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Relative epimorphisms and monomorphisms in homotopy theory

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0. Introduction

In this paper, the methods and results of earlier papers on epimorphisms and monomorphisms in homotopy theory [Hilton and Roitberg, 1984; Roitberg, 1983] are re-examined and refined.

The first point to mention is that only very mild consequences of the notions of epimorphism and monomorphism are utilized in [Hilton and Roitberg, 1984; Roitberg, 1983]. We are thus led to define formally, in an arbitrary category C , the notions of *epimorphism and monomorphism relative to a class S of objects of C* (or, more briefly, *S -epi* and *S -mono*). In similar fashion, we may introduce the notions of *weak S -epi* and *weak S -mono*, where now C is required to be a category with zero objects; see [Hilton, 1965; Roitberg, 1986]. The results of [Hilton and Roitberg, 1984; Roitberg, 1983] may then be conveniently reformulated in the new language, where $C = H$ is the pointed homotopy category of path-connected CW-spaces and S is a suitably chosen class.

The second point is to study in greater depth the map on (integral) homology groups, resp. homotopy groups, induced by a weak S -epi, resp. weak S -mono in H . It is not difficult to see that a weak (absolute) epi $X \xrightarrow{f} Y$ induces a *split epi* $H_n X \xrightarrow{f_*} H_n Y$, $n \geq 0$. For, as noted in [Roitberg, 1986], the cofibration sequence

$$X \xrightarrow{f} Y \xrightarrow{q} C \xrightarrow{\delta} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \rightarrow \quad (0.1)$$

attached to f then has the property that $q = 0$. (In this paper, we blur the distinction, in our notation, between a map and its homotopy class.) Hence $\Sigma X \simeq C \vee \Sigma Y$ and the homology assertion is a ready consequence. This observation, somehow overlooked in ([Roitberg, 1983], see the mystifying Remark (2)), allows both generalization of the results and simplification of the proofs in [Hilton and Roitberg, 1984; Roitberg, 1983]. Of course, the condition $q = 0$ is equivalent to the condition that f is a weak epi.

It turns out that the same homology conclusion can be drawn if it is merely required that f be a weak K -epi, where K is the subclass of H consisting of Eilenberg-MacLane spaces – one may even restrict to the subclass K_0 consisting of $K(\pi, n)$ with π abelian.

The dual situation is quite similar. A weak (absolute) mono $X \xrightarrow{f} Y$ induces a split mono $\pi_n X \xrightarrow{f_*} \pi_n Y$, $n \geq 2$. Indeed, the fibration sequence

$$\rightarrow \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{\partial} F \xrightarrow{i} X \xrightarrow{f} Y \tag{0.1'}$$

attached to f then satisfies $i = 0$ (this condition is equivalent to f being a weak mono), so that $\Omega Y \simeq F \times \Omega X$. Now if f is merely required to be a weak K' -mono, where K' is the class consisting of Moore spaces, it may be inferred that $\pi_n X \xrightarrow{f_*} \pi_n Y$ is split mono, at least provided $n \geq 3$. The case $n = 2$ may be exceptional due to the non-existence of certain $K'(\pi, 1)$ -spaces, but causes no difficulty if $\pi_2 Y$ is finitely generated.

At this juncture, work of [Felix and Lemaire, 1985] may be brought into the picture. Without using the terminology of weak monos, Lemma 1 of [Felix and Lemaire, 1985], together with the proof of Proposition 1 of [Felix and Lemaire, 1985], gives a sufficient condition for a map $X \xrightarrow{f} Y$ of 1-connected spaces, inducing a split mono of higher homotopy groups, to be a weak mono; for conditions (a)–(d) of Lemma 1 of [Felix and Lemaire, 1985] are all equivalent to f being a weak mono. Their result, which is in fact valid even if X, Y are not 1-connected (it suffices to require that $\pi_1 X \xrightarrow{f_*} \pi_1 Y$ be mono), may be recast so as to give a sufficient condition for a weak K' -mono to be a weak mono. Moreover, the original [Felix and Lemaire, 1985] result and its recasting may be dualized, thereby giving a sufficient condition for a weak K_0 -epi to be a weak epi. The work in [Felix and Lemaire, 1985] is applied, in particular, to a map $X \xrightarrow{f} Y$ of rational spaces, and our dual results also apply to this situation.

These results suggest studying rationalizations, or, more generally, P -localizations of (weak) epis and (weak) monos of nilpotent spaces. It is, of course, well-known that, in the category of groups, P -localization of nilpotent groups preserves both epis and monos. It is also true that, for nilpotent groups, the notions of epi and weak epi coincide, as do the notions of mono and weak mono. Thus it is satisfactory that, in H , P -localization of nilpotent spaces preserves weak epis and weak monos. We do not know whether P -localization of nilpotent spaces preserves epis and monos in H , but we do obtain some partial information.

The remainder of the paper is organized as follows. In §1, the notions of (weak) epi and (weak) mono relative to a class S are defined and a number of

examples are given. In §2, we relate weak S -epis and weak S -monos for certain natural classes S of objects of H with the homomorphisms these induce on (integral) homology and homotopy respectively and discuss the work of [Felix and Lemaire, 1985] and its dualization in our framework. In §3, we make explicit how the results of [Hilton and Roitberg, 1984; Roitberg, 1983] may be improved. Finally, in §4, we study the effect of P -localizing (weak) epis and (weak) monos together with various related questions.

