F. J.-B. J. CLAUWENS

The K-groups of λ-rings. Part I. Construction of the logarithmic invariant

Compositio Mathematica, tome 61, nº 3 (1987), p. 295-328
<http://www.numdam.org/item?id=CM_1987__61_3_295_0>

Compositio Mathematica 61: 295–328 (1987)

The K-groups of λ-rings

Part I. Construction of the logarithmic invariant

F.J.-B.J. CLAUWENS
Mathematical Institute, Catholic University, Toernooiveld, 6525 ED Nijmegen, The Netherlands

Received 10 September 1985; accepted in revised form 2 September 1986

Introduction

The starting point of this paper is the following theorem of [Maazen and Stienstra, 1977] and [Keune, 1978].

THEOREM 0.1. *Let R be a commutative ring with unit, and let I be an ideal. Consider the abelian group defined by the following presentation. The generators are the symbols $\langle a, b\rangle$ with $(a, b) \in I \times R \cup R \times I$. The relations are*

$$\langle a, b\rangle + \langle b, a\rangle \quad \textit{for } (a, b) \in I \times R \cup R \times I$$

$$\langle a, b + c - abc\rangle - \langle a, b\rangle - \langle a, c\rangle$$
$$\textit{for } (a, b, c) \in I \times R \times R \cup R \times I \times I$$

$$\langle a, bc\rangle - \langle ab, c\rangle - \langle ac, b\rangle$$
$$\textit{for } (a, b, c) \in I \times R \times R \cup R \times I \times R \cup R \times R \times I$$

Then $K_2(R, I)$ is isomorphic to this group if I is contained in the Jacobson radical of R.

The fact that the middle relation is of nonlinear nature makes it nontrivial to decide whether a given element of $K_2(R, I)$ vanishes or not. One would like to replace this relation by a linear one:

Definition 0.2. If R is a commutative ring and I an ideal then $K_{2,L}(R, I)$ denotes the abelian group defined by the following presentation. The generators are the symbols $\langle a, b\rangle$ with $(a, b) \in I \times R \cup R \times I$. The relations are

$$[a, b] + [b, a] \quad \text{for } (a, b) \in I \times R \cup R \times I$$

$$[a, b + c] - [a, b] - [a, c] \quad \text{for } (a, b, c) \in I \times R \times R \cup R \times I \times I$$

$$[a, bc] - [ab, c] - [ac, b]$$
$$\text{for } (a, b, c) \in I \times R \times R \cup R \times I \times R \cup R \times R \times I$$

Write δ: $R \to \Omega_R$ for the universal derivation on R, and write $\Omega_{R,I}$ for $ker(\Omega_R \to \Omega_{R/I})$. Then there is a homomorphism $K_{2,L}(R, I) \to \Omega_{R,I}/\delta I$ which maps $[a, b]$ to $a\delta b$. It is easily seen to be an isomorphism if $I = R$; so if the projection $R \to R/I$ splits then

$$K_{2,L}(R, I) = ker(K_{2,L}(R, R) \to K_{2,L}(R/I, R/I))$$
$$= ker(\Omega_R/\delta R \to \Omega_{R/I}/\delta(R/I))$$
$$= ker(\Omega_R \to \Omega_{R/I})/ker(\delta R \to \delta R/I) = \Omega_{R,I}/\delta I.$$

For a description of $K_{2,L}(R, I)$ in terms of Ω_R in the nonsplit case see Proposition 7.1.

To gain insight into the difference between K_2 and $K_{2,L}$ it is instructive to look at the situation for K_1. One can reformulate Theorem 3.2. of [Bass and Murphy, 1967] as follows and make the analogy with Theorem 0.1. apparent:

THEOREM 0.3. *Let R be a commutative ring with unit, and let I be an ideal. Consider the abelian group defined by the following presentation. The generators are the symbols $\langle a \rangle$ with $a \in I$. The relations are $\langle a + b - ab \rangle - \langle a \rangle - \langle b \rangle$ for $a \in I$, $b \in I$. Then $K_1(R, I)$ is isomorphic to this group if I is contained in the Jacobson radical of R.*

Indeed for any such (R, I) this group is isomorphic to the multiplicative group of $1 + I$ by the map $\langle a \rangle \to 1 - a$.

Definition 0.4. If R is a commutative ring and I an ideal then $K_{1,L}(R, I)$ denotes the abelian group defined by the following presentation. The generators are the symbols $[a]$ with $a \in I$. The relations are $[a + b] - [a] - [b]$ for $a \in I$, $b \in I$.

Obviously $K_{1,L}(R, I)$ can be identified with the additive group of I by mapping $[a]$ to a.

The above linear K-groups appear in [Kassel and Loday, 1982], where it is explained that the relation between them and the homology of the Lie algebra $gl(R)$ is roughly the same as the relation between ordinary K-groups and the homology of the group $GL(R)$. Moreover these groups can be interpreted as cyclic homology groups of (R, I); see [Loday and Quillen, 1984]. For the present paper, however, no more knowledge of K-theory is necessary then is contained in 0.1.–0.4.

The aim of this paper is to construct a map L from the above K-groups to the corresponding linear K-groups. The idea for this construction is classical (see [Bloch, 1975]):

$$L\langle a \rangle = [-\log(1 - a)],$$
$$L\langle a, b \rangle = [a, -a^{-1} \log(1 - ab)]$$

Here $-a^{-1} \log(1 - ab)$ must be read as $\sum_{n=1}^{\infty} n^{-1}a^{n-1}b^n$. This only has an obvious meaning if each $n^{-1} \in R$. In order to weaken this condition one must somehow read $n^{-1}a^{n-1}b^n$ as a whole, and if needed add correction terms to the $a^{n-1}b^n$ to make it a multiple of n. In this paper this is accomplished by assuming the structure of λ-ring on R, and we will in effect describe the correction terms.

If the ring R has no Z-torsion then a structure of λ-ring on R is equivalent to a sequence of ring homomorphisms $\psi^n : R \to R$ for $n > 1$ such that $\psi^m\psi^n = \psi^{mn}$ and $\psi^p x = x^p \bmod pR$ for each prime p (see Proposition 1.9). In particular if R is an augmented Q-algebra then one can take $\psi^n = 0$ on the augmentation ideal; in that case we recover the above formulas for L.

Since our map L is a kind of power series we also need a topology on R defined by some ideal J. For any functor F from pairs (R, I) as above to abelian groups we use the notation $F^{top}(R, I)$ for the inverse limit of the groups $F(R/J^N, (I+J^N)/J^N)$. We now formulate a preliminary version of our main theorem:

THEOREM. *Let R be a λ-ring and I an ideal; let a topology on R be defined by another ideal J. If (R, J, I) satisfies certain compatibility conditions then there exists a continuous map $L: K_2^{top}(R, I) \to K_{2,L}^{top}(R, I)$ such that $L\langle a, b\rangle = [a, b] +$ higher order terms.* ■

A precise formulation of this theorem is given in Theorem 7.2. Another version is given in Theorem 6.2. where the map has values in $(\Omega_{R,I}/\delta I)^{top}$. The compatibility conditions are listed in Definition 5.5. The main examples on which this theory can be applied are discussed in 5.9 and 5.10.

The proof of Theorem 6.2. consists of constructing a map $\nu: I \times R \cup R \times I \to \Omega_{R,I}^{top}/\delta I^{top}$ and showing that it vanishes on the relations mentioned in Theorem 0.1. The map ν is defined by a formula

$$\nu(a, b) = \sum_{n=1}^{\infty} \nu^n(a, b), \text{ where } \nu^n(a, b) = \sum_{mk=n} \theta^m(a)\phi^m(\delta\eta^k(a, b)).$$

This formula involves certain maps $\theta^m: R \to R$, $\eta^k: R \times R \to R$ and $\phi^m: \Omega_R \to \Omega_R$, which are defined for each λ-ring R. These are defined and studied in §§2, 3 and 4 respectively; in particular Propositions 2.2, 3.3. and 4.4. relate these operations to the maps ψ^n mentioned before. These §§ are preceded by §1 which gives some general background on λ-rings.

In §5 we discuss the conditions on the topology on R which are needed to give meaning to the infinite sums in the above formula. In §6 the Ω_R version of the main theorem is proved, assuming the existence of certain maps $\beta_d^n : R \times \cdots \times R \to R$ such that

$$\sum_{i=1}^{d} \nu^n(a_i, a_1 \cdots a_{i-1}a_{i+1} \cdots a_d) = \delta\beta_d^n(a_1, \ldots, a_d).$$

In §7 we show that L can be lifted to a map with values in $K_{2,L}^{top}(R, I)$. In preparation for the proof of the existence of the β_d^n we introduce in §8 a generating function formalism. In §9 this is used to reduce the problem to the case that $d=2$ and n is a prime power. In §10 the needed primary congruences are checked.

This paper grew out of [Clauwens, 1984] where the special case of a truncated polynomial ring in several variables over the integers was treated. In a sequel to this paper we will do calculations to determine the kernel and image of L in some interesting cases, like rings of formal or convergent power series and abelian group rings. We will also apply this technique to do calculations for cyclotomic extensions of rings of the above type; even when these rings have no λ-structure they can be approximated well by rings that have one. It appears that L is nearly an isomorphism in all these cases. In the future we expect to apply these results to the computation of K_1 of group rings over polynomial rings.

Concerning related work we remark that our result contains the result of [Roberts and Geller, 1978]

$$K_2(Z[t]/t^n) = \bigoplus_{j=2}^{n} Z/j$$

as a special case and clarifies it. Also the result of chapters 1 and 2 of [Oliver, 1985] is a special case of our result. However, most work on the K-theory of truncated polynomial rings and similar rings either avoids nonlinearity problems by restricting to low truncation (see Van der Kallen, 1971; Labute and Russell, 1975]) or avoids divisibility problems e.g. by assuming that the ring contains a field (see [Stienstra, 1980; Van der Kallen and Stienstra, 1984]).

As mentioned before there are relations between the theory presented here and cyclic homology, as developed by A. Connes, T. Goodwillie, D. Kan, M. Karoubi, and others. Particularly suggestive in conjunction with our work is the recent work of T. Goodwillie in which an isomorphism is constructed between the rational K-theory and the rational cyclic homology for a ring with a nilpotent ideal.

§1. Generalities about λ-rings

In this section we review briefly the theory of λ-rings, due essentially to [Grothendieck, 1958], and developed further by [Atiyah and Tall, 1969]. A good introduction is also [Knutson, 1973].

A λ-ring is a commutative ring R with identity, together with maps $\lambda^n : R \to R$ for $n = 0, 1, 2, \ldots$ such that

$$\lambda^0(a) = 1$$

$$\lambda^1(a) = a,$$

$$\lambda^n(a+b) = \sum_{i+j=n} \lambda^i(a)\lambda^j(b),$$

$$\lambda^n(1) = 0 \text{ for } n > 0,$$

$$\lambda^n(ab) = F_n(\lambda^1(a), \lambda^2(a), \dots, \lambda^n(a), \lambda^1(b), \dots, \lambda^n(b)),$$

$$\lambda^m(\lambda^n(a)) = F_{m,n}(\lambda^1(a), \lambda^2(a), \dots, \lambda^{mn}(a)),$$

where the F_n and $F_{m,n}$ are certain universal polynomials. In *loc. cit.* one calls this a special λ-ring, and uses the name λ-ring if only the first three conditions are satisfied.

An example is the representation ring of a finite group, where the λ^n are given by exterior powers. In particular the ring of integers is a λ-ring with $\lambda^n(a)$ given by the binomial symbol $\binom{a}{n}$. If R is a λ-ring then the polynomial ring $R[t]$ has a unique λ-ring structure for which $\lambda^n(at^m) = \lambda^n(a)t^{nm}$ if $a \in R$ (see [Knutson, 1973], p. 23).

A ring-homomorphism $f: R \to S$ between λ-rings such that $f(\lambda^n(a)) = \lambda^n(f(a))$ for all $a \in R$ will be called λ-map.

