COMPOSITIO MATHEMATICA

F. CATANESE

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Compositio Mathematica, tome 61, nº 1 (1987), p. 81-102

<http://www.numdam.org/item?id=CM_1987__61_1_81_0>

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AUTOMORPHISMS OF RATIONAL DOUBLE POINTS AND MODULI SPACES OF SURFACES OF GENERAL TYPE

F. Catanese *

§0. Introduction

Let S be a minimal surface of general type (complete and smooth over $\mathbb{C}$), and let $\mathcal{M}(S)$ be the coarse moduli space of complex structures on the oriented topological 4-manifold underlying S.

By a well known theorem of Gieseker, [Gi], $\mathcal{M}(S)$ is a quasi projective variety, and the number $\nu(S)$ of its irreducible components is bounded by a function $\nu_0(K^2, \chi)$ (unfortunately: an *unknown* one) of the two numerical invariants K_S^2, $\chi(\mathcal{O}_S)$.

This paper is the third of a series (cf. [Ca1], [Ca2], [Ca4]), devoted to the study of general properties of $\mathcal{M}(S)$ through a detailed investigation of certain irreducible components corresponding to special classes of simply-connected surfaces which are somehow a generalization of hyperelliptic curves.

Here we study deformations in the large of these surfaces, and the key tool is to exploit the relation holding between such deformations and degenerations of $\mathbb{P}^1 \times \mathbb{P}^1$ to a normal surface with certain rational singularities which we call 1/2-Rational Double Points.

As it is well known, hyperelliptic curves are double covers of $\mathbb{P}^1$ and all the curves are obtained as deformations of hyperelliptic curves; whereas the surfaces we mainly considered in [Ca1], sections 2–4, were certain coverings of degree 4 of $\mathbb{P}^1 \times \mathbb{P}^1$ obtained as deformations of bidouble covers (i.e., Galois covers with group $(\mathbb{Z}/2)^2$) of $\mathbb{P}^1 \times \mathbb{P}^1$ (we propose to call bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$ "bihyperelliptic surfaces").

To be more explicit and precise, first of all a Galois $(\mathbb{Z}/2)^2$-cover is said to be simple if one of the three non trivial transformations in the Galois group has a fixed set of codimension at least 2: for the sake of simplicity we shall consider here only simple bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$.

Simple bihyperelliptic surfaces are obtained by extracting the square roots of two bihomogeneous forms f, g of respective bidegrees $(2a, 2b)$, $(2n, 2m)$.

* Partly supported by M.P.I and C.N.R. (contract No. 82.00170.01).

If V is the vector bundle whose sheaf of sections is $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a, b) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(n, m)$, a simple bihyperelliptic surface is a subvariety of V defined by equations:

$$\begin{cases} z^2 = f(x, y) \\ w^2 = g(x, y), \quad f \text{ and } g \text{ being as above.} \end{cases} \tag{0.1}$$

Their natural deformations are the surfaces defined in V by equations (cf. ibidem, 2.8.)

$$\begin{cases} z^2 = f(x, y) + w\varphi(x, y) \\ w^2 = g(x, y) + z\psi(x, y) \end{cases} \tag{0.2}$$

where φ, ψ are bihomogeneous forms of respective bidegrees $(2a - n, 2b - m)$, $(2n - a, 2m - b)$, and the 4-fold cover of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by (0.2) is said to be admissible if it has only rational double points as singularities.

We denote by $\hat{\mathcal{N}}_{(a,b)(n,m)}$ the subset of the moduli space obtained by considering the smooth surfaces which are the minimal resolutions of the admissible natural deformations (0.2), and by $\mathcal{N}_{(a,b)(n,m)}$ the subset corresponding to smooth natural deformations.

We proved ([Ca1], Theorem 3.8.):

(0.3) $\mathcal{N}_{(a,b)(n,m)}$ is a Zariski open irreducible subset of the moduli space. In particular, the closure $\overline{\mathcal{N}_{(a,b)(n,m)}}$ is irreducible, and contains $\hat{\mathcal{N}}_{(a,b)(n,m)}$.

It has to be pointed out that two such varieties $\hat{\mathcal{N}}_{(a,b)(n,m)}$ and $\hat{\mathcal{N}}_{(a',b')(n',m')}$ coincide only if either the roles of f and g are exchanged, or of x and y, and that they belong to the same moduli space if and only if the invariants K^2 and χ (given by quadratic polynomials in the integers a, b, n, m) coincide (cf. [Ca1]).

We conjecture the closure of $\mathcal{N}_{(a,b)(n,m)}$ to be indeed a connected component of the moduli space and the main object of this paper is to describe $\overline{\mathcal{N}_{(a,b)(n,m)}}$ when $a > 2n$, $m > 2b$ (under these assumptions the polynomials φ, ψ in (0.2) are identically zero and $\hat{\mathcal{N}}_{(a,b)(n,m)}$ consists thus entirely of admissible bihyperelliptic surfaces).

Since $\mathbb{P}^1 \times \mathbb{P}^1$ is a deformation of the rational ruled surfaces $\mathbb{F}_{2k} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2k))$, one can consider also admissible covers of type $(a, b)(n, m)$ of $\mathbb{F}_{2k}$ (cf. [Ca2] for details), and obtain a larger subset of the moduli space.

Our main goal here is to prove (4.3. + 4.4.) the following

THEOREM: *If* $a > 2n$, $m > 2b$, $\overline{\mathcal{N}_{(a,b)(n,m)}}$ *consists of admissible bidouble covers of some* $\mathbb{F}_{2k}$, *with* $k \leqslant \max(b/(a-1), n/(m-1))$.

In particular $\mathcal{N}_{(a,b)(n,m)}$ *is a closed subvariety of the moduli space if* $a \geqslant \max(2n+1, b+2)$, $m \geqslant \max(2b+1, n+2)$.

We remark (cf. [Ca2]) that a result similar to 0.3 holds true if one enlarges the set $\mathcal{N}_{(a,b)(n,m)}$ to include also smooth covers of type $(a, b)(n, m)$ of $\mathbb{F}_{2k}$; therefore, to prove the above conjecture in the case $a > 2n$, $m > 2b$, it would suffice to prove that the moduli space is analytically irreducible at the points corresponding to non smooth bidouble covers.

The study of the closure of $\mathcal{N}_{(a,b)(n,m)}$ is achieved in the following way: if we have a family $S_t \to S_0$ where S_t is a bidouble cover of $\mathbb{P}^1 \times \mathbb{P}^1$, then the canonical model X_0 of S_0 still admits an action by $(\mathbb{Z}/2)^2$ in such a way that the quotient $Z_0 = X_0/(\mathbb{Z}/2)^2$ is a normal surface which is a degeneration of $\mathbb{P}^1 \times \mathbb{P}^1$, and one essentially would like to have that Z_0 be one of the Segre-Hirzebruch surfaces $\mathbb{F}_{2k}$ which realize all the other possible complex structures on $\mathbb{P}^1 \times \mathbb{P}^1$. This being in general false, we have to use the very special fact that the singularities of X_0 are at most R.D.P.'s (Rational Double Points), and that we take the quotient by a group of commuting involutions (an involution being, as in the classical terminology, an automorphism of order exactly 2).

We define therefore a *1/2 R.D.P.* to be the quotient of a R.D.P. by a group of commuting involutions and, after classifying all the finite automorphism groups of R.D.P.'s in §1, we classify (§2) all the possible actions which give a 1/2 R.D.P. as a quotient of a R.D.P., and compute in §3 the Milnor numbers of the smoothings of 1/2 R.D.P.'s.

This preparatory material occupies the first half of the paper and the results stated in this part, albeit probably known to experts and in any case not dificult to obtain, are essential to prove the following result (Thm. 3.5), which we believe to be of independent interest:

(0.4) a proper flat map over the disk, smooth over the punctured disk with fibre $\mathbb{P}^1 \times \mathbb{P}^1$, and with central fibre reduced irreducible and with singularities at worst 1/2 R.D.P.'s, has a central fibre isomorphic to $\mathbb{F}_{2k}$ or to $\mathbb{F}_2$, $\mathbb{F}_4$ with the negative section blown down.

In turn this result, plus the precise description of the actions of commuting involutions on R.D.P.'s, implies the main theorem of the paper.