1. The fundamental definitions

Let C be a category, S a class of objects of C .

Definition 1.1: An *epimorphism relative to S* , or an *S -epi*, is a morphism $X \xrightarrow{f} Y$ in C such that, whenever $Y \xrightarrow{u, v} Z$ are morphisms in C with Z in S for which $u \cdot f = v \cdot f$, then $u = v$.

Suppose, moreover, that C is a category with zero objects.

Definition 1.2: A *weak epimorphism relative to S* , or a *weak S -epi*, is a morphism $X \xrightarrow{f} Y$ in C such that whenever $Y \xrightarrow{u} Z$ is a morphism in C with Z in S for which $u \cdot f = 0$, then $u = 0$.

The dual notions of *monomorphism relative to S* (*S -mono*) and *weak monomorphism relative to S* (*weak S -mono*) are defined in the evident way.

Note that in Definitions 1.1. and 1.2., u, v are allowed to range over all morphisms in C from Y to Z . Taking S to be a *subcategory* of C rather than a class of objects of C , we could formulate modified versions of the definitions. Thus, if Y is an object of S and u, v are restricted to range over the morphisms in S from Y to Z , then rather different notions of S -epi and weak S -epi would emerge. [For instance, if C is the category with objects subspaces of \mathbb{R} and morphisms all (not necessarily continuous) maps between such spaces and S is the non-full subcategory with objects subspaces of \mathbb{R} and morphisms all continuous maps between such spaces, then the inclusion $\mathbb{Q} \rightarrow \mathbb{R}$, though not an S -epi in the sense of Definition 1.1., would be in the modified version.]

In this paper, C will always be taken to be H , the pointed homotopy category of path-connected CW-spaces. An example of an S -mono in H already explicit in the literature is that of the natural map of a nilpotent space X into the Cartesian product $\prod X_p$ of its p -localizations ([Hilton *et al.*, 1975], Th. II.5.3); here S is the class consisting of spaces of the homotopy type of a finite CW-complex. An example of an S -epi in H is implicit in the final remark of [Roitberg, 1986]. The essence of the latter is that a certain weak epi constructed in ([Roitberg, 1986], Th. 2.1), while not an epi, is in fact an S -epi

where S is the class consisting of spaces with finitely generated second homotopy group. In the sequel, the following classes S will play a prominent role.

Let K be the class consisting of Eilenberg-MacLane spaces $K(\pi, n)$, $n \geq 1$, and let K_0 be the subclass of K consisting of those $K(\pi, n)$ with π abelian. Note that the notions of K_0 -epi and weak K_0 -epi coincide,¹ and that a morphism $X \xrightarrow{f} Y$ is a K_0 -epi precisely when $f^* : H^*(Y; A) \rightarrow H^*(X; A)$ is mono for all constant coefficient groups A . Similarly, a morphism $X \xrightarrow{f} Y$ is a (weak) K -epi precisely when $f^* : H^*(Y; A) \rightarrow H^*(X; A)$ is mono for all constant coefficient groups A and, in addition, $f_* : \pi_1 X \rightarrow \pi_1 Y$ is a (weak) epi of groups. Notice that while, as stated, weak K -epis are actually K_0 -epis, there are weak K -epis which are not K -epis (compare [Roitberg, 1986], Th. 2.1). However, we have:

PROPOSITION 1.1: *If $\pi_1 Y$ is a nilpotent group, e.g. if Y is a nilpotent space, then any K_0 -epi $X \xrightarrow{f} Y$ is K -epi.*

Proof: Assuming $f^* : H^*(Y; A) \rightarrow H^*(X; A)$ for all constant A , we must show $f_* : \pi_1 X \rightarrow \pi_1 Y$. Now the Universal Coefficient Theorem

$$\begin{array}{ccc} H^1(Y; A) \cong \text{Hom}(H_1 Y, A) & & \\ \downarrow f_* & & \downarrow f_* \\ H^1(X; A) \cong \text{Hom}(H_1 X, A), & & \end{array}$$

applied to $A = \text{coker}[H_1 X \xrightarrow{f_*} H_1 Y]$, shows that $f_* : H_1 X \rightarrow H_1 Y$ is epi. Since $H_1 Y$ is the abelianization of $\pi_1 Y$, the fact that $f_* : \pi_1 X \rightarrow \pi_1 Y$ is epi results from the nilpotence of $\pi_1 Y$ (compare [Hilton and Roitberg, 1976], Cor 3.2). Example of K -epis are abundant.

PROPOSITION 1.2: *Let C_α be the mapping cone associated to the homotopy element $\alpha \in \pi_{n-1} S^m$, $m < n - 1$, and $C_\alpha \xrightarrow{f} S^n$ the collapsing map. Then f is always a K -epi but is weak epi $\Leftrightarrow \Sigma\alpha = 0$.*

Proof: Clearly, $f^* : H^*(S^n; A) \rightarrow H^*(C_\alpha; A)$ for all constant A and $f_* : \pi_1 C_\alpha \rightarrow \pi_1 S^n$. On the other hand, since the cofibration sequence attached to f reads

$$C_\alpha \xrightarrow{f} S^n \xrightarrow{\Sigma\alpha} S^{m+1},$$

we may appeal to ([Roitberg, 1986], Prop. 2.2).

¹ In fact, whenever the objects of S are grouplike spaces, the notions of S -epi and weak S -epi coincide. Moreover, if $X \xrightarrow{f} Y$ is a suspension or more generally, a structure-preserving map of cogrouplike spaces, then, for any S , f is an S -epi $\Leftrightarrow f$ is a weak S -epi.