The following proposition is the key to proving identities in λ-rings.

PROPOSITION 1.1. *There exists a λ-ring U and an element $u \in U$ such that for any λ-ring R and element $a \in R$ there is a unique λ-map $f: U \to R$ such that $f(u) = a$. The ring U is the polynomial ring over the integers freely generated by the $\lambda^n(u)$ with $n > 0$.*

Proof. See [Atiyah and Tall, 1969], p. 260 or [Knutson, 1973] p. 25. ■

The proposition implies that to prove an identity between the $\lambda^n(a)$ for an element $a \in R$ of a λ-ring it is sufficient to do so for the universal example $u \in U$. One of the advantages is that U has no torsion.

Remark 1.2. An element $\xi \in U$ defines a map $\xi_R: R \to R$ if R is any λ-ring. It is defined by $\xi_R(a) = f(\xi)$, where f is the λ-map mapping $u \in U$ to a. On the other hand every natural operation on λ-rings arises this way (see [Atiyah and Tall, 1969] p. 265). We will call such an operation a λ-operation. Henceforth we will not distinguish between an element of U and its associated λ-operation.

Definition 1.3. We will write E for the ideal of U generated by the elements $\lambda^i(u)$ with $i > 0$; elements of E^2 will be called decomposable.

Remark 1.4. We also make U a graded ring by declaring $\lambda^i(u)$ to be of degree i. The fact that an operation ξ is of degree n can be recognised by letting it act on $R[t]$ with the aforementioned structure of λ-ring, and observing that $\xi(rt^m)=\xi(r)t^{mn}$ in that case. This implies that the degree of a composition of homogeneous λ-operations is the product of their degrees. Furthermore any operation which is homogeneous of degree n must be a integral multiple of λ^n modulo decomposables.

Remark 1.5. In the same way one proves that there exists a λ-ring U_d and elements $u_1,\dots,u_d$ of U_d such that for any λ-ring R and elements $a_1,\dots,a_d$ of R there is a unique λ-map f such that $f(u_i)=a_i$ for $1\leqslant i\leqslant d$. The ring U_d is the polynomial ring over the integers freely generated by the $\lambda^n(u_i)$; so $U_d=U\otimes\cdots\otimes U$. Elements of U_d give rise to λ-operations in d variables.

Let R be a λ-ring. We write $\lambda_t(a)$ for the formal power series $\sum_{n=0}^{\infty}\lambda^n(a)t^n\in R[[t]]$. Since the constant term is 1 the Adams operations $\psi^n: R\to R$ can be defined by

$$\sum_{n=1}^{\infty}(-1)^{n-1}\psi^n(a)t^n=\lambda_t(a)^{-1}t\frac{\mathrm{d}}{\mathrm{d}t}\lambda_t(a).$$

PROPOSITION 1.6.
a) $\psi^n(1)=1$
b) $\psi^1(a)=a$ *for* $a\in R$
c) *The* ψ^n *are* λ-*maps.*
d) $\psi^n\psi^m=\psi^{nm}$.
e) *If* $a\in R$ *and* p *is prime then* $a^p-\psi^p(a)\in pR$.

Proof. See [Atiyah and Tall, 1969] p. 264 or [Knutson, 1973] p. 48. ■

To prove a converse to Proposition 1.6. we need some identities. By multiplying both sides of the definition of the Adams operations with $\lambda_t(a)$ we get the Newton formula:

$$\psi^n(a)-\lambda^1(a)\psi^{n-1}(a)+\cdots+(-1)^{n-1}\lambda^{n-1}(a)\psi^1(a)=(-1)^{n-1}n\lambda^n(a).$$

In particular $\psi^n(a)=(-1)^{n+1}n\lambda^n(a)+$ *decomposables,* and ψ^n is of degree n. Furthermore if p is prime we see by applying Remark 1.2. to $\xi=p^{-1}(u^p-\psi^p(u))$ that Proposition 1.6.e) implies the existence of a λ-operation θ^p of degree p such that $a^p=\psi^p(a)+p\theta^p(a)$ for any a.

LEMMA 1.7. *If* R *is a* λ-*ring and* $a\in R$ *then*
a) $\theta^p(a)=(-1)^p\lambda^p(a)+$ *decomposables*

b) $\lambda^m(\lambda^n(a)) = (-1)^{mn+m+n+1}\lambda^{mn}(a)$ + *decomposables*
c) $\lambda^n(ab) = \psi^n(a)\lambda^n(b)$ *modulo the ideal generated by decomposables in b.*

Proof of c). By two applications of the Newton formula we have: $(-1)^{n-1}n\lambda^n(ab) = \psi^n(ab)$ + *decomposables in* $ab = \psi^n(a)\psi^n(b)$ *modulo the ideal* = $\psi^n(a)[(-1)^{n-1}\lambda^n(b)$ + *decomposables in* $b]$ *modulo the ideal* = $(-1)^{n-1}n\psi^n(a)\lambda^n(b)$ *modulo the ideal.* Furthermore $nU \cap E = nE$.

For a) and b) see [Wilkerson, 1982] p. 315. ■

From this one can prove the desired converse, which is useful in recognising λ-rings.

Definition 1.8. A ψ-ring is an associative ring with 1, equipped with ring homomorphisms $\psi^n : R \to R$ for $n = 1, 2, \ldots$ such that $\psi^n(1) = 1$, $\psi^1(a) = a$ for $a \in R$, $\psi^m\psi^n = \psi^{mn}$.

PROPOSITION 1.9. *Let R be a commutative ψ-ring which has no Z-torsion and such that* $\psi^p(a) = a^p$ *modulo pR if p is prime. Then there is a unique λ-ring structure on R such that the* ψ^n *are the associated Adams operations.*

Proof. See [Wilkerson, 1982] p. 314. ■

Example 1.10. A λ-ring in which $\psi^n(a) = a$ for all n and a is called a binomial ring. One deduces easily from the Newton formula that in that case $n!\lambda^n(a) = \prod_{i=0}^{n-1}(a - i)$. Proposition 1.9. implies that any Q-algebra can be given a structure of binomial ring. Another example of a binomial ring is the subring of $Q[s]$ consisting of those functions that take integral values at the integers. In an augmented Q-algebra one can also take $\psi^n(x) = 0$ for $n > 1$ and x in the augmentation ideal; in that case $n!\lambda^n(a) = a^n$.

Definition 1.11. Let R be a λ-ring and I be an ideal of R. Then I is called a λ-ideal if $\lambda^n(I) \subseteq I$ for $n > 0$. It is called a ψ-ideal if $\psi^n(I) \subseteq I$ for $n > 0$. Clearly any λ-ideal is a ψ-ideal; the converse is not true.

PROPOSITION 1.12. *Let R be a λ-ring and let I and J be λ-ideals of R; then IJ is also a λ-ideal of R. In particular* I^n *is a λ-ideal for any n.*

Proof. We must show that $\lambda^n(ab) \in IJ$ if $a \in I$, $b \in J$ and $n > 0$; then the statement follows by application of the formula for λ^n of a sum. We use induction. If n is not prime then Lemma 1.7.b) does the job. If n is a prime p

then Lemma 1.7.a) reduces the problem to one for the operation θ^p. For θ^p the statement is true since $\theta^p(ab) = a^p\theta^p(b) + \theta^p(a)b^p - p\theta^p(a)\theta^p(b)$. ∎

PROPOSITION 1.13. *Let R be a λ-ring and I a λ-ideal, and $\xi \in E$ a λ-operation. Then $\xi(a+b) - \xi(a) - \xi(b) \in I^2$ if $a, b \in I$.*

Proof. If $\xi \in E^2$ then all terms are in I^2, so we may assume that ξ is an integral linear combination of the λ^n. But for $\xi = \lambda^n$ the statement is immediate from the λ-ring axioms. ∎

A similar statement is of course true for λ-operations in several variables.

§2. The operations θ^n and the ring W

In this § we introduce λ-operations θ^n generalising the θ^p introduced in §1, and we study their properties.

For p prime denote by v_p the p-adic valuation on $\mathbf{Q}$. The integrality statements in this paper are based on the fact that $v_p\binom{p^n}{i} = n - v_p(i)$.

Let μ be the Mobius function, which is defined on the natural numbers by the properties

$$\mu(1) = 1, \sum_{m\mid n} \mu(n) = 0 \text{ if } n > 1.$$

PROPOSITION 2.1. *There exist a unique λ-operation θ^n such that*

$$n\theta^n(a) = \sum_{m\mid n} \mu(m)\psi^m(a^{n/m})$$

It is homogeneous of degree n.

Proof. In view of Remark 1.2. we only have to show that the right hand side is divisible by n if $a = u \in U$. If n is prime this is Proposition 1.6. If n is a prime power p^k then $\theta^{p^k}(u) = p^{-k}(u^{p^k} - \psi^p u^{p^{k-1}}) =$

$$p^{-k}\left((\psi^p u + p\theta^p u)^{p^{k-1}} - (\psi^p u)^{p^{k-1}}\right) = \sum_{j=1}^{p^{k-1}} p^{j-k}\binom{p^{k-1}}{j}(\psi^p u)^{p^{k-1}-j}(\theta^p u)^j$$

and the numerical factor is an integer since $j - 1 \geqslant v_p(j)$ for $j \geqslant 1$. If n is not a prime power then we can write $n = km$, $xk + ym = 1$ for some integers

$k < n,\ m < n,\ x,\ y$. Then

$$\begin{aligned}\theta^n(u) &= \frac{xk+ym}{km}\sum_{i\mid m}\sum_{j\mid k}\mu(ij)\psi^{ij}(u^{km/ij})\\ &= \frac{x}{m}\sum_{j\mid k}\mu(j)\psi^j\Big(\sum_{i\mid m}\mu(i)\psi^i(u^{mk/ij})\Big)\\ &\quad+\frac{y}{k}\sum_{i\mid m}\mu(i)\psi^i\Big(\sum_{j\mid k}\mu(j)\psi^j(u^{mk/ij})\Big)\\ &= x\sum_{j\mid k}\mu(j)\psi^j(\theta^m(u^{k/j}))+y\sum_{i\mid m}\mu(i)\psi^i(\theta^k(u^{m/i})),\end{aligned}$$

and this belongs to U by induction hypothesis. ■

Now we list some properties of these operations. These can easily be proved by working in the universal λ-ring and using the properties of the Möbius function and of the Adams operations, and by rearranging sums.

LEMMA 2.2. *For an element a of a λ-ring*

$$a^k = \sum_i \frac{k}{i}\psi^i(\theta^{k/i}(a))$$

where the sum extends over all i dividing k.

PROPOSITION 2.3. *For an element a of a λ-ring*

$$\theta^n(a^k) = \sum_i \frac{k}{i}\psi^i(\theta^{kn/i}(a))$$

where the sum extends over all i dividing k which are prime to n.

PROPOSITION 2.4. *For elements a, b of a λ-ring*

$$\theta^n(ab) = \sum_m \theta^m(a^{n/m})\psi^m(\theta^{n/m}(b))$$

where the sum extends over all m dividing n.

COROLLARY 2.5. *The substitution of Proposition 2.3. into Proposition 2.4. yields the symmetric formula*

$$\theta^n(ab) = \sum_{k,\,m}\frac{n}{km}\psi^k(\theta^{n/k}(a))\psi^m(\theta^{n/m}(b))$$

where the sum extends over all $k \mid n$ and $m \mid n$ which are relatively prime.

COROLLARY 2.6. *Iterating formula 2.5. yields the formula*

$$\theta^n(a_1a_2\dots a_k)=\sum\frac{n^{k-1}}{m_1m_2\dots m_k}\psi^{m_1}(\theta^{n/m_1}(a_1))\psi^{m_2}(\theta^{n/m_2}(a_2))\dots$$

$$\times\psi^{m_k}(\theta^{n/m_k}(a_k))$$

where the sum extends over all sequences $m_1, m_2, \dots, m_k$ *of divisors of* n *that have* 1 *as greatest common divisor.*

Remark 2.7. Theorems 2 and 4 of [Metropolis and Rota, 1983] are just Corollary 2.5. and Proposition 2.3. for the special case that R is the ring of ordinary integers.

