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ANTONIO LANTERI
DANIELE STRUPPA

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Projective 7-folds with positive defect

ANTONIO LANTERI [1] & DANIELE STRUPPA [2]

[1] *Dipartimento di Matematica, 'F. Enriques' Università, Via C. Saldini 50, I-20133 Milano, Italy;*
[2] *Scuola Normale Superiore, Piazza dei Cavalieri, 7, I-56100 Pisa, Italy*

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Abstract. New simple proofs are given for the classification theorems of projective k-folds X ($k \leqslant 6$) with defect $\delta > 0$. Moreover 7-folds with $\delta > 1$ and those with $\delta = 1$ and $K_X \otimes \mathcal{O}_X(5)$ spanned are classified. The section of the 10-dimensional spinor variety of $\mathbb{P}^{15}$ by 3 general hyperplanes and Grassmann fibrations over a smooth curve belong to this last class.

0. Introduction

Recently many results on projective manifolds with small dual varieties have been found by [Ein, 1985]. In the first part of this paper (sections 1 and 2) we approach this subject from a topological-adjunction theoretic point of view. The topological basic facts are a formula due to [Landman, 1976] and some results from [Lanteri and Struppa, 1986]. In particular we provide new (and very short) proofs for the classification theorems of projective manifolds with degenerate dual varieties of dimensions 3, 4 and 6, and we partially classify those of dimension 7. In particular we completely classify 7-folds with defect $\delta = 3$: they are scrolls of $\mathbb{P}^5$'s over a smooth surface. An immediate extension of this result to k-folds X ($k \geqslant 7$) is: $\delta > k - 6$ iff X is a scroll of $\mathbb{P}^{(k+\delta)/2}$'s over a $(k-\delta)/2$-fold. This gives an alternate proof of a weaker form of a result of Ein. In the second part of the paper (section 3) we deal with the case $\delta = 1$ and we find a new class of 7-folds with degenerate dual varieties. Actually, under the extra assumption that $K_X \otimes \mathcal{O}_X(5)$ is spanned by global sections, we prove that, besides Mukai 7-folds and scrolls of $\mathbb{P}^4$'s over a 3-fold, X can be a fibration of grassmannians $G(1, 4)$ (of lines of $\mathbb{P}^4$) over a smooth curve. All these cases really occur: indeed the section of the 10-dimensional spinor variety $S \subset \mathbb{P}^{15}$ by three general hyperplanes is an example of Mukai 7-fold with $\delta = 1$; all scrolls as above have $\delta = 1$ and finally all Grassmann fibrations over a smooth curve have $\delta = 1$. This follows from Proposition 3.5, which we owe to the referee. In an earlier version of the paper we only proved this result for Grassmann bundles; our proof consisted of a detailed topological argument taking advantage of the bundle structure and of the homology of Grassmannians.

Both authors are members of the G.N.S.A.G.A. of the Italian C.N.R.

1. Known results (new proofs)

Let $X \subset \mathbb{P}^N$ be a complex connected projective algebraic manifold of dimension dim $X = k$. We always assume that X is not contained in any hyperplane unless X itself is a hyperplane. We are mainly concerned with the class of projective manifolds with degenerate dual varieties:

$$\Delta_k = \{X \subset \mathbb{P}^N : \dim X = k \quad \text{and } \dim X^* < N-1\}.$$

Here $X^* \subset \mathbb{P}^{N*}$ denotes the dual variety of X. As is known dim $X^* \leqslant N-1$, with equality in the general case. Since the class $\mu(X)$ of X is the number of points that a general line of $\mathbb{P}^{N*}$ cuts out on X^*, we have $\mu(X) = 0$ iff $X \in \Delta_k$.