Remark: A similar example is obtained by taking a closed, orientable n -manifold M^n and collapsing the complement of a coordinate chart to a point, thus yielding $M^n \xrightarrow{f} S^n$.

The entire discussion above may be dualized. In place of K , we substitute K' , the class consisting of Moore spaces $K'(\pi, n)$, $n \geq 1$; recall that a Moore space $K'(\pi, 1)$ is a space X such that $\pi_1 X = \pi$ and $H_i X = 0$, $i \geq 2$ (of course, the homotopy type of such a space is not, in general, determined by π). As shown in [Varadarajan, 1966], such a space exists ² if and only if $H_2 \pi = 0$.

An interesting subclass of K' is the class K'_1 consisting of Moore spaces $K'(\pi, n)$ with $n \geq 2$. All such spaces are suspensions. This is obvious if $n \geq 3$, since then $K'(\pi, n) = \Sigma K'(\pi, n - 1)$. However, it is also true if $n = 2$ since we may always find a (connected) space X with $H_1 X = \pi$, $H_i X = 0$, $i \geq 2$, and then $K'(\pi, 2) = \Sigma X$. Such a space X is a Moore space, but not necessarily $K'(\pi, 1)$. Since the objects of K'_1 are suspensions, the notions of K'_1 -mono and weak K'_1 -mono coincide (dualize footnote 1!). It may be mentioned that a dual of Proposition 1.2. may be formulated, involving Eilenberg-MacLane spaces and 2-stage Postnikov systems. Details are omitted.

2. Characterization of relative epis and monos

We begin by characterizing K_0 -epis $X \xrightarrow{f} Y$ in terms of the induced homomorphisms in integral homology $H_* X \xrightarrow{f_*} H_* Y$.

THEOREM 2.1: $X \xrightarrow{f} Y$ is K_0 -epi $\Leftrightarrow H_n X \xrightarrow{f_*} H_n Y$ is split epi for all n .

Proof: The proof is based on a more judicious exploitation of the Universal Coefficient Theorem than that in [Roitberg, 1983]. Thus, for all constant A , there is a commutative diagram

$$\begin{array}{ccccc} \text{Ext}(H_{n-1}Y, A) & \twoheadrightarrow & H^n(Y; A) & \twoheadrightarrow & \text{Hom}(H_n Y, A) \\ \downarrow f_1^* & & \downarrow f^* & & \downarrow f_2^* \\ \text{Ext}(H_{n-1}X, A) & \twoheadrightarrow & H^n(X; A) & \twoheadrightarrow & \text{Hom}(H_n X, A). \end{array} \tag{2.1}$$

If f_* is split epi, then f_1^* and f_2^* in (2.1) are (split) mono. Hence f^* is also mono and f is a K_0 -epi.

Conversely, suppose $f^* : H^n(Y; A) \twoheadrightarrow H^n(X; A)$ for all constant A . Then $q^* : H^n(C; A) \rightarrow H^n(Y; A)$ is the 0 map, where $Y \xrightarrow{q} C$ is as in (0.1). The

² Proposition 1 of [Varadarajan, 1966] only asserts this if π is abelian, but it is, in fact, true in general. The definition of a Moore space $K'(\pi, 1)$ in [Varadarajan, 1966] required π to be abelian, but we do not adopt this restrictive viewpoint.

Universal Coefficient Theorem

$$\begin{array}{ccc} H^n(C; A) & \rightarrow & \text{Hom}(H_n C, A) \\ \downarrow 0 & & \downarrow \\ H^n(Y; A) & \rightarrow & \text{Hom}(H_n Y, A) \end{array} ,$$

applied to $A = H_n C$, shows that $q_* : H_n Y \rightarrow H_n C$ is the 0 map, whence $f_* : H_n X \rightarrow H_n Y$ is epi. If now $K = \text{Ker}[H_{n-1} X \xrightarrow{f_*} H_{n-1} Y]$, so that $K = H_n C$, the short exact sequence

$$K \rightarrow H_{n-1} X \xrightarrow{f_*} H_{n-1} Y \tag{2.2}$$

defines an element ξ in $\text{Ext}(H_{n-1} Y, K)$; ξ is precisely the image of $1_K \in \text{Hom}(K, K)$ under the homomorphism ω in the Hom-Ext exact sequence

$$\rightarrow \text{Hom}(K, K) \xrightarrow{\omega} \text{Ext}(H_{n-1} Y, K) \xrightarrow{f_1^*} \text{Ext}(H_{n-1} X, K) \rightarrow .$$

But since f^* , hence also f_1^* , is mono, $\xi = \omega(1_K) = 0$ and (2.2) splits.

It is possible to dualize Theorem 2.1. but care must be exercised since, unlike cohomology with coefficients $H^n(-; A)$, homotopy with coefficients $\pi_n(-; A)$ is not defined for all integers n and all abelian groups A . The simplest result is as follows.

THEOREM 2.1': $X \xrightarrow{f} Y$ is a K'_1 -mono $\Leftrightarrow \pi_n X \xrightarrow{f_*} \pi_n Y$ is mono for all $n \geq 2$, split mono for all $n \geq 3$.

