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ALGORITHMIC ORBIT CLASSIFICATION FOR SOME BOREL GROUP ACTIONS

Hartmut Bürgstein and Wim H. Hesselink *

1. Introduction

1.1. Conjugacy classes of nilpotent matrices

A square matrix A of order n is said to be nilpotent if $A^r = 0$ for some positive integer r, or equivalently if $A^n = 0$. The group $GL(n)$ of the invertible matrices S of order n acts by conjugation on the set N of the nilpotent matrices. In fact, if A is nilpotent, then SAS^{-1} is also nilpotent. Every nilpotent matrix A has a Jordan canonical form $J = SAS^{-1}$ with the eigenvalues zero on the main diagonal. The Jordan matrix shown here has three Jordan blocks with the sizes

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3, 2, 1, respectively. In general, the sizes of the Jordan blocks form a partition $n = \lambda_1 + \cdots + \lambda_r$ of the order n, and we may assume that $\lambda_1 \geqslant \cdots \geqslant \lambda_r > 0$. Two nilpotent matrices A and B of order n are conjugate if and only if they have the same Jordan matrix, or equivalently the same partition λ. So the $GL(n)$-orbits in the set N are characterized by a discrete invariant, the partition λ.

1.2. Conjugacy classes of strictly upper triangular matrices

Let V be the space of the strictly upper triangular matrices of order n ("strictly" means: with zeros on the main diagonal). If $A \in V$, then A is nilpotent, so by 1.1 there is an invertible matrix S such that $J = SAS^{-1}$ is a nilpotent Jordan matrix which satisfies $J \in V$. The conjugating matrix S however need not be upper triangular. In fact, it seems more

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natural only to consider conjugating matrices belonging to the subgroup

$$G = \{ S \in GL(n) \mid \forall A \in V: SAS^{-1} \in V \}.$$

This groups G consists of the invertible upper triangular matrices. So one might ask: if $A \in V$, does there exist $S \in G$ such that SAS^{-1} is a Jordan matrix? The answer is negative. In fact, it suffices to consider the following counterexample with matrices of order 3.

$$\text{If } A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and } S = \begin{pmatrix} M & K & K \\ 0 & M & K \\ 0 & 0 & M \end{pmatrix},$$

$$\text{then } SAS^{-1} = \begin{pmatrix} 0 & 0 & M \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, each symbol M stands for some non-zero coefficient and each symbol K stands for an arbitrary coefficient. This negative result is the starting point of this paper.

Inital problem: Give normal forms for the G-orbits in V.

This problem was suggested to the first author by W. Borho in 1981. At that time, inspired by [15], the second author was working on the related problem of the description of the irreducible components of the intersection $C \cap V$ where C is a $GL(n)$-orbit in N. In the summer of 1982 the first author solved the above problem for matrices of order $n \leqslant 6$. Then the second author obtained a geometric classification that partitions the set V into finitely many G-invariant classes, cf. [10]. If $n \leqslant 5$ the classes are precisely the orbits. In the fall of 1983 we worked together for six weeks and obtained the results presented here.

1.3. Aspects of the solution

Our solution consists of two parts:

(a) a reduction to the combinatorics of weights and roots,
(b) a computer algorithm for the combinatorial problem.

Informally, the reduction (a) may be described as follows. The action of the group G is reduced to a kind of chess playing. The board corresponds to the set of weights of the G-module V with respect to a maximal torus T of G. The position on the board represents the support set of an element $v \in V$. The moves represent the action of the one-parameter subgroups of G normalized by T.

In the reduction (a), some information concerning the action is neglected. The algorithm (b) is also not conclusive. So, our method yields an approximate solution, together with markings for the classes which are not proved to be normal forms. In fact, a genuine solution of the initial problem is obtained in the cases $n \leqslant 7$. In the case $n = 7$ the algorithm yields 1415 orbits and 15 one-parameter families of orbits. This result is the case A6R of table 1 in section 6.2 below. In the case $n = 8$ the algorithm gives 8302 orbits, 190 parametrized families of orbits, and 8 unresolved classes (case A7R of the same table). In this case each of the unresolved classes is represented by a one-parameter or two-parameter family of representatives. In principle such an unresolved class could be analysed by hand.

The use of a computer may be justified in several ways. First, the algorithm (b) consists of lots of trivial verifications, so that human calculators tend to make mistakes, see 6.6(a) below. Secondly, the computer enabled us quick tests of the various possible reductions (a). Finally, as we have seen just now, the number of classes obtained can be rather large.

1.4. The wider context

From the beginning, we had in mind to analyse a more general situation. Let H be a reductive linear algebraic group over an algebraically closed field K, cf. [3]. Let B be the homogeneous variety of the Borel subgroups of H. If $x \in \boldsymbol{h}$, let B_x be the closed subset of the Borel groups which have x in the Lie algebra. If x is nilpotent, then B_x is connected (Tits), but not necessarily irreducible. In 1970, Brieskorn showed that the variety N of all nilpotent elements of $\boldsymbol{h}$ has a rational surface singularity at a generic point x of the singular locus, cf. [6]. In that case, B_x is the exceptional divisor, a so-called Dynkin curve. In 1976, Springer constructed the irreducible representations of the Weyl group of H in the top cohomology groups of the varieties B_x with $x \in N$, cf. [17] and [18]. Spaltenstein made an extensive analysis of the irreducible components of B_x, cf. [16]. Let us here define the *component configuration* of a variety X to be the smallest set S which contains as members the irreducible components of X, and the irreducible components of all intersections $X_1 \cap X_2$ with X_1, $X_2 \in S$. In view of [11] 6.3, it seems important to consider

Problem B: If $x \in N$, describe the component configuration of B_x.

The variety B of the Borel groups of H may be identified with the quotient H/G where G is a fixed Borel group. Let $\boldsymbol{u}$ be the Lie algebra of the unipotent radical U of G. Up to the action of a finite group, the component configuration of B_x is isomorphic to the component configuration of $X = \boldsymbol{u} \cap Hx$ where Hx is the orbit of x under the adjoint action of H. Every member of the component configuration of X is invariant under the adjoint action of the Borel group G, and hence equal to a union of G-orbits in $\boldsymbol{u}$. This suggests

Problem R: Describe the orbit structure of the G-module $\boldsymbol{u}$.

If $H = GL(n)$, then problem R reduces to the initial problem of section 1.2. In view of Kirillov's method of orbits, cf. [12] §15 and [5], it is also relevant to investigate the orbit structure in the dual module $\boldsymbol{u}^*$. Then we should get a confirmation of the even-dimensionality of the U-orbits in $\boldsymbol{u}^*$, cf. [12] §15, thm. 1., and a confirmation of Pyasetskii's theorem which implies that the sets of G-orbits in $\boldsymbol{u}$ and $\boldsymbol{u}^*$ have equal cardinality, cf. [14]. So we are interested in

Problem RC: Describe the orbit structures of $\boldsymbol{u}$ and $\boldsymbol{u}^*$ under the actions of G, and of U.

In this paper we concentrate on problem RC. An attack of problem B along the present lines would require knowledge of the inclusion relations between closures of G-orbits in $\boldsymbol{u}$, which is not yet available.

1.5. Sheets and systems of representing sections

The geometric aspects of our algorithm can be formulated in a very general setting. Let G be a connected linear algebraic group over an algebraically closed field K. Let G act on a connected variety V. Our problem is to describe the orbit structure of the G-variety V. Since the number of orbits in V may be infinite, we may need some kind of parametrized families of orbits. The most satisfactory description of such an orbit structure is phrased in terms of sheets, cf. [4]. An explicit description of the sheets however, seems to require cross sections of the sheets. A priori, the sheets are unknown. So, we introduce the weaker concept of sections.

DEFINITION: A locally closed irreducible subset C of the G-variety V is called a *section*, if the product set GC is locally closed in V and the multiplication morphism from $G \times C$ to GC is flat. A section C is said to be *final* if we have $Gx \cap C = \{x\}$ for every point $x \in C$. Then all orbits in GC have dimension $\dim(GC) - \dim(C)$, so that GC is contained in some sheet of V.

A finite family of sections $(C_i)_{i \in I}$ is called a *system of representing sections*, if V is the disjoint union of the sets GC_i. Such a system is considered as an orbit classification, if moreover all members C_i are final. In fact, then every orbit in V meets the disjoint union of the sets C_i in exactly one point.

Our algorithm consists of a kind of stepwise refinement of a system of representing sections. The initial system only consists of the section $C = V$ itself. There are two refinement operations: splitting a section and contracting a section. After finitely many steps we hope to accomplish that all sections are final.

1.6. Conditions on the group G and on the G-variety V

We shall use one-parameter subgroups of G to replace certain sections in the G-variety V by smaller ones. So we need some control over the action of these subgroups on V. On the other hand, every proof of the finality of certain sections will require control over all elements of G. So it is natural to impose conditions of the following type. The group G should be directly spanned by a maximal torus T and certain one-dimensional unipotent subgroups U_α. The G-variety V should be a G-module, and all weight spaces V_λ of V with respect to T should be one-dimensional. In fact, we shall assume a little bit more, namely that the group G is simply solvable and that V is a simply weighted G-module. These concepts are introduced as follows.

Let T be a maximal torus of G. Let the Lie algebra $\boldsymbol{g}$ of G have the root space decomposition

$$\boldsymbol{g} = \boldsymbol{g}^T \oplus \sum_{\alpha \in R} \boldsymbol{g}_\alpha$$

The group G is said to be *simply solvable*, if

(a) G is connected and solvable.
(b) $\boldsymbol{g}^T$ is equal to the Lie algebra $\boldsymbol{t}$ of T.
(c) $\dim(\boldsymbol{g}_\alpha) = 1$ for every root $a \in R$.
(d) Every pair of different roots α, $\beta \in R$ is linearly independent.

The general definition of a simply weighted G-module is postponed to 3.2. Here we assume for simplicity that $\mathrm{char}(K) = 0$. Let V be a G-module with the weight space decomposition

$$V = \sum_{\lambda \in S} V_\lambda.$$

Then we have that V is simply weighted, if and only if
a) $\dim(V_\lambda) = 1$ for every weight $\lambda \in S$.
b) If $\lambda \in S$ and $\alpha \in R$ are such that $\lambda + \alpha \in S$, then $\boldsymbol{g}_\alpha V_\lambda = V_{\lambda+\alpha}$.

Examples: In view of 1.4, the main example is as follows. Let G be a Borel subgroup of a reductive linear algebraic group H. Then G is simply solvable. Let $\boldsymbol{u}$ be the Lie algebra of the unipotent radical U of G. If char$(K)=0$, then $\boldsymbol{u}$ and its dual module $\boldsymbol{u}^*$ are simply weighted. In fact, the condition on the characteristic can be weakened considerably, cf. 3.4.

1.7. The game and the strategy

In Chapter 4, the reduction to the combinatorics of weights and roots, cf. 1.3(a), is obtained under the assumptions that the group G is simply solvable and that V is a simply weighted G-module. The representing sections in V are described in terms of the weight spaces V_λ. The main mathematical results are the Theorems 4.3 and 4.7. These theorems are hardly interesting from the mathematical point of view, but they yield conditions which a computer can verify. In fact, these results enable us to perform stepwise refinement of systems of representing sections, cf. 1.5. This gives us the moves of the game of 1.3.

