COMPOSITIO MATHEMATICA

PIERRE LIARDET

Regularities of distribution

Compositio Mathematica, tome 61, nº 3 (1987), p. 267-293

<http://www.numdam.org/item?id=CM_1987__61_3_267_0>

© Foundation Compositio Mathematica, 1987, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (http://http://www.compositio.nl/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Compositio Mathematica 61: 267–293 (1987)
© Martinus Nijhoff Publishers, Dordrecht – Printed in the Netherlands

Regularities of distribution

PIERRE LIARDET

U.E.R. de Mathématiques, Université de Provence, place V. Hugo F-13331 Marseille, Cedex 3, France and Institut für Mathematik, Universität in Salzburg Petersbrunnstrasse 19, A-5020 Salzburg, Austria

Received 30 October 1985; accepted 1 April 1986

dedicated to Jean Coquet †

Résumé. Une partie A de X, espace compact métrisable, est dite à restes bornés pour une suite donnée $x:\mathbb{N}\to X$ s'il existe a, $0\leqslant a\leqslant 1$, tel que la suite $n\to \operatorname{card}\{m<n;\ x_m\in A\}-na$ soit bornée. L'étude de ces ensembles est étroitement liée aux propriétés spectrales du flot associé à x. Les cas particuliers des suites $(n\alpha)$ dans $\mathbb{T}^d$, des suites de Weyl et des suites multiplicatives en base g sont examinés plus en détail.

Abstract. A subset A of a compact metrizable space X is said to be a bounded remainder set for a given sequence x: $\mathbb{N}\to X$, if there exists a, $0\leqslant a\leqslant 1$, such that the sequence $n\to \operatorname{card}\{m<n;\ x_m\in A\}-na$ is bounded. The study of such sets is closely related to spectral properties of the flow associated with x. We particularly investigate sequences $(n\alpha)$ in $\mathbb{T}^d$, Weyl sequences and multiplicative sequences to base g.

I. Introduction

I.1.

The aim of this paper is to study regularities of distribution for sequences $x=(x_n)_{n\geqslant 0}$ with values in a compact metrizable space X. One way to deal with regularities or as well irregularities of distribution is to describe *bounded remainder sets,* abbreviated B.R.S., in X for x. These are subsets A for which there exists a in [0, 1] such that the sequence of remainders $N\to r_N(A;\ a)$ defined by

$$r_N(A;\ a)=\operatorname{card}\{n<N;\ x_n\in A\}-Na$$

is bounded. In this case a is called an *admissible frequency* of x in A.

The first result is about the familiar sequence $(n\alpha)$ on the torus $\mathbb{T}$, with α irrational. It was proved by [Hecke, 1922] that for all arcs I in $\mathbb{T}$ of length $|I|>0$ in $\alpha\mathbb{Z}+\mathbb{Z}$, one has

$$|r_n(I;\ |I|)|\leqslant |h|,$$

where h is the unique integer such that $|I| - h\alpha \in \mathbb{Z}$. The proof derives from the relation

$$1_{[u,v[}(x) - (v-u) = \langle x - v\rangle - \langle x - u\rangle,$$

where 1_I denotes the characteristic function of I in $[0, 1[$ and $\langle y\rangle$ denotes the fractional part of the real number y; here $0 \leqslant x < 1$ and $0 \leqslant u \leqslant v < 1$. The converse, conjectured by [Erdös, 1964] was first proved by [Kesten, 1973]. Proofs and generalisations of this theorem in the framework of both topological dynamics and ergodic theory were given by [Furstenberg, Keynes and Shapiro, 1973; Petersen, 1973; Oren, 1982]. The method of Oren is purely topological. The proof of Petersen is ergodic and shows that the condition $\beta \in \alpha\mathbb{Z} + \mathbb{Z}$ is equivalent to

$$\sum_{m\neq 0}\left(\frac{\sin(m\beta\pi)}{m\sin(m\alpha\pi)}\right)^2 < +\infty. \tag{1}$$

I.2.

In some cases the sequence (x_n) is derived from iteration of a map $T\colon X \to X$. The initial point x_0 is given and

$$x_n = T(x_{n-1}), \quad n \geqslant 1. \tag{2}$$

The fact that $A(\subset X)$ is a B.R.S. depends on x_0, but in many examples this initial value is irrelevant. The subset A will be called *T-admissible* if A is a B.R.S. with the same admissible frequency for all sequences satisfying (2).

In the general case we look at a sequence as a dynamical system. Let S be the one-sided shift on the compact metrizable product space $X^{\mathbb{N}}$, given by

$$S(t_0, t_1, t_2, \ldots) = (t_1, t_2, \ldots).$$

An X-valued sequence x is viewed as a point of $X^{\mathbb{N}}$ and we denote by K_x the orbit closure of x with respect to the shift. We have $S(K_x) \subset K_x$, so that the restriction T of S on K_x gives the flow $\mathscr{K}(x) = (T;\ K_x)$. Now let $I(x)$ be the set of Borel probability measures on $X^{\mathbb{N}}$ which are accumulation points of the sequence

$$N \to \frac{1}{N}\sum_{n<N}\delta_{S^n x}$$

with respect to the weak topology. To each λ in $I(x)$ we associate the measured flow $\mathscr{K}(x;\lambda) = (T;\ K_x, \lambda)$. We shall be concerned with spectral properties of such a flow.

I.3.

In part II we give the ergodic method following the work of [Petersen, 1973] and we derive some general results on B.R.S. called *regular* (Theorem 2). In part III we find all *blocks* $A = \prod_{k=1}^{d} I_k$ with bounded remainders $r_n(A; a)$ for sequences $n \to (n\alpha_1, \ldots, n\alpha_d)$ in π^d (Theorem 3). We also give examples of cylinders which are B.R.S. for these sequences (Theorem 4), generalizing earlier examples of [Szüsz, 1954] and [Rauzy, 1983–1984]. In part IV it is proved in particular that any arc A of $\mathbb{T}$ which is B.R.S. for a *Weyl sequence* of degree $d \geqslant 2$ is in fact trivial, namely is of length 0 or 1. The next part deals with *q-multiplicative sequences* such as $n \to e^{2i\pi\alpha s_g(n)}$, where α is an irrational number and $s_g(n)$ the sum of digits of n to the base g. Spectral properties of these sequences obtained in [Coquet *et al.*, 1977; Queffelec, 1979] are used to describe the related measured flows. In the case of *strongly* q-multiplicative sequences z with additional properties, Theorem 6 says that arcs which are B.R.S. are trivial. This extends a first result of [Queffelec, 1984] about sequences $(\alpha s_g(n))$.

II. Coboundaries

II.1.

Let E be a locally convex linear space (L.C.S.) over $\mathbb{R}$ or $\mathbb{C}$. The Schauder-Tychonoff theorem [Dunford and Schwartz, 1967] says that if K is a compact convex subset of E and if F is a continuous map from K into K, then F has a fixed point. Following the method of [Petersen, 1973] we give a coboundary theorem in a general form using the notion of quasi-complete L.C.R. [Bourbaki, 1964].

COBOUNDARY THEOREM. *Let E be a quasi-complete L.C.R. and let U: $E \to E$ be a linear map, continuous for the weak topology on E. Assume the E-valued sequence $n \to a_n$, $n \geqslant 0$, given by $a_0 = 0$ and*

$$a_n = a + U(a) + \ldots + U^{n-1}(a)$$

is weakly relatively compact. Then there exists b in E such that

$$a = b - U(b).$$

a is called a U-coboundary and moreover for every continuous linear functional L,

$$|L(b)| \leqslant \sup_{n \geqslant 0} |L(a_n)|$$

holds.

Proof. Define $F\colon E \to E$ by $F(x) = a + U(x)$.

This map is weakly continuous. Let K be the convex (weak)-closure in E of the points a_n, $n \geqslant 0$. Since E is quasi-complete we derive from a theorem of Krein [Bourbaki, 1964] that K is weakly compact. Now we note that for all n-tuples $(\lambda_0, \dots, \lambda_{n-1})$ of real numbers $\lambda_i \geqslant 0$ with $|\lambda| = \sum_{i<n} \lambda_i = 1$, one has $F(\sum_{i<n} \lambda_i a_i) = \sum_{i<n} \lambda_i a_{i+1}$. Hence $F(K) \subset K$ follows from the continuity of F and consequently there exists a fixed point b of F in K. Moreover for any continuous linear functional L,

$$|L(b)| \leqslant \sup_{|\lambda|=1} \sum_{i<n} \lambda_i |L(a_i)| \leqslant \sup_{n \geqslant 0} |L(a_n)|. \qquad \square$$

II.2.