Proof: According to the Eckmann-Hilton Universal Coefficient Theorem ([Hilton, 1965]), there is, for all A , a commutative diagram

$$\begin{array}{ccc} \text{Ext}(A, \pi_{n+1} X) & \rightarrow & \pi_n(X; A) \rightarrow \text{Hom}(A, \pi_n X) \\ \downarrow f_{1*} & & \downarrow f_* \quad \downarrow f_{2*} \\ \text{Ext}(A, \pi_{n+1} Y) & \rightarrow & \pi_n(Y; A) \rightarrow \text{Hom}(A, \pi_n Y), \end{array} \tag{2.1'}$$

$n \geq 2$; here $\pi_n(-; A)$ means $[K'(A, n), -]$, $K'(A, n)$ being equipped with its suspension structure (unique if $n \geq 3$).

If $f_* : \pi_m X \rightarrow \pi_m Y$ is mono, then $f_{2*} : \text{Hom}(A, \pi_m X) \rightarrow \text{Hom}(A, \pi_m Y)$ is mono; and if $f_* : \pi_m X \rightarrow \pi_m Y$ is split mono, then $f_{1*} : \text{Ext}(A, \pi_m X) \rightarrow \text{Ext}(A, \pi_m Y)$ is (split) mono. Thus, if $f_* : \pi_n X \rightarrow \pi_n Y$ is mono for $n \geq 2$ and split mono for $n \geq 3$, then $f_* : \pi_n(X; A) \rightarrow \pi_n(Y; A)$ is mono for $n \geq 2$.

Conversely, suppose $f_* : \pi_n(X; A) \rightarrow \pi_n(Y; A)$, $n \geq 2$. Taking $A = \mathbb{Z}$, we infer $f_* : \pi_n X \rightarrow \pi_n Y$, $n \geq 2$. Setting $L = \text{coker} [\pi_{n+1} X \xrightarrow{f_*} \pi_{n+1} Y]$, so that $L =$

$\pi_n F$ where F is the homotopy-fiber of f , and studying the short exact sequence

$$\pi_{n+1} X \xrightarrow{f_*} \pi_{n+1} Y \rightarrow L \tag{2.2'}$$

in a manner dual to that in the proof of Theorem 2.1., we readily find that (2.2') splits if $n \geq 2$.

By adjoining suitable objects to K'_1 , variants of Theorem 2.1'. may be obtained. For instance, if $S = K'_1 \cup \{S^1\}$, then $X \xrightarrow{f} Y$ is an S -mono $\Leftrightarrow f_* : \pi_n X \rightarrow \pi_n Y$ is mono for all $n \geq 1$, split mono for all $n \geq 3$. A more interesting version of Theorem 2.1'. is the following.

ADDENDUM TO THEOREM 2.1': Let $X \xrightarrow{f} Y$ be such that $L = \text{coker} [\pi_2 X \xrightarrow{f_*} \pi_2 Y]$ is finitely generated. Then $X \xrightarrow{f} Y$ is a weak K' -mono $\Leftrightarrow \pi_n X \xrightarrow{f_*} \pi_n Y$ is mono for all $n \geq 1$, split mono for all $n \geq 2$.

Proof: The proof is based on the fact that (2.1') may be replaced, if $n = 1$, by

$$\begin{array}{ccc} \text{Ext}(A_{ab}, \pi_2 X) \rightarrow \pi_1(X; A) \rightarrow \text{Hom}(A, \pi_1 X) & & (2.1'_1) \\ \downarrow & \downarrow & \downarrow \\ \text{Ext}(A_{ab}, \pi_2 Y) \rightarrow \pi_1(Y; A) \rightarrow \text{Hom}(A, \pi_1 Y), & & \end{array}$$

provided that $H_2 A = 0$. Of course, (2.1'_1) is to be interpreted as a map of exact sequences of pointed sets. To prove (2.1'_1), note that $\ker(\pi_1(X; A) \rightarrow \text{Hom}(A, \pi_1 X))$ is just $\pi_1(\tilde{X}; A)$, where \tilde{X} is the universal cover of X . Now let $\Gamma \rightarrow A$ be a free presentation of A , let $F = \Gamma_{ab}$, and let $R = \ker(F \rightarrow A_{ab})$. We have a Puppe sequence

$$K'(\Gamma, 1) \rightarrow K'(A, 1) \rightarrow K'(R, 2) \rightarrow K'(F, 2) \rightarrow \dots$$

giving rise to the exact sequence

$$\dots \rightarrow \pi_2(\tilde{X}; F) \rightarrow \pi_2(\tilde{X}; R) \rightarrow \pi_1(\tilde{X}; A) \rightarrow 0,$$

or

$$\dots \rightarrow \text{Hom}(F, \pi_2 \tilde{X}) \rightarrow \text{Hom}(R, \pi_2 \tilde{X}) \rightarrow \pi_1(\tilde{X}; A).$$

Thus

$$\text{Ext}(A_{ab}, \pi_2 \tilde{X}) \cong \pi_1(\tilde{X}; A);$$

this, together with naturality, establishes (2.1'_1).³

³ Our argument shows that $\pi_1(\tilde{X}; A)$ is an abelian group.

The implication \Leftarrow of the Addendum now follows along standard lines. For the implication \Rightarrow , it remains to establish that

$$\pi_2 X \xrightarrow{f_*} \pi_2 Y \rightarrow L$$

splits. But it follows from (2.1') that $f_{1*} : \text{Ext}(A, \pi_2 X) \rightarrow \text{Ext}(A, \pi_2 Y)$ for A cyclic. As L is finitely generated, we infer that $f_{1*} : \text{Ext}(L, \pi_2 X) \rightarrow \text{Ext}(L, \pi_2 Y)$, hence (as usual) that the short exact sequence in question does indeed split.