We suspect that a generalization of thm. 3.5 should hold true under the less restrictive assumption that the central fibre have only rational singularities, and we refer the reader to a recent paper by Badescu for more general results on degenerations of rational surfaces (cf. [Ba]).

It is a pleasure here to thank F. Lazzeri and especially O. Riemenschneider for useful discussions on deformations of singularities; also, I'm indebted to the referees for suggestions to improve the presentation of the paper.

§1. Automorphisms of R.D.P.'s

In this paragraph we consider the following problem: describing all the finite groups of automorphisms acting on a Rational Double Point (R.D.P.).

We lose generality, but gain a simpler exposition, if we assume that we are working over the complex number field $\mathbb{C}$. In this case, a R.D.P. X is a quotient singularity $X=\mathbb{C}^2/G$, where G is a finite subgroup of SL(2, $\mathbb{C}$) (acting linearly).

Consider the quotient morphism $\pi:\mathbb{C}^2\to\mathbb{C}^2/G=X$: since $G\subset$ SL(2, $\mathbb{C}$), $\forall g\in G-\{\mathrm{Id}\}$, 0 is the only fixed point of g, hence π is ramified only at the origin, and $\pi':\mathbb{C}^2-\{0\}\to X-\{x_0\}$ is a normal unramified covering with group G.

Let τ be an automorphism of the germ (X, x_0): since π' is unramified and normal, there do exist exactly $|G|$ liftings $\tilde{\tau}'$ of $\tau|_{X-\{x_0\}}$ which, by the Riemann-Hartogs theorem, extend to automorphisms $\tilde{\tau}$ of the germ ($\mathbb{C}^2$, 0).

Let now H be a subgroup of Aut(X, x_0), and let Γ be the set of liftings $\tilde{\tau}$ of elements τ of H. Clearly (i) Γ is a subgroup of Aut($\mathbb{C}^2$, 0), containing G, (ii) there is a natural exact sequence of groups

$$1\to G\to\Gamma\to H\to 1.$$

We see in particular that Aut(X, x_0) is the quotient, by the subgroup G, of the normalizer of G in Aut($\mathbb{C}^2$, 0). We observe that this last group is very big, since it contains all the generalized homotheties

$$\{(w_1, w_2)\to(f(w_1, w_2)w_1, f(w_1, w_2)w_2)$$

where $f\in\pi^*(\mathcal{O}^*_{X,x_0})$.

In particular the homotheties $(w_1, w_2)\to(\lambda w_1, \lambda w_2)$ ($\lambda\in\mathbb{C}^*$) give a homomorphism of $\mathbb{C}^*\to$ Aut(X, x_0), with kernel of order 2 or 1 according to whether $-\mathrm{Id}\in G$ or not.

We assume, from now on, that H is a finite subgroup of Aut(X, x_0). In this case we can assume that (iii) Γ is a finite subgroup of GL(2, $\mathbb{C}$) (acting linearly).

This follows from a beautifully simple lemma of H. Cartan ([C], p. 97), which has been fruitfully generalized ([K]) to the case of a compact group.

CARTAN'S LEMMA: *If Γ is a finite subgroup of* $\mathrm{Aut}(\mathbb{C}^n, 0)$, *there exists a new system of coordinates* $(z_1, \dots, z_n) = z$ *such that* Γ *acts linearly in the new system of coordinates.*

PROOF: For $\gamma \in \Gamma$, let γ' be its differential at the origin.

Let w be the original system of coordinates in $\mathbb{C}^n$, $r = |G|$ and set

$$z = \frac{1}{r} \sum_{\gamma \in \Gamma} \gamma'^{-1} \gamma w$$

We obtain thus a transformation of $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^n, 0)$ whose differential at the origin is the identity, hence a new system of coordinates. Now, for $g \in \Gamma$, $g(z)$ equals, in the new set of coordinates, $z(gw) = \frac{1}{r} \sum_{\gamma \in \Gamma} \gamma'^{-1} \gamma g w$ and, if we set $\hat{\gamma} = \gamma g$, we have $\gamma'^{-1} = g'(\hat{\gamma}')^{-1}$, hence $z(gw) = \frac{1}{r} \sum_{\hat{\gamma} \in \Gamma} g'(\hat{\gamma}')^{-1} \hat{\gamma} w = g' z(w)$. Q.E.D.

It follows from the above proof that the action of $G \subset \mathrm{SL}(2, \mathbb{C})$ is the same in the new set of coordinates.

Now, if $X = \mathbb{C}^2/G$, X has an algebraic structure as $\mathrm{Spec}(\mathbb{C}[w_1, w_2]^G)$, where the graded ring of G-invariant polynomials on $\mathbb{C}^2$ is generated by homogeneous elements x, y, z, such that there exists $f(x, y, z)$ with $\mathbb{C}[w_1, w_2]^G \cong \mathbb{C}[x, y, z]/f(x, y, z)$. We have therefore the following

COROLLARY 1.1: *If H is a finite subgroup of* $\mathrm{Aut}(X, x_0)$, *H is contained in the group of graded automorphisms of the graded ring* $\mathbb{C}[w_1, w_2]^G = \mathbb{C}[x, y, z]/f(x, y, z)$, *a group that we shall denote by* $\widehat{\mathrm{Aut}}(X, x_0)$. *In aprticular,* $\forall \tau \in H$, $\tau^*(f(x, y, z)) = \lambda f(x, y, z)$ *where* $\lambda \in \mathbb{C}^*$ *is a root of unity.*

Let us give now a list of the R.D.P.'s (see Table 1) in terms of the degrees of the generators of the graded ring $\mathbb{C}[w_1, w_2]^G$ and in terms of

TABLE 1. R.D.P.'s

Name	Equation	Degree of x	of y	of z
E_8	$z^2 + x^3 + y^5$	10	6	15
E_7	$z^2 + x(y^3 + x^2)$	6	4	9
E_6	$z^2 + x^3 + y^4$	4	3	6
D_n $(n \geqslant 4)$	$z^2 + x(y^2 + x^{n-2})$	2	$n-2$	$n-1$
A_n	$z^2 + x^2 + y^{n+1}$	$n+1$	2	$n+1$
A_n	$uv + y^{n+1}$			

the relation $f(x, y, z) = 0$ holding between them, where the last equation follows if we choose as new generators, beyond y,

$$\begin{cases} u = z + \mathrm{i}x \\ v = z - \mathrm{i}x. \end{cases}$$

We recall that we had a $\mathbb{C}^*$ action on $\mathbb{C}[w_1, w_2]^G$ s.t., for $\lambda \in \mathbb{C}^*$, one multiplies a homogeneous element of degree m by λ^m. One notices immediately that this action is faithful iff the G.C.D. of the respective degrees of x, y, z, is 1; i.e., always except in the case of A_n with n odd.

We are now going to determine the group $\widehat{\operatorname{Aut}}(X, x_0)$ of graded automorphisms τ^* of the graded ring $R = \mathbb{C}[x, y, z]/f(x, y, z)$, keeping in mind that the subspaces $R_m = \{$homogeneous elements of degree $m\}$ are invariant subspaces for τ^*, and, in particular, that if there is a generator of strictly minimal degree, then this generator must be an eigenvector for τ^*.

For A_1, obviously, $\widehat{\operatorname{Aut}}(A_1)$ is the quotient of the conformal group $\mathrm{CO}(3, \mathbb{C})$ by $\{\pm I\}$. In the case of A_n, $n \geqslant 2$, consider the following faithful action of $(\mathbb{C}^*)^2$, such that $(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$ acts by

$$\begin{cases} y \to \lambda_1 y \\ u \to \lambda_2 u \\ v \to \lambda_1^{n+1} \cdot \lambda_2^{-1} v. \end{cases}$$

If you add to these transformations the involution which permutes u with v, you get the action of a semidirect product $(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2$ classified by the involution of $(\mathbb{C}^*)^2$ s.t. $(\lambda_1, \lambda_2) \to (\lambda_1, \lambda_1^{n+1} \cdot \lambda_2^{-1})$.

We can now state the main result of this section, where by $\mathbb{C}^*$ we mean the above described $\mathbb{C}^*$ action.