Now we introduce the subring of U consisting of those λ-operations which are needed in this paper.

Definition 2.8. We write V for the subring of U generated by the elements $\psi^j(\theta^k(u))$ and similarly V_d for the subring of U_d generated by the $\psi^j(\theta^k(u_i))$.

Remark 2.9. If $\xi\in V$ then Corollary 2.5. says that the operation $(a, b)\to\xi(ab)$ is an element of V_2. Repeated application of Proposition 2.3. now says that the operation $a\to\xi(a^n)$ is in V.

Definition 2.10. We write W for the subring of U generated by V and the elements $\psi^j(\lambda^4(u))$. Similarly one defines $W_d\subseteq U_d$.

Remark 2.11. By definition of λ-ring $\lambda^4(ab)$ is a polynomial in the $\lambda^i(a)$ and $\lambda^i(b)$ with $i\leqslant 4$. But $\lambda^1(u)=u\in V$, $\lambda^2(u)=\theta^2(u)\in V$ and $\lambda^3(u)=u\theta^2(u)-\theta^3(u)\in V$. Therefore W has also the properties mentioned in Remark 2.9.

§3. The operations η^n

In this § we introduce λ-operations η^n in two variables which generalise the operations θ^n introduced in §2, and we study their properties.

Definition 3.1. Let R be a λ-ring. The maps $\eta^n: R\times R\to R$ are recursively defined by $\eta^n(a, b)=b$ if $n=1$, and else

$$\eta^n(a, b)=\sum_m\eta^m(a, a^{(n-m)/m})\psi^m(\theta^{n/m}(b))$$

where the sum extends over all $m<n$ dividing n. The operation η^n is homogeneous of degree $n-1$ in the first variable and of degree n in the second variable.

Remark 3.2. Since $\theta^n(1)=0$ if $n>1$ the second formula implies that $\eta^n(1,1)=0$ if $n>1$ and hence that $\eta^n(1,b)=\theta^n(b)$ for all n and $\eta^n(a,1)=0$ for $n>1$.

One sees easily using Remark 2.9. and induction that $\eta^n \in V_2$ for all n.

PROPOSITION 3.3. *For elements a, b in a λ-ring*

$$a^{n-1}b^n = \sum_{m \mid n} \frac{n}{m} a^{m-1}\psi^m\left(\eta^{n/m}(a,b)\right)$$

Proof. By substituting Definition 3.1. into the right hand side and using the induction hypothesis this reduces to Lemma 2.2. ■

If R is a λ-ring then the polynomial ring $R[t]$ has a unique λ-ring structure for which $\lambda^n(at^m)=\lambda^n(a)t^{nm}$. If $f \equiv g \mod t^mR[t]$ then $\lambda^n(f)\equiv\lambda^n(g) \mod t^{nm}R[t]$. Therefore the λ-ring structure extends uniquely to the ring $R[[t]]$ of formal power series. One sees easily from the definitions that

$$\psi^n(ta)=t^n\psi^n(a), \quad \theta^n(ta)=t^n\theta^n(a), \quad \eta^n(ta,b)=t^{n-1}\eta^n(a,b)$$

for $a, b \in R[[t]]$.

Therefore the expression $\eta(ta,b)=\sum_{n=1}^{\infty}\eta^n(ta,b)$ makes sense as an element of $R[[t]]$.

For the next proposition we need the following notations.

Definition 3.4. If the ring R contains the rationals then the map log: $1+tR[[t]] \to tR[[t]]$ is defined by $\log(1-ta) = -\sum_{n=1}^{\infty} n^{-1}t^na^n$.

It has the property that $\frac{\mathrm{d}}{\mathrm{d}t}\log(f)=\frac{1}{f}\frac{\mathrm{d}}{\mathrm{d}t}f$ and therefore satisfies

$$\log(1-ta)+\log(1-tb)=\log(1-ta-tb+t^2ab) \text{ for } a, b \in R.$$

If a λ-ring R contains the rationals and $a \in R$ then the map $G_a: R[[t]] \to R[[t]]$ is defined by

$$G_a(f) = -\sum_{m=1}^{\infty} m^{-1}t^ma^m\psi^m(f).$$

It is easily seen to be an injective homomorphism of additive groups if a is not a zero divisor.

With this preparation we can formulate the principal property of η.

PROPOSITION 3.5. *Let R be a λ-ring and $a, b, c \in R$. Then*

$$\eta(ta, b) + \eta(ta, c) - \eta(ta, b + c - tabc) = 0.$$

Proof. We may assume that $R = \mathbf{Q} \otimes U_3$ and that a, b, c are the canonical elements. Applying G_a to the left hand side of the desired relation and using 3.3. we get

$$\log(1 - tab) + \log(1 - tac) - \log(1 - tab - tac + t^2a^2bc).$$

The proposition follows since this expression vanishes. ■

The remainder of this § is not needed for the proof of the main theorem but is meant to clarify the meaning of the map η.

To formulate the next proposition we need some notation.

Definition 3.6. If R is a λ-ring and $a, b \in R[[t]]$ then we write $\lambda_{ta}(b)$ or $\lambda_a(tb)$ for $\sum_{n=0}^{\infty} t^n a^n \lambda^n(b)$.

LEMMA 3.7. *Let R be a λ-ring containing the rationals. Then* $\log \lambda_{-ta}(b) = G_a(b)$ *for* $a, b \in R[[t]]$.

Proof. First we show that $\log \lambda_{-t}(b) = -\sum_{n=1}^{\infty} \frac{t^n}{n} \psi^n(b)$ for $b \in R$. To do this we apply $\frac{d}{dt}$ to the difference of both sides and note that we get 0 by definition of the ψ^n; moreover both sides have vanishing constant term. By the substitution $t \to ta$ we get from this that $\log \lambda_{-ta}(b) = -\sum_{n=1}^{\infty} \frac{t^n a^n}{n} \psi^n(b)$ for $a, b \in R$.

Now we apply this identity to the ring $R[[s]]$ in place of R; that yields an identity in $R[[s]][[t]]$. Mapping s to t one gets the desired identity in $R[[t]]$. ■

PROPOSITION 3.8. *Let R be a λ-ring and $a, b \in R[[t]]$. Then* $\lambda_{-ta}(\eta(ta, b)) = 1 - tab$.

Proof. We may assume that a, b are the canonical elements in $\mathbf{Q} \otimes U_2$. From the properties of G_a and Lemma 3.7. it follows that

$$\log \lambda_{-ta}(\eta(ta, b)) = G_a(\eta(ta, b)) = \log(1 - tab).$$

Therefore it is sufficient to note that log is injective.

In fact if R is a ring containing the rationals then the map $\exp: tR[[t]] \to 1 + tR[[t]]$ defined by $\exp(ta) = \sum_{m=0}^{\infty} (m!)^{-1} t^m a^m$ has the property that $\frac{\mathrm{d}}{\mathrm{d}t}\exp(f) = \exp(f)\frac{\mathrm{d}}{\mathrm{d}t}f$ and therefore is an inverse to log. ■

Remark 3.9. One can think of λ_{-ta} as a kind of exponential map with base $1 - ta$, since it is homomorphic from addition to multiplication and maps 1 to $1 - ta$. In this interpretation $\eta(ta, b)$ becomes a logarithm of $1 - tab$ with base $1 - ta$.

Indeed we could have taken the above property as definition and define $\eta^n(a, b)$ as the coefficient of t^n in $(\lambda_{-ta})^{-1}(1 - tab)$. To justify this we may assume that a, b are the canonical elements of U_2 and note the following fact which can easily be checked: if R is a λ-ring and $a \in R$ is not a zero divisor then the map $\lambda_{-ta}: R[[t]] \to 1 + taR[[t]]$ is bijective.

The above proposition also means that for the total θ-operation $\theta(tb) = \sum_{n=1}^{\infty} \theta^n(tb)$ one has $\lambda_{-1}\theta(tb) = \lambda_{-1}\eta(1, tb) = \lambda_{-1}t\eta(t, b) = \lambda_{-t}\eta(t, b) = 1 - tb$. So θ can be interpreted as the inverse of $1 - \lambda_{-1}$.

§4. The operations ϕ^n on differential forms

Let R be a λ-ring. Since the Adams operation $\psi^n: R \to R$ is a ring homomorphism it induces a map $\psi^n: \Omega_R \to \Omega_R$ where Ω_R is the R-module of universal differentials. In this section we show that there are maps $\phi^n: \Omega_R \to \Omega_R$ such that $n\phi^n = \psi^n$. In fact we prove something more general.

Definition 4.1. Let R be a ψ-ring. A left R-module P together with maps $\phi^k: P \to P$ such that
1) $\phi^1 = 1_P$,
2) $\phi^k(x + y) = \phi^k(x) + \phi^k(y)$ if $x, y \in P$,
3) $\phi^k(ax) = \psi^k(a)\phi^k(x)$ if $a \in R$, $x \in P$,
4) $\phi^k\phi^m = \phi^{km}$
will be called a ψ-module over R.

PROPOSITION 4.2. *Let R be a λ-ring and I a λ-ideal. Then I/I^2 can be given the structure of ψ-module over R such that $\phi^n(a + I^2) = (-1)^{n-1}\lambda^n(a) + I^2$ for $a \in I$. In particular $n\phi^n(a + I^2) = \psi^n(a) + I^2$.*

Proof. It follows from the axiom for $\lambda^n(a + b)$ that $(-1)^{n-1}\lambda^n$ is well defined on I and additive modulo I^2. By Proposition 1.12. it is well defined on I/I^2. Now condition 4.1.3) is satisfied because of Lemma 1.7.c) and condition 4.1.4) is satisfied because of Lemma 1.7.b). The second statement follows from the Newton formula. ■

LEMMA 4.3. *Let R_1 and R_2 be λ-rings; then there exists a unique λ-ring structure on $R_1 \otimes R_2$ such that $\lambda^n(a \otimes 1) = \lambda^n(a) \otimes 1$ and $\lambda^n(1 \otimes a) = 1 \otimes \lambda^n(a)$.*

Proof. The uniqueness is obvious from the axioms, so we show existence. We may assume that R_1 and R_2 are finitely generated algebras over the integers, since any element of $R_1 \otimes R_2$ is contained in the tensor product of such rings. So there exist surjective λ-maps $f_1 : U_d \to R_1$ and $f_2 : U_e \to R_2$. Then $f_1 \otimes f_2$ is a surjective map $U_d \otimes U_e \to R_1 \otimes R_2$, and the kernel is $ker(f_1) \otimes R_2 + R_1 \otimes ker(f_2)$. Furthermore $U_d \otimes U_e$ can be identified with U_{d+e} so has a λ-ring structure of the desired type. Using the λ-ring axioms one proves easily that the above mentioned kernel is a λ-ideal since $ker(f_1)$ and $ker(f_2)$ are. ■

For a commutative ring R let $M: R \otimes R \to R$ be the map defined by $M(a \otimes b) = ab$. Then Ω_R is defined as $(ker\, M)/(ker\, M)^2$ and the universal derivation $\delta: R \to \Omega_R$ is defined by $\delta(a) = 1 \otimes a - a \otimes 1 + (ker\, M)^2$.

PROPOSITION 4.4. *If R is a λ-ring then Ω_R has the structure of ψ-module over R in such a way that for $a \in R$ one has*
a) $\delta(\psi^n(a)) = n\phi^n(\delta a)$
b) $\delta(\lambda^n(a)) = \sum_{i=1}^{n} (-1)^{i-1}\lambda^{n-i}(a)\phi^i(\delta a)$.

