. If σ is a closed cone spanned by integral vectors, then we have the following well known result on $\sigma \cap \boldsymbol{Z}_+^n$ which we shall later use.

LEMMA 1. *Let $\sigma = \langle a^1, \dots, a^l \rangle$ be a closed cone in $\boldsymbol{R}_+^n$, where each a^i, $1 \leqslant i \leqslant l$, is an integral vector. There are a finite number of integral vectors $w^1, \dots, w^r$ such that*

$$\sigma \cap \boldsymbol{Z}_+^n = \coprod_{i=1}^{r} \left\{ w^i + \sum_{j=1}^{l} \alpha_j a^j \mid \alpha_j \in \boldsymbol{Z}_+ \right\}.$$

Proof: It is well known that σ has a partition into closed simplicial cones where each such cone is spanned by a subset of $\{a^1, \dots, a^l\}$. Thus we can assume σ is simplicial. We form the parallelotope $P_\sigma = \left\{ \sum_{j=1}^{l} \alpha_j a^j \mid 0 \leqslant \alpha_j < 1 \right\}$. Let $w^1, \dots, w^r$ be the points in $P_\sigma \cap \boldsymbol{Z}_+^n$. Then these w^i satisfy the statement of the lemma. Q.E.D.

Associated to any Newton polyhedron Γ_+ we have a partition of $(\boldsymbol{R}_+^n)^*$ into open cones. For $a \in (\boldsymbol{R}_+^n)^*$ we let $m(a) = \inf_{y \in \Gamma_+} \{a \cdot y\}$ and $\tau_a = \{y \in \Gamma_+ \mid y \cdot a = m(a)\}$. τ_a is called the meet locus of a. We define an equivalence relation $\sim$ by $a^1 \sim a^2$ if $\tau_{a^1} = \tau_{a^2}$. This equivalence relation satisfies the following properties:

i) If $a \in (\boldsymbol{R}_+^n)^*$, τ_a is a face of Γ_+.

ii) Let τ be a face of Γ_+. Let $F_1, \ldots, F_r$ be the facets of Γ_+ containing τ. Let a^i denote a vector dual to F_i, $1 \leqslant i \leqslant r$. Then

$$\left\{a \in (\mathbf{R}_+^n)^* | \tau_a = \tau\right\} = \left\{\alpha_1 a^1 + \cdots + \alpha_r a^r | \alpha_i > 0\right\}.$$

We denote the cone in the above formula by $\check{\sigma}_\tau$. Then its closure σ_τ satisfies $\sigma_\tau = \{a \in (\boldsymbol{R}_+^n)^* \mid \tau_a \supseteq \tau\}$. A vector $a = (a_1, \ldots, a_n)$ in $\boldsymbol{Z}_+^n - \boldsymbol{0}$ is called primitive if the greatest common divisor of the a_j, $1 \leqslant j \leqslant n$, is one. For each facet of Γ_+ there is a unique primitive integral vector dual to that facet. The above properties imply each equivalence class under $\sim$ is an open cone spanned by a subset of primitive integral vectors dual to facets.

If f is a homogeneous polynomial of degree m in n variables note that all $I \in \operatorname{Supp}(f)$ lie on the hyperplane $\boldsymbol{1} \cdot x = m$, where $\boldsymbol{1} = (1, \ldots, 1)$. Let F be a face of $\Gamma(f)$. It is straightforward to see that if P is an exposed point of F then $P = I$ for some $I \in \operatorname{Supp}(f)$. Hence $\Gamma(f)$ is a single face with supporting hyperplane $\bar{1} \cdot x = m$. Let $E(\Gamma_+)$ be the exposed points of Γ_+. Every $P \in E(\Gamma_+)$ lies in Γ hence $\boldsymbol{1} \in \sigma P$. We can partition $\overset{v}{\sigma}_P$ into simplicial cones of the form $\{\alpha_1 a^1 + \cdots + \alpha_n a^n \mid \alpha_i \in \boldsymbol{R},\ \alpha_i > 0\}$ where we may assume $a^1 = \boldsymbol{1}$, and $a^2, \ldots, a^n$ are primitive integral vectors dual to noncompact facets of Γ_+ containing P.