The above theorem will be used particularly for $E = L^2$ and U a linear isometry arising from a measure-preserving transformation but we shall need an improvement of it. Let H be a Hilbert space endowed with the scalar product $(.\,|\,.)$, the norm $\|\,.\,\|$ and let U be a linear isometry of H. We do not require U to be unitary. Let x be an element of H. The map

$$(m, n) \to (U^m x \,|\, U^n x)$$

defined on $\mathbb{N} \times \mathbb{N}$ has a constant value $\gamma_x(a)$ on the half-line $m - n = a$, a given in $\mathbb{Z}$, and so $a \to \gamma_x(a)$ is defined on $\mathbb{Z}$. The map γ_x is the well known *correlation function* of x with respect to U; it is a positive-definite function. From the Bochner-Herglotz theorem, it follows that γ_x is the Fourier transform of a Borel measure λ_x on the torus $\mathbb{T}$, called the *spectral measure* of x with respect to U and moreover, if h denotes the Haar measure on $\mathbb{T}$:

$$\lambda_x(\mathrm{d}t) = * - \lim_{N:\infty} \frac{1}{N} \left\| \sum_{n<N} \mathrm{e}^{-2i\pi nt} U^n x \right\|^2 h(\mathrm{d}t)$$

where the limit is taken in the space $\mathscr{M}_*(\mathbb{T})$ of complex measures on $\mathbb{T}$, dual of $\mathscr{C}(\mathbb{T})$, endowed with the weak topology. We recall that if $\mathscr{M}(\mathbb{T})$ denotes the dual Banach space of $\mathscr{C}(\mathbb{T})$, then the map $x \to \lambda_x$ from H to $\mathscr{M}(\mathbb{T})$ is continuous. By means of the above weak limit one gets $\lambda_{cx} = |c|^2 \lambda_x$ for any complex number c and $\lambda_{x+y} \leqslant 2\lambda_x + 2\lambda_y$ for any x and y in H.

THEOREM 1. *Let U be a linear isometry on the Hilbert space H and let x be in H, λ_x its spectral measure with respect to U. Then the following are equivalent:*

i) The sequence $n \to x + U(x) + \dots + U^{n-1}(x)$ is bounded in H.

ii) *x is a U-coboundary; there exist y in H such that* $x = y - U(y)$.
iii) *The map*

$$t \to \frac{1}{\sin^2 \pi t}$$

from $\mathbb{T}$ *into* $\overline{\mathbb{R}}$ (*with the value* $+\infty$ *at* $t = 0$) *is* λ_x*-integrable.*

Proof. The unit ball of H is weakly compact according to the theorem of Alaoglu, therefore *i*) and *ii*) are equivalent by the coboundary theorem.

ii) $\Rightarrow$ *iii*): Assume $x = y - Uy$, then a straightforward computation gives $\gamma_x(n) = 2\gamma_y(n) - \gamma_y(n-1) - \gamma_y(n+1)$ $(n \in \mathbb{Z})$ and using spectral measures we obtain

$$\gamma_x(n) = \int_{\mathbb{T}} 4 \sin^2(\pi t)\, e^{2i\pi n t} \lambda_y(dt)$$

in other words

$$\lambda_x(dt) = 4 \sin^2(\pi t) \lambda_y(dt).$$

Therefore $\lambda_x(\{0\}) = 0$ since λ_x is bounded and we get iii) with the following value:

$$\int_{\mathbb{T}} \frac{1}{4 \sin^2(\pi t)} \lambda_x(dt) = \lambda_y(\mathbb{T}) = \| y \|^2.$$

iii) $\Rightarrow$ *i*): Assume that $\dfrac{1}{\sin^2(\pi t)}$ is λ_x-integrable. A straightforward computation gives

$$\lambda_{x_N}(dt) = \left| \frac{1 - e^{2i\pi N t}}{1 - e^{2i\pi t}} \right|^2 \lambda_x(dt),$$

where $x_N := \sum_{n<N} U^n(x)$, $N \in \mathbb{N}$. In particular,

$$\lambda_{x_N}(\mathbb{T}) = \left\| \sum_{n<N} U^n(x) \right\|^2 \leqslant \int_{\mathbb{T}} \frac{1}{\sin^2(\pi t)} \lambda_x(dt).$$

This proves i). □

Note that the above proof gives

COROLLARY. *With the assumptions of Theorem* 1, *the equality* $x = y - U(y)$ *implies*:

$$\| y \| = \left\| \frac{1}{2 \sin \pi(\cdot)} \right\|_{L^2(\mathbb{T};\lambda_x)}.$$

II.3.

Now we quote a simple but typical fact about bounded remainder sets for sequences arising from regular processes. By a *process* $\mathscr{X}$ we shall mean in this paper a triplet $(T; X, \mu)$ where X is a compact metrizable space, T a Borel transformation from X to X and μ a Borel measure which is preserved by T. We recall that a map defined on X into a topological space is said to be μ-continuous if it is continuous at μ-almost every point of X. A subset Y of X is then said to be μ-continuous if its characteristic function $1Y$ is μ-continuous. The process $\mathscr{X}$ is called *regular* if T is μ-continuous. Let J be an infinite subset of $\mathbb{N}$. A point x in X is called $(\mathscr{X}, J)$-generic if for all continuous complex maps f: $X \to \mathbb{C}$, one has

$$\lim_{N \in J} \frac{1}{N} \sum_{n<N} f(T^n x) = \int_X f(t)\mu(\mathrm{d}t). \tag{3}$$

When the map f: $X \to \mathbb{C}$ is only μ-Riemann-integrable, that is to say bounded and μ-continuous, then (3) is still true whenever $\mathscr{X}$ is regular.

THEOREM 2. *Let $\mathscr{X} = (T; X, \mu)$ be a regular process and x a $(\mathscr{X}, J)$-generic point. If A is a μ-continuous B.R.S. for the sequence $\xi = (T^n x)_n$ of admissible frequency a then there exists F in $L^\infty(X; \mu)$ such that*

$$1_A - a = F \circ T - F \quad \mu - a.e. \tag{4}$$

Moreover $t \to \dfrac{1}{\sin^2(\pi t)}$ is integrable for the spectral measure λ_f of $f = 1_A - a$ with respect to the linear isometry given by T on $L^2(X; \mu)$.

Proof. Assume A is μ-continuous subset of X. Put $f = 1_A - a$, $f_0 = 0$ and $f_m = f + f \circ T + \ldots + f \circ T^{m-1}$ for $m \geqslant 1$. The maps f_m are μ-Riemann-integrable and by assumption on x for all p in $[1, +\infty[$ one has:

$$\lim_{N \in J} \frac{1}{N} \sum_{n<N} |f_m(T^n x)|^p = \| f_m \|^p_{L^p(X;\mu)}.$$

Let A be a B.R.S. and let a be the corresponding admissible frequency. There exists $c \geqslant 0$ such that

$$\sup_m |f_m(x)| \leqslant c,$$

hence for all integers m, $n \geqslant 0$,

$$|f_m(T^n x)| \leqslant 2c.$$

This implies

$$\| f_m \|_{L^p(X;\mu)} \leqslant 2c$$

for all p, $1 \leqslant p < +\infty$ and consequently these inequalities also hold for $p = +\infty$. Now the sequence $m \to f_m$ is strongly bounded in $L^\infty(X; \mu)$, dual of $L^1(X; \mu)$. From the theorem of Alaoglu (f_m) is relatively compact for the weak topology $\sigma(L^\infty, L^1)$ and the coboundary theorem can be applied to $E = L^\infty(X, \mu)$ endowed with the weak topology. Therefore f is a U_T-coboundary where U_T is the linear isometry derived from the map $\varphi \to \varphi \circ T$ defined for complex maps φ: $X \to \mathbb{C}$. □

Remark 1. The equality $1_A - a = F \circ T - F$ μ.a.e. implies that for μ-almost every point y in X, the set A is a B.R.S. for the sequence $\eta = (T^n y)_n$. Conversely, if A is a Borel set and if there exist a Borel subset Y of X with $\mu(Y) > 0$ and A a B.R.S for all sequences $(T^n y)_n$ with $y \in Y$, then there exists F in $L^\infty(X, \mu)$ satisfying (4) with $a = \mu(A)$. This derives directly from the individual ergodic theorem. Note that in [Halász, 1976] Halász gets the same result but the remainder $r_N(y) = \sum_{n<N} 1_A(T^n y) - N\mu(A)$ is only assumed to be bounded below on Y.

Remark 2. Theorem 2 can be viewed as the metric version of the classical theorem of [Gottschalk and Hedlund, 1955].

III. The sequences $(n\alpha)_n$ in $\mathbb{T}^d$

III.1.

Let τ_α: $\mathbb{T}^d \to \mathbb{T}^d$ be the translation given by $\tau_\alpha(x) = x + \alpha$. It is well known that τ_α preserves the Haar measure h on $\mathbb{T}^d$ and so induces a unitary operator U_α: $L^2(\mathbb{T}^d, h) \to L^2(\mathbb{T}^d, h)$ given by $U_\alpha(f) = f \circ \tau_\alpha$.