Our strategy in this game is described in Chapter 5. We use backtracking to visit all members of the resulting system of representing sections. In the search for optimal refinements, our algorithm is greedy, cf. [1] 10.3. In fact, we only use the moderate form of backtracking described in Theorem 4.3. In exceptional cases, even this moderate form of backtracking leads to virtually unbounded search which has to be checked by a time limit, cf. Remark 5.3.

1.8. Application of the algorithm

The algorithm was devised for the problem RC described in 1.4. The degrees of sophistication of the mathematics and of the computer algorithm are also tuned to this problem. Therefore we have refrained from trying other applications.

So, for each simple algebraic group H we fix a Borel subgroup G with unipotent radical U, and its Lie algebra $\boldsymbol{u}$. We have two tasks:

R: describe the G-orbit structure of $\boldsymbol{u}$.
C: describe the G-orbit structure of $\boldsymbol{u}^*$.

In fact, the U-orbit structure is easily reconstructed from the G-orbit structure. We have applied the algorithm to the simple groups H of the types $A_2, \ldots, A_7$, B_2, B_3, B_4, C_3, C_4, D_4, D_5, F_4, G_2. The results are discussed in Chapter 6. Let it here suffice to mention that a complete orbit classification is obtained for the cases $A_2, \ldots, A_6$, B_2, B_3, C_3, G_2, and for the coradical cases A_7C and C_4C. In fact, the algorithm works better and faster for the coradical $\boldsymbol{u}^*$ than for the radical $\boldsymbol{u}$.

2. A geometric algorithm for orbit classification

2.1. Our main references for algebraic groups are [3] and [19]. We also need some isolated results from the theory of flat morphisms, cf. [2] Chap. V, [13] pp. 424–442, or [9].

Let G be a connected algebraic group over an algebraically closed field K. Let G act on an irreducible variety V, say by means of the operating morphism $\mu: G \times V \to V$. We define a *G-section* C of V to be a locally closed irreducible subvariety of V such that the image $GC = \mu(G \times C)$ is locally closed in V and that the restriction of μ is a flat morphism from $G \times C$ to GC.

For example, since the morphism μ is flat, the variety V itself is a G-section.

2.2. LEMMA: *Let C be a G-section in V.*
(a) If C' is a subset of C with $GC' \cap C = C'$, then GC is the disjoint union of GC' and $G(C \setminus C')$.
(b) If U is an non-empty open subset of C, then U is a G-section and GU is open and dense in GC.
(c) Let C' be an irreducible closed subvariety of C such that C' is isolated in the intersection $GC' \cap C$. Then C' is a G-section and

$$\dim(GC') - \dim(C') = \dim(GC) - \dim(C).$$

PROOF: (a) Is trivial.
(b) The restriction $\mu: G \times C \to GC$ is a flat morphism of varieties. So it is an open mapping. Since $G \times U$ is open and dense in $G \times C$, it follows that GU is open and dense in GC. The restriction of μ to $G \times U$ is flat. So U is a section.
(c) As C' is isolated in $GC' \cap C$, the set C has an open subset U such that $GC' \cap U = C'$. By (b) we may replace C by U. So we may assume that $GC' \cap C = C'$. By (a) and (b), the set GC' is closed in GC. Let X be the inverse image $\mu^{-1}(GC')$ in $G \times C$. Since $GC' \cap C = C'$, we have $X = G \times C'$. It follows that the diagram is a pull back diagram. Therefore the operating morphism $\nu: G \times C' \to GC'$ is flat. This proves that C' is a G-section. Since μ is flat, the dimension formula follows from [9] IV 6.1.4.

$$\begin{array}{ccc} X = G \times C' & \xrightarrow{\nu} & GC' \\ \downarrow & & \downarrow \\ G \times C & \xrightarrow{\mu} & GC \end{array}$$

2.3. COROLLARY: *Let C be a G-section in V. Let $x \in C$ be such that x is an isolated point of $Gx \cap C$. Then*

$$\dim(Gx) = \dim(GC) - \dim(C).$$

2.4. Let C be a locally closed irreducible subvariety of V, usually, but not necessarily, a G-section. There are two extreme cases to consider.
(a) The subvariety C is said to be *G-final*, if for every point $x \in C$ we have $Gx \cap C = \{x\}$.
(b) The subvariety C is said to be *G-contractible* to a subset C' of C, if there is a morphism $h: C \to G$ such that $h(x)x \in C'$ for all points $x \in C$, and that $h(x) = 1$ in G for all points $x \in C'$.

LEMMA: *Let the subvariety C be G-contractible to C'.*
(a) C' is an irreducible closed subset of C and $GC' = GC$.
(b) If C is a G-section, then C' is a G-section.

PROOF: (a) It is clear that C' consists of the points $x \in C$ with $h(x) = 1$. Therefore C' is closed. It is irreducible, since it is the image of the morphism $f: C \to C'$ given by $f(x) = h(x)x$. Every G-orbit in V which meets C, also meets C'. This implies that $GC' = GC$.
(b) We consider $X = G \times C'$ as a closed subvariety of $Y = G \times C$. The operating morphism $\mu: Y \to GC$ is flat. Let $q: Y \to X$ be the morphism given by

$$q(g, v) = \left(gh(v)^{-1}, h(v)v\right)$$

Then $q \mid X$ is the identity and the operating morphism $\nu = \mu \mid X$ satisfies $\nu \circ q = \mu$. It follows that the morphism $\nu: X \to GC$ is flat. In fact, let $x \in X$. Assume that x has the local rings A in X and B in Y. So $A = B/I$ where I is some ideal. Since $q(x) = x$, we have $B \cong A \oplus I$. Since $\mu: Y \to GC$ is flat, the ring B is flat over the local ring R of $\mu(x)$ in GC. The isomorphism $B \cong A \oplus I$ respects the R-module structure, so A is flat over R. This proves that $\nu: X \to GC$ is flat. By (a) we have $\nu(X) = GC$. Therefore C' is a G-section.

2.5. The main algorithm

We define a *system of representing sections* of the G-variety V to be a finite family $(C_i)_{i \in I}$ of G-sections C_i, such that every G-orbit in V meets exactly one of the members C_i. The *initial system* of representing sections is defined as the family (V), which only consists of the section V itself, cf. example 2.1. A system of representing sections $(C_i)_{i \in I}$ is said to be *final*, if all its members C_i are G-final.

A final system of representing sections $(C_i)_{i \in I}$ is considered as an orbit classification. In fact, every G-orbit in V has a unique representative in the disjoint union $\cup C_i$ of the sections. Moreover, all orbits $\mathcal{O}$ which meet the same section C_i, have the same dimension

$$\dim(\mathcal{O}) = \dim(GC_i) - \dim(C_i), \text{ cf. } 2.3.$$

If the group G is solvable, then a final system of representing sections should exist according to [8]. Our objective, however, is an algorithm which under certain circumstances will effectively yield such a final system.

In order to proceed from the initial system of representing sections (V) to some final system, we use a kind of stepwise refinement. So, let $(C_i)_{i \in I}$ be some system of representing sections. Fix a member $C = C_j$. Now two operations are available.

(a) *Splitting*. If C has an irreducible closed subvariety $C' \neq C$ with $GC' \cap C = C'$, then by 2.2 the section C may be replaced by the pair of sections C' and $C'' := C \setminus C'$.

(b) *Reduction*. If C contains a section C' with $GC' = GC$, then C may be replaced by C'. In particular, if C is G-contractible to a subset C', then by 2.4 the section C may be replaced by C'.

It is easy to verify that both operations yield a refinement which again is a system of representing sections. As our aim is an effective algorithm, which can be implemented on a computer, we shall introduce special assumptions both on the group and on the variety.

2.6. The case of a semi-direct product

Let the group G be a semi-direct product $G = TU$ of a subgroup T with a normal subgroup U. Let V be a G-module. So V is also a T-module and a U-module, and we may apply the above theory to all three actions.

LEMMA: (*a*) *Let C be a T-stable U-section in V. Let C' be an irreducible subvariety of C such that the operating morphism $\rho: T \times C' \to C$ is surjective and flat. Then C' is a G-section with $GC' = UC$.*

(*b*) *If moreover C is U-final and C' is T-final, then C' is G-final.*

PROOF: (a) Let $\sigma: U \times C \to UC$ be the other operating morphism. Both ρ and σ are surjective and flat. So the composition

$$G \times C' \cong U \times T \times C' \xrightarrow{1 \times \rho} U \times C \xrightarrow{\sigma} UC$$

is surjective and flat. As this composition corresponds to the action of G, it follows that C' is a G-section with $GC' = UC$.

(b) Let $x \in C'$ and $g \in G$ be such that $gx \in C'$. Write $g = tu$ with $t \in T$ and $u \in U$. Since C is T-stable and $ux = t^{-1}gx$, we have $ux \in C$. Since C is U-final, it follows that $ux = x$. This implies that $tx = gx$, and hence $tx \in C'$. By the T-finality of C' we get $tx = x$, and hence $gx = x$. This proves that C' is G-final.

3. Simply solvable groups and simply weighted modules

3.1. Simply solvable groups

Let G be a linear algebraic group. We choose a maximal torus T of G. Let the Lie algebra $\mathbf{g}$ of G have the root space decomposition

$$\mathbf{g} = \mathbf{g}^T \oplus \sum_{\alpha \in R} \mathbf{g}_\alpha .$$

DEFINITION: The group G is said to be *simply solvable*, if
(a) G is connected and solvable.
(b) $\mathbf{g}^T$ is equal to the Lie algebra $\mathbf{t}$ of T.
(c) $\dim(\mathbf{g}_\alpha) = 1$ for every root $\alpha \in R$.
(d) Every pair of different roots α, $\beta \in R$ is linearly independent.
From now, we assume that G is simply solvable with maximal torus T and root system R. Let U be the unipotent radical of G. By [3] 10.6, the group G is the semidirect product TU. By [3] 14.4, the subgroup U is directly spanned (in any order) by certain unique one-dimensional T-stable subgroups U_α with Lie algebras $\mathbf{g}_\alpha$. By [3] 10.10, there are parametrizations $u_\alpha : K \to U_\alpha$ with

$$(*) \quad t u_\alpha(x) t^{-1} = u_\alpha(t^\alpha x), \quad t \in T, \ x \in K.$$

Note that all closed subgroups of G which contain T, are also simply solvable, cf. [3] 14.4. If H is a reductive linear algebraic group, then all its Borel subgroups are simply solvable. This follows from [3] 13.18.

3.2. Simply weighted modules

Let V be a G-module. Let $V = \sum V_\lambda$ be the weight space decomposition of V with respect to the torus T. Let $S = \{\lambda \mid V_\lambda \neq 0\}$ be the set of the weights of V. If $\alpha \in R$ and $v \in V_\lambda$, then by 3.1($*$) we have

$$u_\alpha(x) v = \sum_{i=0}^{\infty} x^i f_{i,\alpha}(v), \quad x \in K$$

for certain linear maps $f_{i,\alpha} : V_\lambda \to V_{\lambda + i\alpha}$. Since $u_\alpha : K \to U_\alpha$ is an isomorphism of groups, we get

$$(*) \quad f_{0,\alpha} = id, \quad u_\alpha(x) f_{i,\alpha} = \sum_{j=i}^{\infty} \binom{j}{i} x^{j-i} f_{j,\alpha} .$$

DEFINITION: A G-module V is said to be *simply weighted*, if
(a) $\dim(V_\lambda) = 1$ for every weight $\lambda \in S$.
(b) For every weight $\lambda \in S$ and every root $\alpha \in R$, the linear subspace spanned by the set $U_\alpha V_\lambda$ is equal to $\sum_{i=0}^{m-1} V_{\lambda+i\alpha}$, where

$$m = \min\{ i > 0 \mid \lambda + i\alpha \notin S \}.$$

The following results are easily checked.