THEOREM 1.2: * *Given a R.D.P. X in the form* $\operatorname{Spec}(\mathbb{C}[x, y, z]/f)$, *the group* $\widehat{\operatorname{Aut}}(X, x_0)$ *has the following structure*:

$$\begin{cases} \mathbb{C}^* & \textit{for } E_8, E_7 \\ \mathbb{C}^* \times \mathbb{Z}/2 & \textit{for } E_6, D_n\ (n \geqslant 5) \\ \mathbb{C}^* \times \mathfrak{S}_3 & \textit{for } D_4 \\ (\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2 & \textit{for } A_n\ (n \geqslant 2). \end{cases}$$

* As noticed by a referee, the quotient of the group $\operatorname{Aut}(\hat{X}, x_0)$ by the connected component of the identity is the group of symmetries of the resolution diagram; indeed our first proof followed this idea.

PROOF: In case of E_7, E_8, it is immediate to see that x, y, z must be eigenvectors for τ^* and moreover, up to multiplying τ with an element of $\mathbb{C}^*$, we can assume $\tau^*(y)=y$, while we denote by λ_x, λ_z the respective eigenvalues of x, z.

Since $\tau^*(f)$ must then equal f, we get that, in the case of E_8, $\lambda_x^3=1$, $\lambda_z^2=1$, hence there exists λ with $\lambda^6=1$ s.t. $\lambda^{-2}=\lambda_x$, $\lambda^3=\lambda_z$, and thus $\tau\in\mathbb{C}^*$. In the case of E_7 we get $\lambda_x=\lambda_x^3=\lambda_z^2$, i.e., $\lambda_x^2=1$, $\lambda_x=\lambda_z^2$; hence there exists $\lambda=\lambda_z$ with $\lambda^4=1$ with $\lambda_x=\lambda^2$, and $\tau\in\mathbb{C}^*$.

The case of E_6: x, y are obviously eigenvectors, while a priori one has $\tau^*(z)=\lambda_z\cdot z+\rho y^2$. Argueing as before we can assume $\tau^*(y)=y$, and then, since

$$\tau^*(f)=y^4+\lambda_x^3\cdot x^3+\lambda_z^2z^2+2\lambda_z\rho zy^2+\rho^2y^4$$

must be a multiple of f, we get $\rho=0$, $\lambda_x^3=1$, $\lambda_z^2=1$. Using again the action of a cubic root of unity in $\mathbb{C}^*$ we can further assume $\lambda_x=1$. What is then left out is the involution $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\to\begin{pmatrix}x\\y\\-z\end{pmatrix}\right.$ and we conclude since the group is now apparently commutative.

DEFINITION 1.3: The involution of a R.D.P. such that $\tau^*(z)=-z$, $\tau^*(x)=x$, $\tau^*(y)=y$ is called *the trivial involution*, since it is defined by the presentation of X as a double cover of $\mathbb{C}^2$ branched on a singular curve. Any involution σ conjugate to τ will also said to be trivial, and will have the property that $X/\sigma\cong(\mathbb{C}^2,0)$.

The case of D_n, $n\geqslant 5$: I claim that x, y, z are eigenvectors. The claim is clear for x, whereas for y, z we must distinguish 2 subcases;

(i) n odd: y is an eigenvector

(ii) n even: z is an eigenvector.

In the case n is odd, $\tau^*(z)=\lambda_zz+\rho x^{(n-1)/2}$, and we argue as in the case of E_6 to infer that $\rho=0$; the case where n is even is treated analogously. We assume then $\tau^*(x)=x$, and we get $\lambda_y^2=\lambda_z^2=1$. The conclusion is that for n odd, as in the case of E_6, our group is the direct product of $\mathbb{C}^*$ and of the cyclic group of order 2 generated by the trivial involution, while for n even the group of order 2 is generated by the involution $(x,y,z)\to(x,-y,z)$.

The case of A_n ($n\geqslant 2$): it is clear that y is an eigenvector, and we can assume (using the $\mathbb{C}^*$ action) that $\tau^*(y)=y$. Since $\tau^*(f)$ must be a multiple of $f=uv+y^{n+1}$, and $\tau^*(f)=\tau^*(u)\tau^*(v)+y^{n+1}$, we have, if n is odd

$$\begin{cases}\tau^*(u)=a_1u+a_2v+\rho_uy^{n+1/2}\\ \tau^*(v)=b_1u+b_2v+\rho_vy^{n+1/2}\end{cases}$$

where the formula makes sense, with $\rho_u = \rho_v = 0$, also in the case when n is even. Looking now at the Taylor expansion of $\tau^*(f)$ we get that $(a_1 u + a_2 v)(b_1 u + b_2 v)$ is a multiple of uv, hence we can assume, using the action of $(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2$, $a_2 = b_1 = 0$, $a_1 = 1$. Then it follows immediately that ρ_u, ρ_v must be zero; moreover, since $\tau^*(f) = b_2 uv + y^{n+1}$ is proportional to f, $b_2 = 1$, what shows that $(\mathbb{C}^*)^2 \rtimes \mathbb{Z}/2 = \widehat{\mathrm{Aut}}(A_n)$.
The case of D_4: We can write f as $f = z^2 + x(y + ix)(y - ix)$ and we notice that, since z has degree 3, z must be an eigenvector. It must then be

$$\begin{cases} \tau^*(x) = a_1 x + a_2 y \\ \tau^*(y) = b_1 x + b_2 y \\ \tau^*(z) = \lambda z \end{cases}$$

Denoting by γ the matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ we see immediately that our sought for group $\widehat{\mathrm{Aut}}(D_4)$ is the following subgroup G' of $\mathrm{GL}(2) \times \mathbb{C}^* : G' = \{(\gamma, \lambda) \mid \gamma^*(x(y^2 + x^2)) = \lambda^2 x(y^2 + x^2)\}$.

The natural homomorphism of GL(2) onto $\mathbb{P}$GL(2) induces a homomorphism $\mathbb{P} : G' \to \mathbb{P}\mathrm{GL}(2)$, whose kernel is the normal subgroup $\left\{\left(\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}, \lambda\right) \mid \mu^3 = \lambda^2\right\}$ which is easily seen to equal $\mathbb{C}^* = \left\{\left(\begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix}, t^3\right) \mid t \in \mathbb{C}^*\right\}$.

The image of $\mathbb{P}$ is the symmetric group $\mathfrak{S}_3$, in fact $\mathbb{P}(\gamma)$ must give a projectivity of $\mathbb{P}^1$ permuting the 3 points $x = 0$, $y = \pm ix$, hence $\mathrm{Im}\,\mathbb{P} \subset \mathfrak{S}_3$; conversely, if $\gamma^*(x(y^2 + x^2))$ is a multiple of $x(y^2 + x^2)$, then one can find $\lambda \in \mathbb{C}^*$ s.t. $(\gamma, \lambda) \in G'$. We have thus gotten an exact sequence $1 \to \mathbb{C}^* \to G' \to \mathcal{S}_3 \to 1$, and we claim that in fact we have a direct product. To see this, let's change coordinates in the $\mathbb{C}^2$ spanned by x and y, where are given three distinct lines.

I view $\mathbb{C}^2$ as the invariant subspace, in $\mathbb{C}^3$, $\mathbb{C}^2 = \{(x_1, x_2, x_3) \mid \sum_{i=1}^{3} x_i = 0\}$ for the permutation representation of $\mathfrak{S}_3$. $\mathfrak{S}_3$ acts on $\mathbb{C}^2$ permuting the 3 lines spanned by the 3 vectors $e_i - e_j$ $(1 \leqslant i < j \leqslant 3)$, which are in fact the locus of zeros of the respective linear forms x_1, x_2, x_3. Therefore the polynomial $P = x(y^2 + x^2)$ becomes, in the new system of coordinates, $P = x_1 x_2 x_3 \pmod{(x_1 + x_2 + x_3)}$, which is left fixed by the action of $\mathfrak{S}_3$ on $\mathbb{C}^2$. Thus, viewing $\mathfrak{S}_3$ as contained in GL(2), we get an embedding of $\mathfrak{S}_3$ in G' simply associating to $g \in \mathfrak{S}_3$ the pair $(g, 1)$. We finally notice that the elements of $\mathbb{C}^*$ are of the form $(t^2\,\mathrm{Id}, t^3)$, hence they commute with $\mathfrak{S}_3$: we conclude then that $G' = \mathbb{C}^* \times \mathfrak{S}_3$. Q.E.D.

