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Arithmetic vector bundles and automorphic forms on Shimura varieties, II

MICHAEL HARRIS [1]

Introduction

Let (G, X) be the datum defining a Shimura variety $M(G, X)$. In Part I of this paper [72] we constructed a family of vector bundles on $M(G, X)$, with homogeneous $G(\mathbf{A}^f)$-action, defined over specified number fields. (We refer to such vector bundles henceforward as *automorphic vector bundles*.) In the present part, we apply the results of Part I to the study of sections of automorphic vector bundles, which (as we recall in 5.3 below) are naturally identified with homomorphic automorphic forms on G. Our main result is Theorem 6.4, which deals with the arithmeticity of Fourier-Jacobi series. In order to explain just what Theorem 6.4 says about Fourier-Jacobi series, we need to introduce some notation. All paragraph numbers beginning with digits $\leqslant 4$ are references to Part I.

It is easy enough to define Fourier-Jacobi series analytically. Let P be a rational maximal parabolic subgroup of G. Then there is a subgroup $G_P \subset P$ and a "boundary component" F_P of X, stable under P, such that (G_P, F_P) is the datum defining a Shimura variety; moreover, the reflex field $E(G_P, F_P)$ is contained in $E(G, X)$ (cf. 5.1, 6.1). Suppose $[\mathcal{V}]$ is one of the arithmetic vector bundles on $M(G, X)$ constructed in §3 and §4, and let $f \in \Gamma(M(G, X), [\mathcal{V}])$. The Fourier-Jacobi series of f, in the sense of Piatetskii-Shapiro [47], is a power series in certain exponentials whose coefficients are theta-functions on a polarized abelian scheme over $M(G_P, F_P)$. In the variables (z, u, t) representing (a connected component of) X as a Siegel domain of the third kind over (a connected component of) F_P, we write

$$f = \sum_{\underline{\alpha}} f_{\underline{\alpha}}(u, t)\, e^{2\pi i \underline{\alpha}(z)}. \tag{0.1}$$

We are concerned primarily with the values of these theta functions along the zero section of this abelian scheme; i.e. the set $\{u = 0\}$. These theta null-values then have the automorphy properties of sections of a vector bundle $[\tilde{V}_{\rho_P}]$ over $M(G_P, F_P)$. However, the identification of an automorphic form with a section of a vector bundle is not determined uniquely. In order to study the

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arithmetic properties of Fourier-Jacobi series, we need to construct directly a map of global sections

$$\mathrm{F.J.}^{P}: \Gamma(M(G, X), [\mathscr{V}]) \to \hat{\bigoplus_{\underline{\alpha}}} \Gamma\left(M(G_P, F_P), \left[\tilde{V}_{\rho_P}\right]\right) \tag{0.2}$$

where $\{\underline{\alpha}\}$ parametrizes the Fourier-Jacobi coefficients along F_P and $\hat{\oplus}$ is the completed direct sum with respect to a natural topology.

In the following discussion we suppress the distinction between the Fourier-Jacobi coefficients and their null values. This can be done by replacing (G, X) by a sub-pair corresponding in Siegel domain coordinates to $\{(z, 0, t)\}$. We assume, therefore, that u is identically zero in what follows.

In order to construct (0.2), we assume $[\mathscr{V}] \subset [\tilde{V}_\rho]$, where $\rho: G \to GL(V_\rho)$ is a finite-dimensional representation (notation 3.4); we actually need a slightly stronger assumption (5.3.6). For simplicity assume ρ is defined over $\mathbf{Q}$. We construct a complex analytic space Ξ which fits naturally into a diagram

$$\begin{array}{ccc} & \Xi & \\ {}^{\pi}\swarrow & & \searrow^{B} \\ M(G_P, F_P)(\mathbf{C}) & & M(G, X)(\mathbf{C}) \end{array}$$

such that there exists a natural imbedding

$$\pi^*\left[\tilde{V}_{\rho_P}\right] \hookrightarrow B^*\left[\tilde{V}_\rho\right]. \tag{0.3}$$

The space Ξ may be regarded as a tubular neighborhood of the boundary component corresponding to $M(G_P, F_P)$ in the Baily-Borel compactification of $M(G, X)$. The imbedding (0.3) can be normalized so that the canonical local system $\pi^* \mathrm{V}_{\rho_P}^{\nabla} \subset \pi^*[\tilde{V}_{\rho_P}]$ is taken to the canonical local system $B^* V_\rho^{\nabla} \subset B^*[\tilde{V}_\rho]$, and is then essentially determined over $E(G, X)$. A linear transformation of $B^*[\tilde{V}_\rho]$ then identifies $B^*[\mathscr{V}]$ with a subbundle of $\pi^*[\tilde{V}_{\rho_P}]$. In this way one easily identifies $\pi_* B^*[\mathscr{V}]$ with a subbundle of $\hat{\bigoplus}_{\underline{\alpha}} [\tilde{V}_{\rho_P}]$, and the morphism (0.2) now has an obvious definition. Our main theorem states

THEOREM 6.4: *The homomorphism* (0.2) *is rational over the field of definition of* $[\mathscr{V}]$.

We remark that Brylinski proved the above theorem in the cases in which $M(G, X)$ is a moduli space for abelian varieties with level structure and a family of absolute Hodge cycles [31]. Our proof, which was suggested by work of Shimura and Garrett, uses only the special case in which $M(G, X)$

parametrizes elliptic curves; in this case, Theorem 6.4 is essentially contained in the work of Deligne and Rapoport [35].