LEMMA: *Let V be a simply weighted G-module with weight set S.*
(*a*) *If W is a submodule, then both W and V/W are simply weighted.*
(*b*) *The dual module V^* is simply weighted with weight spaces $(V^*)_{-\lambda} = (V_\lambda)^*$.*
(*c*) *If K_μ is a one dimensional G-module with character μ, then the tensor product $K_\mu \otimes V$ is simply weighted with weight set $\mu + S$.*
(*d*) *Let V' be another simply weighted G-module, say with weight set S'. Then the direct sum $V \oplus V'$ is simply weighted if and only if for every pair of weights $\lambda \in S$, $\mu \in S'$ we have $\lambda \neq \mu$ and $\lambda - \mu \notin R$ and $\mu - \lambda \notin R$.*
(*e*) *If H is a closed subgroup of G which contains T, then the H-module V is also simply weighted.*

3.3. PROPOSITION: (*a*) *Let V be a simply weighted G-module. Let $\alpha \in R$ and $\lambda \in S$ with $\lambda + \alpha \in S$. Then $f_{1,\alpha} : V_\lambda \to V_{\lambda+\alpha}$ is an isomorphism. The action of the Lie algebra $\mathbf{g}$ on V is such that $\mathbf{g}_\alpha V_\lambda = V_{\lambda+\alpha}$.*
(*b*) *Assume that $\operatorname{char}(K) = 0$. Let V be a G-module with $\dim(V_\lambda) = 1$ for all weights $\lambda \in S$. Assume that $\mathbf{g}_\alpha V_\lambda = V_{\lambda+\alpha}$ holds for every root $\alpha \in R$ and every weight $\lambda \in S$ with $\lambda + \alpha \in S$. Then V is simply weighted.*

PROOF: (a) If $f_{1,\alpha}$ is the zero map, then $V_\lambda + \sum_{i=2}^{\infty} V_{\lambda+i\alpha}$ is a U_α-submodule of V, contradicting Definition 3.2(b). Since $f_{1,\alpha} : V_\lambda \to V_{\lambda+\alpha}$ is a linear map between one dimensional spaces, this proves that $f_{1,\alpha}$ is an isomorphism. Since $f_{1,\alpha}$ is the linear part of the action of U_α, it follows that $\mathbf{g}_\alpha V_\lambda = V_{\lambda+\alpha}$.
(b) We have to prove condition (b) of definition 3.2. Put

$$W = \sum_{i=0}^{m-1} V_{\lambda+i\alpha}, \quad \text{where } m = \min\{ i > 0 \mid \lambda + i\alpha \notin S \}.$$

The binomial coefficients $\binom{j}{i}$ with $j \geqslant i$ are non-zero in the field K. So it follows from formula 3.2($*$), that if $f_{i\alpha} = 0$ then $f_{j\alpha} = 0$ for all integers $j \geqslant i$. This implies that W is a U_α-submodule of V. The assumption

$\mathfrak{g}_\alpha V_\mu = V_{\mu+\alpha}$ for all $\mu \in S$ with $\mu + \alpha \in S$, clearly implies that W is the smallest U_α-submodule of V which contains V_λ.

3.4. The main examples

Let B be a Borel subgroup of a reductive algebraic group G. Then B is a simply solvable group. Let $\boldsymbol{u}$ be the Lie algebra of the unipotent radical U of B. If $\operatorname{char}(K) = 0$ or $\operatorname{char}(K) \geqslant 5$, then $\boldsymbol{u}$ is a simple weighted B-module. This follows from the known values of the structure constants, e.g. cf. [19] 11.2. Using the above lemma 3.2, one may construct lots of other simply weighted modules.

Henceforward, we assume that the G-module V is simply weighted.

3.5. Partial orders on the set S of the weights of V

Let E be a subset of R. If $\lambda \in S$, we define EV_λ to be the smallest linear subspace of V which contains V_λ and which is invariant under all groups U_α with $\alpha \in E$. We define the preorder $\leqslant_E$ on S by

$$\lambda \leqslant_E \mu \Leftrightarrow V_\mu \subset EV_\lambda.$$

We write $\lambda <_E \mu$ to denote $\lambda \leqslant_E \mu$ and $\lambda \neq \mu$. If E is a singleton $\{\alpha\}$, then we write $\leqslant_\alpha$ and $<_\alpha$ instead of $\leqslant_{\{\alpha\}}$ and $<_{\{\alpha\}}$.

PROPOSITION: (*a*) *The preorder* $\leqslant_E$ *is a partial order.*
(*b*) *The partial order* $\leqslant_E$ *is generated by the partial orders* $\leqslant_\alpha$ *with* $\alpha \in E$.
(*c*) *Let* $\alpha \in R$ *and* $\lambda, \mu \in S$. *Then we have* $\lambda \leqslant_\alpha \mu$ *if and only if there is an integer* $n \geqslant 0$ *with* $\mu = \lambda + n\alpha$ *such that* $\lambda + i\alpha \in S$ *for every integer* $i \in [1 \ldots n-1]$.

PROOF: (a) We have to prove asymmetry. So assume $\lambda <_E \mu$. Then we have to prove $\mu \nleqslant_E \lambda$. Let W be the smallest G-submodule of V which contains V_λ. Then we have

$$V_\mu \subset EV_\lambda \subset W.$$

Since G is connected and solvable, the G-module W has a submodule W' of codimension 1. Since $V_\lambda \not\subset W'$, we have

$$W = V_\lambda + W'.$$

Since W and W' are T-modules, it follows that $V_\mu \subset W'$ and hence $EV_\mu \subset W'$. Since $V_\lambda \not\subset W'$, this proves that $\mu \nleqslant_E \lambda$.

b) Let $\leqslant$ be the smallest preorder on S containing $\leqslant_\alpha$ for every root $\alpha \in E$. If $\lambda \leqslant_\alpha \mu$, then

$$V_\mu \subset \{\alpha\} V_\lambda \subset E V_\lambda$$

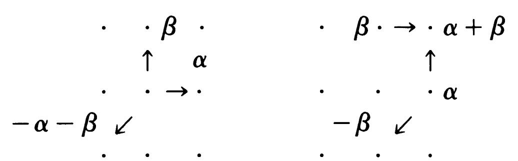
so that $\lambda \leqslant_E \mu$. This implies that $\leqslant_E$ contains $\leqslant_\alpha$ for every root $\alpha \in E$, thus proving that $\leqslant_E$ contains $\leqslant$. Conversely, fix $\lambda \in S$. Put $F = \{\mu \in S \mid \lambda \leqslant \mu\}$ and $P = \sum_{\mu \in F} V_\mu$. Then P is invariant under all groups U_α, $\alpha \in E$. Therefore $EV_\lambda \subset P$. So, if $\lambda \leqslant_E \mu$, then $V_\mu \subset P$ and hence $\mu \in F$. This proves that $\leqslant_E$ is contained in $\leqslant$. Therefore $\leqslant_E$ and $\leqslant$ are equal.

c) This follows immediately from definition 3.2(b).

3.6. EXAMPLE: Let G be the subgroup of $GL(4)$ of the matrices

$$g = \begin{pmatrix} st & 0 & x & y \\ 0 & t^{-1} & 0 & z \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & s \end{pmatrix}, \quad \text{with } s, t, x, y, z \in K \text{ and } st \neq 0.$$

Let T be the torus consisting of the diagonal matrices in G. Let the characters $\alpha, \beta \in X(T)$ be given by $\alpha(g) = s$ and $\beta(g) = t$. The roots of G are α, β, $-\alpha - \beta$. It is clear that G is simply solvable. Let V be the four dimensional vector space K^4, considered as a G-module in the natural way. The weights of V are α, β, $-\beta$, $\alpha + \beta$. Looking at the diagrams of roots and weights, it is easy to verify that V is a simply weighted G-module.



R: roots S: weights

In the diagram of S, the partial order $\leqslant_R$ is indicated using arrows between the points of S. Note that $-\beta \nleqslant_R \beta$, although $\beta \in R$. This fact is related to the fact that the sum of the roots is zero. The partial order $\leqslant_R$ on S is not the restriction of a natural order on the weight lattice $X(T)$.

4. Orbit classification in simply weighted G-modules

In this chapter, and in the next one, we use the following standing conventions. G is a simply solvable algebraic group over an algebraically closed field K, with a maximal torus T, the root system R and the unipotent radical U, and V is a simply weighted G-module with set of weights S.

4.1. Admissible subsets of the G-module V

We now turn to the application of the algorithm of section 2. In view of 2.6, we start by considering systems of representing sections $(C_i)_{i \in I}$ for the action of the unipotent radical U of G. We assume that the sections C_i are stable under the action of the torus T. More precisely, in the algorithm we only consider subsets C of V which are admissible in the following sense.

As the module V is the direct sum of one-dimensional weight spaces V_λ, we may choose basis vectors e_λ, $\lambda \in S$, of V such that $V_\lambda = Ke_\lambda$. Put $M = K \setminus \{0\}$. A subset C of V is called an *admissible* subset of V, if it can be written

$$C = \sum_{\lambda \in S} C_\lambda e_\lambda, \quad \text{with } C_\lambda \in \{0, M, K\}.$$

An admissible subset C of V is completely determined by the *support sets*

$$S_0(C) = \{\lambda \in S \mid C_\lambda = 0\} \quad \text{and } S_m(C) = \{\lambda \in S \mid C_\lambda = M\}.$$

Let $C = \sum C_\lambda e_\lambda$ be an admissible set with contribution $C_\mu = K$ at a given weight μ. Then C is the disjoint union of the two admissible subsets $[0/\mu]C$ and $[M/\mu]C$ defined by

$$[0/\mu]C = C \cap \sum_{\lambda \neq \mu} Ke_\lambda,$$

$$[M/\mu]C = Me_\mu + [0/\mu]C.$$

If moreover C is U-contractible to the admissible subset $[0/\mu]C$, cf. 2.4, then we say that C is *U-contractible at the weight* μ.

REMARKS. An admissible subset C need not be a section. In fact, if $0 \in C$ and $UC \neq C$, then C is not a section, as follows from 2.3. Conversely, not all sections are admissible subsets. Our algorithm uses admissible subsets, since they are obviously T-stable, and they are easy to work with. However, the admissible subsets which appear in the algorithm are sections. This follows by repeated application of the Lemmas 2.2 and 2.4.

4.2. The α-layer of a set of weights

In order to formulate a sufficient condition for U-contractibility of an admissible set C at a given weight, we introduce the concept of α-layer. It is phrased in terms of the partial order $<_\alpha$ of 3.5.