Let X_1 be the section of X with a general hyperplane and consider the class

$$\mathscr{L}_k = \{X \subset \mathbb{P}^N : \dim X = k \quad \text{and } b_{k-1}(X_1) = b_{k-1}(X)\},$$

where $b_i(X)$ is the i-th Betti number of X. Many properties of $\mathscr{L}_k$ are discussed in [Lanteri and Struppa, 1986]. In particular we recall that ([Lanteri and Struppa, 1986], Prop. 3.3)

$$\Delta_k \subseteq \mathscr{L}_k \text{ with equality for } k \text{ odd.} \tag{1.0}$$

Finally we denote by $\Sigma(r, s)$ the class of (r, s)-scrolls $(r+s=k)$; we say that $X \subset \mathbb{P}^N$ is a (r, s)-scroll if i) $X = \mathbb{P}(E)$, E a rank-$(r+1)$ holomorphic vector bundle over some projective manifold of dimension s, ii) the fibers of X are linear spaces and iii) r is the maximum integer with these properties.

Many results on Δ_k are known and are mostly due to [Ein, 1985]. Here we reprove some of them using a topological-adjunction theoretic approach. Let $\chi(X)$ be the Euler-Poincaré characteristic of X and let X_i denote the section of X with i general hyperplanes. The *class formula* ([Lamotke, 1981], p. 25)

$$\chi(X) = 2\chi(X_1) - \chi(X_2) + (-1)^k \mu(X)$$

is the main ingredient in the proof of the following unpublished result of [Landman, 1976], see ([Kleiman, 1986] (II.3.18))

$$\begin{aligned}\mu(X) = {} & (b_k(X) - b_{k-2}(X)) + 2(b_{k-1}(X_1) - b_{k-1}(X)) \\ & + (b_{k-2}(X_2) - b_{k-2}(X)).\end{aligned} \tag{1.1}$$

The three summands in (1.1) are nonnegative numbers due to the strong and the weak Lefschetz theorems. Hence the characteristic condition for X to have degenerate dual variety is

$$b_k(X) - b_{k-2}(X) = b_{k-1}(X_1) - b_{k-1}(X) = b_{k-2}(X_2) - b_{k-2}(X) = 0. \tag{1.2}$$

This immediately shows that $\Delta_2 = \{\mathbb{P}^2\}$, since for a surface $X \in \Delta_2$ the third equality in (1.2) implies $b_0(X_2) = 1$, i.e. that X has degree one.

The following result was first proved by ([Griffiths and Harris, 1979] (3.26)) using differential geometric techniques. Recently ([Ein, 1985], I. Th. 3.3) gave a different proof. Now we deduce it simply from (1.2).

1.3. PROPOSITION: $\Delta_3 = \{\mathbb{P}^3\} \cup \Sigma(2, 1)$.

Proof. We only prove the inclusion $\subseteq$, the other one being easy. Let $X \in \Delta_3$; then $b_1(X_2) = b_1(X)$, by (1.2), and the assertion follows from ([Lanteri and Palleschi, 1984], Th. 3.2), observing that the quadric threefold does not fulfill $b_2(X_1) = b_2(X)$. ¶

As to dimension 4, the following result has been proved by ([Ein, 1985], I, Th. 3.3) and independently by the authors ([Lanteri and Struppa, 1984], (3.3)). Here we provide a third proof stemming from (1.2).

1.4. PROPOSITION: $\Delta_4 = \{\mathbb{P}^4\} \cup \Sigma(3, 1)$.

Proof. As before, we only prove the inclusion $\subseteq$. Let $X \in \Delta_4$; then $X_1 \in \mathscr{L}_3$ by (1.2) and then X_1 is as in (1.3), in view of (1.0). Then either $X = \mathbb{P}^4$, or $X \in \Sigma(3, 1)$ in view of a known result (e.g. see [Bădescu, 1981], §2). ¶

Unfortunately, due to the lack of knowledge of $\mathscr{L}_4$ [Lanteri and Struppa, 1986], (1.2) is not sufficient to recover the following result of Ein:

1.5. PROPOSITION: ([*Ein*, 1985], *II*, *Th.* 5.1) Δ_5 *consists of* $\mathbb{P}^5$, $\Sigma(4, 1)$, $\Sigma(3, 2)$ *and of any nonsingular hyperplane section of the grassmannian G of lines of* $\mathbb{P}^4$ *embedded in* $\mathbb{P}^9$ *via the Plücker embedding.*

In order to deal with higher dimensions we need the following result essentially contained in a paper of [Sommese, 1976].

1.6. PROPOSITION: *Assume* $X_1 \in \Sigma(r, s)$, $r > 2$. *Then* $X \in \Sigma(r+1, s)$; *in particular* $r \geqslant s - 1$.