Let $\sigma = \langle a^1, \ldots, a^n \rangle$ be the closure of one of the maximum dimension cones corresponding to $P \in E(\Gamma_+)$. Write $a^i = (a_{i1}, \ldots, a_{in})$ and let $M = [a_{ij}]$. Then M determines a morphism $\theta \colon K_v^n \to K_v^n$ defined by $\theta(y_1, \ldots, y_n) = (x_1, \ldots, x_n)$ where

$$x_j = y_1^{a_{1j}} \cdots y_n^{a_{nj}}. \tag{1}$$

Let $\mathrm{d}x$ be the differential $\mathrm{d}x_1 \ldots \mathrm{d}x_n$ and $\theta^*(\mathrm{d}x)$ its pullback under θ. Then for $f \in R[x_1, \ldots, x_n]$, Γ_+, and θ as above we have the following result.

PROPOSITION 3:

a) $(f \circ \theta)(y) = y_1^m \prod_{i=2}^n y_i^{m(a^i)} f_\theta(y)$ *where* $f_\theta(y) \in R[y_2, \ldots, y_n]$, $f_\theta(\boldsymbol{0}) \neq 0$.

b) $\theta^*(\mathrm{d}x) = (\det M) y_1^{n-1} \prod_{i=2}^n y_i^{|a^i| - 1} \mathrm{d}y$ *where* $|a^i| = \sum_{j=1}^n a_{ij}$.

c) *Let* $S = \{v \mid \Gamma_+(\bar{f}_v) = \Gamma_+(f),\ \bar{f}_v$ *non-degenerate with respect of* Γ_+, *and*

$\overline{(\det M)_v} \neq 0\}$. *Then for* $v \in S$, $(\bar{f}_\theta)_v(\boldsymbol{0}) \neq 0$, *and if* $b \in k_v^n$ *satisfies* $(\bar{f}_\theta)_v(b) = 0$ *then*

$$y_j \frac{\partial (\bar{f}_\theta)_v}{\partial y_j}(b) \neq 0$$

for some $2 \leqslant j \leqslant n$.

Proof: a) and b) are just specializations of Varchenko's result, Lemma 10.2 in [Varchenko, 1977]. We write $f = a_p x^P + \sum_I$ with P as in the discussion above and $x^I = x_1^{i_1} \cdots x_n^{i_n}$. Then under the map θ the monomial x^I is transformed to $\sum_{i=1}^{n} y_i^{I \cdot a^i}$. For a) we denote that $I \in \Gamma(f)$ implies $I \cdot a^1 = m$ and $I \cdot a^i \geqslant m(a^i)$ for $2 \leqslant i \leqslant n$. Furthermore $P \cdot a^i = m(a^i)$ for all i, and P is the only point of $\Gamma(f)$ having this property, so this gives the above factorization of $(f \circ \theta)(y)$. The formula $\theta^*(\mathrm{d}x)$ is a straightforward consequence of (1).

For c), we first observe that for $v \in S$ we have $(\bar{a}_p)_v \neq 0$, hence $(\bar{f}_\theta)_v(\boldsymbol{0}) \neq 0$. The proof of the rest of c) is identical to Lichtin's proof of Proposition 2.3 in [Lichtin, 1981]. Q.E.D.

Let K_v be the completion of K corresponding to any finite place v of K. Using the same notation as in the introduction, for every such place we fix $\pi_v \in P_v - P_v^2$. Let $U_v = R_v - P_v$. For $x \in K_v^*$ we can write $x = \pi_v^{\mathrm{ord}\, x} u$ where $u \in U_v$. Let $R_v^{(n)} = R_v \times \cdots \times R_v$ (n times) with a similar meaning for $U_v^{(n)}$, $P_v^{(n)}$.