The condition that a map $X \xrightarrow{f} Y$ induces a split mono $\pi_n X \xrightarrow{f_*} \pi_n Y$, $n \geq 2$, appears in a key Proposition of [Felix and Lemaire, 1985] leading to a result about Lusternik-Schnirelmann category. In fact, ([Felix and Lemaire, 1985], Prop. 1) admits a dual which, in conjunction with Theorem 2.1., yields

THEOREM 2.2: *Let $X \xrightarrow{f} Y$ be a weak K -epi and suppose that ΣX splits as a wedge of (1-connected) Moore spaces. Then f is a weak epi.*

Proof: In the cofibration sequence (0.1), it is to be shown that $q = 0$. To that end, it suffices to split δ , that is to find $s : \Sigma X \rightarrow C$ such that $s \cdot \delta = 1_C$. As f is a weak K -epi, hence also a K_0 -epi, we infer from Theorem 2.1. that the homology sequence

$$H_* C \xrightarrow{\delta_*} H_* \Sigma X \xrightarrow{(\Sigma f)_*} H_* \Sigma Y$$

is a split short exact sequence. Thus, $\exists \sigma : H_* \Sigma X \rightarrow H_* C$ such that $\sigma \cdot \delta_*$ is the identity. Now σ is induced by a (not necessarily unique) map $K'(\sigma) : K'(H_* \Sigma X) \rightarrow K'(H_* C)$, i.e. $K'(\sigma)_* = \sigma$; here we use the notation $K'(H_* Z)$ for $\bigvee_{n \geq 2} K'(H_n Z, n)$ where $H_1 Z = 0$. From the hypothesis, we may find a homotopy equivalence $h_1 : \Sigma X \rightarrow K'(H_* \Sigma X)$ inducing the identity on homology. Then the composite

$$C \xrightarrow{\delta} \Sigma X \xrightarrow{h_1} K'(H_* \Sigma X) \xrightarrow{K'(\sigma)} K'(H_* C)$$

also induces the identity on homology. As $f_* : \pi_1 X \rightarrow \pi_1 Y$ is a weak epi, we infer from van Kampen's Theorem that C is 1-connected. It follows that $K'(\sigma) \cdot h_1 \cdot \delta$ is a homotopy equivalence, say h_2 . Setting $s = h_2^{-1} \cdot K'(\sigma) \cdot h_1$, we find $s \cdot \delta = 1_C$, as desired.

Remark: The above proof establishes that $C \simeq K'(H_* C)$, so that C is itself a suspension. Furthermore, since the composite

$$\Sigma X \xrightarrow{\text{comult}} \Sigma X \vee \Sigma X \xrightarrow{s \vee \Sigma f} C \vee \Sigma Y$$

is a homology equivalence, hence a homotopy equivalence, it follows readily that $\Sigma Y \simeq K'(H_*\Sigma Y)$.

Before enunciating the dual of Theorem 2.2., it is convenient to formulate a result which allows us, in many instances, to restrict attention to 1-connected spaces when considering relative weak monos.

LEMMA 2.1: *Let S be any class of objects of H containing S^1 . Then $X \xrightarrow{f} Y$ is a weak S -mono $\Leftrightarrow \pi_1 X \xrightarrow{f_*} \pi_1 Y$ is mono and $\tilde{X} \xrightarrow{\tilde{f}} \tilde{Y}$ is a weak S -mono, where \tilde{X} and \tilde{Y} denote the universal covers of X and Y .*

Proof: For any object W of H , we have a commutative diagram of homotopy sets

$$\begin{array}{ccc} [W, \tilde{X}] & \xrightarrow{\tilde{f}_*} & [W, \tilde{Y}] \\ \downarrow p_{X*} & & \downarrow p_{Y*} \\ [W, X] & \xrightarrow{f_*} & [W, Y], \end{array} \tag{2.3}$$

$\tilde{X} \xrightarrow{p_X} X, \tilde{Y} \xrightarrow{p_Y} Y$ being the respective covering projections. Clearly, p_{X*}, p_{Y*} are mono, since p_X, p_Y are mono.

If f is a weak S -mono, then f_* , in (2.3), is a weak mono for W in S . In particular, $f_*: \pi_1 X \rightarrow \pi_1 Y$. Moreover, by (2.3), \tilde{f}_* is then also a weak mono for W in S , so that \tilde{f} is a weak S -mono.

Conversely, suppose $f_*: \pi_1 X \rightarrow \pi_1 Y$ and \tilde{f} is a weak S -mono. If W is in S and if $f_*(g) = 0, g \in [W, X]$, then $f_*g_*: \pi_1 W \rightarrow \pi_1 Y$ is trivial. Thus $g_*: \pi_1 W \rightarrow \pi_1 X$ is trivial and g lifts to $\bar{g} \in [W, \tilde{X}]$. Since, by (2.3),

$$p_{Y*}\tilde{f}_*(\bar{g}) = f_*p_{X*}(\bar{g}) = f_*(g) = 0,$$

it follows that $\bar{g} = 0$. Thus $g = 0$ and f is a weak S -mono.

Remark: An argument similar to the one in the first paragraph of the above proof shows that f is an S -mono $\Rightarrow f_*: \pi_1 X \rightarrow \pi_1 Y$ and \tilde{f} is an S -mono. Is the converse true?

