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The last coefficient of the Samuel polynomial

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0. Introduction

Let X be an r-dimensional Noetherian scheme, proper over an Artinian ring A. Let Y be a closed subscheme of X defined by a sheaf of ideals I, and let $\pi : \overline{X} \to X$ be the blowing-up of X with respect to I.

For sufficiently large n, Ramanujam [Ramanujam, 1973] proved that

$$S_I(n) = \chi(X, \mathcal{O}_X/I^n) = \sum_{i=0}^{r} (-1)^i \operatorname{length}_A H^i(X, \mathcal{O}_X/I^n)$$

is a polynomial in n of degree $\leqslant r$. Moreover, if E is the exceptional divisor of π, the leading coefficient of this polynomial is

$$\operatorname{degree}\left(I \cdot \mathcal{O}_{\overline{X}} \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{O}_E\right)/r! = -\operatorname{degree}(I \cdot \mathcal{O}_{\overline{X}})/r!$$

Every coefficient of $S_I(n)$, except the last one, can be computed in terms of the exceptional divisor E. In this paper, we prove that the last coefficient of $S_I(n)$ is the difference in the arithmetic genus: $\chi(X, \mathcal{O}_X) - \chi(\overline{X}, \mathcal{O}_{\overline{X}})$. By the way we obtain another proof of Ramanujam's result and a calculation of all the coefficients. The precise result is

Theorem

For sufficiently large n

$$S_I(n) = \chi(X, \mathcal{O}_X) - \chi(\overline{X}, \mathcal{O}_{\overline{X}}) - \sum_{i=1}^{r} (-1)^i \chi(E^i)\binom{n}{i}.$$

The intersections E^i are to be taken in the Grothendieck ring $K^{\cdot}(\overline{X})$ of coherent locally free sheaves on $\overline{X}$.

1. Preliminaries

In this section X will denote a Noetherian r-dimensional scheme, proper over an Artinian ring A.

$K_{\cdot}(X)$ (resp. $K^{\cdot}(X)$) will denote the Grothendieck group of coherent (resp. locally free) sheaves on X. The tensor product gives $K^{\cdot}(X)$ a ring structure and turns $K_{\cdot}(X)$ into a $K^{\cdot}(X)$-module.

We will use freely the following results on these groups:

Given a coherent sheaf (resp. locally free coherent sheaf) $\mathscr{M}$, $cl_{\cdot}(\mathscr{M})$ (resp. $cl^{\cdot}(\mathscr{M})$) will denote the class of $\mathscr{M}$ in $K_{\cdot}(X)$ (resp. $K^{\cdot}(X)$). Also, one writes $1 = cl^{\cdot}(\mathscr{O}_X)$, $\xi = cl_{\cdot}(\mathscr{O}_X)$, so obviously

$$cl_{\cdot}(\mathscr{M}) = cl^{\cdot}(\mathscr{M}) \cdot \xi$$

holds for every locally free sheaf $\mathscr{M}$ on X.

$K_n(X)$ is the subgroup of $K_{\cdot}(X)$ generated by sheaves whose support is of dimension $\leqslant n$, that is sheaves that are concentrated in dimension $\leqslant n$.

Theorem 1.1. (SGA 6, X1.1.2)

For every coherent sheaf $\mathscr{M}$ that is concentrated in dimension n, one has:

$$cl_{\cdot}(\mathscr{M}) = \sum_{Y} l(\mathscr{M}_y) \cdot cl_{\cdot}(\mathscr{O}_Y) + Z$$

where the sum is taken over all the irreducible components Y of the support of $\mathscr{M}$ and $l(\mathscr{M}_y)$ is the length of the $\mathscr{O}_y$-module $\mathscr{M}_y$ (y being the generic point of Y) and $Z \in K_{n-1}(X)$.

A simple consequence of this is:

Corollary 1.2.

If $Z \in K_n(X)$ and $\mathscr{L}$ is an invertible sheaf on X, then

$$(1 - cl^{\cdot}(\mathscr{L})) \cdot Z \in K_{n-1}(X)$$

Notation 1.3.

For every effective Cartier divisor D, we will also denote by D the element

$$D = 1 - cl^{\cdot}(\mathscr{O}_X(-D)) \in K^{\cdot}(X)$$

Corollary 1.4.