REMARK 1.4: SL(2) also goes onto $\mathbb{P}$GL(2), but here the (central) extension

$$1 \to \{\pm \mathrm{Id}\} \to \mathfrak{S}_3' \to \mathfrak{S}_3 \to 1$$

does not split, since a transposition of $\mathfrak{S}_3$ does not lift to SL(2) ($\gamma^2 = \mathrm{Id}$, $\det \gamma = 1 \Rightarrow \gamma = \pm \mathrm{Id}$!). $\mathfrak{S}_3'$ is contained in $\mathbb{Z}/4 \times \mathfrak{S}_3$ and is generated by $x = (i, (1, 2))$, $y = (1, (1, 2, 3))$. $\mathfrak{S}_3'$ is indeed the semidirect product $\mathbb{Z}/3 \rtimes \mathbb{Z}/4$ since $xyx^{-1} = y^2$.

A more important remark is the following: assume H is a finite subgroup of $\mathrm{Aut}(X, x_0)$ where (X, x_0) is a R.D.P., and that we want to describe the singularity $Y = X/H$. Consider then the exact sequence

$$1 \to G \to \Gamma \to H \to 1,$$

and denote by $G' = \Gamma \cap \mathrm{SL}(2, \mathbb{C})$. Since $H' = \Gamma/G' \subset \mathbb{C}^*$ is cyclic, we obtain that $Y = \mathbb{C}^2/\Gamma = X'/H'$ where X' is the R.D.P. $\mathbb{C}^2/G'$, and H' is cyclic.

COROLLARY 1.5: *The singularities that occur as a quotient of a R.D.P. by a finite group are exactly the same that occur as a quotient of a R.D.P. by an automorphism of finite order.*

§2. Quotients of R.D.P.'s by (commuting) involutions

In this paragraph we want to determine the involutions (i.e., automorphisms of order equal to 2) acting on a R.D.P., and, even more, the possible actions of $(\mathbb{Z}/2)^n$ on a R.D.P. We stick to the notation introduced in §1, to call an involution 'trivial' if it is conjugate to the involution $z \to -z$ (here and in the following, the variables which are not mentioned, are to be understood as being left fixed by the involution).

THEOREM 2.1: *The only involution acting on E_7, E_8 is the trivial one. The other R.D.P.'s admit the following non trivial conjugacy classes of involutions*:

(*a*) $y \to -y$ (E_6, D_n, A_{2k+1})

(*b*) $\begin{cases} y \to -y \\ z \to -z \end{cases}$ (E_6, D_n, A_{2k+1})

(*c*) $(u, v, y) \to (-u, v, -y)$ (A_{2n})

(*d*) $\begin{cases} x \to -x \\ z \to -z \end{cases}$ (A_n)

(*e*) $(u, v, y) \to (-u, -v, -y)$ (A_{2k+1})

PROOF: Immediate consequence of theorem 1.

TABLE 2.

Singularity	Type of involution	Quotient
E_6	a	A_2
E_6	b	E_7
D_n	a	A_1
D_n	b	D_{2n-2}
A_{2k+1}	a	A_k
A_{2k+1}	b	D_{k+3}
A_n	d	A_{2n+1}

THEOREM 2.2: *The quotient of a R.D.P. by a non trivial involution not of type* (c); (e), *is again a R.D.P. according to Table* 2. *

PROOF: Let (Y, y_0) be the quotient singularity of (X, x_0). Then the local ring $\mathcal{O}_{Y,y_0}$ is generated by (x, y^2, z) for an involution of type a), by (x, y^2, z^2, yz) for type b), by (x^2, y, z^2, xz) for type d). We set, for convenience, $\eta = y^2$, $\zeta = z^2$, $\xi = x^2$, $u = yz$, $w = xz$.

Since the quotient of $\mathbb{C}^3$ by an involution of type a) is smooth, it suffices in this case to write down the equation of the singularity Y in terms of the coordinates x, η, z: for E_6 we get $z^2 + x^3 + \eta^2$, the equation of A_2, for D_n we get $z^2 + x(\eta + x^{n-2})$, i.e. the equation of A_1 (take new coordinates z, x, $(\eta + x^{n-2})$!), for A_{2k+1} we get $z^2 + x^2 + \eta^{k-1}$, the equation of A_k.

For an involution of type b), the quotient of $\mathbb{C}^3$ is the hypersurface of equation $\eta\zeta = u^2$, which is singular, but the equation of X, written in the (x, η, ζ, u) variables, gives a smooth hypersurface where ζ is a polynomial function of the other 3 variables.

Easy calculations give then the desired result. Q.E.D.

REMARK 2.3: An involution on a smooth point gives either a smooth point or a singularity of type A_1.

Before proceeding to a description of the further singularities occurring as a quotient of a R.D.P. by an involution, let us recall the notion of the Dynkin diagram of a rational singularity (cfr. [A1], [A2]).

Given such a singularity (X, x_0), there exists a minimal resolution of singularities $\pi: S \to X$, which has the property that $\pi^{-1}(x_0)_{\mathrm{red}}$ is a divisor with normal crossings whose components are smooth rational curves with self-intersection $\leqslant -2$.

The Dynkin diagram is an indexed graph whose vertices correspond to the above curves, and are indexed by an integer $-k$ $(k \geqslant 3)$ if the self

* The degree 2 coverings corresponding to the cases b), d) are considered also in [A3] from a different point of view.

intersection of the corresponding curve is $-k$, and whose edges correspond to points of intersections of pairs of curves.

These Dynkin diagrams determine completely the singularity (cf. [B2] 2.12).

REMARK 2.3 bis: One recipe for computing, knowing the Dynkin diagrams for R.D.P.'s, the Dynkin diagram of the quotient singularity $Y = X/\tau$ is the following: τ lifts to an involution σ on S, and, blowing up the isolated fixed points of σ on S, one gets a modification $\tilde{S}$ of S with an involution $\tilde{\sigma}$ acting on $\tilde{S}$ in such a way that the quotient $\tilde{T} = \tilde{S}/\tilde{\sigma}$ is smooth. $\tilde{T}$ is a resolution of Y, and you get the minimal resolution T of Y by blowing down successively all the (smooth rational) curves C with $C^2 = -1$.

THEOREM 2.4: *The quotient B_k of the singularity A_{2k} by an involution of type c) is defined in $\mathbb{C}^4$, with coordinates (u, w, t, η) by the ideal $I_k = (\eta w - t^2,\ uw + t\eta^k,\ ut + \eta^{k+1})$.*

The (reduced) exceptional divisor D of its minimal resolution T has normal crossings, consists of k smooth rational curves, and its Dynkin diagram is

$$\underset{}{\circ} \text{———} \underset{}{\circ} \cdots \underset{}{\circ} \text{———} \underset{}{\circ} \text{———} \underset{-3}{\circ}$$

PROOF: For commodity we assume A_{2k} to be the hypersurface singularity in $\mathbb{C}^3$ of equation $g = uv + y^{2k+1} = 0$, and τ to be the involution

$$\begin{cases} v \to -v \\ y \to -y \end{cases}$$

Therefore the subring of invariants is generated by u, $w = v^2$, $t = vy$, $\eta = y^2$, while the equation $w\eta = t^2$ defines the quotient $Z = \mathbb{C}^3/\tau$.

Consider now a function h on $\mathbb{C}^4$ vanishing on B_k, and let h' be its pull-back to $\mathbb{C}^3$: then h' is a multiple of g, and it is an even function (with respect to the involution τ).

Writing $h' = fg$, we see that f is an odd function, hence f belongs to the ideal (y, v) and there do exist even function a, b such that $h' = a(yg) + b(vg)$. Since $yg = ut + \eta^{k+1}$, $vg = uw + t\eta^k$, this proves that, adding to h a suitable element of the ideal $(ut + \eta^{k+1},\ uw + t\eta^k)$, we get a function h_0 which vanishes on Z, and h_0 is therefore a multiple of $(w\eta - t^2)$. The other assertion follows either from an explicit computation, as indicated in 2.3 bis, or from the fact that (notation as in §1) G is generated by the matrix $\begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}$, where $w = \exp(2\pi i/(2k+1))$, Γ is generated by $\begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, hence B_k is isomorphic to the quotient of $\mathbb{C}^2$ by the cyclic group generated by $\begin{pmatrix} w & 0 \\ 0 & w^{-2} \end{pmatrix}$.