Let $\sigma = \langle a^1, \ldots, a^l \rangle$ be the closure of a cone in the partition corresponding to Γ_+. To each such cone we associate a maximal dimension closed cone $\tilde{\sigma}$ containing σ, and note that it is not unique. For any place v, associated to σ we consider the subset of $R_v^{(n)}$ defined by

$$X_\sigma = \left\{ x \in R_v^{(n)} \middle| (\mathrm{ord}\ x_1, \ldots, \mathrm{ord}\ x_n) \in \sigma \right\}.$$

Let $Y_\sigma = R_v^{(l)} \times U_v^{(n-l)}$ and consider the morphism $\theta|_{Y_\sigma} : Y_\sigma \to R_v^{(n)}$ where θ is the morphism associated to $\tilde{\sigma}$ defined by (1). We observe that $(\mathrm{ord}\ x_1, \ldots, \mathrm{ord}\ x_n) = \sum_{i=1}^{n} (\mathrm{ord}\ y_i) a^i$, hence $\theta(Y_\sigma) \subseteq X_\sigma$. The next Lemma gives the properties of $\theta|_{Y_\sigma}$ and the decomposition of X_σ that were established by Denef, Lemma 3 in [Denef, not yet published]. For $\gamma = (\gamma_1, \ldots, \gamma_n) \in K_v^n$, and T any subset of K_v^n, denote by γT the set $\{(\gamma_1 x_1, \ldots, \gamma_n x_n) \mid (x_1, \ldots, x_n) \in T\}$.

LEMMA 2. *a) The map* $\theta|_{Y_\sigma} : Y_\sigma \to \theta(Y_\sigma)$ *is locally bianalytic and each fiber has cardinality* $\kappa_\theta(v) = \mathrm{card\ ker}\ \theta|_{U_v^{(n)}}$. *b) If* $w^i = (w_{i1}, \ldots, w_{in})$, $1 \leqslant i \leqslant r$, *are the vectors in* $\sigma \cap \boldsymbol{Z}_+^n$ *given by Lemma* 1, *let* π^{w^i} *denote* $(\pi^{w_{i1}}, \ldots, \pi^{w_{in}})$. *Let* $u_1, \ldots, u_{s(v)}$ *be the coset representatives for* $U_v^{(n)}/\theta(U_v^{(n)})$. *Then*

$$X_\sigma = \coprod_{\substack{1 \leqslant i \leqslant s(v) \\ 1 \leqslant j \leqslant r}} u_i \pi^{w^j}(\theta(Y_\sigma)).$$

§2. The degree of $Z(t)$

Let v be a finite place of K. Using the same notation as in the preceding sections, we define an absolute value on K_v^* by $|x|_v = q_v^{-\operatorname{ord} x}$. We let $|dx|_v$ be the Haar measure on K_v normalized so that the measure of R_v is one. Then the measure of $a + P_v$ for any $a \in K_v$ is q_v^{-1}. If $a \in R_v^{(n)}$, $a + P_v^{(n)}$ will denote a coset modulo $P_v^{(n)}$, i.e. $(a_1 + P_v) \times \cdots \times (a_n + P_v)$ where $a = (a_1, \ldots, a_n)$. We shall also use $|dx|_v$, defined above for $n = 1$, to be the measure $\prod_{i=1}^{n} |dx_i|_v$ on $R_v^{(n)}$. When v is fixed we denote π_v, $|dx|_v$ and q_v by π, $|dx|$ and q respectively. Letting $t = q^{-s}$ we have the following basic formulas for N, $n \in \boldsymbol{Z}$; N, $n \geqslant 0$.

$$\begin{aligned} \int_R |x|^{Ns+n-1} |dx| &= \frac{q^n(1-q^{-1})}{q^n - t^N} \\ \int_P |x|^{Ns+n-1} |dx| &= \frac{(1-q^{-1})t^N}{q^n - t^N}. \end{aligned} \tag{2}$$

For $f \in K[x_1, \ldots, x_n]$, and any finite place v, we can consider the zeta function $Z(t)$ associated to f as defined in the introduction. We then have the following result.