For every effective Cartier divisor D on X and every $Z \in K_n(X)$ one has:
i) $D \cdot Z \in K_{n-1}(X)$. *In particular* $D^i = 0$ *in* $K_.(X)$ *for all* $i > r$.
ii) $(1 - D)^{-1} \cdot Z = (1 + D + D^2 + \dots + D^r) \cdot Z$.

Notation 1.5.

The linear difference operator

$$\Delta : \boldsymbol{Q}[X] \to \boldsymbol{Q}[X]$$

is defined by $\Delta P(X) = P(X+1) - P(X)$. If $P(X) = a_r X^r + \dots + a_0$ one has $\Delta^r P(X) = r! \cdot a_r$.

2. Hilbert and Samuel functions

Let X be an r-dimensional scheme, proper over an Artinian ring A, and let I be a coherent ideal of $\mathcal{O}_X$.

Definition 2.1.

The Samuel function with respect to I is:

$$S_I(n) = \chi(X, \mathcal{O}_X/I^n) = \sum_{i \geqslant 0} (-1)^i h^i(X, \mathcal{O}_X/I^n); \quad n \in \boldsymbol{N}.$$

The Hilbert function with respect to I is:

$$H_I(n) = \Delta S_I(n) = S_I(n+1) - S_I(n) = \chi(X, I^n/I^{n+1}).$$

Notation 2.2.

If $\pi : \overline{X} \to X$ is the blowing up of X along the ideal I, E will be the exceptional divisor of π, defined by the sheaf of ideals

$$\mathcal{O}_{\overline{X}}(1) = I \cdot \mathcal{O}_{\overline{X}}.$$

Lemma 2.3. (EGA III, 2.2.1)

For n large enough one has:

$$R^i\pi_*(I^n \cdot \mathcal{O}_{\overline{X}}) = 0 \quad \textit{for all } i > 0$$

$$\pi_*(I^n \cdot \mathcal{O}_{\overline{X}}) = I^n$$

Theorem 2.4.

For n large enough, one has

$$S_I(n) = \chi(X, \mathcal{O}_X) - \chi(\overline{X}, \mathcal{O}_{\overline{X}}) - \sum_{i=1}^{r} (-1)^i \cdot \chi(\overline{X}, E^i) \cdot \binom{n}{i}$$

$$H_I(n) = \Delta S_I(n) = \sum_{i=0}^{r-1} (-1)^i \cdot \chi(\overline{X}, E^{i+1}) \cdot \binom{n}{i}$$

where the self-intersections E^i are taken in $K^{\cdot}(X)$.

Proof

By 2.3. one has $\chi(\overline{X}, I^n \cdot \mathcal{O}_{\overline{X}}) = \chi(X, I^n)$ for n large enough. So

$$S_I(n) = \chi(X, \mathcal{O}_X/I^n) = \chi(X, \mathcal{O}_X) - \chi(X, I^n) = \chi(X, \mathcal{O}_X) - \chi(\overline{X}, I^n \cdot \mathcal{O}_{\overline{X}}$$

On the other hand, one has

$$cl^{\cdot}(I \cdot \mathcal{O}_{\overline{X}}) = 1 - E \text{ and } cl^{\cdot}(I^n \cdot \mathcal{O}_{\overline{X}}) = (1 - E)^n$$

Also

$$cl_{\cdot}(I^n \cdot \mathcal{O}_{\overline{X}}) = \left(1 + \sum_{i=1}^{r} (-1)^i \cdot \binom{n}{i} \cdot E^i\right) \cdot \xi \qquad \text{(see 1.4.i))}$$

One concludes by taking the Euler characteristic.

Corollary 2.5. ([Ramanujam, 1973])

Let U be a separated scheme, of dimension r, and of finite type over an Artinian ring A. Let I be a coherent ideal defining a subscheme of U, proper over A. Then, for sufficiently large n, the Hilbert function $H_I(n)$ is a polynomial of degree $\leqslant r - 1$.

Proof

By a result of [Nagata, 1962], there exists an open immersion $U \hookrightarrow X$ where X is proper over A. One concludes by 2.4.

Now let $\mathcal{L}$ be an invertible sheaf on X and $\mathcal{F}$ a coherent sheaf. Then

$$P(n) = \chi(X, \mathcal{F} \otimes \mathcal{L}^n)$$

where $\mathscr{L}^n$ means $\mathscr{L}^{\otimes n}$, is a polynomial function in n of degree less or equal to the dimension of the support of $\mathscr{F}$ (Snapper; see [Kleiman, 1966]).