This cyclic group is the group $C_{2k+1,2k-1}$ in Brieskorn's notation ([B 2], page 346), and the Dynkin diagram (ibidem, page 345) can be computed from a corresponding partial fraction. Q.E.D.

THEOREM 2.5: *Let Z be the affine cone over the Veronese surface, i.e. the set of symmetric matrices*

$$\begin{pmatrix} x_1 & x_2 & x_6 \\ x_2 & x_3 & x_4 \\ x_6 & x_4 & x_5 \end{pmatrix}$$

of rank $\leqslant 1$.

Then the quotient Y_{k+1} of the singularity A_{2k+1} by the involution e) is the intersection of Z with the hypersurface $\phi = x_6 - x_3^{k+1} = 0$ (in particular Y_{k+1} can also be defined as the singularity in $\mathbb{C}^5$ defined by the ideal

$$J_k = \big(x_1x_3 - x_2^2,\ x_2x_4 - x_3^{k+2},\ x_3x_5 - x_4^2,\ x_1x_4 - x_2x_3^{k+1},$$
$$x_2x_5 - x_3^{k+1}x_4,\ x_1x_5 - x_3^{2k+2}\big).$$

The exceptional divisor D in the minimal resolution T of Y_{k+1} has normal crossings, consists of $(k+1)$ smooth rational curves, and the associated Dynkin diagram is

$$\underset{-4}{\circ} \qquad \textit{for } k = 0$$

$$\underset{-3}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ \cdots \circ\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\underset{-3}{\circ} \qquad \textit{for } k \geqslant 1$$

REMARK 2.6: The (apparently) funny way of numbering the entries of the symmetric (3×3) matrix can be explained as follows: for $k = 0$ you get $x_6 = x_3$, and one immediately sees that Y_1 is the cone over the rational normal curve of degree 4 in $\mathbb{P}^4$.

PROOF OF THEOREM 2.5: The first part is clear, since the map of $\mathbb{C}^3 \to \mathbb{C}^6$ given by $(u, v, y) \to {}^t(u, v, y)(u, v, y)$ is a quotient map of $\mathbb{C}^3$ by e) and maps onto Z, whereas $\phi = uv - y^{2k+2}$ is an even function, expressible as $x_6 - x_3^{k+1}$ in the coordinates of Z.

We can then argue as in the proof of 2.4, observing that Y_k is, in Brieskorn's notation ([B 2], page 346) the quotient of $\mathbb{C}^2$ by the cyclic group $C_{4k,2k-1}$ (cf. also [R]). Q.E.D.

For later use we classify all the possible actions of $(\mathbb{Z}/2)^2$ on a R.D.P., and the respective quotients.

THEOREM 2.7: *Let* (X, x_0) *be a R.D.P. and let* H *be a subgroup of* $\mathrm{Aut}(X, x_0)$, *isomorphic to* $(\mathbb{Z}/2)^2$. *Then* H *is conjugate to a subgroup listed in the following table.*

PROOF: First of all one can easily check, by the method used in the proof of 2.2, that for the actions of $(\mathbb{Z}/2)^2$ listed in the table, the quotients are as stated.

Since conjugate subgroups give isomorphic quotients and the first two actions are not conjugate, (1) inspection on the table shows that to prove the claim it suffices to check that the number of conjugacy classes of subgroups of $\mathrm{Aut}(X, x_0)$, isomorphic to $(\mathbb{Z}/2)^2$, is one for E_6, D_n, 2 for A_{2k}, 5 for A_{2k+1} $(k \geqslant 1)$, 3 for A_1.

This last assertion follows from Theorem 1, except for the case of A_1, by easy algebraic considerations.

For A_1, we have an exact sequence

$$1 \to \mathbb{C}^* \to \mathrm{CO}(3) \xrightarrow{\phi} \mathbb{P}\mathrm{GL}(2) \to 1$$

Clearly $H \ni$ e) $\Leftrightarrow \phi(H)$ has 2 elements.

In this case, we notice that two involutions in $\mathbb{P}\mathrm{GL}(2)$ are always conjugate.

If $\phi(H)$ contains two commuting involutions, we recall that they are completely determined by their pairs of fixed points, forming a harmonic set, which we can therefore assume to be given by (0, 1), (1, 0), (1, 1), (1, -1) on $\mathbb{P}^1$; in the model of $\mathbb{P}^1$ as a conic in $\mathbb{P}^2$ we can assume that these two pairs be cut by the two lines $\{x = 0\}$ and $\{y = 0\}$, hence that x, y, z be eigenvectors for the action of H. It is now easy to check that, up to conjugation, there are only two cases occurring, according to whether a trivial character of $(\mathbb{Z}/2)^2$ occurs or it does'nt. Q.E.D.

REMARK 2.8: There exists a subgroup $H \cong (\mathbb{Z}/2)^b$, in $\mathrm{Aut}(X, x_0)$, with $b \geqslant 3$, iff $b = 3$ and $X = A_{2k+1}$. Clearly the quotient is then smooth. By Theorem 2.7 we conclude also that the quotient of a R.D.P. by $(\mathbb{Z}/2)^b$ $b = 2, 3$ is again a R.D.P., and that we get new singularities only taking the quotient of a R.D.P. by $\mathbb{Z}/2$.

TABLE 3.

Singularity	Involutions in H	Quotient $X/H = Y$
E_6, D_n, A_{2k+1}	$(x, y, z) \to (x, \pm y, \pm z)$	smooth
A_n	$(x, y, z) \to (\pm x, y, \pm z)$	smooth
A_{2k+1}	a), d), e)	A_{2k+1}
A_{2k+1}	trivial, e) $(x, y, z) \to (-x, -y, z)$	A_1
A_{2k+1}	b), d), $(x, y, z) \to (-x, -y, z)$	D_{2k+4}
A_{2k}	d), the two of type c)	A_{2k}

DEFINITION 2.9: A 1/2 R.D.P. is either a R.D.P., or a singularity obtained by taking the quotient of a R.D.P. by an involution (in view of 2.2–2.5, we are only adding the singularities B_k, Y_k).

§3. Milnor numbers of smoothings of 1/2 R.D.P.'s and a rigidity result

We recall now the notion of a smoothing of an isolated singularity and of the associated Milnor fibre and Milnor number.

Let $(X, 0)$ be an isolated singularity in $\mathbb{C}^n$, let (T, t_0) be an irreducible germ of analytic space, and let $f: \mathscr{X} \to T$ be a deformation of the singularity $(X, 0)$: this means that $\mathscr{X}$ is an analytic subspace of $\mathbb{C}^n \times T$, $\mathscr{X} \supset \{0\} \times T$, and the projection of $\mathbb{C}^n \times T$ onto T induces a flat holomorphic map $f: \mathscr{X} \to T$ with $f^{-1}(t_0) = X$. $\mathscr{X}$ is said to be a *smoothing* of $(X, 0)$ if $f^{-1}(t)$ is smooth for t in an open dense set U of T.

One can choose a (sufficiently small) ball B in $\mathbb{C}^n$, with centre 0, such that, for each t (this is possible, shrinking T if necessary), ∂B intersects $f^{-1}(t)$ transversally in a smooth (real) manifold K_t and, for t in U, $X_t = f^{-1}(t) \cap B$ is called the *Milnor fibre* of the smoothing.

For t in U, $\overline{X}_t$ is a smooth manifold with boundary K_t, moreover $\bigcup_{t \in T} K_t = \mathscr{X} \cap (\partial B \times T)$ is a topologically trivial bundle over T, whereas $\bigcup_{t \in U} \overline{\mathrm{X}}_t = \mathscr{X} \cap (\overline{B} \times U)$ is a differentiable fibre bundle on U, hence, U being connected, the pair $(\overline{X}_t, K_t)$ is, up to diffeomorphism, independent of $t \in U$ (cf. [St], [G-S]).

Now, by a theorem of Grauert ([Gr]), an isolated singularity $(X, 0)$ has a versal family of deformations $g: \hat{\mathscr{X}} \to \mathscr{B}$ hence, corresponding to the irreducible components of the base space $\mathscr{B}$ which give a smoothing of X, one gets several Milnor fibres, and, for any smoothing $f: \mathscr{X} \to T$, the associated Milnor fibre is diffeomorphic to one of them. Let d be the dimension of X: then the *Milnor number* μ of the smoothing $f: \mathscr{X} \to T$ is defined by $\mu = \dim_{\mathbb{R}} H_d(X_t, \mathbb{R})$.