Proof. One sees easily from the λ-ring axioms that $M: R \otimes R \to R$ is a λ-map. Therefore Proposition 4.2. yields a structure of ψ-module on $(ker\, M)/(ker\, M)^2$. For this structure one has

$$n\phi^n(\delta a) = n\phi^n\big(1 \otimes a - a \otimes 1 + (ker\, M)^2\big) = \psi^n(1 \otimes a - a \otimes 1) + (ker\, M)^2$$

$$= 1 \otimes \psi^n(a) - \psi^n(a) \otimes 1 + (ker\, M)^2 = \delta(\psi^n(a)).$$

This proves part a).

Let $\delta_R: R[[t]] \to \Omega_R[[t]]$ be defined by $\delta_R \sum_{i=0}^{\infty} x_i t^i = \sum_{i=0}^{\infty} (\delta x_i) t^i$ for $x_i \in R$. Since δ_R commutes with $t\frac{\mathrm{d}}{\mathrm{d}t}$ one checks easily that $t\frac{\mathrm{d}}{\mathrm{d}t}\{\lambda_t(a)^{-1}\delta_R\lambda_t(a)\} = \delta_R\{\lambda_t(a)^{-1} t\frac{\mathrm{d}}{\mathrm{d}t}\lambda_t(a)\}$ for $a \in R$.

By definition of the ψ^n the right hand side can be written as

$$\delta_R\left\{\sum_{i=1}^{\infty} (-1)^{i-1}\psi^i(a)t^i\right\} = \sum_{i=1}^{\infty} (-1)^{i-1}\phi^i(\delta a)it^i$$

$$= t\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{i=1}^{\infty} (-1)^{i-1}\phi^i(\delta a)t^i\right\}$$

Therefore $\lambda_t(a)^{-1}\delta_R\lambda_t(a) = \sum_{i=1}^{\infty}(-1)^{i-1}\phi^i(\delta a)t^i$. Multiplying both sides with $\lambda_t(a)$ and looking at the coefficient of t^n on both sides yields part b). ∎

COROLLARY 4.5. *If R is a λ-ring then there is a structure of ψ-module ϕ_i on the differential forms Ω^i_R such that*

$$\phi^n_{i+j}(\alpha_1 \wedge \alpha_2) = \phi^n_i(\alpha_1) \wedge \phi^n_j(\alpha_2) \quad \textit{and } \delta\phi^n_i(\alpha_1) = n\phi^n_{i+1}(\delta\alpha_1)$$

$$\textit{for } \alpha_1 \in \Omega^i_R,\ \alpha_2 \in \Omega^j_R.$$

Here the version of the Grassmann algebra is meant in which $\alpha \wedge \alpha = 0$ for $\alpha \in \Omega^1_R$.

Example 4.6. If R is a binomial ring then Ω_R is divisible group; indeed ϕ^n acts as division by n.

Remark 4.7. The universal ring U is the polynomial ring generated by the $\lambda^n(u)$, where u is the canonical element. So Ω_U is a free module over U with the $\delta\lambda^n(u)$ as a basis. Therefore formula 4.4.b) expresses that the $\phi^n\delta u$ are also a basis of Ω_U.

Remark 4.8. For a commutative ring R let $D: R \otimes R \otimes R \to R \otimes R$ be defined by $D(x \otimes y \otimes z) = x \otimes yz - xy \otimes z - xz \otimes y$. Then Ω_R can also be defined as $(R \otimes R)/im(D)$ since the inclusion induces an isomorphism $(ker\, M)/(ker\, M)/^2 \to (R \otimes R)/im(D)$.

So a more direct way to prove the existence of the operations $\phi^n: \Omega_R \to \Omega_R$ would have been to define

$$\phi^n(a \otimes b) = (-1)^{n-1}\psi^n(a)\sum_{i=1}^{n}\lambda^{n-i}(-b)\delta\lambda^i(b)$$

and to check that the right hand side is additive in a and b and vanishes on $im(D)$.

§5. Convergence

For the first proposition we introduce the notation $\|n\| = \min\{\sum_i (n_i - 1); n = \prod_i n_i\}$ where n and the n_i are natural numbers. Then it is easily seen that $\|n_1 n_2\| = \|n_1\| + \|n_2\|$ and that $n - 1 \geqslant \|n\| \geqslant {}^2\log(n)$ for all n_1, n_2, n.

PROPOSITION 5.1. *Let R be a λ-ring and let I and J be λ-ideals. If $a \in I$ and $b \in J$ then $\eta^n(a, b) \in I^{\|n\|}J$.*

Proof. Induction on n. If $n=1$ then $\eta^1(a,b)=b\in J$. If $n>1$ then

$$\eta^n(a,b)=\sum_m \eta^m(a,a^{(n-m)/m})\psi^m(\theta^{n/m}(b))$$

$$\in \sum_m I^{\|m\|+(n-m)/m}J\subseteq I^{\|n\|}J$$

by Proposition 1.12. and the induction hypothesis. ■

The above proposition shows that if $a\in I$ and $b\in R$ then $\eta(a,b)=\sum_{n=1}^{\infty}\eta^n(a,b)$ has a meaning in the completion of R with respect to powers of I.

If, however, $a\in R$ and $b\in I$ then $\eta(a,b)$ will in general not converge in the topology given by the powers of I (see Example 5.10). From now on we therefore assume that a topology on R is given by the powers of an ideal J which contains I. The corollary of the next proposition shows that J does not have to be a λ-ideal but only a ψ-ideal; that allows the important case that $J=I+pR$ for some number p.

PROPOSITION 5.2. *Let R be a λ-ring and let J be a ψ-ideal. Then $\theta^n(J^N)\subseteq J^{N-1}$ for all n, N.*

Proof. If n is a prime p then we use induction on N. Since

$$\theta^p(a+b)-\theta^p(a)-\theta^p(b)=\sum_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}a^ib^{p-i}$$

the operation θ^p is additive modulo J^{pN} on J^N. Therefore it is sufficient to check the statement on elements of the form ab where $a\in J^{N-1}$, $b\in J$. From Proposition 2.4. we then get

$$\theta^p(ab)=a^p\theta^p(b)+\theta^p(a)\psi^p(b)$$

where the second term belongs to $J^{N-2}J$ by induction hypothesis, and the first term belongs even to $J^{(N-1)p}$.

If n is a prime power p^k then we have seen in the proof of Proposition 2.1. that

$$\theta^{p^k}(a)=\sum_j p^{j-k}\binom{p^{k-1}}{j}(\psi^pa)^{p^{k-1}-j}(\theta^pa)^j$$

which belongs to J^{N-1} because $\theta^p(a)$ does.

If n is not a prime power then we have seen in the proof of Proposition 2.1. that

$$\theta^n(a) = x \sum_{j|k} \mu(j) \psi^j(\theta^m(a^{k/j})) + y \sum_{i|m} \mu(i) \psi^i(\theta^k(a^{m/i})).$$

where $k < n$, $m < n$, x, y are integers such that $n = km$ and $xk + ym = 1$. This belongs to J^{N-1} because $\theta^m(a^{k/j})$ and $\theta^k(a^{m/i})$ do. ∎

COROLLARY 5.3. *Let R be a λ-ring and let J be a ψ-ideal. Then every λ-operation is continuous for the topology on R defined by J. So the J-completion of R is again a λ-ring.*

Proof. We prove by induction on n that λ^n is continuous. For n prime the statement follows from Propositions 1.7.a) and 5.2. and the induction hypothesis. For n composite it follows from Proposition 1.7.b) and the induction hypothesis. ∎

Remark 5.4. Let R be a λ-ring and J a ψ-ideal; then the operations $\phi^n : \Omega_R \to \Omega_R$ are also continuous. This follows from the ψ-module structure on Ω_R and the fact that

$$\lim_{\leftarrow} \Omega_{R/J^N} = \lim_{\leftarrow} \Omega_R / J^N \Omega_R.$$

Now we are ready to formulate the topological conditions which the ring R and its ideal I in the main theorem should satisfy.

Definition 5.5. A triple (R, J, I) will be called admissible if R is a λ-ring, I is a λ-ideal, and J is a ψ-ideal containing I such that

a) $\forall_N \exists_M \forall_n : n \geqslant M \to \theta^n(I) \subseteq J^N$

b) $\forall_N \exists_M \forall_n : n \geqslant M \to \psi^n(I) \subseteq J^N$

Condition a) is needed to give a meaning to $\theta(a) = \sum_{n=1}^{\infty} \theta^n(a)$ for $a \in I$.

Condition b) is needed to give a meaning to $\eta(a, b) = \sum_{n=1}^{\infty} \eta^n(a, b)$ for $a \in R$ and $b \in I$. Both sums take values in the J-completion of R.

Now we can give a new interpretation of Proposition 3.5.

PROPOSITION 5.6. *If (R, J, I) is admissible then*

$$\theta(a) + \theta(b) = \theta(a + b - ab) \quad \textit{if } a, b \in I,$$

$\eta(a, b) + \eta(a, c) = \eta(a, b + c - abc)$ *if either $a \in I$ and $b, c \in R$ or $a \in R$ and b, $c \in I$.*

Proof. Proposition 3.5. says that for $a,\ b,\ c \in R$

$$\sum_{n=1}^{\infty} \eta^n(a,\ b)t^{n-1} + \sum_{n=1}^{\infty} \eta^n(a,\ c)t^{n-1} = \sum_{n=1}^{\infty} \eta^n(a,\ b+c-tabc)t^{n-1}$$

In the given circumstances all three sums are of the form $\sum_{n=1}^{\infty} f_n t^{n-1}$ where $f_n \in R[t]$ for all n and where for any N one has $f_n \in J^N R[t]$ for large n.

In other words those formal power series are in fact convergent for the topology given by the powers of J. Therefore the substitution of 1 for t makes sense and it yields an equality in R^{top}. That the substitution maps $\eta^n(a,\ b+c-tabc)$ to $\eta^n(a,\ b+c-abc)$ follows from the fact that the ideal $(t-1)R[t] \subseteq R[t]$ is invariant under the λ^m. ■

COROLLARY 5.7. *Proposition* 5.6. *says that the total* θ*-operation is the desired logarithmic map* $L: K_1(R,\ I) \to K_{1,L}^{top}(R,\ I) = I^{top}$

For use in the next § we generalize the considerations in Proposition 5.6. as follows.

Observation 5.8. If $\xi^n \in W$ is homogeneous of degree $\mathrm{d}(n)$ for every n and if $\lim_{n\to\infty} \mathrm{d}(n) = \infty$ then $\lim_{n\to\infty} \xi^n(a) = 0$ if $a \in I$; thus $\sum_{n=1}^{\infty} \xi^n(a)$ is defined in I^{top}.

We now discuss the two types of example for which this whole theory was meant and check that they are admissible.

Example 5.9. Let A be any λ-ring. Then $R = A[t_1, \ldots, t_k]$ has a structure of λ-ring such that $\lambda^n(t_i) = 0$ for $n > 1$. Let $J = I = t_1R + \cdots + t_kR$. Then $\lambda^n(I) \subseteq J^n$ and more generally $\xi(I) \subseteq J^n$ for any operation ξ of degree n, so certainly for $\xi = \psi^n$ or $\xi = \theta^n$; therefore the conditions are satisfied.

From this example one can construct other ones by taking the quotient of R by some λ-ideal e.g. the ideal generated by a number of monomials, or difference of two monomials.