The degree of an invertible sheaf $\mathscr{L}$ *is defined to be*

$$d_r(X, \mathscr{L}) = \Delta^r \chi(X, \mathscr{L}^n)$$

In particular, with the notations of 2.4. and computing in $K^{\cdot}(X)$, one has

$$(-1)^r \chi(\bar{X}, E^r) = d_r(\bar{X}, I \cdot \mathscr{O}_{\bar{X}}) = -d_{r-1}(E, I \cdot \mathscr{O}_{\bar{X}} \otimes \mathscr{O}_E)$$

So from 2.4. one gets that the leading coefficients of the polynomial $S_I(n)$ and $H_I(n)$, for large n, are

$$-d_r(\bar{X}, I \cdot \mathscr{O}_{\bar{X}})/r! \quad \text{and} \quad -d_r(\bar{X}, I \cdot \mathscr{O}_{\bar{X}})/(r-1)! \text{ respectively.}$$

The above result remains true if $\pi: \bar{X} \to X$ is supposed to be a birational proper map, such that $I \cdot \mathscr{O}_{\bar{X}}$ is principal. The reason for this is that π must map through the blowing up of I:

$$\begin{tikzcd} \bar{X} \arrow[rr, "\pi"] \arrow[dr, "h"'] & & X \\ & X' \arrow[ur] & \end{tikzcd}$$

But for every birational map $h: \bar{X} \to X'$, the degree of an invertible sheaf $\mathscr{L}$ verifies that

$$d_r(X', \mathscr{L}) = d_r(\bar{X}, h^*\mathscr{L})$$

([Kleiman, 1966], I. 2 Prop. 6). Taking $\mathscr{L} = I \cdot \mathscr{O}_{X'}$ we are in the case above.

One can get another version of theorem 2.4. when I is an $\mathfrak{m}_x$-primary ideal, where $\mathfrak{m}_x$ is the maximal ideal corresponding to a closed point $x \in X$. Suppose for a moment that X is projective and $\mathscr{L}$ is an ample invertible sheaf on X. Let $\pi: \bar{X} \to X$ be as in 2.2. Changing $\mathscr{L}$ by $\mathscr{L}^n$, for a convenient n, one can suppose that

$$\bar{\mathscr{L}} = \pi^* \mathscr{L} \otimes \mathscr{O}_{\bar{X}}(1)$$

is very ample on $\bar{X}$ ([Hartshorne, 1977], II.7.10). Let H be an effective Cartier divisor corresponding to $\bar{\mathscr{L}}$ (i.e. an hyperplane section).

Theorem 2.6.

With the above notations and hypotheses, and for sufficiently large n one has

$$H_I(n) = \sum_{i=0}^{r-1} \chi(E \cdot H^i)\binom{n+i-1}{i},$$

$$S_I(n) = \chi(X, \mathcal{O}_{\bar{X}}) - \chi(\bar{X}, \mathcal{O}_{\bar{X}}) + \chi(E) + \sum_{i=1}^{r} \chi(E \cdot H^{i-1})\binom{n+i-2}{i}$$

Proof

By 2.3., for large n, one has

$$H_I(n) = \chi(X, I^n/I^{n+1}) = \chi(\bar{X}, I^n \cdot \mathcal{O}_{\bar{X}}/I^{n+1} \cdot \mathcal{O}_{\bar{X}})$$

One then has, computing in $K_.(\bar{X})$, that

$$cl_.(I^n \cdot \mathcal{O}_{\bar{X}}/I^{n+1} \cdot \mathcal{O}_{\bar{X}}) = (1-E)^n \cdot \xi - (1-E)^{n+1} \cdot \xi = E \cdot (1-E)^n \cdot \xi$$

but

$$(1-E) = cl^{\cdot}(\mathcal{O}_{\bar{X}}(1)) = cl^{\cdot}(\mathcal{L} \otimes \pi^*\mathcal{L}^{-1}) = (1-H)^{-1} \cdot cl^{\cdot}(\pi^*\mathcal{L}^{-1})$$

and

$$E \cdot cl^{\cdot}(\pi^*\mathcal{L}^{-1}) = E$$

So

$$cl_.(I^n \cdot \mathcal{O}_{\bar{X}}/I^{n+1} \cdot \mathcal{O}_{\bar{X}}) = E \cdot (1-H)^{-n} \cdot \xi = E \cdot (1 + H + \dots + H^r)^n \cdot \xi$$

$$= \sum_{i=0}^{r-1} \binom{n+i-1}{i} E \cdot H^i \cdot \xi$$

which gives the first equality.