THEOREM. *Let $f(x) = f(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ be a homogeneous polynomial that is non-degenerate with respect to its Newton polyhedron. Then for almost every place v of K, $\deg Z(t) = -\deg f(x)$.*

Proof: Let $\deg f(x) = m$, and Γ_+ be the Newton polyhedron of f. As explained in the previous section, associated to this Newton polyhedron we have a partition of $\boldsymbol{R}_+^n$ into open cones. For P an exposed point of Γ_+, let $\check{\sigma}_P$ be the associated maximal dimension open cone. As previously observed we can partition $\check{\sigma}_P$ into simplicial cones of the form $\{\alpha_1 a^1 + \cdots + \alpha_n a^n \mid \alpha_i > 0\}$ where $a^1 = \boldsymbol{1}$, if $\check{\sigma}_P$ is not already in this form. The a^i, $2 \leqslant i \leqslant n$, are dual to noncompact facets of Γ_+. Repeating this process for all points of $E(\Gamma_+)$ let $\check{\sigma}_1, \ldots, \check{\sigma}_K$ denote the resulting simplicial cones, and let $\sigma_1, \ldots, \sigma_K$ denote the corresponding closed cones. $\boldsymbol{R}_+^n \subseteq \bigcup_{i=1}^{K} \sigma_i$ and if $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots K\}$ then $\bigcap_{j=1}^{k} \sigma_{i_j}$ is a closed cone, which is a face of each σ_{i_j}, hence is a simplicial cone. Furthermore the closed cone $\{\alpha \boldsymbol{1} \mid \alpha \geqslant 0\}$ is contained in every such cone.

Consider

$$\bigcup_{i=1}^{K}\sigma_i - \bigcup_{1\leqslant i_1<i_2\leqslant K}(\sigma_{i_1}\cap\sigma_{i_2}) + \cdots + (-1)^{j-1}$$

$$\times \bigcup_{1\leqslant i_1<\cdots<i_j\leqslant K}(\sigma_{i_1}\cap\cdots\cap\sigma_{i_j}) + \cdots + (-1)^{K-1}(\sigma_1\cap\cdots\cap\sigma_K). \tag{3}$$

Since every $(k_1,\ldots,k_n)\in \boldsymbol{Z}_+^{(n)}$ occurs exactly once in (3) we can write $Z(t)$ as the sum and difference of integrals of the form

$$\int_{X_\sigma}|f(x)|_v^s|\mathrm{d}x|_v \tag{4}$$

where $\sigma=\langle \boldsymbol{1},\ a^2,\ldots,a^l\rangle$ for some l, $1\leqslant l\leqslant n$, where the $l=1$ case is $\sigma=\langle\boldsymbol{1}\rangle$.

For each maximal dimension cone $\sigma_k=\langle \boldsymbol{1},\ a^2,\ldots,a^n\rangle$ write $a^i=(a_{i1},\ldots,a_{in})$, let $M_k=[a_{ij}]$, and let θ_k be the morphism defined by (1) in §1. Let S be the set of places satisfying the conditions in Proposition 3 c) for M_k, $1\leqslant k\leqslant K$.

We now fix $v\in S$, and $\sigma=\langle\boldsymbol{1},\ a^2,\ldots,a^l\rangle$. Choose a maximal dimension cone σ_k, $1\leqslant k\leqslant K$, such that σ_k contains σ. We denote this choice by $\tilde{\sigma}=\langle\boldsymbol{1},\ a^l,\ a^{l+1},\ldots,a^n\rangle$ and let M, θ be the matrix and morphism associated to $\tilde{\sigma}$. Referring to the decomposition of X_σ in Lemma 2 b) we can write (4) as a sum of integrals of the form

$$\int_{u\pi^w\theta(Y_\sigma)}|f(x)|^s|\mathrm{d}x| \tag{5}$$

for some $u=u_i$, $1\leqslant i\leqslant s(v)$, and $w=w^j$, $1\leqslant j\leqslant r$, where $Y_\sigma=R_v^{(l)}\times U_v^{(n-l)}$.