THEOREM 3. *Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be in $\mathbb{T}^d$ such that $\alpha_1, \ldots, \alpha_d$ are $\mathbb{Z}$-independent in $\mathbb{T}$. Let $P = I_1 \times \cdots \times I_d$ be a block with arcs I_j of length $|I_j| \neq 0$. Then P is a bounded remainder set for $(n\alpha)_n$ if and only if there exists an index k such that $|I_k| \in \mathbb{Z}\alpha_k + \mathbb{Z}$ and for all other indices $j \neq k$, one has $|I_j| = 1$.*

Proof. Assume that the given block P satisfies $|I_k| \in \mathbb{Z}\alpha_k + \mathbb{Z}$ for an index k and $|I_j| = 1$ for the other indices j. Clearly we can as well assume $I_j = \mathbb{T}$ so that P is a B.R.S. from Kesten's theorem. Note that the case $h(P) = 0$ is obvious.

Now suppose that P is a B.R.S. for $(n\alpha)_n$ with $h(P)=\prod_{n=1}^{d}|I_n|\neq 0$ and put $f=1_P-h(P)$. Since $(n\alpha)_n$ is uniformely distributed in $\mathbb{T}^d$ we derive as in Theorem 2 that the sequence

$$N\to f+f\circ\tau_\alpha+\cdots+f\circ\tau_\alpha^{N-1}$$

is bounded in $L^2(\mathbb{T}^d;\,h)$ (and also in L^∞). By Theorem 1, the map $\frac{1}{\sin^2\pi(\cdot)}$ is integrable for the spectral measure h_f of f with respect to U_α. Using the Fourier expansion of f we easily verify that

$$h_f=\sum_{\substack{m\in\mathbb{Z}^d\\ m\neq 0}}|(1_P|\mathrm{e}_m)|^2\delta_{(m|\alpha)},$$

where δ_a is the Dirac measure at the point a $(a\in\mathbb{T})$, $(m|\alpha)=m_1\alpha_1+\cdots+m_d\alpha_d$ for $m=(m_1,\dots,m_d)$ in $\mathbb{Z}^d$ and e_m is the character on $\mathbb{T}^d$ given by $\mathrm{e}_m(t)=\mathrm{e}^{2i\pi(m|t)}$. The condition iii) in Theorem 1 is now equivalent to

$$\sum_{\substack{m\in\mathbb{Z}^d\\ m\neq 0}}\frac{|(1_P|\mathrm{e}_m)|^2}{(\sin\pi(m|\alpha))^2}<+\infty. \tag{5}$$

For $d=1$, (5) is the Petersen's condition (1). For $d\geqslant 2$ we get from (5)

$$\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{|(1_{I_j}|\mathrm{e}_m)|^2}{(\sin\pi m\alpha_j)^2}<+\infty,$$

hence

$$|I_j|\in\mathbb{Z}\cdot\alpha_j+\mathbb{Z}$$

for all indices j. Assume that k is an index such that $|I_k|\neq 1$. From (5) we derive now:

$$\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\left(\frac{\sin(\pi m|I_j|)\,\sin(\pi|I_k|)}{m\,\sin\pi(m\alpha_j+\alpha_k)}\right)^2<+\infty \tag{6}$$

for all indexes j, $j\neq k$. By a result of [Kronecker, 1884] there exists a constant $c>0$ such that

$$\liminf_{m:\infty}|m\,\sin\pi(m\alpha_j+\alpha_k)|\leqslant c.$$

In other words, there is a sequence $(m_n)_n$ of integers such that

$$\lim_{n:\infty} (m_n \alpha_j) = -\alpha_k \tag{7}$$

on the torus $\mathbb{T}$ and

$$\frac{\sin(\pi m_n |I_j|)}{m_n \sin \pi(m_n \alpha_j + \alpha_k)} \geqslant \frac{\sin(\pi m_n |I_j|)}{2c}.$$

But now (6) implies

$$\lim_{n:\infty} \sin(\pi m_n |I_j|) = 0.$$

Since there exists $p \in \mathbb{Z}$ with $|I_j| - p\alpha_j \in \mathbb{Z}$, it follows

$$\lim_{n:\infty} (m_n p \alpha_j) = 0$$

in $\mathbb{T}$ and (7) gives

$$p \cdot \alpha_k = 0,$$

so that $p = 0$, hence $|I_j| = 1$. □

III.2.

We now give a construction of bounded remainder sets in d-dimensional torus, $d \geqslant 2$, which are not blocks. To do this we need some definitions. Let $v = (v_1, \ldots, v_d)$ be in $\mathbb{R}^d$ with $v_d \neq 0$ and let ρ: $\mathbb{R}^d \to \mathbb{R}^d/\mathbb{Z}^d$ be the canonical map. τ_v is the translation by v modulo $\mathbb{Z}^d$ to which is associated the *cross translation* θ_v modulo $\mathbb{Z}^d$ given by

$$\theta_v(t_1, \ldots, t_{d-1}, 0) = \left(t_1 + \frac{v_1}{v_d}, \ldots, t_{d-1} + \frac{v_{d-1}}{v_d}, 0\right) \bmod \mathbb{Z}^d.$$

We consider $\mathbb{T}^d$ as $Q_d = \prod_{i=1}^{d} [0, 1[$. Now a subset Σ of Q_d will be called a *section* for v if for any point σ in Σ one has

$$\Sigma \cap (\sigma + \mathbb{R} \cdot v) = \{\sigma\}.$$

Put $H_0 = \{t \in \mathbb{R}^d;\ t_d = 0\}$ and define

$$\Sigma_0 := (\Sigma + \mathbb{R} \cdot v) \cap H_0.$$

Finally, a subset B in $\mathbb{R}^d$ will be called *simple* (with respect to $\mathbb{Z}^d$) if it is bounded and satisfies

$$\forall x,\ y \in B;\ x - y \in \mathbb{Z}^d \Rightarrow x = y.$$

The following theorem extends results of [Szüsz, 1954], its proof follows an idea of [Larcher, 1985].

THEOREM 4. *Let* $\alpha = (\alpha_1, \dots, \alpha_d)$ *be in* Q_d *with* $1, \alpha_1, \dots, \alpha_d$ $\mathbb{Z}$*-independent, let* v *in* $\mathbb{N}\alpha + \mathbb{Z}^d$, $v \notin \mathbb{Z}^d$ *and let* Σ *be a section for* v *such that* $\rho(\Sigma_0)$ *is* θ_v*-admissible. If the cylinder* $C = \Sigma + [0, 1[.\ v$ *is simple then* $\rho(C)$ *is* τ_α*-admissible.*

Proof. We first prove the theorem in a particular case. Assume $\Sigma = \Sigma_0$ and denote by $\langle x \rangle$ the point in Q_d congruent to x $(x \in \mathbb{R}^d)$ modulo $\mathbb{Z}^d$. Let $p \in \mathbb{N}$, $p \neq 0$, and $v = \langle p\alpha \rangle$. For any x in Q_d we denote by $X_n(x)$ the point $x + n.v$ in $\mathbb{R}^d$ and we define the half-straight lines

$$D := \{ y \in \mathbb{R}^d;\ \exists \lambda \geqslant 0,\ y = \lambda X_1(0) \},$$

$$D_x := D + x.$$

Let $d_n(x)$ be the intersection point of D_x with the affine hyperplane $P_n = \{ y \in \mathbb{R}^d;\ y_d = n \}$, $n \in \mathbb{N}$. For any point ξ in Q_d, we have to prove that the sequence $(\xi_N)_N$ given by

$$\xi_N := \sum_{n<N} 1_B(\rho(\xi + n\alpha)) - \langle p\alpha_1 \rangle \cdot \lambda N$$

is bounded, where $B = \rho(C)$ and λ is the admissible frequency of $\Sigma_0 (= \Sigma)$ for the cross translation θ_v. Let k_N be the integer such that $X_{[(N/p)]}(0)$ belongs to the segment $[d_{k_N - 1}(0),\ d_{k_N}(0)]$, then

$$k_N = \frac{N}{p} \langle p\alpha_1 \rangle + \mathcal{O}(1)$$

and with $\Sigma' = \Sigma + \mathbb{Z}^d$:

$$\sum_{n<N} 1_B(\rho(\xi + n\alpha)) = \sum_{r=0}^{p-1} \sum_{k<k_N} c_{k,r} 1_{\Sigma'}\ (d_k(\langle \xi + r\alpha \rangle)) + \mathcal{O}(1)$$

where $c_{k,r}$ is the number of points $X_m(\langle \xi + r\alpha \rangle)$ on the segment $[d_k(\langle \xi + r\alpha \rangle),\ d_{k+1}(\langle \xi + r\alpha \rangle)]$ such that $m < \dfrac{N}{p}$ and

$$\langle \xi_d + r\alpha \rangle + m \langle p\alpha_d \rangle \in [k,\ k + \langle p\alpha_1 \rangle[.$$

But for $k \geqslant 1$, $t \in [0, 1[$ and $a + [0, 1[$ one has

$$\operatorname{card}\{m \in \mathbb{N};\ k \leqslant t + ma < k + a\} = 1$$

hence

$$\sum_{n<N} 1_B(\rho(\xi + n\alpha)) = \sum_{r=0}^{p-1} \sum_{k<k_N} 1_\Sigma \circ \theta_v^{\,k}(\langle \xi + r\alpha \rangle) + \mathcal{O}(1)$$

$$= p\lambda k_N + \mathcal{O}(1) = \langle p\alpha_1 \rangle \lambda N + \mathcal{O}(1),$$

this proves the particular case.