Proof. Let $p: X_1 \to B$ be the projection morphism onto the base B of X_1; since $r > 2$ and by ([Sommese, 1976], Prop. III), p extends to a morphism $\tilde{p}: X \to B$. Let F be a fiber of $\tilde{p}$; then $f = X_1 \cdot F$ is a fiber of p and is an ample divisor in F, since X_1 is ample. But $f \simeq \mathbb{P}^r$ and $\mathcal{O}_{X_1}(1) \otimes \mathcal{O}_f = \mathcal{O}_{\mathbb{P}^r}(1)$. Then $F \simeq \mathbb{P}^{r+1}$ and $\mathcal{O}_X(1) \otimes \mathcal{O}_F = \mathcal{O}_{\mathbb{P}^{r+1}}(1)$ (e.g. see [Sommese, 1976], p. 67). This implies that $X \in \Sigma(r+1, s)$. Furthermore, since $\tilde{p}$ is a surjection and $p = \tilde{p}_{|X_1}$ makes X_1 into a $\mathbb{P}$-bundle over B, it has to be $r \geqslant s - 1$ ([Sommese, 1976], Prop. V]. ¶

In the context of very ample divisors (1.6) extends the above quoted results of Bădescu on ample divisors which are $\mathbb{P}$-bundles over a smooth curve.

Notice also that (1.6) can be viewed as a converse to Proposition 2.2. in [Lanteri and Struppa, 1986].

First of all we use (1.6) jointly with (1.2) to give an alternate proof of a result of ([Ein, 1985], II, Th. 5.2).

1.7. PROPOSITION: Δ_6 *consists of* $\mathbb{P}^6$, $\Sigma(5, 1)$, $\Sigma(4, 2)$ *and of the grassmannian* G.

Proof. That the above classes of manifolds belong to Δ_6 is easily seen (e.g. see [Lanteri and Struppa, 1986]). Now let $X \in \Delta_6$. Once again by (1.2) this implies that $X_1 \in \mathscr{L}_5$ and therefore X_1 is as in (1.5), in view of (1.0). Firstly assume that X_1 is isomorphic to a hyperplane section of G. Let K_X be the canonical bundle of X; since $K_{X_1} = \mathcal{O}_{X_1}(-4)$, by adjunction we get

$$K_X \otimes \mathcal{O}_{X_1} = \mathcal{O}_{X_1}(-5)$$

and then $K_X = \mathcal{O}_X(-5)$, as $\mathrm{Pic}(X) \simeq \mathrm{Pic}(X_1) \simeq \mathbb{Z}$. So X is a 6-dimensional Del Pezzo manifold in the sense of Fujita and therefore $X = G$ in view of Fujita's classification ([Fujita, 1982], (6.3)). Now, if $X_1 \in \Sigma(4, 1) \cup \Sigma(3, 2)$, then X belongs to $\Sigma(5, 1) \cup \Sigma(4, 2)$, by (1.6). Finally, if $X_1 = \mathbb{P}^5$, then $X = \mathbb{P}^6$, trivially. ¶

2. Dimension 7: defects 3 and 5

Just as for Δ_5, the topological-adjunction theoretic method used before does not yield a complete description of Δ_7.

To study the class Δ_7 we need the notion of defect. Recall that the defect of a nonlinear $X \subset \mathbb{P}^N$ is the integer

$$\delta(X) := N - 1 - \dim X^*.$$

We put also $\delta(\mathbb{P}^k) = k$; this is consistent with our general assumption on X. We will need the following facts.

$$\delta(X_1) = \max\{0, \delta(X) - 1\} \text{ ([Hefez and Kleiman, 1985], (5.9))}; \tag{2.1}$$

$$\text{if } X \in \Sigma(r, s) \text{ with } r \geqslant s, \text{ then } \delta(X) = r - s$$
$$\text{([Lanteri and Struppa, 1986], Prop. 5.2).} \tag{2.2}$$

An independent proof of (2.2) will follow from (3.5). Moreover (1.1) implies, by induction,

$$b_{k-i}(X_i) = b_{k-i}(X), \quad \text{for } i = 1, \ldots, \delta(X) + 1 \text{ [Landman, 1976].} \tag{2.3}$$

In view of the parity of $k-\delta$ ([Landman, 1976]; see also [Ein, 1985], I, Th. 2.4), if $X \in \Delta_7$ then either $X = \mathbb{P}^k$, or $\delta(X) = 1, 3, 5$.