THEOREM 2.2': *Let $X \xrightarrow{f} Y$ be a weak K' -mono and suppose that (each path-component of) ΩY splits as a product of Eilenberg-MacLane spaces. If $\pi_1 F$ is finitely generated, where F is as in (0.1'), then f is a weak mono.*

Proof: By Lemma 2.1., it suffices to show that \tilde{f} is a weak mono. Thus we may suppose, without loss of generality, that X and Y are themselves 1-connected

and therefore that all spaces occurring in (0.1') are objects of H , that is, path-connected.

To show $F \xrightarrow{i} X$ in (0.1') is 0, we seek $t: \Omega Y \rightarrow \Omega X$ such that $\Omega f \cdot t = 1_{\Omega Y}$. By the Addendum to Theorem 2.1', $f_*: \pi_n X \rightarrow \pi_n Y$ is split mono, $n \geq 2$. Appeal to ([Felix and Lemaire, 1985], Prop. 1) then allows completion of the proof of Theorem 2.2'.

Interesting consequences of the foregoing results arise when a 'rationality' condition is imposed on the spaces X and Y .

THEOREM 2.3: *Let X and Y be rational, nilpotent spaces. Then $X \xrightarrow{f} Y$ is a weak epi $\Leftrightarrow H_* X \xrightarrow{f_*} H_* Y$ is epi.*

Proof: The implication \Rightarrow is plain so we assume $f_*: H_* X \rightarrow H_* Y$. Since f_* is certainly split epi, Theorem 2.1. shows that f is a K_0 -epi. Moreover, as Y is nilpotent, Proposition 1.1. implies that f is a K -epi. Now ΣX (as well as ΣY) is a (1-connected) rational co- H -space. Hence, according to [Henn, 1983], ΣX is homotopy equivalent to a wedge of rationalized spheres S_0^n , $n \geq 2$, and $S_0^n = K'(\mathbb{Q}, n)$. We conclude from Theorem 2.2. that f is a weak epi.

THEOREM 2.3': *Let X and Y be 'almost' rational spaces in the sense that the universal covers \tilde{X} and \tilde{Y} are rational spaces (compare [Henn, 1983]). Then $X \xrightarrow{f} Y$ is a weak mono $\Leftrightarrow \pi_* X \xrightarrow{f_*} \pi_* Y$ is mono.*

Details of the proof are easily supplied. The important special case of Theorem 2.3' where X and Y are 1-connected is due to [Felix and Lemaire, 1985].

3. Hopfian and co-Hopfian objects

In this brief section, we make a quick review of [Hilton and Roitberg, 1984; Roitberg, 1983], the aim being, on the one hand, to indicate simplifications of the proofs and, on the other, to generalize the central results therein. We recall that a Hopfian object in the category C is an object X such that any epimorphism $f: X \rightarrow X$ in C is invertible; the dual notion is that of a co-Hopfian object.

The first theorem improves both ([Hilton and Roitberg, 1984], Th. 3) and ([Roitberg, 1983], Th. 1.1).

THEOREM 3.1: *If $X \xrightarrow{f} X$ is a K_0 -epi and if, for each n , $H_n X$ is either a Hopfian or a co-Hopfian group, then f is a homology equivalence. If, moreover, X is in NH , the class consisting of nilpotent spaces, then X is a Hopfian object of H .*

Proof: Only the first sentence requires comment. By Theorem 2.1., $H_n X \xrightarrow{f_*} H_n X$ is split epi for all n . Now a group which is either Hopfian or co-Hopfian cannot be a nontrivial direct summand of itself, Hence, $H_n X \xrightarrow{f_*} H_n X$ must be an isomorphism for all n .

Dually, using Theorem 2.1'. and its Addendum, we may improve ([Hilton and Roitberg, 1984], Th. 7).

THEOREM 3.1': *If $X \xrightarrow{f} X$ is either i) a K_1' -mono with $\pi_2 X$ a co-Hopfian group or ii) a K' -mono with $\pi_2 X$ finitely generated, and if, for each $n \geq 3$, $\pi_n X$ is either a Hopfian or a co-Hopfian group, then $f = p \cdot \bar{f}$ with $X \xrightarrow{\bar{f}} \bar{X}$ a homotopy equivalence and $\bar{X} \xrightarrow{p} X$ a covering map. If, moreover, $\pi_1 X$ is a co-Hopfian group, then X is a co-Hopfian object of H .*

Remark: If f , in Theorem 3.1', is required to be a weak mono, then $\pi_2 X$ may be taken to be either a Hopfian or a co-Hopfian group with the same conclusion holding (compare §0).

Finally, we have the following improvement of ([Roitberg, 1983], Th. 2.1).

THEOREM 3.2: *If $X \xrightarrow{f} Y$ is both a K_0 -epi and a K' -mono and if f is a nilpotent map then f is a homotopy equivalence.*

Proof: We merely follow the proof of ([Roitberg, 1983], Th. 2.1), taking into account the fact that $H_* X \xrightarrow{f_*} H_* Y$ is epi to simplify that argument. Notice that the fact that f is nilpotent implies that $f_* : \pi_1 X \rightarrow \pi_1 Y$.

Remark: As ([Roitberg, 1986], Th. 2.1) demonstrates, a map $X \xrightarrow{f} Y$ may be both a K_0 -epi (indeed, a weak epi) and a K' -mono (indeed, mono) without being a homotopy equivalence.