Example 5.10. Let p be a prime number and let A be a λ-ring in which primes $\neq p$ are invertible. Then $A[t_1, \ldots, t_k]$ has a structure of λ-ring such that $\psi^p(t_i) = t_i^p$ and $\psi^q(t_i) = 1$ if q is prime to p; if A is the integers localised away from p this follows from Proposition 1.9. and in general by taking the tensor product of that case with A (see Lemma 4.3). Write $x_i = t_i - 1$ and let $I = x_1R + \cdots + x_kR$ and $J = I + pR$. If n has the form $p^e q$ with p prime to q then $\theta^n(a) = \dfrac{1}{q}\theta^{p^e}(a^q)$ for $a \in I$, and according to the formula in the proof of Proposition 2.1. this belongs to $I^{p^{e-1}q}$. Therefore condition a) is satisfied. In order to show that condition b) is satisfied we note that we only have to consider $\psi^n(x_i)$. This vanishes if n is not a power of p and $\psi^{p^e}(x_i) = (x_i+1)^{p^e} - 1 \in (pR + x_iR)^{e+1} \subseteq J^{e+1}$. Again one can construct suitable quotients from this R, in particular the group ring over A of a finite p-group. In this way the theory of chapters 1 an 2 of [Oliver, 1985] is generalised.

Remark 5.11. Remark 3.9. can be interpreted as saying that $1-\lambda_{-1}$ is an inverse to θ whenever it converges. However the assumptions in this § are not strong enough to enforce that. In the case of Example 5.9. it converges. However in the case of Example 5.10. it follows from Lemma 1.7. and the fact that $\theta^p(t_i) = -t_i$ *modulo* I^2 that $\lambda^{p^n}(t_i) = (-1)^{p-1} t_i$ *modulo* I^2.

Of course it is not surprising that something which generalizes the logarithm converges more easily then something which generalizes the exponential function.

§6. The weak version of the main theorem

We can now state the main theorem. First we introduce some notation.

Definition 6.1. If R is a λ-ring then $\upsilon^n : R \times R \to \Omega_R$ is defined by the formula

$$\nu^n(a, b) = \sum_{mk=n} \theta^m(a)\phi^m(\delta\eta^k(a, b))$$

THEOREM 6.2. *Let (R, J, I) be admissible. Then* $\nu(a, b) = \sum_{n=1}^{\infty} \nu^n(a, b)$ *converges if $a \in I$, $b \in R$ or $a \in R$, $b \in I$. The resulting map $\nu : I \times R \cup R \times I \to \Omega_{R,I}^{top}/\delta I^{top}$ maps the relations in Theorem* 0.1 *to zero. This means that ν induces a continuous homomorphism $K_2^{top}(R, I) \to (\Omega_{R,I}/\delta I)^{top}$.*

In this § we will prove this theorem using the cases $d = 2$ and $d = 3$ of the following proposition, which will be proved in §§8, 9 and 10.

PROPOSITION 6.3. *There exist λ-operations $\beta_d^n \in W_d$, homogeneous of degree n and symmetric in all variables, such that*

$$\sum_{i=1}^{d} \nu^n\Big(a_i, \prod_{j \neq i} a_j\Big) = \delta\beta_d^n(a_1, \ldots, a_d)$$

Most of the proof of Theorem 6.2. is contained in the following two lemmas.

LEMMA 6.4. *Let (R, J, I) be admissible. If $a \in I$ and $b, c \in R$ then* $\nu(a, b) = \sum_{n=1}^{\infty} \nu^n(a, b)$ *converges to an element of $\Omega_{R,I}^{top}$ and* $\nu(a, b + c - abc) = \nu(a, b) + \nu(a, c)$

Proof. There exists a number M such that $\xi(I) \subseteq J^N$ if $\xi \in V$ is homogeneous of degree $> M$. If $k > M$ then

$$\phi^m\left(\delta\eta^k(a, b)\right) \in \phi^m\left(\delta(J^N)\right) \subseteq \phi^m\left(J^{N-1}\Omega_R\right) \subseteq J^{N-1}\Omega_R;$$

if $m > M$ then $\theta^m(a) \in J^N$. Therefore $\nu^n(a, b) \in J^{N\text{-}1}\Omega_R$ if $n > M^2$. This yields that $\nu(a, b)$ not only converges, but that it can even be written as

$$\sum_{m=1}^{\infty} \theta^m(a)\phi^m\left(\delta \sum_{k=1}^{\infty} \eta^k(a, b)\right) = \sum_{m=1}^{\infty} \theta^m(a)\phi^m(\delta\eta(a, b))$$

The addition formula now follows from Proposition 5.6. ■

LEMMA 6.5. *Let (R, J, I) be admissible. If $a \in R$ and $b, c \in I$ then again $v(a, b) = \sum_{n=1}^{\infty} v^n(a, b)$ converges and $v(a, b + c - abc) = v(a, b) + v(a, c)$.*
Moreover $v(a, b) + v(b, a) \in \delta(I^{top})$.

Proof. From Observation 5.8. and Proposition 6.3. we see that $\beta_2(a, b) = \sum_{n=1}^{\infty} \beta_2^n(a, b)$ converges to an element of I^{top}. On the other hand $\nu^n(a, b) = \delta\beta_2^n(a, b) - \nu^n(b, a)$. Therefore Lemma 6.4. implies that $v(a, b)$ converges and that $\nu(a, b) = \delta\beta_2(a, b) - \nu(b, a)$.

To prove the addition formula we show that

$$\sum_{n=1}^{j} \nu^n(a, b + c - abc) - \nu^n(a, b) - \nu^n(a, c)$$

$$= \sum_{mk \leqslant j} \theta^m(a)\phi^m\delta\left[\eta^k(a, b + c - abc) - \eta^k(a, b) - \eta^k(a, c)\right]$$

is in $J^{N-1}\Omega_R$ if $j > M^2$; to do that we split the sum in three parts.

First we take the sum over all $m \leqslant M$ and $k \leqslant M$. On the one hand

$$\sum_{k=1}^{\infty} \eta^k(a, b + c - abc) - \eta^k(a, b) - \eta^k(a, c)$$

$$= \eta(a, b + c - abc) - \eta(a, b) - \eta(a, c) = 0$$

according to Proposition 5.6. On the other hand $\eta^k(a, I) \subseteq J^N$ if $k \geqslant M$. Therefore

$$\sum_{k=1}^{M} \eta^k(a, b + c - abc) - \eta^k(a, b) - \eta^k(a, c) \in J^N$$

for every m. So this part of the sum lands in $\sum_{k=1}^{M} \theta^m(a)\phi^m\delta(J^N) \subseteq J^{N-1}\Omega_R$. Now consider the remaining terms for which $m > M$. From Propositions 1.12. and 1.13. one sees that

$$\eta^k(a, b+c-abc) - \eta^k(a, b) - \eta^k(a, c) \in I^2 \text{ since } b, c \in I.$$

Therefore

$$\theta^m(c)\phi^m\delta\left[\eta^k(c, a+b-abc) - \eta^k(c, a) - \eta^k(c, b)\right] \in R\phi^m\delta(I^2)$$

$$\subseteq R\phi^m(I\Omega_R) \subseteq R\psi^m(I)\Omega_R \subseteq RJ^N\Omega_R.$$

The remaining terms have $k > M$ and therefore $\eta^k(a, b+c-abc)$, $\eta^k(a, b)$, $\eta^k(a, c) \in J^N$. So these terms are in $R\phi^m\delta(J^N) \subseteq J^{N-1}\Omega_R$. ■

Proof of Theorem 6.2.

According to Lemmas 6.4. and 6.5. the map ν is defined and satisfies the first two relations. From Observation 5.8. and Proposition 6.3. we see that $\beta_3(a, b, c) = \sum_{n=1}^{\infty} \beta_3^n(a, b, c)$ converges to an element of I^{top}. Proposition 6.3 also says that

$$\nu^n(a, bc) - \nu^n(ab, c) - \nu^n(ac, b)$$

$$= \delta\{\beta_3^n(a, b, c) - \beta_2^n(ab, c) - \beta_2^n(ac, b)\}.$$

Therefore $\nu(a, bc) - \nu(ab, c) - \nu(ac, b) = \delta\{\beta_3(a, b, c) - \beta_2(ab, c) - \beta_2(ac, b)\}$. Thus the third relation is satisfied. Therefore we have a homomorphism from the group defined in Theorem 0.1. to $\Omega_{R,I}^{top}/\delta I^{top}$ and thus to $(\Omega_{R,I}/\delta I)^{top}$.

Using Proposition 5.2. we see the following. If $a \in J^{N+1}$ then $\theta^m(a) \in J^N$; if $b \in J^{N+2}$ then $\eta^k(a, b) \in J^{N+1}$ and so $\phi^m\delta\eta^k(a, b) \in J^N\Omega_R$. In both cases $\nu(a, b)$ vanishes in the quotient of $(\Omega_{R,I}/\delta I)^{top}$ associated to R/J^N. Therefore we get a map to that quotient which is defined on the group given by Presentation 0.1. for the case of the ring R/J^{N+2} and the ideal $(I+J^{N+2})/J^{N+2}$. Since I is nilpotent in R/J^{N+2} Theorem 0.1. implies that this group can be identified with $K_2(R/J^{N+2}, (I+J^{N+2})/J^{N+2})$. Now we get the map on $K_2(R, I)$ by composition with the projection $R \to R/J^{N+2}$, and this map obviously factorises over $K_2^{top}(R, I)$. ■

§7. The strong version of the main theorem

First we give another description of $K_{2,L}(R, I)$.

PROPOSITION 7.1. *Let R be a commutaive ring and I an ideal. Then $K_{2,L}(R, I)$ can be identified with the cokernel of $\Delta: I \otimes_R I \to I \otimes_R \Omega_R$, where $\Delta(a \otimes b) = a \otimes \delta b + b \otimes \delta a$. The identification maps $[a, b]$ to $a \otimes \delta b$ if $a \in I$.*

Proof. Let $D_I: I \otimes R \otimes R \to I \otimes R$ and $T_I: I \otimes I \to I \otimes R$ be defined by

$$D_I(a \otimes b \otimes c) = a \otimes bc - ab \otimes c - ac \otimes b \quad \text{and } T_I(a \otimes b) = a \otimes b + b \otimes a.$$

Then the common cokernel of D_I and T_I is an abelian group with a presentation which is easily seen to be equivalent to the presentation which defines $K_{2,L}(R, I)$.

From the exactness of $I \otimes_R$ it follows that $I \otimes_R \Omega_R$ is the cokernel of $1_I \otimes D$, which map can be identified with D_I. Therefore $K_{2,L}(R, I)$ can be identified with the quotient of $I \otimes_R \Omega_R$ by the image of the map corresponding to T_I; and that map is just Δ. ■

The multiplication map $I \otimes_R \Omega_R \to \Omega_{R,I}$ induces a surjection $\rho: coker(\Delta) \to \Omega_{R,I}/\delta I$. This map is not always an isomorphism, as can be seen by considering the example $R = Z[t]/(t^3)$, $I = t^2 R$. In this example $coker(\Delta)$ is infinite cyclic with generator $t^2 \otimes \delta t$, where as $\Omega_{R,I}/\delta I$ is cyclic of order 3. However, if the canonical surjection $R \to R/I$ splits then ρ is an isomorphism, as noted in the introduction. So in the split case the map in Theorem 6.2. can be viewed as one with values in $K_{2,L}(R, I)^{top}$. We now use this fact to accomplish the same in the nonsplit case.

THEOREM 7.2. *Let (R, J, I) be admissible. Then there is a continuous map $L: K_2^{top}(R, I) \to K_{2,L}^{top}(R, I)$. Under the identification in Proposition 7.1. this map is given by*

$$L\langle a, b\rangle = \sum_{n=1}^{\infty} \theta^n(a) \otimes \phi^n(\delta\eta(a, b)) \quad \textit{if } a \in I.$$

Proof. We define a new λ-ring $\tilde{R}$ and ideals $\tilde{I}$ and $\tilde{J}$ by

$$\tilde{R} = \{(x, y) \in R \times R;\ x - y \in I\}$$

with all operations componentwise,

$$\tilde{I} = (0 \times I),\ \tilde{J} = \{(x, y) \in J \times J;\ x - y \in I\}.$$

Furthermore we write $R_N = R/J^N$, $I_N = (I + J^N)/J^N$, $\tilde{R}_N = \tilde{R}/\tilde{J}^N$, $\tilde{I}_N = (\tilde{I} + \tilde{J}^N)/\tilde{J}^N$. Then it is straightforward to check that $\tilde{R}$ is a λ-ring and that $(\tilde{R}, \tilde{J}, \tilde{I})$ is admissible. Therefore Theorem 6.2. yields a map $v: K_2(\tilde{R}, \tilde{I})^{top} \to (\Omega_{\tilde{R},\tilde{I}}/\delta\tilde{I})^{top}$. More precisely one gets for each N a map $K_2(\tilde{R}_{N+2}, \tilde{I}_{N+2}) \to \Omega_{\tilde{R}_N,\tilde{I}_N}/\delta\tilde{I}_N$.