The case $\delta(X) = 5$ is settled by the following.

2.4. PROPOSITION: *Let $k \geqslant 3$. Then $\delta(X) = k-2$ iff $X \in \Sigma(k-1, 1)$.*

Proof. If $X \in \Sigma(k-1, 1)$, then $\delta(X) = k-2$ (e.g. see [Kleiman, 1977], p. 363). Assume $\delta(X) = k-2$; then (2.3) gives $b_1(X_{k-1}) = b_1(X)$ and the assertion follows now by ([Lanteri and Palleschi, 1984], Th. 3.2). Notice that quadrics are hypersurfaces, hence $\delta = 0$. ¶

Different proofs of (2.4) have already been given by ([Ein, 1985], I. Th. 3.2 and II. Th. 3.1) and by the authors ([Lanteri and Struppa, 1984], Cor. 3.4). More generally ([Ein, 1985], II, Th. 4.1) has proved that if $\delta(X) \geqslant k/2$, then $X \in \Sigma((k+\delta)/2, (k-\delta)/2)$. Unfortunately for $k = 7$ and $\delta = 3$ this result does not apply; in spite of this we can prove by our method that X belongs indeed to $\Sigma(5, 2)$.

2.5. PROPOSITION: *Let $X \in \Delta_7$ with $\delta(X) = 3$. Then $X \in \Sigma(5, 2)$.*

Proof. We have $\delta(X_1) > 0$, by (2.1), i.e. $X_1 \in \Delta_6$. However it cannot be that $X_1 = G$, since the grassmannian G cannot be an ample divisor ([Fujita, 1981], (5.2)). Then the assertion follows from (1.6), (1.7). ¶

An obvious inductive step based on (1.6), (2.3) and (2.5) shows that:

2.6. PROPOSITION: *Let $k \geqslant 7$; then $\delta(X) = k-4$ iff $X \in \Sigma(k-2, 2)$.*

For $k \geqslant 8$, (2.6) is absorbed in the more general result of Ein quoted before.

In higher dimensions a new interesting manifold arises: the 10-dimensional spinor variety $S \subset \mathbb{P}^{15}$, which parametrizes each one of the two disjoint families of 4-planes lying on a smooth 8-dimensional hyperquadric ([Lazarsfeld and Van de Ven, 1984], p. 16). Such a manifold is known to be self-dual, i.e. $S \simeq S^*$; hence $\delta(S) = 4$. Therefore S_2, the section of S by two general hyperplanes has dimension $k = 8$ and defect $\delta = 2$. Since $S \notin \Sigma(7, 3)$ it follows from (1.6) that $S_2 \notin \Sigma(5, 3)$; this shows that a result like (2.4) or (2.6) cannot hold for $\delta = k-6$.

3. Dimension 7: defect 1

We finally look at the case $\delta(X) = 1$. We first note that $\Sigma(4, 3)$ does not exhaust the class of 7-folds with $\delta(X) = 1$. Indeed S_3, the section of the spinor

variety S by three general hyperplanes, is such a manifold, by (2.1). In order to extend an argument of Ein, we confine ourselves to the class $\Delta'_7 = \{X \in \Delta_7 : \delta(X) = 1$ and $K_X \otimes \mathcal{O}_X(5)$ is spanned by global sections$\}$.
To determine Δ'_7 we need some preliminary discussion. First of all, if $X \in \Delta'_7$, the linear system $|K_X \otimes \mathcal{O}_X(5)|$ defines a morphism $f: X \to f(X)$. Now we use two results of Ein:

$$\text{through a general point } p \in X \text{ there passes a 3-dimensional family of lines } \{\ell\}, \text{ ([Ein, 1985], I, Th. 2.3);} \tag{3.1}$$

$$K_{X|\ell} = \mathcal{O}_\ell(-5) \text{ for every } \ell \in \{\ell\}, \text{ ([Ein, 1985], I, Th. 2.4).} \tag{3.2}$$

Therefore by (3.2) the cone spanned by $\{\ell\}$ is contracted by f and since $\dim(f^{-1}(f(p))) \geqslant 4$ in view of (3.1), we conclude that

$$r = \dim f(X) \leqslant 3.$$

Let $r = 0$; then, since $K_X \otimes \mathcal{O}_X(5)$ is spanned, we have $K_X = \mathcal{O}_X(-5)$, i.e. X is a Mukai 7-fold [Mukai, 1985].