By Poincaré duality $H_d(X_t, \mathbb{R}) \cong H_c^d(X_t, \mathbb{R})$ and, if d is even, on $H_d(X_t, \mathbb{R})$ is therefore defined a symmetric quadratic form q (intersection form) and one can write $\mu = \mu_0 + \mu_+ + \mu_-$, where μ_0 (resp.: μ_+, μ_-) is the number of zero (resp.: positive, negative) eigenvalues for q.

The description of the Milnor fibre (which, being a Stein manifold, has non zero homology only up to its complex dimension) is rather easy if one has a simultaneous resolution, i.e. if, $(S, E) \xrightarrow{\pi} (X, x_0)$ being the minimal resolution of the singularity, $(E = \pi^{-1}(x_0)_{\mathrm{red}})$, one can find

(i) a smoothing $(\mathscr{X}, X) \xrightarrow{f} (T, t_0)$ with $\dim_{\mathbb{C}} T = 1$, $X_t \cong$ given Milnor fibre

(ii) a proper holomorphic map $\pi': \mathscr{S} \to \mathscr{X}$, biholomorphic on $\mathscr{X} - \{x_0\}$, and with $\pi': \pi'^{-1}(X) \to X$ isomorphic to $\pi: S \to X$.

In fact, in this case, X_t is diffeomorphic to a neighbourhood of E in S, and homotopically equivalent to E.

In particular, if (X, x_0) is a surface singularity, μ is the number of components of E, and $\mu = \mu_-$, since the intersection form on S is negative definite, by the theorem of Mumford ([M]).

We shall also make use of the following result of Steenbrink ([St], Thm. 2.24): given a smoothing of a normal surface singularity, $\mu_0 + \mu_+ = \dim_{\mathbf{C}} H^1(\mathcal{O}_S) = p_g$, the geometric genus of the singularity.

Hence rational singularities ($p_g = 0$) are exactly the ones for which the intersection form q is negative definite. Moreover, the singularities considered in this paragraph being rational, we shall only limit ourselves to compute μ for smoothings which do not admit simultaneous resolution, recalling that R.D.P.'s admit a simultaneous resolution by the results of Brieskorn, Tjurina ([B], [T]).

The singularities B_k, Y_{k+1}, considered in theorems 2.4, 2.5, have been studied by O. Riemenschneider ([R]), who proved the following results ([R], theorems 10, p. 234, 12, p. 238, 13, p. 243):

(3.1) The singularity B_k has a versal family with smooth base, giving a smoothing of B_k which admits a simultaneous resolution.

(3.2) The singularity Y_{k+1} has a versal family with base $\mathscr{B}$ consisting of two smooth components T_1, T_2 intersecting transversally. Both components T_1, T_2 give a smoothing of Y_{k+1}, but only T_1 gives one which admits a simultaneous resolution.

(3.3) We want to describe the Milnor fibre of the smoothing corresponding, in (3.2), to the component T_2.

In the notations of theorem 2.5, Z is the cone of symmetric (3×3) matrices of rank $\leqslant 1$, $\phi : \mathbf{C}^6 \to \mathbf{C}$ is the function given by $x_6 - x_3^{k+1}$, $Y_{k+1} = Z \cap \phi^{-1}(0)$, and, since the origin is the only singular point of Z, by Sard's theorem $\phi : (Z, Y_{k+1}) \to (\mathbf{C}, 0)$ gives a smoothing of Y_{k+1}, corresponding to the component T_2 ([R], (63), (64), p. 242, where T_2 is denoted by Σ_2).

In fact the deformation with base T_2 can be easily described as follows: let $i : \mathbf{C}^3 \to \mathbf{C}^3$ the involution (of type e)) such that $i(y_0, y_1, y_2) = (-y_0, -y_1, -y_2)$.

The semiuniversal deformation of the singularity A_{2k+1} is given by

$$\psi(y_0, y_1, y_2, \lambda_0, \ldots, \lambda_{2k}) = y_0 y_2 + y_1^{2k+2} + \sum_{i=0}^{2k} \lambda_i y_1^i = 0$$

The subfamily $\mathscr{X}$ given by $\psi(y_0, y_1, y_2, \lambda_0, \lambda_2, \ldots, \lambda_{2k}) = y_0 y_2 + y_1^{2k+2} + \sum_{i=0}^{k} \lambda_{2i} y_1^{2i} = 0$ is given by a family X_μ of surfaces stable by i, and our family on T_2 is just $\mathscr{X}/i$, and our previous smoothing the restriction to the line $\lambda_2 = \ldots \lambda_{2k} = 0$.

PROPOSITION 3.4: *The Milnor number* μ *of the above smoothing* $\phi: Z \to \mathbb{C}$ *of* Y_{k+1} *equals* k *(notice that the Milnor number of the other smoothing is instead equal to* $(k+1)$*).*

PROOF: Since, with the notations just introduced, $Z = \mathbb{C}^3/i$, and the pull back to $\mathbb{C}^3$ of the function ϕ is just the function $\psi(y_0, y_1, y_2) = y_0 y_2 + y_1^{2k+2}$, we have that $\phi^{-1}(\epsilon) = \psi^{-1}(\epsilon)/i$, moreover, if we intersect the ball of radius δ in $\mathbb{C}^6$ with Z, its pullback to $\mathbb{C}^3$ contains the ball of radius $\sqrt{\delta}$ and is contained in the ball of radius $\sqrt{2\delta}$. Hence the Milnor fibre of ϕ, F_ϵ, is diffeomorphic to the quotient of the Milnor fibre G_ϵ of ψ by the involution i (which has no fixed points on G_ϵ!) Since G_ϵ is a connected 4-manifold homotopically equivalent to a bouquet of $(2k+1)$ S^2's, we obtain

(i) $\pi_1(F_\epsilon) = \mathbb{Z}/2$
(ii) $e(F_\epsilon) = \frac{1}{2}e(G_\epsilon) = k+1$

(where e denotes the topological Euler-Poincaré characteristic). F_ϵ is obviously connected, and, by i), ii), $\mu = b_2(F) = k$. Q.E.D.

After all this preparatory material, we are now in a position to prove a theorem about normal degenerations of $\mathbb{P}^1 \times \mathbb{P}^1$, a theorem which, though it could possibly be given in a more general form, is sufficient for our present purposes.

THEOREM 3.5: *Let* Δ *be the disk of radius* 1 *in* $\mathbb{C}$, $\Delta = \{t \mid |t| < 1\}$, *and let* $f: \mathscr{X} \to \Delta$ *be a proper flat holomorphic map, smooth over* $\Delta^* = \Delta/\{0\}$, *such that moreover*

(i) $\mathscr{X}$ *is a normal* 3*-dimensional complex space*
(ii) $Z_0 = f^{-1}(0)$ *is normal, reduced, and with singularities at most* $1/2$ *R.D.P.'s*
(iii) $Z_t = f^{-1}(t) = \mathbb{P}^1 \times \mathbb{P}^1$ *for* $t \neq 0$.

Then either Z_0 *is smooth, hence a Segre-Hirzebruch surface* $\mathbb{F}_{2m}$, *or* Z_0 *is* $\mathbb{F}_2$ *or* $\mathbb{F}_4$ *with the section of negative self-intersection blown down.*

PROOF: Let $P_1, \ldots, P_r$ be the singular points of Z_0, and let $F_1, \ldots, F_r$ be the Milnor fibres of the smoothing f of Z_0. We clearly have an embedding $i: F \to Z_t \cong \mathbb{P}^1 \times \mathbb{P}^1$, where F is the disjoint union of the F_i's. We recall that our singularities are rational, hence $\mu = \mu_-$, we notice moreover that the homomorphism

$$i_*: H_2(F, \mathbb{Z}) = \overset{\perp}{\bigoplus} H_2(F_i, \mathbb{Z}) \to H_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$$

is an isometry (and thus, in particular, it is injective). But the lattice

$H_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$ is the hyperbolic lattice associated to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, therefore

(I) the rank of $H_2(F, \mathbb{Z})$ is at most 1, and the intersection form is even.