Furthermore the canonical surjection $\tilde{R} \to \tilde{R}/\tilde{I}$ can be identified with the projection $\tilde{R} \to R$ on the first factor and therefore it splits by the map

$x \to (x, x)$. Similarly the maps $\tilde{R}_N \to \tilde{R}_N/\tilde{I}_N$ split, so ρ is an isomorphism for each N and we get maps $L: K_2(\tilde{R}_{N+2}, \tilde{I}_{N+2}) \to K_{2,L}(\tilde{R}_N, \tilde{I}_N)$ that satisfy the stated formula.

Now consider the homomorphism $K_2(\tilde{R}_{N+2}, \tilde{I}_{N+2}) \to K_2(R_{N+2}, I_{N+2})$ associated to the projection $\tilde{R} \to R$ on the second factor. One deduces easily from Theorem 0.1. that this map is surjective and that that its kernel is generated by the elements $\langle a, b\rangle$ with $a \in \tilde{I}_{N+2}$ and

$$a \text{ or } b \in ker\left(\tilde{R}_{N+2} \to R_{N+2}\right) = (I \times 0 + \tilde{J}^{N+2})/\tilde{J}^{N+2}.$$

Using the formula it is straightforward to check that the composition of L and the map $K_{2,L}(\tilde{R}_N, \tilde{I}_N) \to K_{2,L}(R_N, I_N)$ associated to that projection maps these elements to zero. Therefore one gets maps $K_2(R_{N+2}, I_{N+2}) \to K_{2,L}(R_N, I_N)$. By taking limits the theorem follows. ■

The following theorem connects our invariant ν with the chern class $c_i: K_i(R) \to \Omega^i_R$ defined by Gersten (see [Bloch, 1972]). On Dennis-Stein symbols these maps satisfy (see [Loday, 1981]):

$$c_i\langle a_1, \dots, a_i\rangle = (-1)^i (i-1)!(1 - a_1 \dots a_i)^{-1} \delta a_1 \wedge \cdots \wedge \delta a_i.$$

THEOREM 7.3. *If (R, J, I) is admissible then the map $\delta\nu$: $K_i(R, I) \to (\Omega^i_R)^{top}$ is equal to*

$$(-1)^i \sum_{k=1}^{\infty} \mu(k) \phi^k_i c_i \quad \textit{for } i = 1, 2.$$

Proof. Let a, b be the canonical elements of U_2. In the language used here one can write the formula in the first half of the proof of Proposition 8.6. as

$$n^2 \nu^n(a, b) = \delta\left\{ \sum_{mk=n} m\theta^m(a) \psi^m(k^2 \eta^k(a, b))\right\}$$
$$- \sum_{mk=n} m\mu(m) \phi^m\left(kb(ab)^{k-1} \delta a\right).$$

Therefore

$$\begin{aligned} n^2 \delta\nu^n(a, b) &= -\delta \sum_{mk=n} m\mu(m) \phi^m\left(kb(ab)^{k-1} \delta a\right) \\ &= -\sum_{mk=n} m^2 \mu(m) \phi^m_2 \delta(ka^{k-1} b^k \delta a) \\ &= -\sum_{mk=n} m^2 \mu(m) \phi^m_2 (k^2 a^{k-1} b^{k-1} \delta b \wedge \delta a) \\ &= +\sum_{mk=n} n^2 \mu(m) \phi^m_2 (a^{k-1} b^{k-1} \delta a \wedge \delta b) \end{aligned}$$

Dividing by n^2 and summing gives the desired result for K_2; the proof for K_1 is similar. ■

§8. Formal Dirichlet series

Since Proposition 6.3. is about operations of fixed degree n its proof is a purely combinatorial matter. Situations of this kind are most clearly described using a language of generating functions. This motivates the following definition.

Definition 8.1. Let R be a ψ-ring. Then $DS(R)$ is the set of all formal combinations $\sum_{n=1}^{\infty} a_n n^s$ with $a_n \in R$, equipped with the following addition and multiplication

$$\sum a_n n^s + \sum b_n n^s = \sum (a_n + b_n) n^s$$

$$\left(\sum a_n n^s\right) \cdot \left(\sum b_n n^s\right) = \sum \left(\sum_{m \mid n} a_m \psi^m (b_{n/m}) \right) n^s$$

It is an associative ring with 1. In a similar way one defines $DS(P)$ if P is a left ψ-module over R; it is a left module over $DS(R)$. In particular if R is a λ-ring then $DS(R)$ is an associative ring and $DS(\Omega_R)$ is a left module over it using the operations constructed in §4.

Furthermore there is an additive map $\delta : DS(R) \to DS(\Omega_R)$ defined by $\delta \sum a_n n^s = \sum (\delta a_n) n^s$. It is not an honest derivation, however; we need the following definition to explain the situation.

Definition 8.2. Let R be a ψ-ring. Then T: $DS(R) \to DS(R)$ is defined by $T(\sum a_n n^s) = \sum n a_n n^s$. Obviously T is a ring homomorphism. Similarly one defines T: $DS(P) \to DS(P)$ if P is a ψ-module over R.

Now Proposition 4.4. can be reformulated in this language as

$$\delta(a \cdot b) = \delta a \cdot b + T(a) \cdot \delta b.$$

An element $\xi = \sum \xi_n n^s$ of $DS(U)$ can be viewed as an operation which maps $a \in R$ to $\sum \xi_n(a) n^s \in DS(R)$ for any λ-ring R. If ξ_n is homogeneous of degree $d + en$ for every n then we call ξ homogeneous of degree (d, e). If α_1 is of degree (d, e) and α_2 is of degree $(-e, f)$ then $\alpha_1 \cdot \alpha_2$ is of degree (d, f).

Similarly elements of $DS(U_d)$ and $DS(\Omega_U)$ can be viewed as operations.

Definition 8.3. Let R be a λ-ring and $a, b \in R$. Then

$$Y(a) = \sum a^{n-1} n^s, \quad H(a, b) = \sum \eta^n(a, b) n^s,$$

$$N(a, b) = \sum \nu^n(a, b) n^s.$$

Then $Y(a)$ is invertible and the following relations are satisfied:

$$\delta(aY(a)) = \delta a \cdot TY(a)$$

$$bY(ab) = Y(a) \cdot TH(a, b) \text{ (this is Proposition 3.3)}$$

$$N(a, b) = H(1, a) \cdot \delta H(a, b)$$

The use of these relations is facilitated by the following proposition.

PROPOSITION 8.4.
a) *Let R be a ψ-ring. If R has no Z-torsion then T is injective. If R has no divisors of zero and ψ^n is injective for every n then $DS(R)$ has no divisors of zero.*
b) *The ring U has the properties mentioned in a).*

Proof. We only prove that ψ^n is injective on U; the other statements are obvious. Suppose that $a \in E^k$ has $\psi^n a = 0$. Then $a = \sum c_{i_1 \dots i_k} \lambda^{i_1}(u) \dots \lambda^{i_k}(u)$ *modulo* E^{k+1} for certain integers $c_{i_1 \dots i_k}$. Therefore

$$0 = \sum c_{i_1 \dots i_1} \psi^n \lambda^{i_1}(u) \dots \psi^n \lambda^{i_k}(u) \textit{ modulo } E^{k+1}.$$

But

$$\psi^n \lambda^i(u) = n(-1)^{(n+1)i} \lambda^{ni}(u) \textit{ modulo } E^2.$$

So

$$0 = \sum c_{i_1 \dots i_k} n^k (-1)^{(n+1)\sum i_k} \lambda^{ni_1}(u) \dots \lambda^{ni_k}(u) \textit{ modulo } E^{k+1}$$

which is only possible if all coefficients vanish, so that $a \in E^{k+1}$. ∎

Even in $DS(U)$ however one has to be careful. If $a^{-1}b$ is defined and a and b are both in the image of T then $a^{-1}b$ is not necessarily in the image of T: take $a = 2 \cdot 1^s + 2 \cdot 2^s$ and $b = 2 \cdot 1^s$.

COROLLARY 8.5. *If R is a λ-ring and $a, b, c \in R$ then $H(a, bc) = H(a, b) \cdot H(ab, c)$.*

Writing this out yields

$$\eta^n(a, bc) = \sum_{mk=n} \eta^m(a, b) \psi^m \eta^k(ab, c).$$

In particular

$$\theta^n(bc) = \sum_{mk=n} \theta^m(b)\psi^m\eta^k(b, c).$$

Now we can write the left hand side of Proposition 6.3. in this language.

PROPOSITION 8.6. *Let R be a λ-ring and $a_1, \ldots, a_d$ elements of R. Then*

$$\sum_{i=1}^{d} TN\left(a_i, \prod_{j\neq i} a_j\right) = \delta\left\{\sum_{i=1}^{d} H(1, a_i)\cdot TH\left(a_i, \prod_{j\neq i} a_j\right) - H\left(1, \prod_{j=1}^{d} a_j\right)\right\}$$

Proof. If R is any λ-ring and $a, b \in R$ then

$$\begin{aligned}
T^2N(a, b) &= T^2H(1, a)\cdot\delta T^2H(a, b)\\
&= TY(1)^{-1}\cdot aTY(a)\cdot\delta\left\{TY(a)^{-1}\cdot bTY(ab)\right\}\\
&= \delta\left\{Y(1)^{-1}\cdot aY(a)\cdot TY(a)^{-1}\cdot bTY(ab)\right\}\\
&\quad -\delta\left\{Y(1)^{-1}\cdot aY(a)\right\}\cdot TY(a)^{-1}\cdot bTY(ab)\\
&= \delta\left\{TH(1, a)\cdot T^2H(a, b)\right\}\\
&\quad - TY(1)^{-1}\cdot\delta a\cdot TY(a)\cdot TY(a)^{-1}\cdot bTY(ab)\\
&= \delta\left\{TH(1, a)\cdot T^2H(a, b)\right\} - TY(1)^{-1}\cdot\delta a\cdot bTY(ab).
\end{aligned}$$

So if $u_1, \ldots, u_d$ are the canonical elements in U_d then

$$\begin{aligned}
\sum_{i=1}^{d} T^2N\left(u_i, \prod_{j\neq i} u_j\right) &= \sum_{i=1}^{d} \delta\left\{TH(1, u_i)\cdot T^2H\left(u_i, \prod_{j\neq i} u_j\right)\right\}\\
&\quad - TY(1)^{-1}\cdot\delta\left(\prod_{j=1}^{d} u_j\right)\cdot TY\left(\prod_{j=1}^{d} u_j\right)
\end{aligned}$$

where the last term can be rewritten as

$$\delta\left\{Y(1)^{-1}\cdot\left(\prod_{j=1}^{d} u_j\right)Y\left(\prod_{j=1}^{d} u_j\right)\right\} = \delta TH\left(1, \prod_{j=1}^{d} u_j\right).$$

Now remove the T on both sides and map u_i to a_i. ∎

From this proposition we see that to prove Proposition 6.3, we must show that there exist $B_d \in DS(W_d)$ such that

$$TB_d(a_1,\dots,a_d) = \sum_{i=1}^{d} H(1,\ a_i)\cdot TH\left(a_i,\prod_{j\neq i} a_j\right) - H\left(1,\prod_{j=1}^{d} a_j\right).$$

This will be done in §9 and §10. It is easy to check that $H(a,\ b)$ is of degree $(-1,\ 1)$ in a and of degree $(0,\ 1)$ in b; thus B_d will be of degree $(0,\ 1)$ in all entries.