DEFINITION: Let α be a root. Let Q be a set of weights. Then the *α-layer* of Q is defined as the set

$$L(\alpha, Q) = \{\mu \in S \mid \forall \lambda \in S\colon \lambda <_\alpha \mu \Rightarrow \lambda \in Q\}.$$

LEMMA: *Let C be an admissible set. Let $Q \subset S_0(C)$ and $\mu \in S$.*
(a) If $\mu \in L(\alpha, Q)$, then the group U_α does not change the μ-coordinate of the elements of C.
(b) Assume that $\mu - \alpha \in L(\alpha, Q) \cap S_m(C)$. Then there is a unique morphism $h\colon C \to U_\alpha$ such that the μ-coordinate of $h(x)x$ vanishes for every element $x \in C$. The morphism h is a U_α-contraction of C to $[0/\mu]C$. The weight $\mu - \alpha$ is called the pivot weight.

PROOF: Put $F = \{\lambda \in S \mid \lambda \not\leq_\alpha \mu\}$. Considering the U_α-submodules

$$W = \sum_{\lambda \in F} V_\lambda \quad \text{and } W_1 = V_\mu + W.$$

(a) Since $\mu \in L(\alpha, Q)$, the set C is contained in W_1. So the assertion follows from the fact that U_α acts trivially on W_1/W.
(b) In this case the set C is contained in the U_α-submodule

$$W_2 = V_{\mu-\alpha} + V_\mu + W.$$

By 3.3. (a), we have with a suitable choice of the basis vectors:

$$u_\alpha(t)(me_{\mu-\alpha} + se_\mu) \in me_{\mu-\alpha} + (s + tm)e_\mu + W.$$

So the morphism $h\colon C \to U_\alpha$ has to be prescribed by

$$h\big(me_{\mu-\alpha} + se_\mu + W\big) = u_\alpha(-s/m).$$

This morphism h satisfies the requirements.

4.3. A sufficient condition for U-contractibility

The above lemma is sharp enough to perform stepwise U-contraction. A priori, however, the order in which to perform the steps, is unknown. As we want to use a prescribed order for a linear scan of the weights, our criterion for U-contractibility at μ will admit the possibility of first damaging other weights which are restored later.

Let C be an admissible set with a contribution $C_\mu = K$ at a given weight μ. The aim of the contraction is the admissible set $C' = [0/\mu]C$. Our criterion is formulated in terms of the support sets $S_0 = S_0(C')$ and $S_m = S_m(C')$ of the class C'.

THEOREM: *A sufficient condition for U-contractibility of C at μ is the existence of a sequence of weights $\mu_1, \ldots, \mu_t$ and a sequence of roots $\alpha_1, \ldots, \alpha_t$ with $t \geqslant 1$, such that*

(*a*) $\mu_1 = \mu$

(*b*) *For all indices $i \in [1 \ldots t]$ we have $\mu_i \in S_0$ and $\mu_i - \alpha_i \in S_m$.*

(*c*) *If we put $Q_1 := S_0 \setminus \{\mu\}$ and*

$$Q_{i+1} := \{\mu_i\} \cup (Q_i \cap L(\alpha_i, Q_i)),$$

then $Q_{t+1} = S_0$ and for all indices $i \in [1 \ldots t]$ we have $\mu_i \notin Q_i$ and $S_m \subset L(\alpha_i, Q_i)$.

PROOF: Since all weights μ_i belong to S_0, the sets Q_i are contained in S_0. Put $F(i) := S_0 \setminus Q_i$ and $C(i) := C' + \sum_{\lambda \in F(i)} V_\lambda$. Then $C(1) = C$ and $C' = C(t+1)$. For every index i we have

$$S_0(C(i)) = Q_i \quad \text{and} \quad S_m(C(i)) = S_m \subset L(\alpha_i, Q_i).$$

So by Lemma 4.2(b), there is a morphism

$$h_i : C(i) \to U_{\alpha_i}$$

such that the μ_i-coordinate of $h_i(x)x$ vanishes for every element $x \in C(i)$ and that $h_i(x) = 1$ whenever $x \in C'$. By 4.2 (a), the coordinates of $h_i(x)x$ are unchanged at all weights $\lambda \in L(\alpha_i, Q_i)$. It follows that $h_i(x)x \in C(i+1)$ for all $x \in C(i)$. By a transitivity argument we obtain the required contracting morphism.

4.4. Splitting by means of U-finality at a given weight

An admissible set $C = \sum C_\lambda e_\lambda$ is defined to be *U-final at a weight $\mu \in S$*, if for every element $x \in C$ and every element $g \in U$ with $gx \in C$ the μ-coordinates of x and gx are equal. It follows that U-finality of C at all weights $\mu \in S$ is equivalent to U-finality as defined in 2.4.

U-finality at a given weight is a sufficient condition for splitting. In fact, if C is an admissible U-section with $C_\mu = K$ which is U-final at μ, then C splits into the admissible U-sections $C' := [0/\mu]C$ and $C'' := [M/\mu]C$, cf. 4.1 and 2.5. The set UC'' is open and dense in UC and

$$\dim(UC') = \dim(UC) - 1, \text{ cf. } 2.2.$$

In order to get a condition sufficient for U-finality at a given weight, we have to investigate the elements $g \in U$ such that the set gC meets C.

4.5. Amendability

DEFINITION: Let $C = \sum C_\lambda e_\lambda$ be an admissible subset of V with support sets $S_0 = S_0(C)$ and $S_m = S_m(C)$. A subset E of R is said to be *C-amendable*, if for every root $\alpha \in E$ and every weight $\mu \in S_0$ such that $\mu - \alpha \in S_m$ there exists a weight $\lambda <_E \mu$ with $\lambda \notin S_0$ and $\lambda \neq \mu - \alpha$.

LEMMA: *Let $g \in U$ be such that the sets C and gC meet. Then we have*

$$g = \prod_{\alpha \in E} u_\alpha(s_\alpha), \quad s_\alpha \in K$$

where E is some C-amendable subset of R.

PROOF: Since the group U is directly spanned by the subgroups U_α, cf. 3.1, we have an expression

$$g = \prod_{\alpha \in R} u_\alpha(s_\alpha), \quad s_\alpha \in K.$$

Let E be the set of the roots α with $s_\alpha \neq 0$. We have to prove that E is C-amendable. So, let $\alpha \in E$ and $\mu \in S_0$ be such that $\mu - \alpha \in S_m$, and assume that for all weights $\lambda \neq \mu - \alpha$ with $\lambda <_E \mu$ we have $\lambda \in S_0$. We shall derive a contradiction.

Put $F := \{\lambda \in S \mid \lambda \not\leqslant_E \mu\}$ and $W = \sum_{\lambda \in F} V_\lambda$. If $\lambda \in F$ and $\lambda \leqslant_E \lambda'$, then $\lambda' \in F$. It follows that W is stable under the groups U_α with $\alpha \in E$. In particular, we have $gW = W$. Since C and gC are not disjoint, we can choose $x \in C$ with $gx \in C$. Then we have

$$x \in me_{\mu-\alpha} + W \text{ with } m \in M.$$

Therefore

$$gx \in M \prod_{\beta \in A} u_\beta(s_\beta) e_{\mu-\alpha} + W$$

where

$$A = \{\beta \in E \mid \mu - \alpha + \beta \leqslant_E \mu\}.$$

Since $\alpha \in A$, the set A is non-empty. So we may choose an element $\beta \in A$ such that $\mu - \alpha + \beta$ is minimal with respect to the partial order $\leqslant_E$. It follows from this minimality that the vectors gx and $mu_\beta(s_\beta)e_{\mu-\alpha}$ have the same coordinate at the weight $\lambda := \mu - \alpha + \beta$. Since $s_\beta \neq 0$, it follows with 3.3(a) that gx has a non-zero coordinate at weight λ. Since $gx \in C$, it follows that $\lambda \notin S_0$. In particular $\lambda \neq \mu$. Since $\beta \in A$, it follows that $\lambda <_E \mu$. So the assumption implies that $\lambda = \mu - \alpha$, so that $\beta = 0$, a contradiction.

4.6. The amendator

DEFINITION: A subset A of R is called an *amendator* of the admissible set C, if it is C-amendable and contains all other C-amendable subsets of R.

LEMMA: *The admissible set C has a unique amendator A, which can be obtained from the set of all roots R by successive omission of the roots α which violate the amendability condition.*

PROOF: Uniqueness is obvious. Since R is a finite set, it suffices to prove the following assertion:
$(*)$ Let A be a subset of R. Let $\alpha \in A$ and $\mu \in S_0$ be such that $\mu - \alpha \in S_m$ and that for all weights $\lambda \neq \mu - \alpha$ with $\lambda <_A \mu$ we have $\lambda \in S_0$. Then every C-amendable subset of A is contained in $A \setminus \{\alpha\}$.
Proof of $(*)$: Assume $\alpha \in E$. Since E is C-amendable, there exists $\lambda <_E \mu$ with $\lambda \notin S_0$ and $\lambda \neq \mu - \alpha$. Since $E \subset A$, we have $\lambda <_A \mu$, contradicting the assumption of $(*)$.

4.7. A sufficient condition for U-finality at μ

THEOREM: *Let C be an admissible subset of V. Let E be a subset of R which contains the amendator of C. Let $\mu \in S$ be such that $\lambda \in S_0(C)$ for every weight $\lambda <_E \mu$. Then C is U-final at μ.*

PROOF: Suppose that $x \in C$ and $g \in U$ are such that $gx \in C$. By lemma 4.5 we have

$$g = \prod_{\alpha \in E} u_\alpha(s_\alpha), \quad s_\alpha \in K.$$

At all weights $\lambda <_E \mu$ the vector x has coordinate 0. So, at μ the vectors x and gx have the same coordinate.

4.8. Normalization by means of the torus action

The action of the torus T enables us to normalize certain coefficients of a given admissible set $C = \Sigma\, C_\lambda e_\lambda$. If N is a subset of the support set $S_m(C)$, put

$$C_N := \sum_{\lambda \in N} e_\lambda + \sum_{\lambda \in S \setminus N} C_\lambda e_\lambda .$$

LEMMA: *The following conditions are equivalent*:

(*a*) *The weights* $\lambda \in N$ *are linearly independent over* $\mathbb{Z}$ *in the weight lattice of* T.

(*b*) *The operating morphism* $\rho : T \times C_N \to C$ *is flat and surjective.*

(*c*) $TC_N = C$.

PROOF: The implication $b \Rightarrow c$ is trivial. Put $C' = \sum_{\lambda \in N} Me_\lambda$ and $C'' = \sum_{\lambda \in S \setminus N} C_\lambda e_\lambda$, so that C may be identified with the product $C' \times C''$. Put $e = \Sigma_{\lambda \in N}\, e_\lambda$, so that $C_N = e + C''$. The operating morphism ρ can be factorized according to diagram 1,

$$\begin{array}{ccc} T \times C_N & \xrightarrow{\rho} & C \\ {\scriptstyle q}\downarrow & & \| \\ T \times C'' & \xrightarrow{f \times 1} & C' \times C'' \end{array}$$

Diagram 1

where the mapping q is defined by

$$q(t, x) = (t, t(x - e))$$

and the mapping f: $T \to C'$ is defined by $f(t) = te$. Now the mapping q is easily seen to be an isomorphism. We may identify C' with the torus M^N. Since $f(t) = \Sigma\, \lambda(t) e_\lambda$, the mapping $f : T \to C'$ is a homomorphism of tori, and condition (a) is equivalent to the surjectivity of f.

$c \Rightarrow a$. If (c) holds, then ρ is surjective; therefore f is surjective; so (a) follows.

$a \Rightarrow b$. If (a) holds, then $f : T \to C'$ is a surjective homomorphism of tori. Therefore f is flat, and hence ρ is surjective and flat.