Since the intersection form is not even for the B_k's, we can immediately exclude that Z_0 contains singular points of type B_k.

Hence (I) implies that Z_0 contains at most one singular point whose associated smoothing has Milnor number 1 and, in view of our classification of 1/2 R.D.P.'s and of their smoothings, there are only 3 possibilities for this singular point, i.e.

(1) Y_2, with a smoothing not admitting simultaneous resolution

(2) A_1

(3) Y_1, with a smoothing admitting simultaneous resolution.

Moreover, again by (I) and the above remark, the singular points of Z_0 with smoothing of Milnor number zero can a priori only be of type Y_1, with smoothing not admitting simultaneous resolution. The next lemma shows that there is at most one singular point, with Milnor number 1, and that case (1) does not occur.

LEMMA 3.6: *The smoothings which do not admit simultaneous resolution don't occur.*

PROOF OF THE LEMMA: Let P be a singular point of type Y_1, or Y_2, where the smoothing does not admit a simultaneous resolution.

Though the second case can be treated is the same way as the first, it is quicker to show that the intersection form is odd. In the notations of prop. 3.4 let $\pi: G_\epsilon \to F_\epsilon$ be the quotient map by the involution i.

$H_2(G_\epsilon, \mathbb{Z})$ is a lattice with intersection form $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$, and if e_j is one of the two generators, we know that

$$2\pi_*(e_j)^2 = (\pi^*\pi_* e_j)^2 = (e_j + i_*(e_j))^2 = -2 + 2e_j \cdot i_* e_j - 2$$

since i_* is an isometry. Hence if $\pi_*(e_j)^2$ is even, $i_* e_j$ is a vector with square (-2), and even scalar product with e_j, and then it must be $\pm e_j$; since $(i_* e_1)(i_* e_2) = e_1 \cdot e_2$, i_* should then be $\pm$Identity, against the fact that $b_1(F_\epsilon) = 1$.

We can thus assume that, locally around p, $\mathscr{X}$ be isomorphic to a neighbourhood of the origin in the cone Z over the Veronese surface, and, by Riemennschneider's result, that f is given, in local coordinates, by $x_6 - x_3^{k+1}$ ($k = 0$, but we like to see what happens in general).

Performing now a blow-up with centre p, we get $\pi: \tilde{\mathscr{X}} \to \mathscr{X}$, where $\tilde{\mathscr{X}}$ is a smooth 3-fold which is an open neighbourhood of the zero section in the total space of the line bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$.

To read out $\tilde{f} = f \circ \pi$ on $\tilde{\mathscr{X}}$, we consider coordinates (y_0, y_1, y_2, z), weighted homogeneous of respective weights $(1, 1, 1, -2)$ (the projection of $\tilde{\mathscr{Z}}$ to $\mathbb{P}^2$ being given by (y_0, y_1, y_2)). $\tilde{f}$ is the function (homogeneous of weight zero) given by

$$\tilde{f} = y_0 y_2 z - \left(y_1^2 z\right)^{k+1} = z\left(y_0 y_2 - z^k y_1^{2k+2}\right). \tag{3.7}$$

On the other hand π is given by $(y_0, y_1, y_2, z) \to (zy_i y_j)$. Then $\tilde{f}^{-1}(0)$ consists of $\mathbb{P}^2$, of equation $z = 0$, and of another component W. If $k = 0$, W intersects $\mathbb{P}^2$ in the conic $(y_0 y_2 - y_1^2) = 0$, if $k \geqslant 1$ $\mathbb{P}^2 \cap W = \{y_0 y_2 = 0\}$.

Clearly W is smooth iff $k \leqslant 1$ (in fact the only singular point is $y_0 = y_2 = 0$, where the local equation is $(y_0 y_2 - z^k) = 0$, i.e., a point of type A_{k-1}).

Therefore, if we blow-up all the singular points whose smoothing has $\mu = 0$, say $p_1, \ldots, p_r$, we obtain $\tilde{f}: \tilde{\mathscr{Z}} \to \Delta$, where $\tilde{\mathscr{Z}}$ has at most one singular point p, and $\tilde{\mathscr{Z}} = W \cup \mathbb{P}_1^2 \cup \ldots \cup \mathbb{P}_r^2$, where W has at most a singular point, in p.

Consider now the restriction $f^* : \mathscr{Z}^* \to \Delta^*$: there exists an effective divisor $\mathscr{L}^*$ s.t. $\mathscr{L}^* \mid_{Z_t}$ is of type (1,1). In fact the monodromy automorphism, being holomorphic, leaves the class (1,1) invariant, and the Leray spectral sequence tells us that the associated invertible sheaf has a section on $\mathscr{Z}^*$.

Let $\mathscr{L}$ be the closure of $\mathscr{L}^*$, let further $\omega_{\tilde{\mathscr{Z}}}$ be the dualizing sheaf on $\tilde{\mathscr{Z}}$, and $K_{\tilde{\mathscr{Z}}}$ the associated Weil divisor. Since, for $t \neq 0$, $K_{\tilde{\mathscr{Z}} \mid \mathscr{Z}_t} \equiv K_{Z_t}$, we have

$$-K_{\tilde{\mathscr{Z}}} \equiv 2\mathscr{L} + B, \tag{3.8}$$

where B is a divisor supported on the special fibre $\tilde{Z}_0$. Let's finally consider a component $\mathbb{P}^2 = \mathbb{P}_i^2$ of $\tilde{Z}_0$: by the adjunction formula $K_{\mathbb{P}^2} \equiv (K_{\tilde{\mathscr{Z}}} + \mathbb{P}^2)_{\mid \mathbb{P}^2}$ and since $\tilde{Z}_0 \equiv 0$, we can write

$$K_{\mathbb{P}^2} \equiv (-2\mathscr{L} + tW) \mid_{\mathbb{P}^2}, \quad \text{for a suitable integer } t.$$

This is however a contradiction, since $W \mid_{\mathbb{P}^2}$ is a conic, hence the right term is divisible by 2. Q.E.D. for Lemma 3.6.

I'm now in the situation where $f: \mathscr{Z} \to \Delta$ is such that Z_0 has at most 1 singular point p, either of type A_1, or of type Y_1 where f is a smoothing admitting a simultaneous resolution. Hence, by a base change $\Delta' \to \Delta$, I have $f' : \mathscr{Z}' \to \Delta'$ with $\mathscr{Z}_0'$ a minimal resolution of Z_0.

Then, since it is well known that $\mathscr{Z}_0' \cong \mathbb{F}_{2m}$, moreover the exceptional curve has self-intersection either (-2) or (-4), while on $\mathbb{F}_{2m}$ there is only one curve with negative selfintersection, equal to $-2m$, we conclude that if p is singular Z_0 is $\mathbb{F}_2$ or $\mathbb{F}_4$ with the negative section blown down (i.e., Z_0 is the projective cone over either a conic or over a rational normal curve of degree 4). Q.E.D.

§4. Limits of bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$

As in [Ca1], we define a bidouble cover $\pi: S \to X$ to be a finite Galos cover with group $(\mathbb{Z}/2)^2$, moreover we shall say that the bidouble cover is smooth if S, X are smooth.

THEOREM 4.1: *Assume that $f: \mathscr{S} \to \Delta$ is a smooth proper 1-dimensional family, such that for $t \in \Delta^* = \Delta - \{t_0\}$ $S_t = f^{-1}(t)$ is a smooth minimal bidouble cover of $\mathbb{P}^1 \times \mathbb{P}^1$ of general type. Then S_0 is the minimal resolution of a bidouble cover of $\mathbb{F}_{2m}$ with at most R.D.P.'s as singularities.*

PROOF: Let $\mathscr{X} \xrightarrow{g} \Delta$ be the corresponding family of canonical models (the fibres of g have only R.D.P.'s as singularities). Then $(\mathbb{Z}/2)^2$ acts biregularly on $\mathscr{X}$, and preserving the fibres of g (cf., e.g. [Ca2] Thm. 1.8.) (though, cf. [B-R], $(\mathbb{Z}/2)^2$ does not necessarily act biregularly on $\mathscr{S}$!).

Let's denote by $\mathscr{Z}$ the quotient $\mathscr{Z} = \mathscr{X}/(\mathbb{Z}/2)^2$, and by π the quotient-map. Then we have $f: \mathscr{Z} \to \Delta$ with $Z_t \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $t \neq t_0$, and the hypotheses of Theorem 3.5 are clearly satisfied.