A number of consequences of Theorem 6.4 are derived in the text. One easy consequence is a (characteristic zero) *q-expansion principle* (Theorem 6.9). A more interesting consequence is a theorem on the rationality of holomorphic Eisenstein series (Theorem 8.5), which generalizes the main result of [12]. Roughly speaking, if $f \in \Gamma(M(G_P, F_P), [\tilde{V}_{\rho_P}])$ is a cusp form, and if $[\tilde{V}_{\rho_P}]$ satisfies certain hypotheses, then we can define an absolutely convergent homomorphic Eisenstein series $E(f) \in \Gamma(M(G, X), [\mathscr{V}])$. Our theorem states, among other things, that the homomorphism $f \mapsto E(f)$ is rational over the field of definition of $[\mathscr{V}]$.

Section 7 is not related to Fourier-Jacobi series at all. Its subject is the explicit classification of some of the holomorphic differential operators provided by Theorem 4.8. The classification depends on the theory of modules of homomorphic type over enveloping algebras, and is a generalization of the techniques introduced in [37].

The main technical tools for the study of Fourier-Jacobi series are developed in §5. It was Garrett who explained to me the importance of the domain $\Delta(P)$, and I thank him for his observations which, in conjunction with unpublished notes of Deligne, made possible such a straightforward development of the theory. Section 5 also contains a very brief discussion of Shimura's method of determining rationality of automorphic forms by their values at CM points. The contents of §6–§8 have been discussed above. The final section contains a list of questions not treated in this paper.

I have already expressed my gratitude to Garrett for his suggestions. Otherwise I have nothing to add to the acknowledgments already noted in Part I, except to repeat my thanks to the Institute for Advanced Study and Columbia University for their hospitality.

Notations and conventions

The notation of Part I remains in force. We find it convenient occasionally to write $z \in \mathbf{G}_m$ instead of saying "z is a geometric point of $\mathbf{G}_m$."

References to sections or formulas in Part I will be given simply by the section numbers, without any further comment.

§5. Boundary components and trivializations

This section is primarily intended as a compilation of technical results relevant to the theory of Fourier-Jacobi expansions, which will be treated in §6. Most of these results are familiar from the standard literature on boundary components of Hermitian symmetric spaces as developed, for example, in [26,1,49,65]. However, these sources only carry out many of these constructions over **R** or

C, whereas we need information over number fields. Our method is to rely on results available in the literature and explain, as briefly as possible, how to derive what we need from them. Concepts not explained below are discussed in the articles cited above.

Our presentation has been influenced primarily by the treatment in [26] and by unpublished notes of Deligne [34]. Some of the contents of the latter can be found in the thesis of Brylinski [31]. In particular, Theorem 5.1.3, due to Deligne, has not been published anywhere, but is cited in [31], and in general can be seen as a reformation of the results of the standard sources mentioned above.

Conversations with P. Garrett were very helpful in clarifying the contents of this section, especially 5.1.

5.1: We begin as always by choosing a pair (G, X), satisfying (1.1.1) – (1.1.4). We assume G to have $\mathbf{Q}$-rank $\geqslant 1$. It will be convenient to fix a connected component X^+ of X.

Let P be a rational maximal parabolic subgroup of G, with unipotent radical $W = W_P$; let $U = U_P$ be the center of W. The subgroup $P(\mathbf{R})^0 \subset G(\mathbf{R})^0$ fixes a unique boundary component $F^+ = F_P^+$ of X^+, in the sense of the references recalled above. By definition, F^+ is a rational boundary component of X^+. Let $A = A_P$ be a fixed split component of P; it is a $\mathbf{Q}$-rational torus in P, one-dimensional and split modulo Z_G.

For any $h \in X$, let $w_h = h \circ w : \mathbf{G}_{m,\mathbf{R}} \to G_{\mathbf{R}}$ be the weight morphism; we know by (1.1.1) that it does not depend on h, and by (1.1.3) that it is defined over $\mathbf{Q}$. We denote it $w_0 : \mathbf{G}_m \to G$.

We recall that if $\rho : G \to GL(V)$ is a rational representation, then each $h \in X$ determines a Hodge filtration

$$F_h^p(V) \subset F_h^{p-1}(V) \subset \dots \subset V$$

as in 3.1.3. On the other hand, if $\psi : \mathbf{G}_m \to A_P$ is a homomorphism, one can define an increasing filtration

$$W_i^{\psi} V \overset{\text{def}}{=} \bigoplus_{j \leqslant i} V_{j,\psi} \subset W_{i+1}^{\psi} V \dots$$

where

$$V_{j,\psi} = \{ v \in V \mid \psi(t) v = t^j v,\ t \in \mathbf{G}_m \}.$$

Following Deligne, we call ψ *admissible* (for P) if

$$(5.1.1) \quad W_0^{\psi} = \mathrm{Lie}(P); \quad \psi \circ w_0^{-1}(\mathbf{G}_m) \subset G^{\mathrm{der}}.$$

the first equality refers to the adjoint representation of G on $\mathfrak{g}$. We call ψ a *Cayley morphism* if the corresponding filtration W_i^{ψ} has the property

(5.1.2) (W_ι^ψ, F_h^P) define a mixed Hodge structure (cf. [69]) on V for any $h \in X$ and any rational representation $\rho: G \to GL(V)$.

(Actually, Deligne works with the P-conjugacy class of the morphism ψ, but we prefer to fix A_P.) Deligne proved

5.1.3 THEOREM: (*Deligne*, [34], §3.1; *cf.* [31]). *There is a unique admissible Cayley morphism* $w_P: \mathbf{G}_m \to A_P$.

We denote the corresponding filtration $W_\cdot^P$ instead of $W_\cdot^{w_P}$.