DEFINITION: If N is a maximal linearly independent subset of $S_m(C)$, then C_N is called a *normalization* of C.

REMARKS: Let C_N be a normalization of C.

(a) If C is a U-section in V, then C_N is a G-section, cf. 2.6(a).

(b) It need not be true that C is T-contractible to C_N.

(c) If K has positive characteristic, it may happen that the operating morphism $T \times C_N \to C$ is inseparable.

(d) If $S_0(C) \cup S_m(C) = S$, then every T-orbit in C meets C_N in only finitely many points. If moreover $S_m(C)$ is contained in the lattice spanned by the set N, then every T-orbit in C meets C_N in precisely one point.

EXAMPLE: Let T be a torus with a character λ. Let V be a two-dimensional T-module with basis vectors $e \in V_{4\lambda}$ and $f \in V_{6\lambda}$. The admissible set $C = Me + Mf$ is a section with two normalizations, namely $C' = e + Mf$ and $C'' = Me + f$. If char$(K) \neq 2, 3$, then every T-orbit in C meets C' in two points and C'' in three points. One may prefer the T-section $C''' = M(e+f)$. In fact, every T-orbit in C meets C''' in precisely one point. The section C is not T-contractible to any of these subsections. If char$(K) = 2$, all three operating morphisms are inseparable.

4.9. Description of the algorithm and its results

The algorithm starts with the admissible set $V = \Sigma\, Ke_\lambda$, which is a U-section, cf. 2.1. So we have an admissible set C which is a U-section. Now the weights $\mu \in S$ are inspected one after the other. If C is U-contractible at μ according to 4.3, then C is replaced by $[0/\mu]C$. If C is U-final at μ according to 4.7, then C is replaced by the sections $[0/\mu]C$ and $[M/\mu]C$. At the end, after all applications of 4.3 and 4.7, we choose normalizations of the representing sections obtained. So it results a system of representing sections for the action of G which consists of normalizations of admissible sets.

The admissible sets C which occur in the algorithm, have a common property carried over. Let an admissible set C be called *good*, if it is U-final at all weights $\mu \in S_m(C)$. Since $S_m(V)$ is empty, the initial section V is good. If C is good, then $[0/\mu]C$ is also good. If C is good, and C is U-final at weight μ, then $[M/\mu]C$ is good. This shows that the condition is carried over.

Every member C' of a resulting system of representing sections is a normalization of a good admissible set C. So it can be represented as $C' = \Sigma\, C'_\lambda e_\lambda$ with $C'_\lambda \in \{0, 1, M, K\}$ where $C = \Sigma\, C_\lambda e_\lambda$ and $C_\lambda = MC'_\lambda$. The set C' is a singleton if $C'_\lambda \in \{0, 1\}$ for all weights $\lambda \in S$.

A resulting section $C' = \Sigma\, C'_\lambda e_\lambda$ is said to be a *series*, if it is not a singleton and $C'_\lambda \in \{0, 1, M\}$ for all weights $\lambda \in S$. Then the corre-

sponding admissible set C has $S_0(C) \cup S_m(C) = S$. As C is good, it follows that C and C' are U-final. Using remark 4.8(d) and the arguments of lemma 2.6(b) we get:
(*) If C' is series and $x \in C'$, then $Gx \cap C'$ is finite.

*4.10. Example: the Borel action on **sl**(2)*

Assume char$(K) \neq 2$. Let G be the Borel group of SL(2). Let V be the Lie algebra **sl**(2), considered as a G-module by means of the adjoint action. This is not one of our main cases cf. 1.4. However, as the zero weight space of V is one-dimensional, it fits in our framework. To fix the notation, let G be the group of the matrices

$$g(m, t) = \begin{pmatrix} m & t \\ 0 & m^{-1} \end{pmatrix}, \quad m \in M, \ t \in K.$$

Let the torus T consist of the diagonal matrices in G. The root system R of G consists of one root α, which satisfies

$$\alpha(g(m, t)) = m^2.$$

The group G is simply solvable. The module V has basis vectors

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with the weights $-\alpha$, 0, α, respectively. As char$(K) \neq 2$, the G-module V

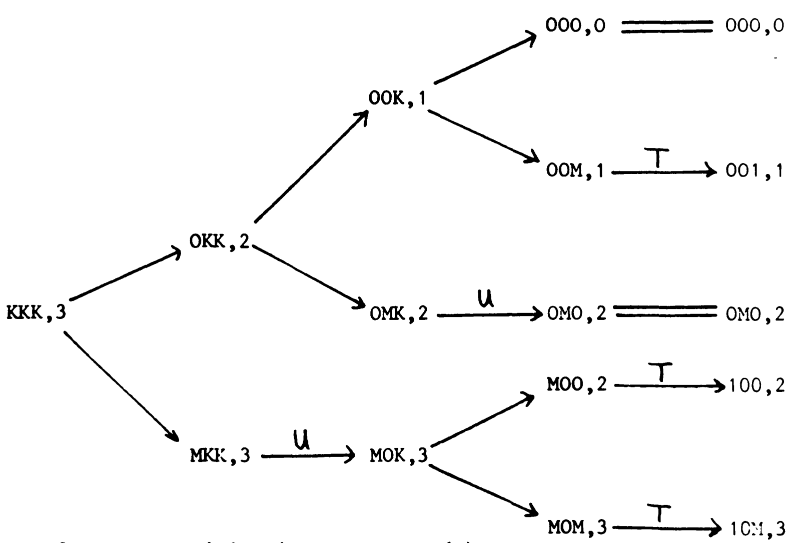


Diagram 2. The classification tree of $\underline{s\ell}(2)$.

is simply weighted. Since $\dim(G) < \dim(V)$, there are infinitely many G-orbits in V. In this simple case, the algorithm is easily performed by hand. The steps are shown in the classification tree of Diagram 2. A section C is represented by a string "pqr,d" where $C = pY + qH + rX$, and d is the represented dimension $\dim(GC)$. The tree contains four splittings, two U-contractions indicated by a U on the arrow, and three normalizations indicated by a T. The only non-trivial point is the splitting of the section "M0K,3". The result is that there are three separate orbits, namely the orbits of 0, of X, and of Y, and two one-dimensional series, which consist of the orbits of mH, and of $Y + mX$, $m \in M$. The calculation of the represented dimensions is based on 4.4. It follows, that the orbits X and of mH, $m \in M$, are one-dimensional, and the orbits of Y and of $Y + mX$ are two-dimensional. Of course, in this case other methods are available. For example, the two series consist of semi-simple matrices, which may be parametrized by the determinant, and a square root of the determinant, respectively.

5. A description of the computer algorithm

5.1. The choice of an order

We choose an enumeration $\mu_1, \ldots, \mu_s$ of the weights $\mu \in S$ such that $\mu_i \leqslant_R \mu_j$ implies $i \leqslant j$. Admissible sets tend to be U-final at the weights with the lower indices. So after the splitting procedure some of these weights can be used as pivot weights in tests of U-contractibility of other weights. Therefore, the algorithm is based on a linear scan of the weights from μ_1 and μ_s. This idea is due to Brodskii, cf. [7].

5.2. The central routine "conjugate"

The algorithm is described here in a kind of programming language which is a mixture of english, pascal and algol 68.

The central routine of the algorithm is a recursive procedure "conjugate" which depends on three value parameters: the admissible section C, the represented dimension $d = \dim(UC)$, and the index j of the weight where the procedure has to start.

Procedure: conjugate $(C, d, j) =$
(for i from j to s do
if C is seen to be U-contractible at μ_i
then replace C by $[0/\mu_i]C$
else if C is seen to be U-final at μ_i
then replace C by $[M/\mu_i]C$ and
also start a new process by the
call of conjugate $([0/\mu_i]C, d-1, i+1)$

else do nothing; (end of the loop)
finally replace C by a normalisation of C
and report C and d).

The tests whether C is seen to be U-contractible or U-final at the weight μ_i, are calls of Boolean functions "ucon" and "ufin" specified by

ucon(C, μ) := (C satisfies the assumptions of 4.3 at μ),
ufin(C, μ) := (C satisfies the assumptions of 4.7 at μ).

The procedure "conjugate" is activated once by the call of

conjugate(V, dim(V), 1).

After that, it calls itself recursively each time splitting occurs.

REMARK: A priori, it is not clear whether it is better to test first for U-finality or first for U-contractibility. In our implementation, the order proposed here required much less computer time than the reverse order.

5.3. The Boolean function "ucon"

This function has to test whether a given class C satisfies the assumptions of 4.3 at a given weight μ. As this might require an infinite search procedure, we force termination of the search by imposing the extra condition that all roots in the sequence $\alpha_1, \dots, \alpha_t$ are different. So actually, we are testing for a slightly stronger condition than the one described in 4.3. However, in view of the direct spanning property, cf. 3.1, we assume to miss hardly any case of observable U-contractibility in this way.

The function "ucon" is implemented by means of recursive Boolean function "accept" which depends on three value parameters: on a weight $\lambda = \mu_i$, a set of weights $Q = Q_i$ and a set of used roots $E = \{\alpha_1, \dots, \alpha_{i-1}\}$. The function "accept" first tries to use a new root $\alpha = \alpha_i$. It determines the corresponding set $Q' = Q_{i+1}$. It verifies whether $t = i$ would be sufficient. If not, it tries to use a new weight $\nu = \mu_{i+1}$ and it calls itself recursively.

FUNCTION: accept (λ, Q, E) =
(use a Boolean variable found with initial value false;
for $\alpha \in R \setminus E$ while not found do
if $\lambda - \alpha \in S_m \subset L(\alpha, Q)$ then
(put $Q' := \{\lambda\} \cup (Q \cap L(\alpha, Q))$;
found := ($Q' = S_m$);
for $\nu \in S_0 \setminus Q'$ while not found do
found := accept (ν, Q', $E \cup \{\alpha\}$));
finally, deliver the value of found).

The function "ucon" itself consists of the initialisation of the support sets S_0 and S_m of $[0/\mu]c$. It contains a subroutine for the calculation of the α-layers $L(\alpha, Q)$, and it delivers the value of

$$\text{accept}(\mu, S_0 \setminus \{\mu\}, \text{empty set}).$$

REMARK: The termination condition of the recursion is so weak, that we have chosen to force termination by the introduction of a time limit. At every call, the function "ucon" gets two seconds central processor time. The influence of this restriction is discussed in section 6.2.

5.4. The Boolean function "ufin"

The test for U-finality, cf. 4.7, is more straightforward but longer than the one for U-contractibility. In fact, we introduce two Boolean functions, specified mathematically by

$$sha(\lambda, \rho, E) := (\exists \nu \in S \setminus S_0 : \nu \neq \rho \ \&\ \nu \leqslant_E \lambda).$$

$$ex(\alpha, E) := (\alpha \in E \ \&\ \exists \lambda \in S_0 : \lambda - \alpha \in S_m \ \&\ sha(\lambda, \lambda - \alpha, E)).$$

The amendator A can be constructed by

> $A := R$;
> while A contains a root α such that $ex(\alpha, A)$ do
> *replace* A *by* $A \setminus \{\alpha\}$.

After this loop the existence of a weight $\lambda \in S \setminus S_0$ with $\lambda <_A \mu$ can be tested by calling $sha(\mu, \mu, A)$. A negative answer implies U-finality. We prefer to call $sha(\mu, \mu, A)$ each time a new set A is constructed in the above algorithm. In fact, a negative answer may be obtained earlier (even with $A = R$), and that would be sufficient for U-finality, cf. 4.7.