Now let A be any τ_α-admissible part of $\mathbb{T}^d$ and let B be a subset of A. Then for any β in $\mathbb{Z}\alpha$, the set $(A \setminus B) \cup (\tau_\beta B)$ is τ_α-admissible. From this we derive the theorem in its general form. □

IV. Weyl sequences

IV.1. Weyl flows

Let $P(X) = a_0 X^d + a_1 X^{d-1} + \cdots + a_d$ be a polynomial of degree $d \geqslant 2$ with real coefficients a_i. When a_0 is irrational P is said to be a *Weyl polynomial.* Put $\alpha = d!a_0$. Following [Furstenberg, 1981] we define the flow $W_\alpha : \mathbb{T}^d \to \mathbb{T}^d$ by

$$W_\alpha(t_1, \ldots, t_d) = (t_1 + \alpha,\ t_2 + t_1, \ldots, t_d + t_{d-1}).$$

This flow is uniquely ergodic, preserving the Haar measure. Let Δ be the transformation defined on the space of real polynomials by

$$\Delta Q(X) = Q(X+1) - Q(X).$$

The following formula will be useful:

$$W_\alpha^m(\Delta^{d-1}p(0), \ldots, \Delta p(0),\ p(0)) = (\Delta^{d-1}p(m), \ldots, \Delta p(m),\ p(m)) \quad (8)$$

for all $m \in \mathbb{Z}$.

IV.2.

The next theorem derives from spectral properties of W_α.

THEOREM 5. *Let A be a h-continuous subset of $\mathbb{T}$ for the Haar measure h. Let P be a Weyl polynomial of degree $d \geqslant 2$. If A is a B.R.S. for the sequence* $(p(n) \bmod 1)_n$ *then the Haar measure $h(A)$ of A is* 0 *or* 1.

Proof. Define $B := \{(t_1, \ldots, t_d) \in \mathbb{T}^d;\ t_d \in A\}$ and put $f := 1_B - h(B)$. By assumption and (8) there exists $C > 0$ such that

$$\left| \sum_{m<M} f \circ W_\alpha^m(\Delta^{d-1}p(0), \ldots, \Delta p(0),\ p(0)) \right| \leqslant C.$$

But any point x in $\mathbb{T}^d$ is $(W_\alpha, \mathbb{N})$ – generic hence by Theorem 2 the map $t \to \dfrac{1}{\sin^2 \pi t}$ on $\mathbb{T}$ is integrable for the spectral measure h_f of f with respect to W_α. The Fourier expansion

$$f = \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} (1_A \mid e_m)\, e_m$$

holds in $L^2(\mathbb{T}^d, h)$ with e_m viewed as the character

$$(t_1, \ldots, t_d) \to e^{2i\pi m t_d}$$

on $\mathbb{T}^d$. By finite induction we easily obtain that for m in $\mathbb{Z}$, and n in $\mathbb{N}$ there exists a continuous map

$$\psi_{m,n} : \mathbb{T}^{d-2} \to \mathbb{C},$$

which is constant if $d = 2$ (and in this case, we put $\mathbb{T}^0 = \{0\}$) such that

$$e_m \circ W_\alpha^n(t_1, \ldots, t_d) = \psi_{m,n}(t_1, \ldots, t_{d-2})\, e_{mn}(t_{d-1})\, e_m(t_d)$$

on $\mathbb{T}^d$. Hence

$$(e_m \circ W_\alpha^n \mid e_{m'}) = \begin{cases} 0 & \text{if } m \neq m' \quad \text{or } n \neq 0, \\ 1 & \text{if } m = m' \quad \text{and } n = 0, \end{cases}$$

so that

$$(f \circ W_\alpha^n \mid f) = \begin{cases} 0 & \text{if } n \neq 0, \\ \| f \|^2 & \text{if } n = 0. \end{cases}$$

It follows

$$h_f(dt) = \| f \|^2 h(dt).$$

Since $\| f \|^2 = h(A)(1 - h(A))$, the integrability condition iii) of Theorem 1 implies $h(A) = 0$ or 1. □

V. q-Multiplicative sequences

V.1.

Let $q=(q_n)_{n\geqslant 1}$ be a sequence of integers $q_n \geqslant 2$. We put $q_0=1$ and $p_n = q_0 q_1 \dots q_n$. A complex valued sequence $z:\mathbb{N}\to\mathbb{C}$ is said to be q-multiplicative if for all integers $b\geqslant 0$, $n\geqslant 0$ and $0\leqslant a<p_n$, one has

$$z(a+bp_n)=z(a)z(bp_n). \tag{9}$$

The special case where $z(bp_n)=\mathrm{e}^{2\mathrm{i}\pi b/p_{n+1}}$ for $0\leqslant b<q_{n+1}$ corresponds to a generalisation of Halton sequences. The set of arcs of bounded remainder for such sequences is almost completely determined by [Hellekalek, 1984]. Let g be an integer $\geqslant 2$ and assume that in place of (9) one has

$$z(a+bp_n)=z(a)z(b) \tag{10}$$

where $p_n=g^n$. Then z is said to be a *strongly multiplicative* sequence to base g. An example is provided by the sequence $n\to \mathrm{e}^{2\mathrm{i}\pi\alpha s_g(n)}$ given in the introduction and for which the corresponding dynamical system is known to be a uniquely ergodic flow [Kamae, 1977]. Another example is the Kakutani sequence studied in part V.4. below.

From now on we assume that the q-multiplicative sequences take their values in the group of complex numbers of modulus 1. This group will be denoted by $\mathbb{U}$.

THEOREM 6. *Let z be a strongly multiplicative ($\mathbb{U}$-valued) sequence to base g. The following hold:*

i) The associated flow $\mathscr{K}(z)$ is uniquely ergodic and z is well-distributed in the closed subgroup $\mathbb{U}(z)$ generated by all the values of z.

ii) Assume moreover that z takes a value which is not a root of unity (hence $\mathbb{U}(z)=\mathbb{U}$) and suppose there exists an integer $m\neq 0$ prime to g such that:

$$(*)\qquad \left|\sum_{j<g}(z(j))^m\right|>1.$$

Then an arc I of $\mathbb{U}$ is a B.R.S. for z if and only if the length of I is 0 or 2π.

Before we are going to prove Theorem 6 we will study q-multiplicative flows.

V.2. q-Multiplicative flows

Let z: $\mathbb{N}\to\mathbb{U}$ be a q-multiplicative sequence and let $\mathscr{K}(z;\,\nu)$ be the corresponding measured flow with $\nu\in I(z)$. For any integer $k\geqslant 0$ we define the sequences $q^{(k)}$, $p^{(k)}$ and $z^{(k)}$ respectively by

$$q_0^{(k)}=1, \quad q_n^{(k)}=q_{n+k} \quad (n \geqslant 1)$$

$$p_n^{(k)}=q_0^{(k)} \ldots q_n^{(k)}$$

and

$$z^{(k)}(n)=z(np_k) \quad (n \in \mathbb{N}).$$

Clearly $z^{(k)}$ is a $q^{(k)}$-multiplicative sequence. Finally for each $m=(m_0,\ldots,m_s)$ in $\mathbb{Z}^{s+1}$ we associate the character χ_m on $\mathbb{U}^{\mathbb{N}}$ given by

$$\chi_m(u_0, u_1, u_2, \ldots):=u_0^{m_0} \ldots u_s^{m_s}$$

and for short, we put

$$\Omega:=\mathbb{U}^{\mathbb{N}}$$

$$\chi_m(j):=\chi_m(S^j(z)),$$

$$|m|:=m_0+\cdots+m_s.$$

We prove the following extension of earlier results [Coquet *et al.*, 1977; Liardet, 1980; Queffelec, 1984].