Assume now that $r > 0$ and consider the Stein factorization

$$X \xrightarrow{g} B \to f(X)$$

of f. The general fibre D of g is a (7-r)-fold, by generic smoothness and its normal bundle $N_{D|X}$ is trivial. Hence

$$K_D = K_{X|D'}$$

by adjunction; moreover, since f is constant on D by (3.2), this implies

$$K_D = \mathcal{O}_D(-5).$$

Hence D is a Fano (7-r)-fold of index 5 for $r = 2, 3$ and a Del Pezzo 6-fold in the sense of Fujita, for $r = 1$. Let $\Lambda = \langle D \rangle$ be the linear space spanned by D in $\mathbb{P}^N$. Then we have only the following possibilities for $D \subset \Lambda$, according to the values of r.

i) Let $r = 3$; then $D = \Lambda = \mathbb{P}^4$, in view of [Ochiai and Kobayashi, 1973]; thus X is a $\mathbb{P}^4$-bundle over B and $X \in \Sigma(4, 3)$, since the fibres are embedded linearly.
ii) Let $r = 2$; then $D \subset \Lambda$ is a quadric hypersurface of $\mathbb{P}^6$, by [Ochiai and Kobayashi, 1973].
iii) Let $r = 1$; then the Fujita classification of Del Pezzo manifolds ([Fujita, 1982] (6.3)) implies that $D \subset \Lambda$ is either
   a) a cubic hypersurface of $\mathbb{P}^7$,

b) a complete intersection of type (2,2) of $\mathbb{P}^8$,
c) the grassmannian G embedded in $\mathbb{P}^9$ via the Plücker embedding.

We can now state the main result of this section:

3.3. THEOREM: *Let $X \in \Delta_7'$. Then, either X is a Mukai 7-fold, $X \in \Sigma(4, 3)$,* or *there exists a morphism $g: X \to B$ over a smooth curve B, whose general fibre is the grassmannian G, and $\mathcal{O}_X(1)$ embeds it into a $\mathbb{P}^9$ via the Plücker embedding.*

This latter case will be referred to as a G-fibration.

Proof. In view of the previous discussion, it clearly suffices to show that cases ii) and iii) a), b) cannot occur. To deal with cases iii), take a general point p of D and a general hyperplane Π tangent to X at p. As $\delta(X) = 1$, we know from [Kleiman, 1986] that Π is tangent to X along a line ℓ_0 on which g is constant by (3.2); On the other hand, $\Lambda = \langle D \rangle$ cannot be contained in Π since otherwise one would have $D \subset \Pi \cap X$: this would imply that D is a component of $\Pi \cdot X$; then, since Π is general, D would coincide with $\Pi \cdot X$ and hence D would be singular at p, contradiction. So $\Lambda \not\subset \Pi$, and, by restricting to Λ, we conclude that $\Pi \cap \Lambda$ is a hyperplane of Λ tangent to D along ℓ_0; but this excludes a) and b) since in those cases any tangent hyperplane is tangent at a single point. As far as case ii) is concerned, the proof runs as above if we know that $\Lambda \not\subset \Pi$; this however cannot be proven with the argument used before, since now codim $D = 2$. So we have only to consider the following case.

3.4. ASSUMPTION: Every hyperplane tangent to X at a general point $x \in X$ contains the linear span $\langle D \rangle$ of the fibre D of g through x.