§9. Reduction to a special case

In this § we show that operations β_d^n with the desired properties exist if they exist for $d=2$ and n a prime power. First we treat the reduction to $d=2$.

PROPOSITION 9.1. *Suppose that there exists* $B_2 \in DS(W_2)$ *such that*

$$TB_2(a,\ b) = H(1,\ a)\cdot TH(a,\ b) + H(1,\ b)\cdot TH(b,\ a) - H(1,\ ab).$$

Then there exist $B_d \in DS(W_d)$ *such that*

$$TB_d(a_1,\dots,a_d) = \sum_{i=1}^{d} H(1,\ a_i)\cdot TH\left(a_i,\prod_{j\neq i} a_j\right) - H\left(1,\prod_{j=1}^{d} a_j\right).$$

Proof. Define $B_d \in DS(W_d)$ by

$$B_d(a_1,\dots,a_d) = \sum_{e=2}^{d} B_2\left(\prod_{i=1}^{e-1} a_i,\ a_e\right)\cdot H\left(\prod_{i=1}^{e} a_i,\ \prod_{i=e+1}^{d} a_i\right).$$

Then $B_d(a_1,\ a_2,\dots,a_d) = B_{d-1}(a_1a_2,\ a_3,\dots,a_d) + B_2(a_1,\ a_2)$ $\cdot H(a_1a_2,\prod_{i=3}^{d} a_i)$ for $d \geqslant 3$.

If we write $b_1 = a_1a_2$ and $b_i = a_{i+1}$ for $2 \leqslant i \leqslant d-1$ we get therefore by induction on d

$$\sum_{i=1}^{d} H(1,\ a_i)\cdot TH\left(a_i,\prod_{j\neq i} a_j\right) - H\left(1,\prod_{j=1}^{d} a_j\right)$$

$$= \left\{\sum_{i=1}^{d-1} H(1,\ b_i)\cdot TH\left(b_i,\prod_{j\neq i} b_j\right) - H\left(1,\prod_{j=1}^{d-1} b_j\right)\right\}$$

$$+ \left\{H(1,\ a_1)\cdot TH\left(a_1,\ a_2\prod_{j=3}^{d} a_j\right) + H(1,\ a_2)\cdot TH\left(a_2,\ a_1\prod_{j=3}^{d} a_j\right)\right.$$

$$\left. - H(1,\ a_1a_2)\cdot TH\left(a_1a_2,\prod_{j=3}^{d} a_j\right)\right\}$$

$$= TB_{d-1}(b_1,\ldots,b_{d-1}) + \{H(1,a_1)\cdot TH(a_1,a_2)$$

$$+ H(1,a_2)\cdot TH(a_2,a_1) - H(1,a_1a_2)\}\cdot TH\left(a_1a_2,\prod_{j=3}^{d} a_j\right)$$

$$= TB_{d-1}(a_1a_2,a_3,\ldots,a_d) + TB_2(a_1,a_2)\cdot TH\left(a_1a_2,\prod_{j=3}^{d} a_j\right)$$

$$= TB_d(a_1,a_2,\ldots,a_d). \quad \bullet$$

Next we show that we only have to check a suitable local version for each prime p. Fix the prime p. We write P for the set of powers of p and Q for the set of numbers indivisible by p. If $\Xi = \sum \xi_n n^s$ is a Dirichlet series then we write

$$\Xi_P = \sum_{n\in P} \xi_n n^s, \quad \Xi_Q = \sum_{n\in Q} \xi_n n^s, \quad T_P\Xi = \sum p^{v_p(n)}\xi_n n^s,$$

so that $T_P\Xi_P = T\Xi_P$. Obviously T_P and the maps $\Xi \to \Xi_P$ and $\Xi \to \Xi_Q$ are ring homomorphisms.

LEMMA 9.2. *If R is a λ-ring and $a,\ b\in R$ then*
1. $TH(1,a) = Y_Q(1)^{-1}\cdot \sum_{q\in Q} TH_P(1,a^q)q^s$
2. $TH(a,b) = [\sum_{q\in Q} TH_P(a,a^{q-1})q^s]^{-1}\cdot[\sum_{q\in Q} TH_P(a,a^{q-1}b^q)q^s]$

Proof.
1. First we note that

$$Y(a) = \sum a^{n-1}n^s = \sum_{q\in Q} a^{q-1}\left[\sum_{m\in P}(a^q)^{m-1}m^s\right]\cdot q^s$$

$$= \sum_{q\in Q} a^{q-1}Y_P(a^q)\cdot q^s.$$

Therefore

$$TH(1,a) = Y(1)^{-1}\cdot Y(a) = Y_Q(1)^{-1}\cdot Y_P(1)^{-1}\cdot \sum_{q\in Q} a^q Y_P(a^q)\cdot q^s$$

which equals the stated formula since $TH_P(1,a^q) = Y_P(1)^{-1}\cdot a^qY_P(a^q)$.

2. From the above formula and Corollary 8.5. we get

$$TH(1,ab) = Y_Q(1)^{-1}\cdot \sum_{q\in Q} TH_P(1,a^qb^q)\cdot q^s$$

$$= Y_Q(1)^{-1}\cdot TH_P(1,a)\cdot \sum_{q\in Q} TH_P(a,a^{q-1}b^q)\cdot q^s$$

Similarly $TH(1, a) = Y_Q(1)^{-1} \cdot TH_P(1, a) \cdot \sum_{q \in Q} TH_P(a, a^{q-1}) \cdot q^s$.

Now the stated formula follows from $TH(1, a) \cdot TH(a, b) = TH(1, ab)$ ■

PROPOSITION 9.3. *If there exists* $B_{2,P} \in DS(W_2)$ *such that*

$$TB_{2,P}(a, b) = H_P(1, a) \cdot TH_P(a, b)$$
$$+ H_P(1, b) \cdot TH_P(b, a) - H_P(1, ab)$$

then the operation $(a, b) \to TH(1, a) \cdot T^2H(a, b) + TH(1, b) \cdot T^2H(b, a) - TH(1, ab)$ *is in* $T_P^2(DS(W_2))$.

Proof. By Proposition 9.1. the existence of $B_{2,P}$ implies the existence of $B_{d,P}$ such that in particular

$$TB_{q,P}(a, a, \dots, a) = qH_P(1, a) \cdot TH_P(a, a^{q-1}) - H_P(1, a^q),$$

$$TB_{2q,P}(a, a, \dots, b, b) = qH_P(1, a) \cdot TH_P(a, a^{q-1}b^q)$$
$$+ qH_P(1, b) \cdot TH_P(b, a^q b^{q-1}) - H_P(1, a^q b^q)$$

where a, b are the universal elements in U_2. By Lemma 9.2. one has

$$TH(1, a) \cdot T^2H(a, b) - Y_Q(1)^{-1} \cdot TH_P(1, a)$$

$$\cdot \left[\sum_{q \in Q} qT^2H_P(a, a^{q-1}b^q) q^s \right] = Y_Q(1)^{-1} \cdot \left[\sum_{q \in Q} TH_P(1, a^q) q^s \right]$$

$$\cdot \left[\sum_{q \in Q} qT^2H_P(a, a^{q-1}) q^s \right]^{-1} \cdot \left[\sum_{q \in Q} qT^2H_P(a, a^{q-1}b^q) q^s \right]$$

$$- Y_Q(1)^{-1} \cdot \left[\sum_{q \in Q} qTH_P(1, a) T^2H_P(a, a^{q-1}) q^s \right]$$

$$\cdot \left[\sum_{q \in Q} qT^2H_P(a, a^{q-1}) q^s \right]^{-1} \cdot \left[\sum_{q \in Q} qT^2H_P(a, a^{q-1}b^q) q^s \right]$$

$$= Y_Q(1)^{-1} \cdot \left[\sum_{q \in Q} - T^2B_{q,P}(a, a, \dots, a) q^s \right]$$

$$\cdot \left[\sum_{q \in Q} qT^2H_P(a, a^{q-1}) q^s \right]^{-1} \cdot \left[\sum_{q \in Q} qT^2H_P(a, a^{q-1}b^q) q^s \right]$$

$$= T_P^2\left\{ Y_Q(1)^{-1} \cdot \left[\sum_{q\in Q} - B_{q,P}(a, a, \dots, a) q^s \right] \right.$$

$$\left. \cdot \left[\sum_{q\in Q} qH_P(a, a^{q-1}) q^s \right]^{-1} \cdot \left[\sum_{q\in Q} qH_P(a, a^{q-1}b^q) q^s \right] \right\}$$

So up to terms in $T_P^2(DS(W_2))$ one has

$$TH(1, a) \cdot T^2H(a, b) + TH(1, b) \cdot T^2H(b, a) - TH(1, ab)$$

$$= Y_Q(1)^{-1} \cdot TH_P(1, a) \cdot \left[\sum_{q\in Q} qT^2H_P(a, a^{q-1}b^q) q^s \right]$$

$$+ Y_Q(1)^{-1} \cdot TH_P(1, b) \cdot \left[\sum_{q\in Q} qT^2H_P(b, a^q b^{q-1}) q^s \right]$$

$$- Y_Q(1)^{-1} \cdot \left[\sum_{q\in 0} TH_P(1, a^q b^q) q^s \right]$$

$$= Y_Q(1)^{-1} \cdot \left[\sum_{q\in Q} T^2 B_{2q,P}(a, a, \dots, b, b) q^s \right]$$

$$= T_P^2\left\{ Y_Q(1)^{-1} \cdot \left[\sum_{q\in Q} B_{2q,P}(a, a, \dots, b, b) q^s \right] \right\} \quad \blacksquare$$

The proposition says that the coefficient of n^s is element of $p^{2e}W_2$ whenever p^e divides n. However, if that is the case for every prime p then by the Chinese Remainder Theorem it follows that the coefficient of n^s is in fact in n^2W_2 or equivalently that the operation is in $T^2(DS(W_2))$. By Proposition 9.1. we know that this is sufficient to prove Proposition 6.3. What remains is to show the existence of the $B_{2,P}$; this will be done in the final §.

§10. Treatment of the special case

Fix the prime p. We must show that the operation

$$(a, b) \to H_P(1, a) \cdot TH_P(a, b) + H_P(1, b) \cdot TH_P(b, a) - H_P(1, ab)$$

is in $T(DS(W_2))$ or equivalently that the coefficient of $(p^e)^s$ in this Dirichlet series is an element of p^eW_2. To do that we compute the three terms modulo p^eW_2.

We start by considering the first term and give it another description which is more suitable for the computation.

Definition 10.1. The operations $\tau^n \in W_2$ are defined by $\sum \tau^n(a, b)n^s = H(1, a) \cdot TH(a, b)$. In other words $\tau^n(a, b) = \sum_{m \mid n} \frac{n}{m} \theta^m(a) \psi^m \eta^{n/m}(a, b)$.

PROPOSITION 10.2. *The operations τ^n satisfy $\tau^1(a, b) = ab$ and*

$$\tau^n(a, b) = \theta^n(a)\psi^n(b) + \sum \frac{n}{m} \tau^m(a, a^{(n-m)/m}) \psi^m \theta^{n/m}(b)$$

where the sum extends over all $m < n$ dividing n.

Proof. Substitute the Definition 3.1. of η in the above definition, rearrange the sum, and use the induction hypothesis. ■

We will use Proposition 10.2. to prove a formula for τ^{p^e} *modulo* $p^e W_2$ by induction on e. For this we first need a few lemmas.