THEOREM 7. *Let* $z\colon \mathbb{N} \to \mathbb{U}$ *be a q-multiplicative sequence. For any m in* $\mathbb{Z}^{s+1}$ *the spectral measure* $\Lambda_{z,m}$ *of* χ_m *with respect to* $\mathscr{K}(z;\nu)$ *does not depend on the choice of* ν *in* $I(z)$. *Moreover, let* $P_{m,N}$ *be the polynomial*

$$P_{m,N}(t):=\frac{1}{N}\left|\sum_{j<N} \chi_m(j)\, \mathrm{e}^{-2\mathrm{i}\pi jt}\right|^2,$$

then

i) $\Lambda_{z,m}$ *is the weak limit of* $\mu_{m,N}(\mathrm{d}t)=P_{m,N}(t)h(\mathrm{d}t)$.

ii) Let λ *be a Borel measure on* $\mathbb{T}$ *and let p be an integer* >0.

We denote by λ^p *the Borel measure given by*

$$\int_{\mathbb{T}} f(t)\lambda^p(\mathrm{d}t)=\int_{\mathbb{T}}\left(\frac{1}{p}\sum_{l<p} f\left(\frac{u}{p}+\frac{l}{p}\right)\right)\lambda(\mathrm{d}u)$$

for all f in $\mathscr{L}(\mathbb{T})$. *Then* $\Lambda_{z,m}$ *is the limit in the sense of total variance (i.e. in* $\mathscr{M}(\mathbb{T})$*) of the sequence of measures*

$$\sigma_{m,k}(\mathrm{d}t)=P_{m,p_k}(t)(\Lambda_{z^{(k)},|m|})^{p_k}(\mathrm{d}t).$$

Proof. Let k be a given integer and define $\chi_m^{(k)}\colon \mathbb{N} \to \mathbb{U}$ by

$$\chi_m^{(k)}(j) := \left(z\left(\left[\frac{j}{p_k} \right] p_k \right) \right)^{|m|} \chi_m^*(j) \tag{11}$$

where $[x]$ is the integral part of x and $\chi_m^*(\cdot)$ is the periodic sequence of period p_k given by

$$\chi_m^*(j) = \chi_m(j)$$

for $j = 0, 1, \ldots, p_k - 1$. By construction $\chi_m^{(k)}(j) = \chi_m(j)$ for all integers j which do not belong to

$$A_k = \bigcup_{l=1}^{\infty} ([lp_k - s, lp_k[\cap \mathbb{N}).$$

From now on we assume $s < p_k$ so that A_k has the density s/p_k in $\mathbb{N}$.

Let J be an infinite part of $\mathbb{N}$ such that ν is the weak limit of $N \to \frac{1}{N} \sum_{j<N} \delta_{S^j z}$ $(N \in J)$. Hence

$$\lim_{N \in J} \frac{1}{N} \sum_{j<N} \chi_m(j) = \int_\Omega \chi_m(u) \nu(\mathrm{d}u).$$

Assume $|m| = 0$, then

$$\lim_{N:\infty} \sup \frac{1}{N} \sum_{j<N} |\chi_m(j) - \chi_m^*(j)| \leqslant \frac{s}{p_k}$$

and χ_m^* has the mean

$$M_k = \frac{1}{p_k} \sum_{j<p_k} \chi_m(j).$$

This proves that

$$\lim_{k:\infty} M_k = \int_\Omega \chi_m(u) \nu(\mathrm{d}u)$$

and implies finally:

LEMMA 1. *If $|m| = 0$, then for any ν in $I(z)$:*

$$\int_\Omega \chi_m(u) \nu(\mathrm{d}u) = \lim_{N:\infty} \frac{1}{N} \sum_{j<N} \chi_m(j).$$

Now let γ_m be the correlation function

$$\gamma_m(n) := \int_\Omega \chi_m(S^n u)\overline{\chi_m(u)}\nu(\mathrm{d}u) = \lim_{N\in J} \frac{1}{N} \sum \chi_m(j+n)\overline{\chi_m(j)}.$$

On the other hand there exist $s' \in \mathbb{N}$ and $m' \in \mathbb{Z}^{s'+1}$ such that $|m'| = 0$ and $\chi_m(j+n)\overline{\chi_m(j)} = \chi_{m'}(j)$ for all j, hence from the above Lemma $\gamma_m(n)$ does not depend on ν and the same is true for $\Lambda_{z,m}$ with Fourier coefficients given by

$$\hat{\Lambda}_{z,m}(n) = \lim_{N:\infty} \frac{1}{N} \sum_{j<N} \chi_m(j+n)\overline{\chi_m(j)}.$$

For n fixed, a straightforward computation leads to

$$\int_{\mathbb{T}} P_{m,N}(t)\, \mathrm{e}^{2i\pi nt}\, \mathrm{d}t = \frac{1}{N} \sum_{\substack{u,\, v<N \\ u=n+v}} \chi_m(u)\overline{\chi_m(v)}$$

$$= \frac{1}{N}\left(\sum_{v<N} \chi_m(n+v)\overline{\chi_m(v)} + o(1)\right)$$

hence i) holds.

Set for short

$$\sigma_k(\mathrm{d}t) = P_{m,p_k}(t)\left(\Lambda_{z^{(k)},|m|}\right)^{p_k}(\mathrm{d}t)$$

where $\Lambda_{z^{(k)},|m|}$ is the spectral measure of the $q^{(k)}$-multiplicative sequence

$$n \to \left(z(np_k)\right)^{|m|} = \chi_{|m|}(np_k).$$

By a classical result [Coquet *et al.*, 1977], $\Lambda_{z^{(k)},|m|}$ is the weak limit of the sequence

$$N \to \frac{1}{N}\left| \sum_{j<N} \chi_{|m|}(jp_k)\, \mathrm{e}^{-2i\pi jt}\right|^2 \mathrm{d}t.$$

We claim that, in the sense of total variance (i.e. for the norm $|||\cdot|||$ in the dual of $\mathscr{C}_{\mathbb{C}}(\mathbb{T})$) the inequality

$$|||\, \sigma_k - \Lambda_{z,m}\, ||| \leqslant 4\left(\frac{s}{p_k} + \left(\frac{s}{p_k}\right)^{1/2}\right) \tag{12}$$

holds. In fact, put $P_{m,N}^{(k)}(t)=\frac{1}{N}\left|\sum_{j<N}\chi_m^{(k)}(j)\ \mathrm{e}^{-2i\pi jt}\right|^2$, so that σ_k is the weak limit of

$$P_{m,Np_k}^{(k)}(t)\ \mathrm{d}t=\frac{1}{N}P_{m,p_k}(t)\left|\sum_{j<N}\chi_{|m|}(jp_k)\ \mathrm{e}^{-2\mathrm{i}\pi jp_k t}\right|^2\ \mathrm{d}t,$$

then from i):

$$|||\,\sigma_k-\Lambda_{z,m}\,|||\leqslant\liminf_{N:\infty}\int_{\mathbb{T}}|\,P_{m,Np_k}(t)-p_{m,Np_k}^{(k)}(t)\,|\mathrm{d}t.$$

The equality

$$\chi_m(j)=\chi_m(j')\chi_{|m|}(lp_k)(=\chi_m^{(k)}(j))$$

with $j=j'+lp_k$, $0\leqslant j'<p_k$ is true for all j in $\mathbb{N}\setminus A_k$, therefore, using the inequality

$$|\,|\alpha|^2-|\beta|^2\,|\leqslant|\alpha-\beta|^2+2|\alpha|\,|\alpha-\beta|$$

satisfied for all complex numbers α, β one gets:

$$\begin{aligned}&|\,P_{m,N}(t)-P_{m,N}^{(k)}(t)\,|\\&\quad\leqslant\frac{1}{N}\left|\sum_{\substack{j<N\\ j\in A_k}}\left(\chi_m(j)-\chi_m^{(k)}(j)\right)\mathrm{e}^{-2\mathrm{i}\pi jt}\right|^2\\&\qquad+\frac{2}{N}\left|\sum_{j<N}\chi_m(j)\ \mathrm{e}^{-2\mathrm{i}\pi jt}\right|\left|\sum_{j<N}\left(\chi_m(j)-\chi_m^{(k)}(j)\right)\mathrm{e}^{-2\mathrm{i}\pi jt}\right|.\end{aligned}$$

Integration on the torus and Schwarz inequality lead to

$$\begin{aligned}&\int_{\mathbb{T}}|\,P_{m,N}(t)-P_{m,N}^{(k)}(t)\,|\mathrm{d}t\\&\quad\leqslant\frac{1}{N}\sum_{\substack{j<N\\ j\in A_k}}|\chi_m(j)-\chi_m^{(k)}(j)|^2\\&\qquad+2N^{-1/2}\left(\int_{\mathbb{T}}P_{m,N}(t)\ \mathrm{d}t\right)^{1/2}\left(\sum_{\substack{j<N\\ j\in A_k}}|\chi_m(j)-\chi_m^{(k)}(j)|^2\right)^{1/2},\end{aligned}$$

but $\int_{\mathbb{T}} P_{m,N}(t)\,\mathrm{d}t = 1$, hence we get in fact

$$\lim_{N:\infty} \sup \int_{\mathbb{T}} | P_{m,N} - P_{m,N}^{(k)} | \mathrm{d}t \leqslant 4\left(\frac{s}{p_k} + \left(\frac{s}{p_k} \right)^{1/2} \right).$$

Taking Np_k in place of N above we derive (12). □

Remark 3. Assume $|m| = 0$, then $\Lambda_{z^{(k)},0} = \delta_0$ (the Dirac measure at 0 on $\mathbb{T}$) and

$$P_{m,p_k}(t) \cdot (\delta_0)^{p_k}(\mathrm{d}t) = \frac{1}{p_k} \sum_{l<p_k} P_{m,p_k}\left(\frac{l}{p_k} \right) \delta_{l/p_k},$$

hence: *if* $|m| = 0$ *then*:

$\Lambda_{z,m}$ *is discrete, supported by the group* $G_q = \bigcup_{k=0}^{\infty} \left(\frac{1}{p_k}\mathbb{Z} + \mathbb{Z} \right)$ modulo 1.