Since the Hodge structure $F_{ih}^\cdot \mathfrak{g}$ is of type $(-1, 1) + (0, 0) + (1, -1)$, it follows easily from (5.1.2) that the weight filtration on $\mathfrak{g}$ is of the form

$$(5.1.4) \quad \{0\} = W_{-3}^P \mathfrak{g} \subset W_{-2}^P \mathfrak{g} \subset \ldots \subset W_2^P \mathfrak{g} = \mathfrak{g}.$$

We let $\mathfrak{g}^i \subset \mathfrak{g}$ be the t^i eigenspace of $w_P(t)$, $t \in \mathbf{G}_m$. Then we have

$\mathfrak{g}^{-2} = \text{Lie } U$; $\mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} = \text{Lie } W$;

$\mathfrak{g}^0$ is the centralizer of A_P: i.e., a Levi component of Lie P.

We make the following definitions:

(5.1.5)
$\mathfrak{g}_l^0 = [\mathfrak{g}^{-2}, \mathfrak{g}^2]$; $\mathfrak{g}_l = \mathfrak{g}^{-2} \oplus \mathfrak{g}_l^0 \oplus \mathfrak{g}^2$

$\mathfrak{g}_h$ = the orthogonal complement of $\mathfrak{g}_l^0$ in $\mathfrak{g}^0$, with respect to the Killing form.

G_l^0, G_l, and G_h are the connected subgroups of G whose Lie algebras are $\mathfrak{g}_l^0$, $\mathfrak{g}_l$, and $\mathfrak{g}_h$, respectively.

These Lie algebras and Lie groups are all defined over $\mathbf{Q}$ and are all reductive. It is easy to check that G_l and G_h are mutual centralizers in G (cf. [31], 4.1.2). We let $G^{\text{even}} = G_P^{\text{even}} = G_l \cdot G_h$ be the centralizer of $w_P(-1)$. Let $\Delta(P)^+$ be the subset of X^+ fixed by $w_P(-1)$.

In what follows, we will frequently make use of a weakened version of axiom (1.1.2). We denote this weakened version (1.1.2)*:

(1.1.2)* The automorphism $\text{ad}(h(i))$ of $G(\mathbf{R})$ induces a Cartan involution on $G^{\text{der}}(\mathbf{R})^0$. (Cf. Remark 5.1.6.3, below.)

5.1.6 LEMMA:

(*i*) *The set* $\Delta(P)^+$ *is non-empty, and consists of homomorphisms* $h: S \to G_{\mathbf{R}}^{\text{even}}$.

(*ii*) *Let* $\Delta(P)$ *be the* $G^{\text{even}}(\mathbf{R})$*-conjugacy class of homomorphisms* $h: S \to G_{\mathbf{R}}^{even}$ *containing* $\Delta(P)^+$. *Then* $(G^{even}, \Delta(P))$ *satisfies* (1.1.1), (1.1.2)*, (1.1.3), *and* (1.1.4).

(*iii*) *Let* $h \in \Delta(P)$, *and let* $\tilde{G}$ *be the derived subgroup of* G^{even}. *Then* $ad(h(i))$ *is a Cartan involution of* $\tilde{G}(\mathbf{R})^0$.

PROOF: Assuming (i), it is clear that (1.1.3) and (1.1.4) hold for $(G^{\text{even}}, \Delta(P))$. The remaining statements concern only the structure of the real group $G(\mathbf{R})$. Moreover, if $G_{\mathbf{R}} = G_1 \times \ldots \times G_n$, and if $X = X_1 \times \cdots \times X_n$ correspondingly, then the truth of the lemma for (G, X) follows from the corresponding statements for (G_i, X_i), $i = 1, \ldots, n$. Thus we may assume G is an almost simple group over $\mathbf{R}$. If X is a point, then the lemma is obvious; thus we may exclude that case.

We first prove (i). Let $h \in X^+$, and define $\mathfrak{p}^+$ as in (3.1.1). The Borel-Harish-Chandra imbedding corresponding to h identifies X^+ with a bounded domain in $\mathfrak{p}^+$ in such a way that h is identified with the origin in $\mathfrak{p}^+$ (cf. [26], p. 170). Changing h if necessary, we may assume that F^+ is one of the standard boundary components with respect to this imbedding; i.e., that F^+ is one of the boundary components denoted F_S in [26], p. 196. In particular, if $\sigma_h = \operatorname{ad}(h(i))$ is the Cartan involution of $G^{\text{der}}(\mathbf{R})^0$ corresponding to h, as in (1.1.3), then $\sigma_h \circ w_P = w_P^{-1}$ (cf. [34], 3.1.4). Thus $w_P(-1)$ is fixed by σ_h, i.e., it commutes with $h(\underline{S})$. In other words, $h \in \Delta(P)^+$. The second part of (i) is obvious.

The statement (ii) follows immediately from the corresponding statements for (G, X). To prove (iii), we argue as follows. Let $h \in \Delta(P)$, let K_h be the centralizer of h in G, and let

$$(5.1.6.1) \qquad \mathfrak{g}^{\text{der}} = \left(\mathfrak{k}_h \cap \mathfrak{g}^{\text{der}}\right) \oplus \mathfrak{p}_h$$

be the corresponding Cartan decomposition. By hypothesis, (5.1.6.1) is the eigenspace decomposition for $\sigma_h = \operatorname{ad}(h(i))$. Let $\tau = w_P(-1)$. Since σ_h and τ commute, each term in (5.1.6.1) is the direct sum of its (± 1)-eigenspaces for τ:

(5.1.6.2)

$$\mathfrak{g}^{\text{der}} = \left[\left(\mathfrak{k}_h \cap \mathfrak{g}^{\text{der}}\right)^{\tau = +1} \oplus \mathfrak{p}_h^{\tau = +1}\right]\left[\left(\mathfrak{k}_h \cap \mathfrak{g}^{\text{der}}\right)^{\tau = -1} \oplus \mathfrak{p}_h^{\tau = -1}\right].$$

The sum of the first two terms in the right-hand side of (5.1.6.2) is the sum of $\tilde{\mathfrak{g}}$ and a subalgebra of the center of $\mathfrak{g}^{\text{even}}$. Thus (iii) is a consequence of (5.1.6.2).