If Z_0 is smooth, then there is nothing to prove, since S_0 is minimal too (by the stability of exceptional curves of the I kind, cf. [Ko]).

Assume that Z_0 has a singular point p. We claim that it cannot be a singular point of type Y_1, quotient of a singular point q of type A_1 by the involution (e). In fact, if F_ϵ is the Milnor fibre of the smoothing f of p, $\pi^{-1}(F_\epsilon)$ is an unramified double cover (we are restricting ourselves to a neighbourhood of q) of F_ϵ, which therefore must be disconnected, since F_ϵ is homotopically equivalent to S^2. This is a contradiction, since, by arguments similar to those used in the proof of 3.4., we see that $\pi^{-1}(F_\epsilon)$ is diffeomorphic to the Milnor fibre of the smoothing of q, which is connected (note that the essential fact which is being used is that if $\mathbb{Z}/2$ preserves the fibres, there can be no simultaneous resolution downstairs).

Therefore p can only be a singular point of type A_1. Assume again that $\pi(q) = p$: we claim that q cannot be a smooth point (the quotient being given by an involution having q as an isolated fixed point on X_0).

In fact, in this case, q would be a smooth point of $\mathscr{X}$, and the involution, since it preserves the fibres of g, would have a 1-dimensional fixed locus in $\mathscr{X}$ near q, hence also Z_t would be singular for $t \neq t_0$.

Second, we claim that, if $\pi(q) = p$, the isotropy subgroup of q cannot be the whole of $(\mathbb{Z}/2)^2$.

In fact, cf. Theorem 2.7., the only possibility is to have an action of the fourth type (since, for $k = 0$, the action of the third type is a conjugate one).

In this case the same argument as before applies, since there is a trivial involution σ acting on q, hence $\mathscr{X}/\sigma \to \mathscr{Z}$ is such that p is the quotient of a smooth point, contradicting the nonsingularity of Z_t for $t \neq t_0$.

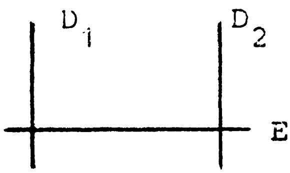
So the only remaining possibility is that above p lie two points q_1, q_2, which are (both) either of type A_3 or of type D_n, and are fixed by an involution of type (a) (cf. Thm. 2.1.).

We have a diagram

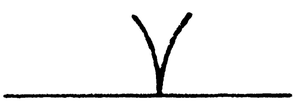
$$\begin{tikzcd} S_0 \arrow[r, dashed, "\tilde{\pi}"] \arrow[d, "\tau"] & \mathbb{F}_2 \arrow[d, "\sigma"] \\ X_0 \arrow[r, "\pi"] & Z_0 \end{tikzcd}$$

and we have to check only that $\tilde{\pi}$ is indeed a morphism. In fact, locally around p, X_0 is obtained from Z_0 by taking the square root y of η, where the local equation of Z_0 is either $z^2 + x^2 + \eta^2 = 0$ or $z^2 + x(\eta + x^{n-2}) = 0$. On the blow-up $\mathbb{F}_2$ of Z_0 the divisor σ^* $(\mathrm{div}(\eta))$ is of the form $E + D$, where E is the exceptional curve of σ, and $D \cdot E = 2$, and, moreover, in the first case $D = D_1 + D_2$ with the D_i's crossing E transversally in two distinct points, in the second case D has a double point of type $(v^2 - x^{n-1}) = 0$.

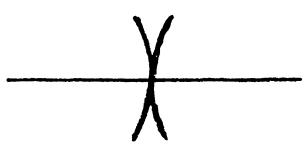
Case 1:



Case 2: (n even)



Case 2: (n odd)



The double covering W of a neighbourhood of E, branched on $E + D$, has 2 A_1 singularities, respectively 1 D_{n-1} singularity ($D_3 = A_3$), and it is easy now to see that the minimal resolution of W equals the minimal resolution of (X_0, q_i) (cf. [B-P-V], III §7). Q.E.D.

Remark 4.2: We have given some indication about the proof that $\tilde{\pi}$ is a morphism because we wanted to single out an important piece of information; i.e., in the case when $X_0/(\mathbb{Z}/2)^2$ is a quadratic cone, hence S_0 can be gotten as the minimal resolution of a Galois cover W_0 of $\mathbb{F}_2$, we know that, B_1, B_2, B_3 being the 3 branch loci of the 3 non-trivial involutions of $(\mathbb{Z}/2)^2$, $B_1 = E + D$, with $E \cdot D = 2$, moreover $E \cdot B_2 = E \cdot$

$B_3 = 0$. It follows then immediately that, σ_0 and f ($\sigma_0^2 = 2$, $f^2 = 0$, $\sigma_0 \cdot f = 1$) being the standard basis for Pic($\mathbb{F}_2$), since $E \cdot \sigma_0 = 0$, $E \cdot f = 1$, $B_2 \equiv d_2\sigma_0$, $B_3 \equiv d_3\sigma_0$, $D \equiv 2f + (d_1 - 1)\sigma_0$, so that, finally, $\forall i\ B_i \equiv d_i\sigma_0$. Hence S_t, for $t \neq 0$, is a bidouble cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched on 3 divisor of respective types (d_1, d_1), (d_2, d_2), (d_3, d_3). This allows us to prove the desired result

THEOREM 4.3: *Let $\mathcal{N}_{(a,b)(n,m)}$ be the family of smooth simple bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$ branched on 2 curves of resp. bidegrees $(2a, 2b)$, $(2n, 2m)$, and assume that $a > 2n$, $m > 2b$. Then the closure $\overline{\mathcal{N}_{(a,b)(n,m)}}$ consists of surfaces whose canonical model is a simple bidouble cover of some $\mathbb{F}_{2k}$, with $k \leqslant \max\left(\dfrac{b}{a-1}, \dfrac{n}{m-1}\right)$.*

PROOF: Let X_0 be a canonical a model of a surface whose isomorphism class belongs to $\overline{\mathcal{N}_{(a,b)(n,m)}}$: then we can find a family $\mathscr{X} \xrightarrow{g} \Delta$ as in theorem 4.1. s.t. X_t, for $t \neq 0$, is a smooth simple bidouble cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched on two curves of resp. bidegrees $(2a, 2b)$, $(2n, 2m)$, B_t^1, B_t^2. Then the minimal resolution of X_0 equals, by Theorem 4.1, the minimal resolution of a simple bidouble cover of some $\mathbb{F}_{2k}$ branched on two divisors B_0^1, B_0^2, such that either

$$\begin{cases} B_0^1 \equiv 2(a\sigma_0 + (b - ak)f) \\ B_0^2 \equiv 2(n\sigma_0 + (m - nk)f) \end{cases}$$

or

$$\begin{cases} B_0^1 \equiv 2(b\sigma_0 + (a - bk)f) \\ B_0^2 \equiv 2(m\sigma_0 + (n - mk)f) \end{cases}$$

(cf. e.g. [Ca2], §2). Since B_0^1, B_0^2 are effective and reduced, in the first case one must have $b \geqslant (a-1)k$ (equality holds iff the section at infinity σ_0 is contained in B_0^1: since then B_0^1 and B_0^2 have no common component, it must be $m \geqslant nk$).

Anyhow, we remark that we have the following inequalities: $\dfrac{m}{2} \geqslant b \geqslant (a-1)k \geqslant 2nk \Rightarrow m \geqslant 4nk$, therefore in our range $b \geqslant (a-1)k$ is the only inequality which has to hold effectively.

In the second case, a symmetrical argument gives $k \leqslant \dfrac{n}{m-1}$.

Q.E.D.

COROLLARY 4.4: *The family $\hat{\mathcal{N}}_{(a,b)(n,m)}$ of simple bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$ with branch curves of bidegrees $(2a, 2b)(2n, 2m)$, and with at most R.D.P.'s as singularities, when $\begin{cases} a > 2n \\ a - 1 > b \end{cases}$, $\begin{cases} m > 2b \\ m - 1 > n \end{cases}$, is a closed component of the moduli space.*

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(Oblatum 21-VI-1985 & 11-VIII-1986)

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