5.5. The data structure

In the above descriptions of the functions "ucon" and "ufin" we used three primitive concepts: the α-layer $L(\alpha, Q)$, the orders $\leqslant_E$ and the α-translations $\lambda \to \lambda - \alpha$ in S. Now the α-layer and the partial orders are easily expressed in the α-translations. Therefore, as a supporting data structure we first build a translation table "minus" satisfying

$$\text{minus}[i][j] = k \Leftrightarrow \mu_j - \alpha_i = \mu_k,$$

$$\text{minus}[i][j] = 0 \Leftrightarrow \mu_j - \alpha_i \notin S.$$

So, "minus" is a variable of the type

ARRAY[rootnr] OF weightmap, where

weightmap = ARRAY[weightnr] OF (weightnr $\cup$ {0}).

The weight vectors themselves are only needed for the normalization procedure, cf. 4.8. The choice of a $\mathbb{Q}$-basis in S_m is easily implemented with Gausz elimination. As the weight vectors have integer coefficients with respect to some natural basis, the euclidean algorithm can be used to construct suitable pivot entries.

6. Discussion of results

6.1. Our methods might be tried in every case of a simply weighted module over a simply solvable group. For reasons explained in the introduction we have concentrated on the following case. The group G is a Borel subgroup of a simple algebraic group H, and the G-module V is either the Lie algebra $\boldsymbol{u}$ of the unipotent radical U of G, or V is the dual module $\boldsymbol{u}^*$. We asume that V is a simply weighted G-module. A sufficient condition is that the base field K has characteristic $\neq 2, 3$, see 3.4.

We may assume that the group H is adjoint. We have four families of simple groups A_n, B_n, C_n, D_n, and five exceptional groups E_6, E_7, E_8, F_4, and G_2. In each case, we may consider the radical $V = \boldsymbol{u}$ and the coradical $V = \boldsymbol{u}^*$. We use the names XnR and XnC to denote the cases of the group $H = H_n$ with the modules $V = \boldsymbol{u}$, and $V = \boldsymbol{u}^*$, respectively. For example, A3C denotes the coradical case $V = \boldsymbol{u}^*$ for the group $H = A_3$. As A_3 is the group PGL(4), this is the case of the group G of the invertible upper triangular matrices of order four, acting by conjugation on the dual of the space of the strictly upper triangular matrices of order four. So, A3C is the case studied by Brodskii, cf. [7].

6.2. Overview of the results

In Table 1, we have collected all global information concerning the cases we have investigated. The first column gives the case. The second and the third column give the respective dimensions of the group G and the module V. The fourth column gives the number of representing sections obtained. An overwhelming majority of the sections is formed by the singletons. Therefore, the non-singleton sections are described in the columns 5 and 6. Column 5 gives the number of series, cf. 4.9. The sixth column gives the number of representing sections $C = \sum C_\lambda e_\lambda$ where a contribution $C_\lambda = K$ occurs. For these sections the algorithm is inconclusive. In the columns 5 and 6, we indicate between brackets the maximal dimension of the occuring sections.

TABLE 1. Global results.

Case	dim(G)	dim(V)	sections	series	not-final	msecs	ratio
A2R	5	3	5	0	0	6	1.2 (Bü, 82)
A2C	5	3	5	0	0	6	1.2
B2R	6	4	7	0	0	11	1.6
B2C	6	4	7	0	0	12	1.7
G2R	8	6	13	1 (1)	0	24	1.8
G2C	8	6	13	1 (1)	0	24	1.8
A3R	9	6	16	0	0	32	2.0 (Bü, 82)
A3C	9	6	16	0	0	31	1.9 (Br, 69)
B3R	12	9	35	1 (1)	0	104	3.0
B3C	12	9	35	1 (1)	0	89	2.5
C3R	12	9	35	1 (1)	0	94	2.7
C3C	12	9	35	1 (1)	0	89	2.5
A4R	14	10	61	0	0	181	3.0 (Bü, 82)
A4C	14	10	61	0	0	179	2.9
D4R	16	12	98	2 (1)	3 (2)	451	4.6
D4C	16	12	99	2 (1)	1 (3)	362	3.7
A5R	20	15	275	1 (1)	0	1680	6.1 (Bü, 82)
A5C	20	15	275	1 (1)	0	1157	4.2
B4R	20	16	225	15 (2)	5 (2)	2237	9.9
B4C	20	16	228	16 (2)	1 (3)	1365	6.0
C4R	20	16	226	13 (2)	2 (3)	2060	9.1
C4C	20	16	229	16 (2)	0	1554	6.8
D5R	25	20	707	25 (2)	35 (3)	12965	18.3
D5C	25	20	720	32 (2)	13 (3)	7056	9.8
A6R	27	21	1430	15 (1)	0	22678	15.9
A6C	27	21	1430	15 (1)	0	12069	8.4
F4R	28	24	824	158 (4)	39 (5)	36572	44.4
F4C	28	24	873	194 (4)	6 (4)	20986	24.0
A7R	35	28	8500	190 (2)	8 (2)	438890	51.6
A7C	35	28	8506	198 (2)	0	179562	21.1

A complete classification is obtained for the cases A_n with $n \leqslant 6$, for B_2, G_2, B_3, C_3, and for C4C and A7C. In the cases B_4, C4R, D_4, A7R, it seems feasible to conclude the classification by hand. In five cases a classification by hand was done first. In fact, a classification for A3C with two omissions was published in 1969, cf. [7]. The cases A2R, ..., A5R were settled by the first author in 1982.

The last two columns of Table 1 indicate the efficiency or complexity of the computer algorithm. Column 7 gives the central processor time used for the classification on the local computer, a CDC-Cyber. This time is measured in msecs. The quotient of the time used and the number of sections may be considered as an indication of the relative complexity. This ratio is given in the last columns. In comparison with a mean time consumption of 51.6 msecs for A7R, the time limit of two seconds for each call of the procedure "ucon", cf. 5.3, seems to be rather generous.

This time limit only effected six sections, 3 in case F4R and 3 in case A7R.

6.3. The classification in the case B2R

Case B2R is sufficiently simple to construct the complete classification tree as an example of the algorithm. The group G is the Borel subgroup of a simple group of type B_2, say of the special orthogonal group SO(5, ℂ). The G-module V is the Lie algebra $\boldsymbol{u}$ of the unipotent radical U of G. We assume that char(K) ≠ 2. Then G is a simply solvable group and V is a simply weighted G-module. The system of roots of G is equal to the system of weights of V. It consists of the four positive roots of the root system B_2. The roots are ordered as α, β,

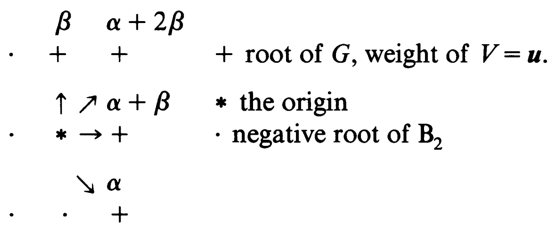


Diagram 1. roots and weights.

$\alpha + \beta$, $\alpha + 2\beta$. An admissible subset $C = \sum C_\lambda e_\lambda$ of V is represented by the string "$pqrs, d$" where $p = C_\alpha$, $q = C_\beta$, $r = C_{\alpha+\beta}$, $s = c_{\alpha+2\beta}$ and d is

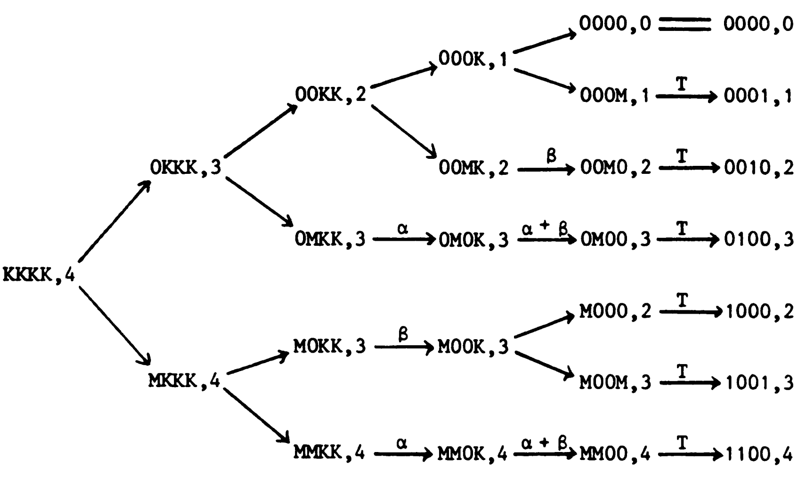


Diagram 2. The classification tree of B2R.

the represented dimension dim(GC). Diagram 2 shows the classification tree. The root of the tree is the module $V = \boldsymbol{u}$, represented as "KKKK,4". Admissible subsets of $\boldsymbol{u}$ are always U-final at the simple roots. So the tree starts by splitting in four branches. The most interesting branch is the third one, headed by "M0KK,3". This representing section is U-contractible to "M00K,3" by means of the one-parameter group u_β. Now u_β has been used and the other root groups centralise. Therefore the section "M00K,3" is U-final, so it splits. The arrows in the diagram with an associated root, say γ, indicate U-contraction by means of u_γ. The arrows with an associated T are normalisations by means of the torus action. The diagram shows that the group G has seven orbits in $\boldsymbol{u}$.

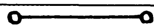
6.4. Detailed results

In Table 2 we give the detailed results in the cases A2, B2, G2, A3, B3, C3, A4. In these cases a complete classification is obtained and the number of classes is not too big.

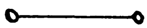
In each case we use the simple roots as a basis of the root lattice. So the weights of the G-module V ($\boldsymbol{u}$ or $\boldsymbol{u}^*$) are expressed in terms of the simple roots. The order of the simple roots is given by the Dynkin diagram. So each list of results starts with the case name, the Dynkin diagram and the ordered list of the weights of V. The representing sections obtained are listed in the order in which the program delivers them. Each section gets an identification number. If the weights are listed in the order $\mu_1 \ldots \mu_s$, the admissible set $\Sigma\, C_i e_{\mu_i}$ is represented by the string $\mathrm{C}_1 \ldots \mathrm{C}_s$ with a space inserted after every fifth character. Just as in 6.3, this string is followed by a comma and a number which is the represented dimension dim(GC). The dimension of the admissible set C itself is of course equal to the number of occurring characters 'K' and 'M'. Açtually, in the present cases almost all representing sections are singletons. The non-singleton sections are marked by the number dim(C) between brackets, followed by the character 'F' which indicates that the section is a series.

In view of 1.4, it may be useful to describe for each G-orbit in $\boldsymbol{u}$ the corresponding H-orbit in $\boldsymbol{h}$. In Table 2, this is done in the cases of the classical Lie algebras $A_n = \boldsymbol{sl}(n+1)$, $B_n = \boldsymbol{so}(2n+1)$, $C_n = \boldsymbol{sp}(2n)$. The H-orbit in $\boldsymbol{h}$ is characterized by the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of the sizes of the blocks in the Jordan normal form of a corresponding nilpotent matrix, cf. [20] IV. In Table 2, these partitions are given between brackets in the last column. In case of a series, the partition is given only for a generic value $m \in \mathrm{M}$. The extra effort to calculate the partitions is not incorporated in the time consumption reported in Table 1.