We show that this leads to a contradiction. To do this consider the correspondence

$$\begin{tikzcd} & \mathcal{I} = \{(\Pi, L) \in \mathbb{P}^{N*} \times G(6, N) : \Pi \supset L\} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}^{N*} & & G(6, N). \end{tikzcd}$$

The second projection gives $\mathcal{I}$ the structure of a $\mathbb{P}^{N-7}$-bundle over the grass mannian $G(6, N)$ of 6-planes of $\mathbb{P}^N$. Of course we have $\dim\langle D_b \rangle = 6$ for every $b \in B$. So there is an injection $j: B \to G(6, N)$, defined by $j(b) = \langle D_b \rangle$. Let $\mathcal{I}_B$ be the pull-back of $\mathcal{I}$ via j and identify

$$\mathcal{I}_B = \{(\Pi, L, b) \in \mathbb{P}^{N*} \times G(6, N) \times B : \Pi \supset L = \langle D_b \rangle\}$$

with its image projected isomorphically into $\mathbb{P}^{N*} \times B$,

$$\mathscr{I}'_B = \{(\Pi, b) \in \mathbb{P}^{N*} \times B : \Pi \supset \langle D_b \rangle\}.$$

Now let Π be a hyperplane tangent to X at a general point x. As before, since $\delta(X) = 1$, Π is tangent to X along a line $\ell_0 \subset X$ which, by (3.2), is contained in a single fibre D_b of g; moreover, $\Pi \supset \langle D_b \rangle$, by (3.4). Then letting $\varphi(\Pi) = (\Pi, b)$ one defines a rational map $\varphi: X^* \dashrightarrow \mathscr{I}^*_B$, which is birational between X^* and $\overline{\varphi(X^*)}$. Hence

$$\dim X^* \leqslant \dim \mathscr{I}'_B = N - 7 + 2 = N - 5.$$

But this implies $\delta(X) \geqslant 4$, contradiction. ¶

Manifolds as in (3.3) really occur in Δ'_7. To prove it we recall that a complete classification of Mukai manifolds is not yet known; anyway, for $k = 7$, in addition to the quartic hypersurfaces and to the complete intersections of type (2, 3) and (2, 2, 2), which however are not in Δ_7, this class contains the section S_3 of the spinor variety $S \subset \mathbb{P}^{15}$ by three general hyperplanes. Actually, since $K_S = \mathcal{O}_S(-8)$, we have, by adjunction, $K_{S_3} = \mathcal{O}_{S_3}(-5)$. Moreover $\delta(S_3) = 1$, by (2.1). As to the class $\Sigma(4, 3)$ there is nothing to say in view of (2.2).

We conclude the paper by showing that all G-fibrations over a smooth curve are in Δ'_7. Let $g: X \to B$ be such a fibration. First of all notice that $K_X \otimes \mathcal{O}_X(5)$ is spanned by global sections. This follows from the fact that the rational map Φ associated with $|K_X \otimes \mathcal{O}_X(5)|$ factors through g and $\dim B = 1$; indeed, by adjunction, Φ is constant along the fibres of g.

Now, in view of (2.4), (2.5) it is enough to show that $\delta(X) \geqslant 1$. This follows immediately from the following general proposition, which we owe to the referee.

3.5. PROPOSITION: *Let $X \subset \mathbb{P}^N$ be a projective k-fold such that through its general point there passes a submanifold Y of dimension h and defect θ. Then* $\delta(X) \geqslant \theta - k + h$ (*i.e.* $\dim X + \delta(X) \geqslant \dim Y + \delta(Y)$).

Proof. Let Π be a hyperplane tangent to X at a general point $x \in X$. Then, since the defect is the dimension of the contact locus, Π is tangent to Y along a subvariety Z containing x and of dimension θ. Let $f = 0$ be a local equation for Π at x. In a neighbourhood U of x in X, the differential df annihilates the tangent spaces $T_{X,x}$ and $T_{Y,z}$ for every $z \in Z \cap U$. Hence df defines on Z a 'co-section' of the rank-$(k-h)$ bundle $(T_X/T_Y)_Z$, vanishing at x. But then a local computation shows that on $(T_X/T_Y)_{Z \cap U}$ this co-section vanishes on the zero locus of $k - h$ functions and therefore Π is tangent to X along a subvariety of Z of codimension less than or equal to $k - h$. This means that $\delta(X) \geqslant \theta - (k - h)$. ¶

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