Write $f=\sum_I a_Ix^I$, then $f(u\pi^wx)=\sum_I a_Iu^I\pi^{w\cdot I}x^I$. We have $w\cdot I\geqslant m(w)$ for all $I\in\Gamma_+$, so we let

$$f_{u,w}(x)=\sum_I a_Iu^I\pi^{w\cdot I-m(w)}x^I. \tag{6}$$

Then the integral in (5) equals

$$q^{-|w|}t^{m(w)}\int_{\theta(Y_\sigma)}|f_{u,w}(x)|^s|\mathrm{d}x|.$$

By applying a) and b) in Proposition 3, in addition to the above observations, we have that the integral in the above is

$$\frac{1}{\kappa_\theta(v)}\int_{R_v}|y_1|^{ms+n-1}|\mathrm{d}y_1|\cdot\int_{Y'_\sigma}\prod_{i=2}^{n}|y_i|^{m(a^i)s+|a^i|-1}|g_{u,w}(y)|^s|\mathrm{d}y|$$

where $Y'_\sigma = R_v^{(l-1)}\times U_v^{(n-l)}$ and $g_{u,w}(y)\in R_v[y_2,\ldots,y_n]$. Applying (2) to the first integral we have that the contribution to $Z(t)$ from (5) is

$$q^n(1-q^{-1})(q^n-t^m)^{-1} \tag{7}$$

times

$$\frac{q^{-|w|}}{\kappa_\theta(v)}t^{m(w)}\int_{Y'_\sigma}\prod_{i=2}^{n}|y_i|^{m(a^i)s+|a^i|-1}|g_{u,w}(y)|^s|\mathrm{d}y_2\cdots\mathrm{d}y_n|. \tag{8}$$

By our observations above the factor (7) occurs for any integral of the form (5), so we can write

$$Z(t)=\frac{q^n(1-q^{-1})}{q^n-t^m}\tilde{Z}(t)$$

where $\tilde{Z}(t)$ is the sum and difference of expressions in the form of (8) for all possible σ, u, w. We shall show that (8) can be written in the form $P_{\sigma,u,w}(t)/Q(t)$ where $Q(t)=(q-t)\Pi(q^{|a^i|}-t^{m(a^i)})$ and the product is over all a^i dual to a noncompact facet of Γ_+. We then write $\tilde{Z}(t)=P(t)/Q(t)$ and

$$P(t)=\sum_\sigma(\operatorname{sign}\sigma)\sum_{u,w}P_{\sigma,u,w}(t) \tag{9}$$

where $\operatorname{sign}\sigma=\pm1$ is the coefficient of σ in the decomposition (3). Let $D=1+\Sigma m(a^i)=\deg Q(t)$. We shall show that after possibly excluding an additional finite set of places in S, that $\deg P(t)=D$, in which case the theorem follows.

Now consider $g_{u,w}(y)$. Referring back to $f_{u,w}(x)$ as given in (6) we see that $\bar{f}_{u,w}(x)=\sum_{I\in\tau_w}\bar{a}_i\bar{u}^Ix^I$, where τ_w is a face of Γ_+. We have that $\bar{f}_{u,w}$ is nondegenerate with respect to its Newton polyhedron since if τ' is a face of $\tau_{w'}$ and $b\in(\bar{k}_v\text{-}\{0\})^n$ is a solution to

$$\left(x_j\frac{\partial\bar{f}_{u,w}}{\partial x_j}\right)_{\tau'}=0\quad 1\leqslant j\leqslant n$$

then $\bar{u}b$ would be a solution to

$$\left(x_j\frac{\partial\bar{f}}{\partial x_j}\right)_{\tau'}=0\quad 1\leqslant j\leqslant n.$$

But τ' is a face of Γ_+, hence this contradicts the non-degeneracy of $\bar{f}$. Thus by applying Proposition 3 we have $\bar{g}_{u,w}(b) = 0$ implies

$$\left(y_j \frac{\partial \bar{g}_{u,w}}{\partial y_j}\right)(b) \neq 0 \tag{10}$$

for some j, $2 \leqslant j \leqslant n$.