LEMMA 10.3. *Let $u \in U$ be the canonical element. Then modulo pV one has*

$$\theta^{p^e}(u) \equiv \begin{cases} u^{p^e - p}\theta^p(u) & \text{if } p > 2 \text{ or } e = 1 \\ u^{2^e - 2}\theta^2(u) - u^{2^e - 4}\theta^2(u)^2 & \text{if } p = 2 \text{ and } e \geqslant 2 \end{cases}$$

Proof. One has

$$\begin{aligned} p^e \theta^{p^e}(u) &= (\psi^p(u) + p\theta^p(u))^{p^{e-1}} - (\psi^p(u))^{p^{e-1}} \\ &= \sum_{i \geqslant 1} \binom{p^{e-1}}{i} p^i \psi^p(u)^{p^{e-1} - i} \theta^p(u)^i \end{aligned}$$

The term associated to i has $e - 1 - v_p(i) + i$ factors p. This is at least $e + 1$ if $i - 2 \geqslant v_p(i)$, which is the case if $p > 2$ and $i \geqslant 2$, or $p = 2$ and $i \geqslant 3$. Moreover $\psi^p(u)^{p^{e-1} - i}\theta^p(u)^i \in V$ for all i. So only the terms with $i = 1$ *or* 2 can give a nontrivial contribution. In those terms we have replaced $\psi^p(u)^{p^{e-1}}$ by u^{p^e} which is allowed since both are equivalent modulo pV. ■

Definition 10.4. If $m \mid n$ then we write $\chi^{n,m}$ for the operation defined by

$$\chi^{n,m}(a) = \theta^n(a) - a^{n-m}\theta^m(a).$$

LEMMA 10.5. *Let $u \in U$ be the canonical element, and let $f < e$. Then modulo $p^f V$ one has*

$$\chi^{p^e, p^f}(u) \equiv \begin{cases} 0 & \text{if } p > 2 \text{ or } f = 0 \\ 2^{f-1} u^{2^e - 4} \theta^2(u)^2 & \text{if } p = 2 \text{ and } f = 1 \\ 2^{f-1}\left\{ u^{2^e-4}\theta^2(u)^2 + u^{2^e-8}\theta^2(u)^4 \right\} & \text{if } p = 2 \text{ and } f \geqslant 2 \end{cases}$$

Proof. We may assume that $f \geqslant 1$. Then

$$p^e \theta^{p^e}(u) = \left(\psi^p(u)^{p^{f-1}} + p^f \theta^{p^f}(u)\right)^{p^{e-f}} - \psi^p(u)^{p^{e-1}}$$

$$= \sum_{i \geqslant 1} \binom{p^{e-f}}{i} p^{fi} \theta^{p^f}(u)^i \psi^p(u)^{p^{f-1}(p^{e-f}-i)}$$

The term associated to i has $e - f - v_p(i) + fi$ factors p. This is at least $e + f$ if $(i-2) f \geqslant v_p(i)$, which is the case if $p > 2$ and $i \geqslant 2$, or $p = 2$ and $i \geqslant 3$. Moreover $\theta^{p^f}(u)^i \psi^p(u)^{p^{f-1}(p^{e-f}-i)} \in V$ for every i. So only the terms with $i = 1$ *or* 2 can give a nontrivial contribution. As in Lemma 10.3. we replace $\psi^p(u)^{p^{f-1}}$ by u^{p^f}; then the term with $i = 1$ becomes just $p^e \theta^{p^f}(u) u^{p^e - p^f}$.

This finishes the proof if $p > 2$. In case $p = 2$ we substitute the result of Lemma 10.3. for $\theta^{2^f}(u)$, we replace $\binom{2^{e-f}}{2} 2^{2f}$ by 2^{e+f-1}, and we replace $\{u^{2^f-2}\theta^2(u) + u^{2^f-4}\theta^2(u)^2\}^2$ by $u^{2^{f+1}-4}\theta^2(u)^2 + u^{2^{f+1}-8}\theta^2(u)^4$. ■

LEMMA 10.6. *Let $u \in U$ be the canonical element and let $e \geqslant 2$ then*

$$\sum_{f=1}^{e-1} \psi^{2^f} \theta^{2^{e-f}}(u) \equiv u^{2^e-4}\theta^2(u)^2 \text{ modulo } 2W.$$

Proof. If $f \leqslant e - 2$ then the term $\theta^{2^{e-f}}(u)$ is by Lemma 10.3. equivalent to

$$u^{2^{e-f}-2}\theta^2(u) - \psi^2\left(u^{2^{e-f-1}-2}\theta^2(u)\right) \text{ modulo } 2V$$

Now all terms in the sum cancel, except the first one. So we get $\psi^2(u^{2^{e-1}-2}\theta^2(u))$ *modulo* $2V$. We may replace $\psi^2(u)$ by u^2, and modulo $2W$ we may replace $\psi^2\theta^2(u)$ by $\theta^2(u)^2$. ●

Definition 10.7. The operation ϵ^{p^e} is defined by

$$\epsilon^{p^e}(a, b) = \begin{cases} 0 & \text{if } p > 2 \text{ or } e \leqslant 1 \\ 2^{e-1}(ab)^{2^e-4}\theta^2(a)^2\theta^2(b)^2 & \text{if } p = 2 \text{ and } e \geqslant 2 \end{cases}$$

PROPOSITION 10.8. *Let a, b be the canonical elements of U_2. Then modulo $p^e W_2$ one has*

$$\tau^{p^e}(a, b) \equiv \theta^{p^e}(a) b^{p^e} + \epsilon^{p^e}(a, b)$$

Proof. Induction on e. For $e = 0$ the statement reads $ab \equiv ab + 0$; so assume $e > 0$. By Proposition 10.2. one has

$$\tau^{p^e}(a, b) = \theta^{p^e}(a)\psi^{p^e}(b) + \sum_{f<e} p^{e-f}\tau^{p^f}\left(a, a^{p^{e-f}-1}\right)\psi^{p^f}\theta^{p^{e-f}}(b)$$

$$= \theta^{p^e}(a)\psi^{p^e}(b)$$

$$+ \sum_{f<e} p^{e-f}\left[\theta^{p^f}(a) a^{p^e - p^f} + \epsilon^{p^f}\left(a, a^{p^{e-f}-1}\right) \textit{ modulo } p^f W\right] \psi^{p^f}\theta^{p^e}$$

$$\equiv \theta^{p^e}(a)\psi^{p^e}(b) + \sum_{f<e} p^{e-f}\theta p^e(a)\psi^{p^f}\theta p^{e-f}(b)$$

$$+ \sum_{f<e} p^{e-f}\left[-\chi^{p^e, p^f}(a) + \epsilon^{p^f}\left(a, a^{p^{e-f}-1}\right)\right]\psi^{p^f}\theta^{p^{e-f}}(b)$$

modulo $p^e W_2$

By Lemma 2.2. the terms containing $\theta^{p^e}(a)$ add up to $\theta^{p^e}(a) b^{p^e}$. If $p > 2$ or $f = 0$ the remaining terms are in $p^e W_2$ according to Lemma 10.5. and Definition 10.7. and we are finished. So assume $p = 2$ and $f > 0$ and consider the expression $-\chi^{2^e, 2^f}(a) + \epsilon^{2^f}(a, a^{2^{e-f}-1})$ modulo $2^f V$. If $f = 1$ we get $-2^{f-1}a^{2^e-4}\theta^2(a)^2$ and if $f \geqslant 2$ we get $-2^{f-1}a^{2^e-4}\theta^2(a)^2 - 2^{f-1}a^{2^e-8}\theta^2(a)^4 + 2^{f-1}(a^{2^{e-f}})^{2^{e-f}}\theta^2(a)^2\theta^2(a^{2^{e-f}-1})^2$ which is also equivalent to $2^{f-1}a^{2^e-4}\theta^2$-$(a)^2$. This follows from the fact that $\theta^2(a^j) \equiv ja^{2j-2}\theta^2(a)$ *modulo* $2V$.

So the remaining terms add up to

$$\sum_{f=1}^{e-1} 2^{e-1}a^{2^e-4}\theta^2(a)^2\psi^{2^f}\theta^{2^{e-f}}(b) \textit{ modulo } 2^e V_2$$

and according to Lemma 10.6. this is modulo $2^e W_2$ equal to $2^{e-1}a^{2e-4}\theta^2(a)^2 b^{2e-4}\theta^2(b)^2 = \epsilon^{2^e}(a, b)$. ■

COROLLARY 10.9. *If a, b are the canonical elements of U_2 then*

$$\tau^{p^e}(a, b) + \tau^{p^e}(b, a) - \theta^{p^e}(ab)$$

$$\equiv \theta^{p^e}(a)b^{p^e} + \theta^{p^e}(b)a^{p^e} - \theta^{p^e}(ab) = p^e\theta^{p^e}(a)\theta^{p^e}(b) \equiv 0 \textit{ modulo } p^e W_2.$$

This proves the claim made at the start of this § and thus finishes the proof of Proposition 6.3.

Acknowledgements

The author wishes to thank J. Stienstra who suggested the notion of ring of formal Dirichlet series associated to a λ-ring and indicated how by the use of that formalism the exposition of a number of combinatorial identities in the original manuscript could be shortened substantially.

References

Atiyah, M.F. and Tall, D.O.: Group representations, λ-rings and the J-homomorphism. *Topology* 8 (1969) 253–297.

Bass, H. and Murphy, M.P.: Grothendieck groups and Picard groups of abelian group rings. *Ann. of Math.* 86 (1967) 16–73.

Bloch, S.: On the tangent space to Quillen K-theory. In: *Algebraic K-Theory I. Lecture Notes* 341. Springer (1972) 205–210.

Bloch, S.: K_2 of Artinian Q-algebras with application to algebraic cycles. *Comm. in Alg.* 3 (1975) 405–428.

Clauwens, F.: *The K_2 of truncated integral polynomial rings.* Dept. of Math. Catholic University Nijmegen, report 8416.

Grothendieck, A.: La theorie des classes de Chern. *Bull. Soc. Math.* 86 (1958) 137–154.

Van der Kallen, W.: Le K_2 des nombres duaux. *C.R. Acad. Sci. Paris* 273 (1971) 1204–1207.

Van der Kallen, W. and Stienstra, J.: The relative K_2 of truncated polynomial rings. *J. Pure and Appl. Alg.* 34 (1984) 277–289.

Kassel, C. and Loday, J.-L.: Extensions centrales d'algebres de Lie. *Ann. Inst. Fourier* 32 (1982) 119–142.

Keune, F.: The relativization of K_2. *J. of Alg.* 54 (1978) 159–177.

Knutson, D.: λ-rings and the representation theory of the symmetric group. *Lecture Notes* 308. Springer Verlag, New York, (1973).

Labute, J. and Russell, P.: On K_2 of truncated polynomial rings. *J. of Pure and Appl. Alg.* 6 (1975) 239–251.

Loday, J.-L.: Symboles en K-theorie superieure. *C.R. Acad. Sci. Paris* 292 (1981) 863–866.

Loday, J.-L. and Quillen, D.: Cyclic homology and the Lie algebra homology of matrices. *Comm. Math. Helv.* 59 (1984) 565–591.

Maazen, H. and Stienstra, J.: A presentation for K_2 of split radical pairs. *J. of Pure and Appl. Alg.* 10 (1977) 271–294.

Metropolis, N. and Rota, G.C.: Witt vectors and the algebra of necklaces. *Adv. in Math.* 50 (1983) 95–125.

Oliver, R.: *K_2 of p-adic group rings of abelian p-groups.* Matematisk Institut, Aarhus Universitet, preprint 1984/85 no 12.

Roberts, L. and Geller, S.: K_2 of some truncated polynomial rings. In: *Ring theory, Waterloo 1978. Lecture Notes in Math.* 734. Springer Verlag (1978) 249–279.

Stienstra, J.: On K_2 and K_3 of Truncated Polynomial Rings. In: *Algebraic K-theory, Evanston 1980. Lecture Notes in Math.* 854. Springer Verlag (1980) 409–455.

Wilkerson, C.: λ-rings, binomial domains, and vector bundles over $CP(\infty)$. *Comm. in Alg.* 10 (1982) 311–328.