Remark 4. In the general case for any α in $\mathbb{T}$:

$$\sigma_{m,k}(\{\alpha\}) = \frac{1}{p_k} P_{m,p_k}(\alpha) \Lambda_{z^{(k)},|m|}(\{p_k\alpha\}),$$

and

$$\left(\Lambda_{z^{(k)},|m|}(\{p_k\alpha\}) \right)^{1/2} = \lim_{l:\infty} \frac{1}{p_l^{(k)}} \left| \sum_{u<p_l^{(k)}} \chi_{|m|}(p_k u)\, \mathrm{e}^{-2\mathrm{i}\pi u p_k \alpha} \right|$$

by a classical result in [Coquet *et al.,* 1977], therefore

$$\left(\sigma_{m,k}(\{\alpha\}) \right)^{1/2} = \lim_{l:\infty} \frac{1}{p_l} \left| \sum_{j<p_l} \chi_m^{(k)}(j)\, \mathrm{e}^{-2\mathrm{i}\pi j\alpha} \right|$$

and since $\chi_m^{(k)} = \chi_m$ on $\mathbb{N} \setminus A_k$ we get

$$\left(\Lambda_{z,m}(\{\alpha\}) \right)^{1/2} = \lim_{l:\infty} \frac{1}{p_l} \left| \sum_{j<p_l} \chi_m(j)\, \mathrm{e}^{-2\mathrm{i}\pi j\alpha} \right|$$

$$= \lim_{N:\infty} \sup \frac{1}{N} \left| \sum_{j<N} \chi_m(j)\, \mathrm{e}^{-2\mathrm{i}\pi j\alpha} \right|, \qquad (13)$$

the last equality following from a result of J.P. Bertrandias [Bertrandias, 1964; Coquet *et al.,* 1977]. The next result supplies to a misstatement in ([Liardet, 1980], thm 4) and furnishes a proof.

THEOREM 8. *Let $z\colon \mathbb{N} \to \mathbb{U}$ be a q-multiplicative sequence. The flow $\mathscr{K}(z)$ is uniquely ergodic if and only if for all integers $s \geqslant 0$ and m in $\mathbb{Z}^{s+1}$ one has*:

$$\Lambda_{z,m}(\{0\}) > 0 \Rightarrow \lim_{k:\infty}\left(\sup_{j\geqslant 0} |(z(p_k j))^{|m|} - 1|\right) = 0. \tag{14}$$

Proof. Since the functions χ_m, $m \in \mathbb{Z}^{s+1}$, $s \in \mathbb{N}$ generate a dense subspace in $\mathscr{C}_{\mathbb{C}}(\mathbb{U}^{\mathbb{N}})$, the unique ergodicity of $\mathscr{K}(z)$ is equivalent to the uniform convergence in n of the following sequences of means:

$$N \to \frac{1}{N}\sum_{j<N} \chi_m(j+n).$$

Assume this condition holds. Let $m = (m_0, \ldots, m_s)$ be in $\mathbb{Z}^{s+1}$. For $\epsilon > 0$ given there exists N_0 such that

$$\left|\sum_{j<N} \chi_m(j+n) - \sum_{j<N} \chi_m(j)\right| \leqslant \epsilon N \tag{15}$$

for all $N \geqslant N_0$ and all $n \in \mathbb{N}$. In particular, take $n = p_k l$, $l \in \mathbb{N}$ with $p_k \geqslant N + |m|$, then (15) gives

$$|\chi_{|m|}(p_k l) - 1|\left|\sum_{j<N} \chi_m(j)\right| \leqslant \epsilon N.$$

Suppose $\Lambda_{z,m}(\{0\}) = c_m > 0$ and choose $N = p_s$ large enough such that by (13):

$$\left|\sum_{j<p_s} \chi_m(j)\right| \geqslant \tfrac{1}{2} p_s (c_m)^{1/2},$$

then

$$|\chi_{|m|}(p_k l) - 1| \leqslant 2\epsilon (c_m)^{1/2}$$

therefore

$$\sup_l |\chi_{|m|}(p_k l) - 1| \leqslant 2\epsilon (c_m)^{1/2}$$

for any k such that $p_k \geqslant p_s + |m|$. This proves (14).

Conversely, assume that (14) holds. Let $\chi_m^{(k)}(\cdot)$ be defined as in Theorem 7. By a straightforward computation one has

$$\left|\sum_{j<N} (\chi_m(j+n) - \chi_m^{(k)}(j+n)\right| \leqslant \left(\frac{N}{p_k} + 1\right) s$$

for all integers $n \geqslant 0$. Let n, p_k and N be given and let a, b integers defined by

$$ap_k \leqslant n < (a+1)p_k, \quad bp_k \leqslant n+N-1 < (b+1)p_k.$$

Since $\chi_m^{(k)}(r + lp_k) = \chi_{|m|}(lp_k)\chi_m(r)$ for $0 \leqslant r < p_k$,

one has

$$\left| \sum_{n \leqslant j < n+N} \chi_m^{(k)}(j) - \left(\sum_{r<p_k} \chi_m(r) \right) \left(\sum_{a \leqslant l < b} \chi_{|m|}(p_k l) \right) \right| \leqslant 2p_k$$

and then

$$\left| \sum_{n \leqslant u < n+N} \chi_m^{(k)}(u) - \sum_{0 \leqslant v < N} \chi_m^{(k)}(v) \right| \leqslant 4p_k + \left| \sum_{r<p_k} \chi_m(r) \right| (\Delta_m(a, b) + 1)$$

with

$$\Delta_m(a, b) = \left| \sum_{a \leqslant l < b} \chi_{|m|}(p_k l) - \sum_{0 \leqslant l' < b-a} \chi_{|m|}(p_k l') \right|,$$

hence

$$\left| \sum_{j<N} \chi_m(j+n) - \sum_{j<N} \chi_m(j) \right| \leqslant 2\left(\frac{N}{p_k} + 1 \right) s + 4p_k + p_k(\Delta_m(a, b) + 1). \tag{16}$$

Let $\epsilon > 0$ and assume $\Lambda_{z,m}(\{0\}) > 0$. By (14) there exists t such that $k \geqslant t$ implies $|\chi_{|m|}(p_k l) - 1| \leqslant \epsilon$ for all l, therefore

$$\Delta_m(a, b) \leqslant 2\epsilon(b-a) \leqslant 2\epsilon\left(\frac{N}{p_k} + p_k \right)$$

and (16) now gives

$$\left| \sum_{j<N} (\chi_m(j+n) - \chi_m(j)) \right| \leqslant N\left(\frac{2s}{p_k} + 2\epsilon \right) + 2\epsilon p_k^2 + 5p_k + 2s.$$

Choose $k \geqslant t$ such that $\dfrac{s}{p_k} \leqslant \epsilon$, then for

$$N \geqslant 2p_k^2 + \frac{5p_k + 2s}{\epsilon}$$

we get for all integers n:

$$\left| \frac{1}{N} \sum_{j<N} (\chi_m(j+n) - \chi_m(j)) \right| \leqslant 5\epsilon,$$

so that $(1/N) \sum_{j<N} \chi_m(j+n)$ converges uniformly in n.