5.1.6.3 REMARK: We see that $(G^{\text{even}}, \Delta(P))$ satisfies the axioms for the data defining a Shimura variety, except that $G^{\text{even}}(\mathbf{R})$ may have a semisimple compact factor defined over $\mathbf{Q}$. However, one still has canonical models in this case. Suppose, in fact, that G' is the maximal quotient of G^{even}, defined over $\mathbf{Q}$, such that for every $h \in \Delta(P)$, the image of h in G' is trivial. Let $G'' \subset G^{\text{even}}$ be the kernel of the map $G^{\text{even}} \to G'$; then $(G'', \Delta(P))$ is a pair satisfying (1.1.1)–(1.1.4). Thus the canonical model $M(G'', \Delta(P))$ exists, as do the arithmetic vector bundles and the canonical local systems of Part I. We let

$$M(G^{\text{even}}, \Delta(P)) = M(G, \Delta(P)) \times^{G''(\mathbf{A}^f)} G^{\text{even}}(\mathbf{A}^f),$$

and define arithmetic vector bundles and canonical local systems similarly. It is easy to see that the objects thus constructed have the same properties as canonical models, etc., constructed in the case of pairs satisfying (1.1.1-4) and we take this for granted henceforward.

In preparation for the remainder of this section, we recall the realization of X^+ as a *Siegel domain of the third kind* over F^+ [47,66,49]. Following Deligne [34] (cf. [26], p. 227 ff.) we may identify X^+ with the domain

(5.1.7)

$$\mathcal{S}_P = \{(z, u, t) \in \mathfrak{g}^{-2}(\mathbf{C}) \times \mathfrak{g}^{-1}(\mathbf{R}) \times F_P^+ \mid \mathrm{Im}\, z - B_t(u, u) \in C\}.$$

Here C is a self-dual cone in $\mathfrak{g}^{-2}(\mathbf{R})$, homogeneous under the adjoint action of $G_l^0(\boldsymbol{R})$, and B_t is a certain real bilinear form on $\mathfrak{g}^{-1}(\mathbf{R})$ with values in $\mathfrak{g}^{-2}$, depending real analytically on $t \in F_P^+$.

5.1.8 LEMMA:

(*i*) *The Hermitian symmetric space $\Delta(P)^+$ decomposes as a product $F^+ \times D_P^+$, where D_P^+ is a tube domain. The group $G_h(\mathbf{R})^0$ acts trivially on D_P^+ and transitively on F^+, and $G_l(\mathbf{R})^0$ acts trivially on F^+ and transitively on D_P^+.*

(*ii*) *The tube domain D_P^+ is rational with respect to the quotient G^{even}/G_h; i.e., it has a zero-dimensional boundary component whose normalizer is a rational minimal parabolic subgroup of G^{even}/G_h.*

PROOF:

(i) We identify X^+ with $\mathcal{S}_P$, as in (5.1.7). It follows from the Korányi-Wolf theory of Siegel domains of the third kind [66] that $w_P(\mathbf{G}_m)$ preserves the three factors in (5.1.6), and that

$$w_P(a)(z, u, t) = (a^{-2}z, a^{-1}u, t).$$

In other words

$$\Delta(P)^+ \approx \{(z, u, t) \in \mathcal{S}_P \mid u = 0\} = D_P^+ \times F_P^+$$

where $D_P^+ = \mathfrak{g}^{-2}(\mathbf{R}) \oplus iC$ is a tube domain in $\mathfrak{g}_{\mathbf{C}}^{-2}$. The remaining assertions are also consequences of the general theory.

(ii) Let $G_l' = G^{\mathrm{even}}/G_h$, and let D be the image in $G_l'(\mathbf{R})$ of $\Delta(P)$, under the natural morphism. Then (G_l', D) is a pair satisfying (1.1.1)–(1.1.4) (sic). Let $D^+ \subset D$ be the image of $\Delta(P)^+$ in $G_l'(\mathbf{R})$. It follows from (i) that D^+ may be identified with D_P^+. This realization of D^+ as a tube domain corresponds to the maximal parabolic subgroup P_l of G_l', whose Lie algebra is $\mathfrak{g}^{-2} \oplus \mathfrak{g}_l^0$. Since P_l is rational, the assertion is clear.

5.1.9 Let $\rho: G \to GL(V)$ be a faithful representation, and let $\mathrm{Gr}^P(V) = \bigoplus_l W_l^P V / W_{l-1}^P V$ be the graded vector space associated to the weight filtration on V. Then $\mathrm{Gr}^P(V)$ is a faithful representation space for P/W_0. Let $Q_P \subset GL(V)$ be the parabolic subgroup stabilizing the weight filtration $W_\cdot^P V$; then P/W_0 is naturally a subgroup of $Q_P/R_u Q_P$.

If $h \in X$, then $\{F_h^\cdot, W_\cdot^P\}$ determine a mixed Hodge structure on V, and therefore a Hodge structure on $\mathrm{Gr}^P(V)$. We denote by $\bar{h}_P$ the corresponding homomorphism $\bar{h}_P: \underline{S} \to GL(\mathrm{Gr}^P(V_{\mathbf{R}}))$. Let M be the centralizer of A_P in Q_P. Then $\bar{h}_P(\underline{S}) \subset Q_P(\mathbf{R})/R_u Q_P(\mathbf{R})$, and $\bar{h}_P$ lifts uniquely to a homorphism $h_P: \underline{S} \to M_{\mathbf{R}}$.