First consider the case where $\sigma = \langle \boldsymbol{1}, a^2, \ldots, a^l \rangle$ with $l \geqslant 2$. We shall show that deg $P_{\sigma,u,w}(t) \leqslant D$. Then writing the coefficient $c_{\sigma,u,w}$ of t^D in $P_{\sigma,u,w}(t)$ as $(\kappa_\theta(v))^{-1}\tilde{c}_{\sigma,u,w}$ we show $q^{n-1}\tilde{c}_{\sigma,u,w} \equiv 0 \bmod q$.

If $w \neq \boldsymbol{0}$, since $w \in \sigma \cap \boldsymbol{Z}_+^{(n)}$ by permuting the vectors $\{a^2, \ldots, a^l\}$ we may suppose $w = \alpha_1 \boldsymbol{1} + \alpha_2 a^2 + \cdots + \alpha_k a^k$ where $0 < \alpha_i < 1$, $2 \leqslant i \leqslant k$, $0 \leqslant \alpha_1 < 1$ and $k \leqslant l$. When $w = \boldsymbol{0}$ set $k = 1$. Then we write the integral in (8) as

$$\int_{R_v^{(l-k)} \times U_v^{(n-l)}} \int_{R_v^{(k-1)}} \prod_{i=2}^{l} |y_i|^{m(a^i)s + |a^i| - 1} |g_{u,w}(y)|^s |\mathrm{d}y|. \tag{11}$$

We have

$$g_{u,w}(y) = \sum_I a_I u^I \pi^{w \cdot I - m(w)} y_2^{I \cdot a^2 - m(a^2)} \cdots y_n^{I \cdot a^n - m(a^n)}.$$

Observing that $I \in \tau_w$ implies $I \cdot a^i = m(a^i)$, $2 \leqslant i \leqslant k$, we have $\bar{g}_{u,w} \in k_v[y_{k+1}, \ldots, y_n]$. Thus in this case (10) specializes to $\bar{g}_{u,w}(b) = 0$, $b \in \bar{k}_v^n$ implies $(y_j\, \partial \bar{g}_{u,w}/\partial y_j)(b) \neq 0$ for some j, $k < j \leqslant n$; which implies the system of congruences

$$\begin{aligned} &g_{u,w}(y) \equiv 0 \bmod P_v \\ &\left(y_j \frac{\partial g_{u,w}}{\partial y_j}\right)(y) \equiv 0 \bmod P_v, \quad k < j \leqslant n \end{aligned} \tag{12}$$

has no solution in $R_v^{(n)}$.

For any subset $J \subseteq \{k+1, \ldots, l\}$ consider cosets $(c_{k+1}, \ldots, c_n) + P_v^{(n-k)_v}$ of $R_v^{(l-k)} \times U_v^{(n-l)}$ satisfying

$$\begin{aligned} &c_i \equiv 0 \bmod P_v \quad i \in J \\ &c_i \not\equiv 0 \bmod P_v \quad i \notin J \end{aligned} \tag{13}$$

and call these cosets of type J. We distinguish the cosets of type J further by saying a coset is of type J_1 if it satisfies $g_{u,w} \not\equiv 0 \bmod P_v$ in addition to the above conditions and say it is of type J_2 if it satisfies $g_{u,w} \equiv 0 \bmod P_v$ in addition to the above conditions. We then write (11) as a sum over varying J

of integrals of type

$$\int_{C_J}\int_{R_v^{(k-1)}}\prod_{i=2}^{l}|y_i|^{m(a^i)s+|a^i|-1}|g_{u,w}(y)|^s|dy| \tag{14}$$

where C_J is a coset of type J.