TABLE 2

Case A2R 

There are 3 weights:
(1 0)(0 1)(1 1)
Nr 1: 000, 0, (111)
Nr 2: 001, 1, (21)
Nr 3: 010, 2, (21)
Nr 4: 100, 2, (21)
Nr 5: 110, 3, (3)

Case A2C 

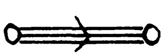
There are 3 weights:
(−1 −1)(0 −1)(−1 0)
Nr 1: 000, 0
Nr 2: 001, 1
Nr 3: 010, 1
Nr 4: 011, 2
Nr 5: 100, 3

Case B2R 

There are 4 weights:
(1 0)(0 1)(1 1)(1 2)
Nr 1: 0000, 0, (11111)
Nr 2: 0001, 1, (221)
Nr 3: 0010, 2, (311)
Nr 4: 0100, 3, (311)
Nr 5: 1000, 2, (221)
Nr 6: 1001, 3, (311)
Nr 7: 1100, 4, (5)

Case B2C 

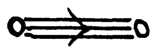
There are 4 weights:
(−1 −2)(−1 −1)(0 −1)(−1 0)
Nr 1: 0000, 0
Nr 2: 0001, 1
Nr 3: 0010, 1
Nr 4: 0011, 2
Nr 5: 0100, 3
Nr 6: 1000, 3
Nr 7: 1001, 4

Case G2R 

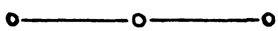
There are 6 weights:
(1 0)(0 1)(1 1)(1 2)(1 3)(2 3)
Nr 1: 00000 0, 0
Nr 2: 00000 1, 1
Nr 3: 00001 0, 2
Nr 4: 00010 0, 3
Nr 5: 00100 0, 3
Nr 6: 00101 0, 4
Nr 7: 01000 0, 4

TABLE 2(continued)

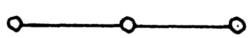
Nr 8: 01000 1, 5
Nr 9: 10000 0, 3
Nr 10: 10001 0, 4
Nr 11: 10010 0, 4
Nr 12: 1001M 0, 5 (1)F
Nr 13: 11000 0, 6

Case G2C 

There are 6 weights:
(−2 −3)(−1 −3)(−1 −2)(−1 −1)(0 −1)(−1 0)
Nr 1: 00000 0, 0
Nr 2: 00000 1, 1
Nr 3: 00001 0, 1
Nr 4: 00001 1, 2
Nr 5: 00010 0, 3
Nr 6: 00100 0, 3
Nr 7: 00100 1, 4
Nr 8: 01000 0, 3
Nr 9: 01000 1, 4
Nr 10: 01010 0, 4
Nr 11: 01010M, 5 (1)F
Nr 12: 10000 0, 5
Nr 13: 10001 0, 6

Case A3R 

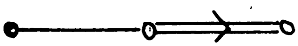
There are 6 weights:
(1 0 0)(0 1 0)(0 0 1)(1 1 0)(0 1 1)(1 1 1)
Nr 1: 00000 0, 0, (1111)
Nr 2: 00000 1, 1, (211)
Nr 3: 00001 0, 2, (211)
Nr 4: 00010 0, 2, (211)
Nr 5: 00011 0, 3, (22)
Nr 6: 00100 0, 3, (211)
Nr 7: 00110 0, 4, (31)
Nr 8: 01000 0, 3, (211)
Nr 9: 01000 1, 4, (22)
Nr 10: 01100 0, 5, (31)
Nr 11: 10000 0, 3, (211)
Nr 12: 10001 0, 4, (31)
Nr 13: 10100 0, 4, (22)
Nr 14: 10101 0, 5, (31)
Nr 15: 11000 0, 5, (31)
Nr 16: 11100 0, 6, (4)

Case A3C 

There are 6 weights:
(−1 −1 −1)(0 −1 −1)(−1 −1 0)(0 0 −1)(0 −1 0)(−1 0 0)
Nr 1: 00000 0, 0
Nr 2: 00000 1, 1
Nr 3: 00001 0, 1
Nr 4: 00001 1, 2

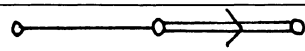
TABLE 2(continued)

Nr 5: 00010 0, 1
Nr 6: 00010 1, 2
Nr 7: 00011 0, 2
Nr 8: 00011 1, 3
Nr 9: 00100 0, 3
Nr 10: 00110 0, 4
Nr 11: 01000 0, 3
Nr 12: 01000 1, 4
Nr 13: 01100 0, 4
Nr 14: 01100 1, 5
Nr 15: 10000 0, 5
Nr 16: 10001 0, 6

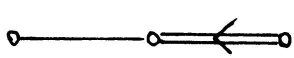
Case B3R 

There are 9 weights:
(1 0 0)(0 1 0)(0 0 1)(1 1 0)(0 1 1)(1 1 1)(0 1 2)(1 1 2)(1 2 2)
Nr 1: 00000 0000, 0, (1111111)
Nr 2: 00000 0001, 1, (22111)
Nr 3: 00000 0010, 2, (22111)
Nr 4: 00000 0100, 3, (22111)
Nr 5: 00000 1000, 3, (31111)
Nr 6: 00000 1100, 4, (322)
Nr 7: 00001 0000, 4, (31111)
Nr 8: 00001 0010, 5, (322)
Nr 9: 00010 0000, 3, (22111)
Nr 10: 00010 0010, 4, (31111)
Nr 11: 00010 0100, 5, (331)
Nr 12: 00011 0000, 5, (322)
Nr 13: 00011 0010, 6, (331)
Nr 14: 00100 0000, 5, (31111)
Nr 15: 00100 0001, 6, (322)
Nr 16: 00110 0000, 7, (511)
Nr 17: 01000 0000, 4, (22111)
Nr 18: 01000 0010, 5, (331)
Nr 19: 01000 0100, 5, (31111)
Nr 20: 01000 0110, 6, (331)
Nr 21: 01000 1000, 5, (322)
Nr 22: 01000 1010, 6, (331)
Nr 23: 01000 1100, 6, (331)
Nr 24: 01000 11M0, 7, (331) (1)F
Nr 25: 01100 0000, 8, (511)
Nr 26: 10000 0000, 4, (22111)
Nr 27: 10000 0001, 5, (31111)
Nr 28: 10000 0100, 6, (331)
Nr 29: 10001 0000, 7, (511)
Nr 30: 10100 0000, 6, (322)
Nr 31: 10100 0001, 7, (331)
Nr 32: 10101 0000, 8, (511)
Nr 33: 11000 0000, 7, (331)
Nr 34: 11000 0100, 8, (511)
Nr 35: 11100 0000, 9, (7)

TABLE 2(continued)

Case B3C 

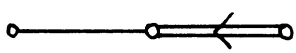
There are 9 weights:
(−1 −2 −2)(−1 −1 −2)(0 −1 −2)(−1 −1 −1)(0 −1 −1)
(−1 −1 0)(0 0 −1)(0 −1 0)(−1 0 0)
Nr 1: 00000 0000, 0
Nr 2: 00000 0001, 1
Nr 3: 00000 0010, 1
Nr 4: 00000 0011, 2
Nr 5: 00000 0100, 1
Nr 6: 00000 0101, 2
Nr 7: 00000 0110, 2
Nr 8: 00000 0111, 3
Nr 9: 00000 1000, 3
Nr 10: 00000 1100, 4
Nr 11: 00001 0000, 3
Nr 12: 00001 0001, 4
Nr 13: 00001 1000, 4
Nr 14: 00001 1001, 5
Nr 15: 00010 0000, 5
Nr 16: 00010 0010, 6
Nr 17: 00100 0000, 3
Nr 18: 00100 0001, 4
Nr 19: 00100 0010, 4
Nr 20: 00100 0011, 5
Nr 21: 00100 1000, 6
Nr 22: 00110 0000, 6
Nr 23: 00110 0010, 7
Nr 24: 01000 0000, 5
Nr 25: 01000 0010, 6
Nr 26: 01000 1000, 6
Nr 27: 01000 1010, 7
Nr 28: 01001 0000, 6
Nr 29: 01001 0010, 7
Nr 30: 01001 1000, 7
Nr 31: 01001 10M0, 8 (1)F
Nr 32: 10000 0000, 7
Nr 33: 10000 0001, 8
Nr 34: 10000 0100, 8
Nr 35: 10000 0101, 9

Case C3R 

There are 9 weights:
(1 0 0)(0 1 0)(0 0 1)(1 1 0)(0 1 1)(1 1 1)(0 2 1)(1 2 1)(2 2 1)
Nr 1: 00000 0000, 0, (111111)
Nr 2: 00000 0001, 1, (21111)
Nr 3: 00000 0010, 2, (2211)
Nr 4: 00000 0100, 2, (21111)
Nr 5: 00000 0101, 3, (2211)
Nr 6: 00000 1000, 3, (2211)
Nr 7: 00000 1100, 4, (222)
Nr 8: 00001 0000, 4, (2211)

TABLE 2(continued)

Nr 9: 00001 0001,	5, (222)
Nr 10: 00010 0000,	4, (2211)
Nr 11: 00010 0100,	5, (222)
Nr 12: 00011 0000,	6, (33)
Nr 13: 00100 0000,	3, (21111)
Nr 14: 00100 0001,	4, (2211)
Nr 15: 00100 0010,	5, (222)
Nr 16: 00100 0100,	5, (2211)
Nr 17: 00100 0101,	6, (222)
Nr 18: 00110 0000,	6, (411)
Nr 19: 00110 0100,	7, (42)
Nr 20: 01000 0000,	5, (2211)
Nr 21: 01000 0001,	6, (222)
Nr 22: 01000 1000,	7, (33)
Nr 23: 01100 0000,	7, (411)
Nr 24: 01100 0001,	8, (42)
Nr 25: 10000 0000,	5, (2211)
Nr 26: 10000 0100,	6, (411)
Nr 27: 10001 0000,	6, (33)
Nr 28: 10001 0100,	7, (42)
Nr 29: 10100 0000,	6, (222)
Nr 30: 10100 0100,	7, (42)
Nr 31: 10101 0000,	7, (33)
Nr 32: 10101 0*M*00,	8, (42) (1)*F*
Nr 33: 11000 0000,	7, (33)
Nr 34: 11000 0100,	8, (42)
Nr 35: 11100 0000,	9, (6)

Case C3C 

There are 9 weights:

(−2 −2 −1)(−1 −2 −1)(0 −2 −1)(−1 −1 −1)(0 −1 −1)
(−1 −1 0)(0 0 −1)(0 −1 0)(−1 0 0)

Nr 1: 00000 0000, 0
Nr 2: 00000 0001, 1
Nr 3: 00000 0010, 1
Nr 4: 00000 0011, 2
Nr 5: 00000 0100, 1
Nr 6: 00000 0101, 2
Nr 7: 00000 0110, 2
Nr 8: 00000 0111, 3
Nr 9: 00000 1000, 3
Nr 10: 00000 1100, 4
Nr 11: 00001 0000, 3
Nr 12: 00001 0001, 4
Nr 13: 00001 1000, 4
Nr 14: 00001 1001, 5
Nr 15: 00010 0000, 5
Nr 16: 00010 0010, 6
Nr 17: 00100 0000, 3
Nr 18: 00100 0001, 4
Nr 19: 00100 0100, 4
Nr 20: 00100 0101, 5