5.1.10 Lemma: *The image of h_P is contained in $G_{h,\mathbf{R}} \cdot A_{P,\mathbf{R}}$.*

Proof: We first prove that $h_P(\underline{S}) \subset G_{\mathbf{R}}$. It suffices to prove that, if $\alpha \in V^{a,b} \overset{\text{def}}{=} V^{\otimes a} \otimes (V^*)^{\otimes b}$ is a rational invariant with respect to G, then it is invariant with respect to $h_P(\underline{S})$. It is easy to see that $h_P(\underline{S})$ respects the weight filtration $W_\cdot^P(V^{a,b})$, and reduces to the Hodge structure induced by $F_h^\cdot$ on $\mathrm{Gr}^P(V^{a,b})$. Now the Hodge structure induced by h on the $\mathbf{Q}$-linear span of α is of type (0, 0); moreover, $\alpha \in W_0^P(V^{a,b})$. It follows that $\bar{h}_P(\underline{S})$ acts trivially on the image of α in $W_0^P(V^{a,b})/W_{-1}^P(V^{a,b})$. Thus α is invariant with respect to $h_P(\underline{S})$.

We see that the image of h_P is contained in the centralizer $G_h \cdot G_l^0$ of A_P in G. Now the Hodge structure induced by $\{F_h^\cdot, W_\cdot^P\}$ on $W_{-2}^P\mathfrak{g}$ is necessarily of type $(-1, -1)$. It follows that, under the adjoint representation, $h_P(\underline{S})$ acts on $\mathfrak{g}_{\mathbf{R}}^{-2}$ via real homotheties. But G_h acts trivially on $\mathfrak{g}^{-2}$, whereas the representation of $\mathfrak{g}_l^0$ on $\mathfrak{g}^{-2}$ is faithful. The subgroup of $G_{h,\mathbf{R}} \cdot G_{l,\mathbf{R}}^0$ which acts on $\mathfrak{g}_{\mathbf{R}}^{-2}$ via real homotheties is thus $G_{h,\mathbf{R}} \cdot A_{P,\mathbf{R}}$.

We let $G_P = G_h \cdot A_p$. Let F_P be the $G_P(\mathbf{R})$-conjugacy class of homomorphisms $\underline{S} \to G_{P,\mathbf{R}}$ containing h_P, for $h \in X$. Let $\pi_P: X \to F_P$ be the map which sends $h \in X$ to $h_P \in F_P$. The following proposition is essentially due to Deligne.

5.1.11 Proposition:

(*i*) *The image $\pi_P(X^+) \subset F_P$ is a connected component of F_P, and is analytically isomorphic to F_P^+. In the coordinates* (5.1.7), *$\pi_P((z, u, t)) = t \in F_P^+$. The morphism π_P is $P(\mathbf{R})$-equivariant.*

(*ii*) *The pair (G_P, F_P) satisfies* (1.1.1), (1.1.2)*, (1.1.3), *and* (1.1.4). *Moreover, if $h \in F_P$, then $ad(h(i))$ is a Cartan involution of $G_P^{\mathrm{der}}(\mathbf{R})^0$.*

Proof: The assertions in (i) are established, though not precisely in the present form, in 3.3.7 of Deligne's notes [34]. The proof of (ii) is analogous to the proof of Lemma 5.1.6 above, and is omitted.

5.1.12 REMARK: The argument of Remark 5.1.6.3 applies to the pair (G_P, F_P), and implies that the canonical model $M(G_P, F_P)$ exists. It also implies that the results of §3 and §4 are valid for (G_P, F_P).

5.2. Canonical automorphy factors

Unfortunately, the standard treatments of canonical automorphy factors are carried out with **R** as ground field. We explain how to obtain results over the reflex field of a CM point, which is the appropriate context in which to study the values of automorphic forms at CM points.

5.2.1 Let $h \in X$, and let $\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_h \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$, as in (3.1.1); let K_h, P_h^+, and P_h^- be the corresponding subgroups of G. Let E_h be a field of definition for the subgroup $K_h P^+ \subset G$. It is well known that every element g of $G(\mathbf{R})^0$ may be represented in the form $g^+ J_h(g) g^-$, with $g^+ \in P_h^+(\mathbf{C})$, $g^- \in P_h^-(\mathbf{C})$, and $J_h(g) \in K_h(\mathbf{C})$. Moreover, we know

(5.2.1.1) The multiplication map $P_h^+ \times K_h \times P_h^- \to G$
$(p^+, k, p^-) \mapsto p^+ k p^-$
is injective.

(For these facts, cf. [26], p. 170). Thus g^+, $J(g)$, and g^- are uniquely determined.

Let X^+ be a connected component of X and assume $h \in X^+$. If $x \in X^+$, $g \in G(\mathbf{R})^0$, we write

(5.2.1.2) $J^{h,0}(g, x) = J_h(g g_x) \cdot J_h(g_x)^{-1}$

where g_x is any element of $G(\mathbf{R})^0$ such that $g_x(h) = x$. Then $J^{h,0}(g, x)$ is well-defined, and satisfies the following relations (we write J for $J^{h,0}$):

(5.2.1.3) $J(gg', x) = J(g, g'(x)) \cdot J(g', x)$;

(5.2.1.4) $J(k, h) = k \quad \forall k \in K_h(\mathbf{R})^0$;

(5.2.1.5) For any $g \in G(\mathbf{R})^0$, the function $J(g, \cdot): X^+ \to K_h(\mathbf{C})$ is holomorphic.

We call $J^{h,0}$ the canonical (bounded) automorphy factor for h.