For the second one, observe that applying the difference operator Δ, the second equality gives the first. So the difference between the two sides of the second equality is constant. But both sides have equal coefficients in degree zero, so they are equal.

Note 2.7.

Observe that $\chi(E \cdot H^i)$ does not depend on the choice of the ample invertible sheaf $\mathcal{L}$.

Note also, that 2.6. gives, in particular, that the multiplicity of the ideal I in $\mathcal{O}_{X,x}$ is the same as the degree of the exceptional divisor E, as is well known. Another result related with 2.6 can be found in [Severi, 1958], p. 71.

3. Applications

The coefficient of degree zero in $S_I(n)$ is the difference in the Euler characteristic when blowing up the ideal I (cf. 2.4). We will use this fact to compute this difference in some examples.

a) Let H be an hypersurface in a smooth ambient Z, proper over a field k. Let ν be the multiplicity of H at a closed point x. If m and $\overline{m}$ are the maximal ideals corresponding to x in Z and H, then there is an exact sequence for $n \geqslant \nu$:

$$0 \to \mathcal{O}_Z(-H) \otimes \mathcal{O}_Z/m^{n-\nu} \to \mathcal{O}_Z/m^{\nu} \to \mathcal{O}_H/\overline{m}^{\nu} \to 0$$

Taking the Euler characteristic one has

$$S_{\overline{m}}(n) = \chi(\mathcal{O}_H/\overline{m}^n) = \chi(\mathcal{O}_Z/m^n) - \chi(\mathcal{O}_Z(-H) \otimes \mathcal{O}_Z/m^{n-\nu})$$

$$= \binom{n+d-1}{d} - \binom{n-\nu+d-1}{d}$$

where d is the dimension of Z at x.

So the coefficient of degree zero in $S_{\overline{m}}(n)$ is

$$(-1)^{d-1}\binom{\nu}{d}$$

To conclude we put all this together in the following

Theorem 3.1.

Let H be an hypersurface of a smooth ambient variety Z, proper over k. Let $\pi : \overline{H} \to H$ be the blowing up with center a closed point x of H. If d is the dimension of Z at x and ν is the multiplicity of H at x then

$$\chi(H, \mathcal{O}_H) - \chi(\overline{H}, \mathcal{O}_{\overline{H}}) = (-1)^{d-1}\binom{\nu}{d}$$

b) Let X be an r-dimensional scheme, proper over an Artinian ring A, and let $I \subseteq \mathcal{O}_X$ be a coherent ideal. By 2.4., $H_I(n)$ is a polynomial function for $n \gg 0$. Denote $P(n)$ this polynomial, then

$$H_I(n) = P(n) \quad \text{for } n \gg 0$$

Theorem 3.2.

With the notation of 2.4.,

$$\chi(X, \mathcal{O}_X) - \chi(\bar{X}, \mathcal{O}_{\bar{X}}) = \sum_{n \geqslant 0} [H_I(n) - P(n)]$$

Proof

Let $P(n) = a_0\binom{n}{r-1} + a_1\binom{n}{r-2} + \ldots + a_{r-1}$. By 2.4., we have

$$S_I(n) = a_0\binom{n}{r} + \ldots + a_{r-1}\binom{n}{1} + \chi(X, \mathcal{O}_X) - \chi(\bar{X}, \mathcal{O}_{\bar{X}}) \quad \text{for } n \gg 0$$

A simple computation gives

$$\sum_{i=0}^{n-1} P(i) = a_0\binom{n}{r} + \ldots + a_{r-1}\binom{n}{1}$$

Therefore

$$S_I(n) = \sum_{i=0}^{n-1} P(i) + \chi(X, \mathcal{O}_X) - \chi(\bar{X}, \mathcal{O}_{\bar{X}}) \quad \text{for } n \gg 0$$

From the definitions, we obtain

$$S_I(n) = \sum_{i=0}^{n-1} H_I(n)$$

and we conclude easily.

Corollary 3.3.

If the Hilbert function $H_I(n)$ is a polynomial for all $n \geqslant 0$, then

$$\chi(X, \mathcal{O}_X) = \chi(\bar{X}, \mathcal{O}_{\bar{X}})$$

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