Assume now that $\Lambda_{z,m}(\{0\}) = 0$, then with the above notations one has

$$\begin{aligned}\left| \sum_{j<N} \chi_m(j+n) \right| &\leqslant 2p_k + \left| \sum_{ap_k \leqslant u < bp_k} \chi_m^{(k)}(u) \right| + \left(\frac{N}{p_k} + 1 \right) s \\ &\leqslant 2p_k + \left(\frac{N}{p_k} + 1 \right) s + \left| \sum_{r<p_k} \chi_m(r) \right| (b-a) \\ &\leqslant 2p_k + \left(\frac{N}{p_k} + 1 \right) s + \left(N + p_k^2 \right) \left| \frac{1}{p_k} \sum_{r<p_k} \chi_m(r) \right|.\end{aligned}$$

Let k be such that $\dfrac{s}{p_k} \leqslant \epsilon$ and by (13)

$$\left| \frac{1}{p_k} \sum_{r<p_k} \chi_m(r) \right| \leqslant \epsilon.$$

Now with $N \geqslant p_k^2 + \dfrac{2p_k + s}{\epsilon}$ we obtain

$$\left| \frac{1}{N} \sum_{j<N} \chi_m(j+n) \right| \leqslant 3\epsilon$$

for all integers n. This finishes the proof. □

V.3. Proof of Theorem 6

We continue the study of flows $\mathscr{K}(z)$ but now we assume that $z: \mathbb{N} \to \mathbb{U}$ is a strong multiplicative sequence to base g.

V.3.1. $\mathscr{X}(z)$ is uniquely ergodic. Let m be a rational integer. By (13) one has

$$\Lambda_{z,m}(\{0\})^{1/2} = \lim_{k:\infty} \frac{1}{g^k}\left|\sum_{j<g^k} (z(j))^m\right| = \lim_{k:\infty}\left|\frac{1}{g}\sum_{j<g}(z(j))^m\right|^k$$

$$= \lim_{N:\infty} \sup\left|\frac{1}{N}\sum_{j<N}(z(j))^m\right|, \tag{17}$$

hence $\Lambda_{z,m}(\{0\})$ is 0 or 1 and the value 1 is taken if and only if z^m is the constant sequence $n \to 1$ since $z(0)=1$.

Now let m be in $\mathbb{Z}^{s+1}$. By assumption on z, the equality $z^{(k)}=z$ holds for any integer k, so that by Theorem 7:

$$\sigma_{m,k}(\{0\}) = \left|\frac{1}{g^k}\sum_{j<g^k}\chi_m(j)\right|^2 \Lambda_{z,|m|}(\{0\}).$$

If $z^{|m|}$ is a constant sequence then the implication (14) of theorem 8 is obvious. If $z^{|m|}$ is not constant then $\sigma_{m,k}(\{0\})=0$ and by Theorem 7, part ii):

$$\Lambda_{z,m}(\{0\}) = 0, \tag{18}$$

hence (14) holds again. This proves the unique ergodicity of $\mathscr{X}(z)$. Let λ be the corresponding unique invariant measure on $\mathscr{X}(z)$. The projection $\lambda_{|1}$ of λ onto the first factor of $\Omega(=\mathbb{U}^{\mathbb{N}})$ is carried by $\mathbb{U}(z)$ and from (17) we easily derive that $\lambda_{|1}$ is the Haar measure on $\mathbb{U}(z)$, as expected to complete the proof of i), Theorem 6.

V.3.2. $\Lambda_{z,1}$ is g-invariant. For short we put $\Lambda_z := \Lambda_{z,1}$ and $P := P_{1,g}$ (see Theorem 7). The g-invariant property of Λ_z means that Λ_z is an invariant measure for the transformation $t \to g.t$ on the torus. This result is contained in [Queffelec, 1979], it derives from i) Theorem 7 and the identity

$$\frac{1}{g}\sum_{l<g} P\left(\frac{t}{g}+\frac{l}{g}\right) \equiv 1. \tag{19}$$

V.3.3. A spectral criterium. Define $t \to \|t\|$ on $\mathbb{T}$ by

$$\|t\| = \mathrm{Min}\{\,|\theta|;\quad t=\theta+\mathbb{Z}\}.$$

LEMMA 2

$$\left|\sum_{j<g} z(j)\right| > 1 \Rightarrow \int_{\mathbb{T}} \frac{1}{\|t\|^2}\Lambda_z(\mathrm{d}t) = +\infty.$$

Proof. We follow an idea of [Queffelec, 1984]. Let $I_k = \left[\frac{1}{g^k}, \frac{1}{g^{k-1}}\right[$ with $k \geqslant 2$. The g-invariant property of Λ_z implies

$$J_K := \int_{\mathbb{T}} 1_{I_k}(t) \cdot \frac{1}{\| t \|^2} \Lambda_z(\mathrm{d}t) = \int_{\mathbb{T}} 1_{I_k}(gt) \frac{1}{\| gt \|^2} \Lambda_z(\mathrm{d}t)$$

$$= \int_{\mathbb{T}} \sum_{j<g} 1_{I_{k+1}}\left(u + \frac{j}{g}\right) \frac{1}{\| gu \|^2} \Lambda_z(\mathrm{d}u)$$

$$= \frac{1}{g^2} \int_{I_{k+1}} \frac{1}{\| t \|^2} \left(\sum_{j<g} \delta_{\{j/g\}} * \Lambda_z \right)(\mathrm{d}t).$$

But from i) Theorem 7 and (19) we get

$$\frac{1}{g} \sum_{j<g} \delta_{\{j/g\}} * \Lambda_z = \lim_{k:\infty} \left[\left(\frac{1}{g} \sum_{j<g} P\left(t + \frac{j}{g}\right) \right) P(gt) \ldots P(g^k t) h(\mathrm{d}t) \right]$$

$$= (\Lambda_z)^g(\mathrm{d}t) = \frac{1}{P(t)} \Lambda_z(\mathrm{d}t),$$

hence

$$J_k = \int_{\mathbb{T}} 1_{I_{k+1}}(t) \cdot \frac{1}{gP(t) \| t \|^2} \Lambda_z(\mathrm{d}t).$$

Assume that $gP(0) > 1$, then for k large enough the inequality $J_k \leqslant J_{k+1}$ holds and consequently the integral $\int_{\mathbb{T}} \frac{1}{\| t \|^2} \Lambda_z(\mathrm{d}t)$ diverges.

V.3.4. End of the proof. Let I be an arc of $\mathbb{U}$ and suppose that I is a B.R.S. for z satisfying the assumptions of ii). Define f: $\Omega \to \mathbb{C}$ by

$$f(\omega) = 1_I(\omega_0) - | I |$$

and let the above invariant measure λ on $\mathscr{K}(z)$ be viewed as a measure on Ω. By i) $\lambda_{|1}$ is equal to the Haar measure h on $\mathbb{U}$, hence the Fourier expansion of f in $L^2(\Omega;\, \lambda)$ is given by

$$f = \sum_{m \in \mathbb{Z}^*} (1_I \,|\, \chi_m) \chi_m$$

where $(1_I \mid \chi_m)$ is the ordinary Fourier coefficient $\int_I t^m h(\mathrm{d}t)$. On the other hand

$$(\chi_m \circ S^n \mid \chi_{m'}) = \lim_{N:\infty} \frac{1}{N} \sum_{k<N} (z(k+n))^m \overline{z(k)^{m'}}$$

$$= \int_\Omega \chi_M(t)\lambda(\mathrm{d}t) = \Lambda_{z,M}(\{0\})$$

with $M = (-m', 0, \ldots, 0, m)$ in $\mathbb{Z}^{n+1}$. Therefore $(\chi_m \circ S^n \mid \chi_{m'}) = 0$ by (18) if $m \neq m'$ and finally it follows

$$\lambda_f = \sum_{m \in \mathbb{Z}^*} |(1_I \mid \chi_m)|^2 \lambda_m$$

in $\mathscr{M}(\mathbb{T})$ where λ_m is set for the spectral measure $\Lambda_{z,m}$. Theorem 2 implies that $1/\sin^2\pi(\cdot)$, or equivalently $1/\|\cdot\|^2$, is integrable for λ_m if $(1_I \mid \chi_m) \neq 0$.