5.2.2 Let P be a maximal rational parabolic subgroup of G, as in 5.1. Let $\Delta(P)^+$ be a connected component of $\Delta(P)$, and let $h \in \Delta(P)^+$; let K_h be the centralizer of $h \in G$. (Most of the construction which follows is valid for any $h \in X^+$, but it is convenient to assume $h \in \Delta(P)^+$). A *canonical automorphy* *factor* for the pair P, h is a morphism

$$J : G(\mathbf{R})^0 \times X^+ \to K_h(\mathbf{C})$$

satisfying (5.2.1.2)–(5.2.1.4) and the additional hypotheses

(5.2.2.1) Let $N_P \subset P$ be the subgroup with Lie algebra $\mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}_l^0$, in the notation of 5.1. Then the map $J(\cdot, x) : N_P(\mathbf{R})^0 \to K_h(\mathbf{C})$ comes from a homomorphism $N_P \to K_h$ of algebraic groups, and is independent of $x \in X^+$. This homomorphism is *trivial* on U_P.

(5.2.2.2) The restriction of J to $G^{\text{even}}(\mathbf{R})^0 \times \Delta(P)^+$ takes values in $K_h(\mathbf{C}) \cap G^{\text{even}}(\mathbf{C})$. Denote this function

$$J' : G^{\text{even}}(\mathbf{R})^0 \times \Delta(P)^+ \to K_h(\mathbf{C}) \cap G^{\text{even}}(\mathbf{C}).$$

(5.2.2.3) Let J'' be the composite of J' with projection $K_h'' \stackrel{\text{def}}{=} K_h(\mathbf{C}) \cap G^{\text{even}} / K_h(\mathbf{C}) \cap G_\ell(\mathbf{C})$. Then

$$J''(g, (z, t)) = J^{\pi_P(h),0}(g, t)$$

$$g \in G_h(\mathbf{R})^0, \ z \in D_P^+, \ t \in F_P^+.$$

5.2.3 We construct the canonical automorphy factor for P, h, as follows. First assume F^+ is a point, and that X^+ is a rational tube domain. Thus $G^{\text{der}} = G_l$, and $D_P^+ = X^+$. Associated to K_h, a maximal torus $T \subset K_h$, and the choice of a maximal set of strongly orthogonal non-compact roots for $T(\mathbf{C})$ in $\mathfrak{g}_{\mathbf{C}}$, one obtains a standard point boundary component F_∞ of X^+ (cf. [26], Chapter III, §3). The stabilizer of F_∞ is a maximal parabolic $P_\infty \subset G$, defined over $\mathbf{R}$. We also have the Cayley transform $\bar{c}_\infty \in G^{\text{ad}}(\mathbf{C})$, corresponding to the realization of D_P^+ as a tube domain over F_∞ ([66], §2). Let c_∞ be a lifting of $\bar{c}_\infty$ to $G(\mathbf{C})$. We have $c_\infty(K_h P_h^+)c_\infty^{-1} = P_\infty$. Now $K_h(\mathbf{R})$ acts transitively on the set of point boundary components of X^+; thus there exists $k \in K_h(\mathbf{R})$ such that, if $c = kc_\infty$, then $c(K_h P_h^+)c^{-1} = P$. On the other hand, $K_h P_h^+$ is a parabolic subgroup, hence equal to its own normalizer in G. Thus if $d(K_h P_h^+)d^{-1} = P$, for some $d \in G(\mathbf{C})$, then $d = cp$ for some $p \in K_h(\mathbf{C})P_h^+(\mathbf{C})$. It follows easily from [30], 4.13(c) that we may choose $c = c_{h,P} \in G(E_h)$ such that, for some $p \in K_h(\mathbf{C})P_h^+(\mathbf{C})$, $k \in K_h(\mathbf{R})$, we have

(5.2.3.1) $\quad cp = kc_\infty$ and $\quad c(K_h P_h^+)c^{-1} = P.$

Now we consider the general case. We have assumed $h \in \Delta(P)^+$. The image of $\Delta(P)$ under the natural map $G^{\text{even}} \to G_l^{\text{ad}}$ is contained in a $G_l^{\text{ad}}(\mathbf{R})$-conjugacy class D of homomorphisms $\underline{S} \to G_{l,\mathbf{R}}^{\text{ad}}$. It follows from Lemma 5.1.7 that

(G_l^{ad}, D) satisfies (1.1.1) and (1.1.2)*, and that each connected component of D is analytically equivalent to the rational tube domain D_P^+. Let P_l, $K_{h,l}$, and $P_{h,l}^+$ be the subgroups of G_l^{ad} with Lie algebras $\mathfrak{g}_l^0 \oplus \mathfrak{g}^{-2}$, $\mathfrak{k}_h \cap \mathfrak{g}_l$, and $\mathfrak{p}^+ \cap \mathfrak{g}_l$, respectively. Let $\overline{c_{\infty,l}}$ be the standard Cayley transform in $G_l^{\text{ad}}(\mathbf{C})$, as above. There is an element $\bar{c}_l \in G_l^{\text{ad}}(E_h)$ satisfying the analogue of (5.2.3.1): for some $P \in K_{h,l}(\mathbf{C}) P_{h,l}^+(\mathbf{C})$, $k \in K_{h,l}(\mathbf{R})$, we have

$$(5.2.3.2) \quad \bar{c}_l p = k\overline{c_{\infty,l}} \text{ and } \bar{c}_l(K_{h,l}P_{h,l}^+)\bar{c}_l^{-1} = P_l.$$

Now let c be any lifting of $\bar{c}_l$ to G_l. Up to replacing $\bar{c}_l$ by another solution of (5.2.3.2), we can find a lifting $c = c_{h,P} \in G_l^{\text{der}}(E_h)$, satisfying

$$(5.2.3.3) \quad cp = kc_{\infty,l} \quad c(K_h P_h^+ \cap G_l)c^{-1} = P \cap G_l$$

for some $p \in K_h(\mathbf{C})P_h^+(\mathbf{C}) \cap G_l(\mathbf{C})$, $k \in K_h(\mathbf{R}) \cap G_l(\mathbf{R})$, and some lifting $c_{\infty,l}$ of $\overline{C_{\infty,l}}$ to $G_l(\mathbf{C})$.