TABLE 2(continued)

Nr 21: 00100 1000, 4
Nr 22: 00100 1001, 5
Nr 23: 00100 1100, 5
Nr 24: 00100 110*M*, 6 (1)*F*
Nr 25: 00110 0000, 6
Nr 26: 00110 0001, 7
Nr 27: 01000 0000, 7
Nr 28: 01000 0100, 8
Nr 29: 10000 0000, 5
Nr 30: 10000 0010, 6
Nr 31: 10000 0100, 6
Nr 32: 10000 0110, 7
Nr 33: 10001 0000, 8
Nr 34: 10100 0000, 8
Nr 35: 10100 0100, 9

Case A4R 

There are 10 weights:
(1 0 0 0)(0 1 0 0)(0 0 1 0)(0 0 0 1)(1 1 0 0)(0 1 1 0)
(0 0 1 1)(1 1 1 0)(0 1 1 1)(1 1 1 1)
Nr 1: 00000 00000, 0, (11111)
Nr 2: 00000 00001, 1, (2111)
Nr 3: 00000 00010, 2, (2111)
Nr 4: 00000 00100, 2, (2111)
Nr 5: 00000 00110, 3, (221)
Nr 6: 00000 01000, 3, (2111)
Nr 7: 00000 01100, 4, (221)
Nr 8: 00000 10000, 3, (2111)
Nr 9: 00000 10001, 4, (221)
Nr 10: 00000 11000, 5, (221)
Nr 11: 00001 00000, 3, (2111)
Nr 12: 00001 00010, 4, (221)
Nr 13: 00001 01000, 5, (311)
Nr 14: 00001 10000, 5, (221)
Nr 15: 00001 11000, 6, (32)
Nr 16: 00010 00000, 4, (2111)
Nr 17: 00010 00100, 5, (311)
Nr 18: 00010 10000, 6, (311)
Nr 19: 00011 00000, 5, (221)
Nr 20: 00011 00100, 6, (311)
Nr 21: 00011 10000, 7, (32)
Nr 22: 00100 00000, 4, (2111)
Nr 23: 00100 00001, 5, (221)
Nr 24: 00100 00010, 6, (221)
Nr 25: 00101 00000, 6, (311)
Nr 26: 00101 00010, 7, (32)
Nr 27: 00110 00000, 7, (311)
Nr 28: 00111 00000, 8, (41)
Nr 29: 01000 00000, 4, (2111)
Nr 30: 01000 00001, 5, (221)
Nr 31: 01000 00100, 6, (221)

TABLE 2(continued)

Nr 32: 01000 01000, 6, (311)
Nr 33: 01000 01100, 7, (32)
Nr 34: 01010 00000, 6, (221)
Nr 35: 01010 00100, 7, (32)
Nr 36: 01010 01000, 7, (311)
Nr 37: 01010 01100, 8, (32)
Nr 38: 01100 00000, 7, (311)
Nr 39: 01100 00001, 8, (32)
Nr 40: 01110 00000, 9, (41)
Nr 41: 10000 00000, 4, (2111)
Nr 42: 10000 00010, 5, (311)
Nr 43: 10000 01000, 5, (221)
Nr 44: 10000 01010, 6, (311)
Nr 45: 10000 10000, 6, (311)
Nr 46: 10000 11000, 7, (32)
Nr 47: 10010 00000, 6, (221)
Nr 48: 10010 00010, 7, (311)
Nr 49: 10010 10000, 8, (41)
Nr 50: 10100 00000, 6, (221)
Nr 51: 10100 00010, 7, (32)
Nr 52: 10100 10000, 7, (311)
Nr 53: 10100 10010, 8, (32)
Nr 54: 10110 00000, 8, (32)
Nr 55: 10110 10000, 9, (41)
Nr 56: 11000 00000, 7, (311)
Nr 57: 11000 01000, 8, (41)
Nr 58: 11010 00000, 8, (32)
Nr 59: 11010 01000, 9, (41)
Nr 60: 11100 00000, 9, (41)
Nr 61: 11110 00000, 10, (5)

Case A4C 

There are 10 weights:
(−1 −1 −1 −1)(0 −1 −1 −1)(−1 −1 −1 0)
(0 0 −1 −1)(0 −1 −1 0)(−1 −1 0 0)
(0 0 0 −1)(0 0 −1 0)(0 −1 0 0)(−1 0 0 0)
Nr 1: 00000 00000, 0
Nr 2: 00000 00001, 1
Nr 3: 00000 00010, 1
Nr 4: 00000 00011, 2
Nr 5: 00000 00100, 1
Nr 6: 00000 00101, 2
Nr 7: 00000 00110, 2
Nr 8: 00000 00111, 3
Nr 9: 00000 01000, 1
Nr 10: 00000 01001, 2
Nr 11: 00000 01010, 2
Nr 12: 00000 01011, 3
Nr 13: 00000 01100, 2
Nr 14: 00000 01101, 3
Nr 15: 00000 01110, 3
Nr 16: 00000 01111, 4

TABLE 2(continued)

Nr 17: 00000 10000, 3
Nr 18: 00000 10100, 4
Nr 19: 00000 11000, 4
Nr 20: 00000 11100, 5
Nr 21: 00001 00000, 3
Nr 22: 00001 00001, 4
Nr 23: 00001 01000, 4
Nr 24: 00001 01001, 5
Nr 25: 00001 10000, 4
Nr 26: 00001 10001, 5
Nr 27: 00001 11000, 5
Nr 28: 00001 11001, 6
Nr 29: 00010 00000, 3
Nr 30: 00010 00001, 4
Nr 31: 00010 00010, 4
Nr 32: 00010 00011, 5
Nr 33: 00010 10000, 6
Nr 34: 00011 00000, 4
Nr 35: 00011 00001, 5
Nr 36: 00011 00010, 5
Nr 37: 00011 00011, 6
Nr 38: 00011 10000, 7
Nr 39: 00100 00000, 5
Nr 40: 00100 00010, 6
Nr 41: 00100 01000, 6
Nr 42: 00100 01010, 7
Nr 43: 00110 00000, 6
Nr 44: 00110 00010, 7
Nr 45: 00110 01000, 7
Nr 46: 00110 01010, 8
Nr 47: 01000 00000, 5
Nr 48: 01000 00001, 6
Nr 49: 01000 00100, 6
Nr 50: 01000 00101, 7
Nr 51: 01000 10000, 6
Nr 52: 01000 10001, 7
Nr 53: 01000 10100, 7
Nr 54: 01000 10101, 8
Nr 55: 01100 00000, 8
Nr 56: 01100 00001, 9
Nr 57: 10000 00000, 7
Nr 58: 10000 00010, 8
Nr 59: 10000 00100, 8
Nr 60: 10000 00110, 9
Nr 61: 10001 00000, 10

6.5. The non-trivial sheets

In [4], the sheets of a G-module V are defined as the maximal locally closed irreducible subsets of V which consists of orbits of a fixed

TABLE 3. The non-trivial sheets.

Case	dim(W)	codim(W)	section C	other orbits in sheet S
G2R	5	1	12	6, 10, 11.
G2C	5	1	11	7, 9, 10.
B3R	7	2	24	13, 20, 22, 23.
B3C	8	1	31	23, 27, 29, 30.
C3R	8	1	32	19, 28, 30, 31.
C3C	6	3	24	14, 20, 22, 23.

dimension. It follows that every sheet which contains more than one orbit, contains infinitely many orbits. On the other hand, every G-final section is contained in one sheet, cf. 1.5. So, in the cases where a final system of representing sections is obtained, the determination of the sheets of V is rather straightforward.

If the number of orbits is finite, the sheets coincide with the orbits. So we restrict ourselves to the cases of Table 2 where infinitely many orbits occur. These are the cases of G2, B3 and C3. Each of these cases has one series C with $\dim(C) = 1$. In each of these cases, the set GC is open and dense in a G-submodule W of V. So, the corresponding sheet S is open and dense in W and it consists of the orbits in W of dimension equal to $\dim(W) - 1$. It is easy to determine the remaining orbits in S. The results are given in Table 3. The sections are characterized by the identification number given in Table 2. In these six cases, every other orbit is equal to the sheet in which it is contained.

REMARK: In the case G2R the occurrence of a series can be nicely explained. Here G is the Borel subgroup of a simple group of type G_2, and $V = \boldsymbol{u}$ is the Lie algebra of the unipotent radical U of G. The system of roots of G and the system of weights of V are both equal to the system of positive roots in the root system G_2, see Diagram 3. By Table 2 the algorithm yields a classification of the G-orbits in V. There is one series, namely number 12, given by

$$C = e_\alpha + e_{\alpha+2\beta} + Me_{\alpha+3\beta}.$$

Since $\dim(GC) = 5$, it is clear that GC is open and dense in the ideal W spanned by the root vectors e_γ at the five positive roots $\gamma \neq \beta$; here β is the short simple root. The fact that G has infinitely many orbits in W, has a simple explanation. It suffices to explane, that G has infinitely many orbits in the factor module W/W_1 where W_1 is the ideal

$$W_1 = [W, W] = Ke_{2\alpha+3\beta}.$$

The image of G in the group $\mathrm{GL}(W/W_1)$ is equal to G/Q where Q is

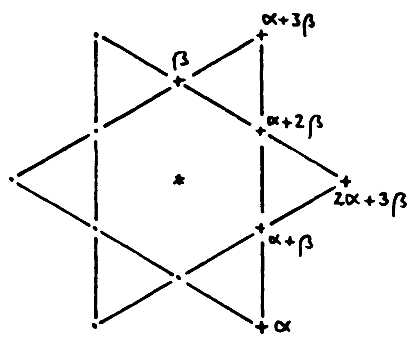


Diagram 3. The root system G_2.

+ positive root

* origin

. negative root

the unipotent subgroup of G corresponding to the ideal W. Since

$$\dim(G/Q) = 3 < 4 = \dim(W/W_1),$$

the group G does not have a dense orbit in W/W_1. Therefore G has infinitely many orbits in W/W_1 and hence in W.

6.6. Remarks

(a) Brodskii [7] investigates the orbit structure for A3C with the action of the maximal unipotent group U instead of the Borel group. He obtains 14 families of orbits instead of 16. So he misses two families of orbits. This illustrates the point that even in cases where the calculation can be done by hand, a computer verification may be useful.

(b) If a connected algebraic group G has finitely many orbits in a G-module V, then the number of orbits in V is equal to the number of orbits in the dual module V^*. This theorem of Pyasetskii, cf. [14], is illustrated here by the cases A2, B2, A3, A4. It seems that even in the case of infinitely many orbits the orbit structures are closely related. The relation between the dimensions of the orbits in V and V^* is not clear; see B2R and B2C.

(c) In [10], a classification for the cases AnR is proposed by means of upper triangular Boolean matrices (called typrices). The methods used here and there are completely different and the resulting classes have a different meaning. Unfortunately, the first differences in the tables of results occur for A5R. In fact, the matrices x_t, $t \in \mathbb{C}$, all have the same typrix A. However,

$$x_t = \begin{pmatrix} 0 & 0 & 0 & 1 & t & 0 \\ & 0 & 1 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 1 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 & 0 & 1 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}.$$

this family intersects two different conjugacy classes for the action of the invertible upper triangular matrices: x_t is conjugate to x_1 if and only if $t \neq 0$.

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