Assume $|I| \neq 0$ and $\neq 2\pi$. We are going to derive a contradiction proving that $1/\sin^2\pi(\cdot)$ is not integrable for λ_f. The functional equation (4) in Theorem 2 is satisfied with $a = |I|/2\pi$ since z is uniformly distributed in $\mathbb{U}$ and I is a B.R.S. Take the exponential of each member of (4), we obtain φ in $L^2(\Omega; \lambda)$, $\varphi \neq 0$, such that $U_s(\varphi) = \mathrm{e}^{\mathrm{i}|I|}\varphi$ and in terms of spectral measures $\lambda_\varphi = \delta_{\{a\}}$. Now let $\psi = \sum_{\chi \in F} c_\chi \chi$ be a finite sum over the characters χ on Ω, with complex coefficients c_χ, then by induction on the cardinal $|F|$ of F one has

$$\lambda_\psi \leqslant 2^{|F|} \sum_{\chi \in F} |c_\chi|^2 \lambda_\chi. \tag{20}$$

Since the subspace generated by characters χ is dense in $L^2(\Omega, \lambda)$, the continuity of $x \to \lambda_x$ from $L^2(\Omega, \lambda)$ to $\mathscr{M}(\mathbb{T})$ and (20) imply that there exists an integer $s \geqslant 0$ and M in $\mathbb{Z}^{s+1}$, $M \neq 0$, such that

$$\lambda_M(\{a\}) > 0.$$

If $|M| = 0$, the Remark 3 §V.2. means that there exists $k > 0$ and $q \in \{1, \ldots, g^k - 1\}$ with $a = \frac{q}{g^k}$. If $|M| \neq 0$, Theorem 7 implies that there exists $k > 0$ such that

$$\sigma_{M,k}(\{a\}) > 0$$

and Remark 4, §V.2. gives $\lambda_{|M|}(\{g^k a\}) > 0$. Put $\alpha = g^k a$ and use formula (13), then the infinite product

$$\prod_{n=0}^{\infty} \left| \frac{1}{g} \sum_{j<g} (z(j))^{|M|} e^{-2i\pi j g^n \alpha} \right|$$

converges to $\lambda_{|M|}(\{\alpha\})$ (> 0) so that

$$\lim_{n:\infty} \frac{1}{g} \left| \sum_{j<g} (z(j))^{|M|} e^{-2i\pi j g^n \alpha} \right| = 1.$$

It follows by convexity that

$$\lim_{n:\infty} (z(j))^{|M|} e^{-2i\pi j g^n \alpha} = 1$$

for $j = 1, \ldots, g-1$ (the case $j = 0$ being obvious), hence $(g^n \alpha)_n$ converges modulo 1 to β such that $g\beta \equiv \beta$ mod 1 and $(z(j))^{|M|} = e^{2i\pi j\beta}$. This leads to a contradiction, namely all the values of z are roots of unity. Finally we have just proved that $|I|/2\pi$ is the g-adic number q/g^k. Now let m, $m \neq 0$, be an integer prime to g such that $\left| \sum_{j<g} (z(j))^m \right| > 1$. Then $m|I|/2\pi \notin \mathbb{Z}$. By Lemma 2, $1/\sin^2 \pi(\cdot)$ is not integrable for λ_m and consequently is also not integrable for λ_f. □

For $g = 2$ there is an odd integer m satisfying (*). In the general case, it can only be proved there exists an integer $q \geqslant 1$ such that (*) holds for many integers $m = qm'$ with m' prime to g. From the above proof we derive:

Corollary. *For any strongly multiplicative* $\mathbb{U}$*-valued sequence to base* g, *the set of admissible frequencies is finite* (*reduced to the set* $\{0, 1\}$ *for* $g = 2$).

V.4. Applications

Let g be again an integer $\geqslant 2$.

Corollary 2. *The only arcs* I *of* $\mathbb{T}$ *which are B.R.S. for the sequence* $n \to \alpha s_g(n)$ mod 1, *with a given irrational number* α, *are the trivial ones, that is to say* $|I| = 0$ *or* 1.

Proof. One has

$$\left| \sum_{j<g} e^{2i\pi\alpha jm} \right| = \left| \frac{\sin \pi g m\alpha}{\sin \pi m\alpha} \right|$$

and there exists a sequence (m_k) of integers such that g is prime to each m_k and $(m_k\alpha)_k$ converges to 0 modulo 1. In particular $\lim_k \left| \dfrac{\sin \pi g m_k \alpha}{\sin \pi m_k \alpha} \right| = g$ so that the required inequality (*) in Theorem 6 holds and the corollary follows. □

We apply the above method directly to the study of the Kakutani sequence ξ introduced in [Kakutani, 1967]. Let p be a prime number $\geqslant 3$, set $\varphi = e^{2i\pi/(p-1)}$ and let L be the logarithmic function which identifies the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$, with generator e, to the additive group $\mathbb{Z}/(p-1)\mathbb{Z}$ in such a way that $L(e) = 1$. Finally let $\nu_p(m)$ be the p-adic valuation of m. The sequence ξ is defined by

$$\xi(n) := \varphi^{L\left(\frac{n!}{p^{\nu_p(n!)}}\right)}.$$

It is proved in [Coquet *et al.,* 1977] that ξ is a strongly multiplicative sequence to base p^2 and that all sequences ξ^j, for $j = 1, \dots, p-2$, are pseudo-random. This implies that the measure $\Lambda_{\xi,m}$, for m in $\mathbb{Z}^{s+1}$, $s \geqslant 0$, is:

- *continuous if* $|m| \not\equiv 0 \bmod (p-1)$,
- *discrete, carried by* G_{p^2} *if* $|m| \equiv 0 \bmod (p-1)$.

Since ξ is uniformly distributed in $\mathbb{U}(\xi) = \{1, \varphi, \dots, \varphi^{p-2}\}$, an admissible frequency for a B.R.S. has necessary the form $\dfrac{a}{p-1}$ with a in $\{0, 1, \dots, p-1\}$. But measures $\delta_{\{(a)/p-1\}}$ with $a \in \{1, \dots, p-2\}$ are not spectral measures for $\mathscr{K}(\xi)$, hence we have:

COROLLARY 3. *The only subsets A in* $\mathbb{U}$ *which are B.R.S. for the Kakutani sequence* ξ *are the trivial ones (i.e. either A contains all the* $p-1$*-th roots of unity or A contains none of these).*

Acknowledgements

The author wants to thank Peter Hellekalek whose interest in this work has been helpful during the one semester stay at the University of Salzburg.

References

Bertrandias, J.-P.: Suites pseudo-aléatoires et critères d'équirépartition modulo 1. *Compositio Math.* 16 (1964) 23–28.

Bourbaki, N.: *Espaces vectoriels topologiques,* ch III., *Actualités scientifiques et industrielles.* Hermann (1964).

Coquet, J., Kamae, T. and Mendès-France, M.: Sur la mesure spectrale de certaines suites arithmétiques. *Bull. Soc. Math. France* 105 (1977) 369–384.

Dunford, N. and Schwartz, J.T.: *Linear operator, part I, Pure and Applied Mathematics,* vol. VII. Interscience Publishers, New York (1967).

Erdös, P.: Problems and results on diophantine approximations. *Compositio Math.* 16 (1964) 52–65.

Furstenberg, H.: *Recurrence in Ergodic Theory and Combinatorial Number Theory.* Princeton Univ. Press (1981).

Furstenberg, H., Keynes, M. and Shapiro, L.: Prime flows in topological dynamics. *Israel J. Math.* 14 (1973) 26–38.

Gottschalk, W.H. and Hedlund, G.A.: *Topological Dynamics, Colloquium Publication* XXXVI. American Math. Soc., Providence, R.I. (1955).

Halász, G.: Remarks on the remainder in Birkhoff's ergodic theorem. *Acta Mathematica Acad. Sc. Hungaricae* 28 (1976) 389–395.

Hecke, E.: Analytische Funktionen und die Verteilung von Zahlen mod. eins. *Abh. Math. Semin. Hamburg Univ.* (1922) 54–76.

Hellekalek, P.: Regularities in the distribution of special sequences. *J. Number Theory* 18 (1984) 41–55.

Kakutani, S.: Ergodic theory of shift transformations, *Proceedings of the 5th Berkeley symposium on mathematical statistics and probability* [1965, Berkeley], vol. 2, part 2. University of California Press, Berkeley (1967) 405–414.

Kamae, T.: Mutual singularity of spectra of dynamical systems given by 'sum of digits' to different bases, Société Math. France. *Astérisque* 49 (1977) 109–114.

Kesten, H.: On a conjecture of Erdös and Szüsz related to uniform distribution mod 1. *Acta Arith.* 14 (1973) 26–38.

Kronecker, L.: *Berliner Sitzungsberichte,* 11 Déc. 1884. *Werke* 3 (1884) 49.

Larcher, G.: Private communication, Salzburg (May 1985).

Liardet, P.: Propriétés dynamiques de suites arithmétiques. *Journées Mathématiques SMF-CNRS, Théorie Analytique et élémentaire des nombres.* Caen (1980) 58–73.

Oren, I.: Admissible functions with multiple discontinuities. *Israel J.Math.* 42 (1982) 353–360.

Petersen, K.: On a series of cosecants related to a problem in ergodic theory. *Compositio Math.* 26 (1973) 313–317.

Queffelec, M.: Mesures spectrales associées à certaines suites arithmétiques. *Bull. Soc. Math. France* 107 (1979) 385–421.

Queffelec, M.: *Contribution à l'étude spectrale de suites arithmétiques.* Thèse d'Etat, Université Paris-Nord (1984).

Rauzy, G.: Ensembles à restes bornés. *Sém. Théorie des nombres.* Bordeaux exp. 24 (1983–1984) 1–12.

Szüsz, P.: Über die Verteilung der Vielfachen einer komplexen Zahl nach dem Modul des Einheitsquadrates. *Acta Math. Acad. Sci. Hungaricae* 5 (1954) 35–39.