It is known ([66], §6) that

$$c_{\infty,l}^{-1} G(\mathbf{R})^0 \subset P_h^+(\mathbf{C}) K_h(\mathbf{C}) P_h^-(\mathbf{C})$$

and it follows from (5.2.3.2) that

$$(5.2.3.4) \quad c^{-1}G(\mathbf{R})^0 \subset P_h^+(\mathbf{C})K_h(\mathbf{C})P_h^-(\mathbf{C}).$$

In particular, $c^{-1} = c^+ c_0 c^-$ with $c^+ \in P_h^+(\mathbf{C})$, $c_0 \in K_h(\mathbf{C})$, and $c^- \in P_h^-(\mathbf{C})$. It follows easily from (5.2.1.1) that c^+, c_0, $c^- \in G(E_h)$. Let $J = J_{h,P} : G_E \to K_h$ be the rational map, defined on the open subset $cP_h^+K_hP_h^- \subset G_E$ by the formula

$$(5.2.3.5) \quad J(cp^+kp^-) = c_0^{-1}k \quad p^+ \in P_h^+ \ k \in K_h, \ p^- \in P_h^-.$$

Then J is well-defined, is rational over E_h, and (by (5.2.3.4)) has no pole on $G(\mathbf{R})^0$.

5.2.3.6 Remark: Most often, we assume h to be a CM point; i.e., there is a CM pair $(T, h) \subset (G^{\text{even}}, \Delta(P))$. It then follows from (3.5.1) (a), applied to the adjoint representation, that we may take $E_h = E(T, h)$.

Now we define the *canonical automorphy factor* for P, h:

$$(5.2.3.7) \quad J^{h,P}(g, x) = J(gg_x)J(g_x)^{-1} \quad g \in G(\mathbf{R})^0, \ x \in X^+$$

where as before, $g_x \in G(\mathbf{R})^0$ is a solution to the equation $g_x(h) = x$. The definition of $J^{h,P}$ depends *a priori* on the choice of an element c satisfying (5.2.3.3). If d is another one, then as in the argument preceding (5.2.3.1), we know that $d = cp$, with $p \in K_h(\mathbf{C})P_h^+(\mathbf{C})$. It is thus clear that J, defined as in (5.2.3.5), is independent of the choice of c, and we are justified in calling $J^{h,P}$ the canonical automorphy factor. One verifies immediately that $J^{h,P}$ has the properties listed in 5.2.2.

5.3. *Canonical trivializations*

Let $\mathscr{V}$ be a homogeneous vector bundle over $\check{M} = \check{M}(G, X)$, defined over the number field L. Pick a CM pair $(T, h) \subset (G, X)$. Let β_X be the Borel imbedding of X of $\check{M}(C)$, as in 3.1. We denote the point $\beta_X(h) = (\mathfrak{P}_h, \mu_h(\text{mod } R_h\mathfrak{P}_h)) \in \check{M}(E(T, h))$ by the symbol $\check{h}$. We let $\tau_h : \mathfrak{P}_h \to GL(\mathscr{V}_{\check{h}})$ be the isotropy representation of the subgroup $\mathfrak{P}_h \subset G$ on the fiber $\mathscr{V}_{\check{h}}$ of $\mathscr{V}$ at $\check{h}$. *Assume that* τ_h *is trivial on* $R_u\mathfrak{P}_h$; i.e., it factors through K_h, in the notation of 5.2.1. We denote the homomorphism $K_h \to GL(\mathscr{V}_{\check{h}})$ by the same symbol τ_h.

Let X^+ be a connected component of X, as before. Let $J = J^{h,0}$ or $J^{h,P}$ be one of the canonical automorphy factors constructed in 5.2. The restriction of $\mathscr{V}(\mathbf{C})$ to $\beta_X(X^+)$ can be trivialized as follows: There is an isomorphism

$$(5.3.1) \quad \begin{aligned} &X^+ \times \mathscr{V}_{\check{h}}(\mathbf{C}) \to \mathscr{V}(\mathbf{C})\,|_{\beta_X(X^+)} \\ &(x, v) \mapsto g_x^*\big(J_\tau(g_x, h)^{-1}v\big) \end{aligned}$$

where $J_\tau(g, z) = \tau_h(J(g, z)) \in GL(\mathscr{V}_{\check{h}})$, $g_x \in G(\mathbf{R})^0$ satisfies $g_x(h) = x$, and

$$(g, v) \mapsto g^*(v) \qquad g \in G(\mathbf{C}), \quad v \in \mathscr{V}(\mathbf{C})$$

is the given action of G on v. It follows from (5.2.1.4) that (5.3.1) is well-defined. With respect to this trivialization, the action of $g \in G(\mathbf{R})^0$ on $\mathscr{V}(\mathbf{C})\,|_{\beta_X(X^+)}$ is given by the formula

$$(5.3.2) \quad g(x, v) = (gx, J_\tau(g, x)v).$$

We know that analytically, $[\mathscr{V}](\mathbf{C})$ is isomorphic to

$$\lim_{\substack{\leftarrow \\ K}} G(\mathbf{Q}) + \backslash \mathscr{V}(\mathbf{C})\,|_{\beta_X(X^+)} \times G(\mathbf{A}^f)/K \quad (cf.\ 3.6.1).$$