4. Localization

In this final section, we explore the result of P -localizing a map $X \xrightarrow{f} Y$ which is either a (weak) epi or a (weak) mono relative to some class S of H ; here P denotes some collection of primes. Of course, while we continue to work in H , X and Y should, throughout this section, be assumed to be in NH in order to guarantee meaning to the P -localized map $X_P \xrightarrow{f_P} Y_P$.

We first consider the case $S = K_0$ (dually, K_1').

THEOREM 4.1: If $X \xrightarrow{f} Y$ is a K_0 -epi, then $X_P \xrightarrow{f_P} Y_P$ is a K_0 -epi.

THEOREM 4.1': If $X \xrightarrow{f} Y$ is a K_1' -mono, then $X_P \xrightarrow{f_P} Y_P$ is a K_1' -mono.

The proofs of Theorems 4.1. and 4.1'. are easily carried out using Theorems 2.1. and 2.1'. together with ([Hilton *et al.*, 1975], Th. II 3B).

In the special case $P = \emptyset$, Theorems 4.1. and 4.1'. may be strengthened by appealing to Theorems 2.3. and 2.3'. Thus:

THEOREM 4.2: If $X \xrightarrow{f} Y$ is a K_0 -epi, then the rationalized map $X_0 \xrightarrow{f_0} Y_0$ is a weak epi.

THEOREM 4.2': If $X \xrightarrow{f} Y$ is a $(K_1' \cup \{S^1\})$ -mono, then the rationalized map $X_0 \xrightarrow{f_0} Y_0$ is a weak mono.

Henceforth, we focus attention on the absolute case $S = H$.

THEOREM 4.3: If $X \xrightarrow{f} Y$ is a weak epi, then $X_P \xrightarrow{f_P} Y_P$ is a weak epi.

Proof: In the cofibration sequence (0.1), all spaces are in NH (it has been remarked in the course of proving Theorem 2.2. that C is 1-connected). Hence (0.1) may be P -localized, producing a cofibration sequence ([Hilton, *et al.*, 1975], Th. II 3.13)

$$X_P \xrightarrow{f_P} Y_P \xrightarrow{q_P} C_P \rightarrow$$

But $q = 0 \Rightarrow q_P = 0$ and so f_P is a weak epi.

THEOREM 4.3': If $X \xrightarrow{f} Y$ is a weak mono, then $X_P \xrightarrow{f_P} Y_P$ is a weak mono.

Proof: To show f_P is a weak mono, it suffices, by Lemma 2.1., to show that $\pi_1 X_P \xrightarrow{f_{P*}} \pi_1 Y_P$ is mono and that $\tilde{X}_P \xrightarrow{\tilde{f}_P} \tilde{Y}_P$ is a weak mono. The former is clear from the (left) exactness of P -localization of nilpotent groups and the latter is proved as in Theorem 4.3., using the P -localization of the fibration sequence (0.1') associated with $\tilde{X} \xrightarrow{\tilde{f}} \tilde{Y}$.

We have not succeeded in establishing a version of Theorem 4.3 (4.3') where $X \xrightarrow{f} Y$ is merely supposed to be an epi (mono). Such a version does exist, however, for a fairly broad family of epis (monos).

Namely, say that a map $X \xrightarrow{f} Y$ is *coinduced* if \exists a map $W \xrightarrow{g} X$ in H such that $X \xrightarrow{\bar{q}} C_g$, the map of X to the homotopy-cofiber of g , is equivalent to $X \xrightarrow{f} Y$, that is, there is a commutative diagram

$$\begin{array}{ccc}
 & Y & \\
 X \xrightarrow{f} & \nearrow & \\
 & Y & \\
 & \downarrow h & \\
 & C_g & \\
 X \xrightarrow{q} & \searrow & \\
 & C_g &
 \end{array}
 \tag{4.1}$$

with h a homotopy equivalence. [If f is coinduced and a cofibration, then we recover the classical notion of *induced cofibration*.]

Dually, say that a map $X \xrightarrow{f} Y$ is *induced* if \exists a map $Y \xrightarrow{g} B$ in H such that $F_g \xrightarrow{\bar{i}} Y$, the map of the homotopy-fiber of g to Y , is equivalent to $X \xrightarrow{f} Y$, that is, there is a commutative diagram

$$\begin{array}{ccc}
 F_g & & \\
 \downarrow h & \searrow \bar{i} & \\
 X & \xrightarrow{f} & Y
 \end{array}$$

with h a homotopy equivalence.

Then we may state

THEOREM 4.4: *If $X \xrightarrow{f} Y$ is an epi coinduced by $W \xrightarrow{g} X$, with W nilpotent, then $X_p \xrightarrow{f_p} Y_p$ is a coinduced epi.*

THEOREM 4.4': *If $X \xrightarrow{f} Y$ is an induced mono, then $X_p \xrightarrow{f_p} Y_p$ is an induced mono.*

The proof of Theorem 4.4. utilizes a dualization of a result of ([Ganea, 1967], Prop. 2.1). Let $W \xrightarrow{g} X$ be a map coinducing $X \xrightarrow{f} Y$. Identifying Y with C_g as in (4.1), we have two maps $\iota, \gamma: Y \rightarrow Y \vee \Sigma W$, the inclusion and the coaction (or cooperation) map. It is readily seen that

$$\iota \cdot f = \gamma \cdot f.
 \tag{4.2}$$

Moreover, if $Y \xrightarrow{u,v} Z$ satisfy $u \cdot f = v \cdot f$, then $\exists \Sigma W \xrightarrow{\mu} Z$ such that

$$v = \langle u, \mu \rangle \cdot \gamma,
 \tag{4.3}$$

where $\langle u, \mu \rangle: Y \vee \Sigma W \rightarrow Z$ is the map with components u, μ .