If C_J is a coset of type J_1, by applying the formulas (2), we have that the integral in (14) is of the form $P_1(t)/Q_1(t)$, where $\deg P_1(t)=\sum_{i\in J} m(a^i)$ and $\deg Q_1=\sum_{i=2}^{k} m(a^i)+\sum_{i\in J} m(a^i)$. If C_J is of type J_2 by (12) we can choose $j, k<j\leqslant n$, such that $y_j\,\partial g_{u,w}/\partial y_j\not\equiv 0 \bmod P_v$. We then make the change of variables $\tilde{y}_j=g_{u,w}$, $\tilde{y}_i=y_i$, $i\neq j$. Then the integral (14) is of the form $P_2(t)/Q_2(t)$, where $\deg P_2(t)=1+\sum_{i\in J} m(a^i)$ and $\deg Q_2(t)=1+\sum_{i=2}^{k} m(a^i)+\sum_{i\in J} m(a^i)$. In either case we have $P_i(t)/Q_i(t)=R_i(t)/Q(t)$ where $\deg R_i(t)=D-\sum_{i=2}^{k} m(a^i)$. Thus (11) is the sum of rational functions with this property, hence referring to (8) we see that for $w\neq \boldsymbol{0}$

$$\deg P_{\sigma,u,w}(t)\leqslant D+m(w)-\sum_{i=2}^{k} m(a^i).$$

Moreover the coefficient of the highest degree term in $P_{\sigma,u,w}(t)$ is

$$\pm\kappa_\theta(v)^{-1}q^{-|w|+\Sigma_{i=2}^{k}|a^i|}\sum_J(-1)^{|J|}(1-q^{-1})^{|J|+k-1}q^{-(n-k-|J|-1)}$$
$$\times\left[N_{J_1}q^{-1}+N_{J_2}(1-q^{-1})\right],$$

where N_{J_i} is the number of cosets of type J_i.

If $w=\boldsymbol{0}$, we have $\deg P_{\sigma,u,w}(t)\leqslant D$. If $w\neq\boldsymbol{0}$ in order to show this we must show $m(w)\leqslant\sum_{i=2}^{k} m(a^i)$. We have $w_j=\alpha_1+\sum_{i=2}^{k}\alpha_i a_{ij}$ where $\alpha_i<1$, $1\leqslant i\leqslant k$, hence $w_j<1+\sum_{i=2}^{k} a_{ij}$, and $w_j\in\boldsymbol{Z}$ implies $w_j\leqslant\sum_{i=2}^{k} a_{ij}$. Now let $P\in E(\Gamma_+)$ be such that $\tilde{\sigma}$ is obtained from the partition of $\check{\sigma}_P$ into simplicial cones. Write $P=(P_1,\dots,P_n)$. We have $m(w)=P\cdot w$ and $m(a^i)=P\cdot a^i$, $2\leqslant i\leqslant k$. Hence

$$m(w)\leqslant\sum_{j=1}^{n}P_j\left(\sum_{i=2}^{k}a_{ij}\right)=\sum_{i=2}^{k}P\cdot a^i.$$

Thus $m(w)\leqslant\sum_{i=2}^{k} m(a^i)$, which implies $\deg P_{\sigma,u,w}(t)\leqslant D$.

Now consider $c_{\sigma,u,w}$. If deg $P_{\sigma,u,w}(t) < D$ then $c_{\sigma,u,w} = 0$. If deg $P_{\sigma,u,w}(t) = D$ then by observing that $N_{J_1} + N_{J_2} = (q-1)^{n-k-|J|}$ we have

$$q^{n-1}\tilde{c}_{\sigma,u,w} = \pm q^{-|w|+\sum_{i=2}^{k}|a^i|}\sum_J (-1)^{|J|}(q-1)^{|J|+k-1}$$
$$\times\left[(q-1)^{n-k-|J|} - N_J - N_J(q-1)\right],$$

where we let $N_J = N_{J_2}$. If $|w| < \sum_{i=2}^{k} |a^i|$ then $q^{n-1}\tilde{c}_{\sigma,u,w}$ is clearly congruent to zero mod q, but if $|w| = \sum_{i=2}^{k} |a^i|$ then

$$q^{n-1}\tilde{c}_{\sigma,u,w} \equiv \pm \sum_J (-1)^{|J|} \bmod q \equiv 0 \bmod q. \tag{15}$$

This proves our assertion about the case $\sigma = \langle \mathit{1}, a^2, \ldots, a^l \rangle$ with $l \geqslant 2$.