LEMMA 4.1: If $X \xrightarrow{f} Y$ is coinduced by $W \xrightarrow{g} X$, then $X \xrightarrow{f} Y$ is epi $\Leftrightarrow \iota = \gamma$.

Now if f is epi, then $\iota = \gamma$. If W is nilpotent, then $W_p \xrightarrow{g_p} X_p$ coinduces f_p and $\iota_p = \gamma_p$, so that f_p is epi.

The proof of Theorem 4.4' is almost precisely dual to that of Theorem 4.4. In place of ι, γ , we have $\pi, \rho: X \times \Omega B \rightarrow X$, the projection and the action map (see Ganea, 1967) where $Y \xrightarrow{g} B$ induces f . There are, however, two points to be noticed, First, since $X \xrightarrow{f} Y$ is mono, it follows that

$$\pi_n X \xrightarrow{f_*} \pi_n Y \xrightarrow{g_*} \pi_n B$$

is a short exact sequence for each $n \geq 1$. Thus B is in NH since Y is in NH . Second, let $(\Omega B)_0$ be the path-component of ΩB containing the constant loop. Let π, ρ restrict to $\pi_0, \rho_0: X \times (\Omega B)_0 \rightarrow X$. Then, if f is mono, $\pi_0 = \rho_0$. But, in this dual situation, our map $Z \xrightarrow{\mu} \Omega B$, arising from a pair of maps $Z \xrightarrow{u, v} X$ with $f \cdot u = f \cdot v$, actually maps Z into $(\Omega B)_0$, since Z is path-connected, so that we may assert

LEMMA 4.1': If $X \xrightarrow{f} Y$ is induced by $Y \xrightarrow{g} B$, then $X \xrightarrow{f} Y$ is mono $\Leftrightarrow \pi_0 = \rho_0$.

Now we have $\pi_0 = \rho_0$, so that $\pi_{0p} = \rho_{0p}$. Since $X_p \xrightarrow{f_p} Y_p$ is induced by $Y_p \xrightarrow{g_p} B_p$ and $(\Omega B_p)_0 \simeq (\Omega B)_{0p}$, it follows that f_p is mono.

Remark: Theorems 4.3' and 4.4' apply, in particular, to the various Hopf fibrations (see [Ganea, 1967]). For instance, the Hopf map $S^3 \rightarrow S^2$ rationalizes to a mono $K(\mathbb{Q}, 3) \rightarrow S_0^2$.

We next consider natural companion problems to those just treated.

THEOREM 4.5: Suppose $X \xrightarrow{f} Y$ is such that, for each prime p , $X_p \xrightarrow{f_p} Y_p$ is a weak epi. If Y has the homotopy type of a finite complex, then $X \xrightarrow{f} Y$ is a weak epi. If each $X_p \xrightarrow{f_p} Y_p$ is epi, then $X \xrightarrow{f} Y$ is an NH -epi.

Proof: Since, for each p , $f_{p*}: \pi_1 X_p \rightarrow \pi_1 Y_p$ is a weak epi, hence epi ($\pi_1 Y_p$ being nilpotent), it follows from ([Hilton *et al.*, 1975], Th. I.3.12) that $f_*: \pi_1 X \rightarrow \pi_1 Y$ is epi. Thus the homotopy-cofiber C of f is 1-connected and the cofibration sequence (0.1) may be p -localized to a cofibration sequence

$$X_p \xrightarrow{f_p} Y_p \xrightarrow{q_p} C_p \rightarrow$$

Since, for each p , $q_p = 0$, we infer from ([Hilton *et al.*, 1975], Cor. II.5.12) that $q = 0$, hence that f is a weak epi.

For the final statement, let $Y \xrightarrow{u, v} Z$, Z in NH , be such that $u \cdot f = v \cdot f$. Then, for each p , $u_p \cdot f_p = v_p \cdot f_p$ and since f_p is an epi, $u_p = v_p$. By ([Hilton *et al.*, 1975] Cor. II.5.12) once again, $u = v$ and f is an NH -epi.

THEOREM 4.5': *If $X \xrightarrow{f} Y$ is such that, for each prime p , $X_p \xrightarrow{f_p} Y_p$ is a (weak) mono, then $X \xrightarrow{f} Y$ is a (weak) F -mono, where F is the class consisting of spaces of the homotopy type of a finite complex.*

Proof: We are content to deal with the situation where each f_p is a mono. Let then $W \xrightarrow{u, v} X$, W in F , be such that $f \cdot u = f \cdot v$. Then $f_p \cdot e_p \cdot u = f_p \cdot e_p \cdot v$, $X \xrightarrow{e_p} X_p$ denoting p -localization. Since each f_p is mono, $e_p \cdot u = e_p \cdot v$. By ([Hilton *et al.*, 1975], Th. II. 5.3), $u = v$ and f is mono.

Remark: Plainly it would have sufficed to assume each f_p in Theorem 4.5'. an F -mono rather than a mono. Similarly, it would have sufficed in the second part of Theorem 4.5. to assume each f_p an NH_p -epi, where NH_p is the class of p -local nilpotent spaces.

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