The only remaining cases to consider are those where u varies and $\sigma = \langle \mathit{1} \rangle$. In this case we show that (8) can be written as $P_{1,u}(t)/Q(t)$ where deg $P_{1,u}(t) \leqslant D$. Denoting the coefficient of t^D by $c_{1,u}$ and defining $\tilde{c}_{1,u}$ as in the previous case we show $q^{n-1}\tilde{c}_{1,u}$ is an integer and $q^{n-1}\tilde{c}_{1,u} \not\equiv 0 \bmod q$.

In this case the integral in (8) is

$$\int_{U_v^{(n-1)}} |g_u(y)|^s |\mathrm{d}y|.$$

Consider the cosets mod $P_v^{(n-1)}$ of $U_v^{(n-1)}$. Letting N be the number of cosets satisfying $g_u \equiv 0 \bmod P_v$ and applying entirely similar reasoning as before we have that the above integral equals

$$(q^{-1})^{(n-2)}\left[\left((q-1)^{n-1} - N\right)q^{-1} + \frac{N(1-q^{-1})t}{(q-t)}\right].$$

Then examination of the above shows deg $P_{1,u}(t) \leqslant D$ and

$$q^{n-1}\tilde{c}_{1,u} = (-1)^D\left[(q-1)^{n-1} - N - N(q-1)\right].$$

Hence $q^{n-1}\tilde{c}_{1,u}$ is an integer and

$$q^{n-1}\tilde{c}_{1,u} \equiv \pm 1 \bmod q. \tag{16}$$

Furthermore we note that the value on the right of the congruence is independent of u.

Let c_v denote the coefficient of t^D in $P(t)$, which we wish to show is nonzero for almost all v. We have $c_v = \sum_\sigma (\text{sign } \sigma) \sum_{u,w} c_{\sigma,u,w}$. Recalling that the morphisms associated to the maximal dimension cones were denoted $\theta_1, \ldots, \theta_K$ we define $\kappa(v) = \prod_{i=1}^{K} \kappa_{\theta_i}(v)$. If σ is a cone, and the morphism associated to the maximal dimension cone $\tilde{\sigma}$ is θ_j, define $\kappa_\sigma = \prod_{\substack{i=1\\ i\neq j}}^{K} \kappa_{\theta_i}(v)$. We assume θ_1 is the morphism associated to $\langle \mathbf{\mathit{1}} \rangle$, and let $\kappa_1(v) = \prod_{i=2}^{K} \kappa_{\theta_i}(v)$. Then

$$q_v^{n-1}\kappa(v)c_v = \sum_{\sigma \neq \langle \mathbf{\mathit{1}} \rangle} (\text{sign } \sigma) \sum_{u,w} q_v^{n-1}\kappa_\sigma(v)\tilde{c}_{\sigma,u,w} \pm \sum_{u} q_v^{n-1}\kappa_1(v)\tilde{c}_{1,u}.$$

Let $s_1(v)$ denote the number of coset representatives in $U_v^n/\theta_1(U_v^n)$. Then the congruences in (15) and (16) give

$$q_v^{n-1}\kappa(v)c_v \equiv \pm\kappa_1(v)s_1(v) \bmod q_v.$$

Now

$$\kappa_1(v) \leqslant \prod_{i=2}^{K} \text{card } W_{v,|M_i|}^{(n)} \leqslant n^{K-1} \prod_{i=2}^{K} |M_i|$$

where $|M_i|$ is the determinant of the matrix M_i associated to θ_i and $W_{v,|M_i|}$ is the $|M_i|$-th roots of unity in U_v. We also have

$$s_1(v) \leqslant n \cdot \left[U_v : U_v^{|M_1|}\right]$$

where $[U_v : U_v^{|M_1|}] = \text{card } W_{v,|M_1|}$ for almost all v. Hence for almost all v

$$\kappa_1(v)s_1(v) \leqslant n^K \prod_{i=1}^{K} |M_i|$$

which implies $\kappa_1(v)s_1(v) \not\equiv 0 \bmod q_v$ for almost all v. Therefore $c_v \neq 0$ for almost all v, which concludes the proof. Q.E.D.

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Addendum. J. Denef has recently given a proof of Igusa's conjecture in the general case.