Using (5.3.1), we rewrite this

$$[\mathscr{V}](\mathbf{C}) = \lim_{\substack{\leftarrow \\ K}} G(\mathbf{Q})_+ \backslash X^+ \times \mathscr{V}_{\check{h}}(\mathbf{C}) \times G(\mathbf{A}^f)/K$$

where the action of $G(\mathbf{Q})_+$ on $X^+ \times \mathscr{V}_{\check{h}}(\mathbf{C}) \times G(\mathbf{A}^f)$ is determined by (5.3.2):

$$\gamma(x, v, g) = (\gamma x, J_\tau(\gamma, x)v, \gamma g).$$

It follows as usual (cf. [12], 2.1) that an automorphic form $f \in \Gamma(M(G, X), [\mathscr{V}])^K$, for some open compact $K \subset G(\mathbf{A}^f)$, lifts to a function

$$\tilde{f}: X^+ \times G(\mathbf{A}^f)/K \to \mathscr{V}_{\check{h}}(\mathbf{C})$$

which satisfies the following conditions:

(5.3.3 (i)) $\tilde{f}$ is holomorphic in X^+

(5.3.3 (ii))

$$\tilde{f}(\gamma(x, g)) = J_\tau(\gamma, x)\tilde{f}(x, g) \;\; \forall \gamma \in G(\mathbf{Q})_+, \; x \in X^+, \; g \in G(\mathbf{A}^f).$$

Let $M_{\mathscr{V}}(K) = M_{\mathscr{V}}(K, G, X, J)$ be the space of $\mathscr{V}_{\check{h}}(\mathbf{C})$-valued functions on $X^+ \times G(\mathbf{A}^f)/K$, satisfying (5.3.3 (i)) and (5.3.3 (ii)). We have defined an isomorphism

(5.3.4) $\Gamma(M(G, X), [\mathscr{V}])^K \xrightarrow{\sim} M_{\mathscr{V}}(K)$.

If we define $\Gamma(M(G, X), [\mathscr{V}]) = \varinjlim_K \Gamma(M(G, X), [\mathscr{V}])^K$, $M_{\mathscr{V}} = \varinjlim_K M_{\mathscr{V}}(K)$, then of course we have

(5.3.5) $\Gamma(M(G, X), [\mathscr{V}]) \xrightarrow{\sim} M_{\mathscr{V}}$.

In order to say something non-trivial, it is convenient to make the following hypothesis:

(5.3.6) There exists a representation $\rho: G \to GL(V_\rho)$, defined over L, and an imbedding $\mathscr{V} \hookrightarrow \tilde{V}_\rho$ (notation 3.4) such that $\mathscr{V}_{\check{h}}$ is the subspace of V_ρ fixed by $R_u \mathfrak{P}_h$.

In other words, Hypothesis (5.3.6) is that τ_h is the lowest K_h-type of a finite-dimensional representation of G. It seems that such $\mathscr{V}$ are the only ones that have interesting spaces of sections. In any event, the following construction can easily be extended to the general case by observing that every $\mathscr{V}$ is contained in the tensor category generated by those satisfying (5.3.6) and their duals.

We have, by hypothesis, an imbedding $\mathscr{V}_{\check{h}} \hookrightarrow V_\rho$, defined over $L_h = L \cdot E(T, h)$. The torus T is a subgroup of K_h, and therefore acts on τ_h through the restriction of τ_h to T. Thus there exists a T-invariant, L_h-rational decomposition

$$V_\rho = \mathscr{V}_{\check{h}} \oplus \mathscr{V}'_{\check{h}}.$$

Let τ'_h denote the representation of T on $\mathscr{V}'_{\check{h}}$. In the notation of (4.1.7), we have compatible L_h-rational decompositions

$$(5.3.8)\quad H_*(\mathfrak{M}(\rho)/M(T, h)) = H_*(\mathfrak{M}(\tau_h)/M(T, h)) \oplus H_*(\mathfrak{M}(\tau'_h)/M(T, h))$$

where * is either B or DR.

Let $r = \dim \mathscr{V}_{\check{h}}$, and let $\omega_1, \ldots, \omega_r$ (resp. $c_1, \ldots, c_r$) be a basis of L_h-rational global sections of $\Gamma(M(T, h), [\mathscr{V}]|_{M(T,h)}) = H_{DR}(\mathfrak{M}(\tau_h)/M(T, h))$ (resp. of $H_B(\mathfrak{M}(\tau_h)/M(T, h))$). Let

$$p(\mathscr{V}, h) \in \mathrm{Aut}([\mathscr{V}]|_{M(T,h)}) = \mathrm{Aut}(H_B(\mathfrak{M}(\tau_h)/M(T, h)) \otimes \mathbf{C})$$

be defined by the relation $p(\mathscr{V}, h)c_i = \omega_i$, $i = 1, \ldots, r$. We call $p(\mathscr{V}, h)$ the *period* matrix of the vector bundle $\mathscr{V}$ at h. It is an invariant of the "motive" $\mathfrak{M}(\tau_h)$, and is well-defined only up to right-multiplication (resp. left-multiplication) by elements of $\mathrm{Aut}(H_B(\mathfrak{M}(\tau_h))/M(T, h))$ (resp. $\mathrm{Aut}(H_{DR}(\mathfrak{M}(\tau_h)/M(T, h)))$ with coefficients in L_h.

Note that $p(\mathscr{V}, h) \in \mathrm{End}([\mathscr{V}]|_{M(T,h)})$ is an automorphic form on $M(T, h)$, with values in $\mathrm{End}([\mathscr{V}])$. By applying (5.3.5), we may thus identify $p(\mathscr{V}, h)$ canonically with a function

$$\tilde{p}(\mathscr{V}, h): \{h\} \times T(\mathbf{A}^f) \to \mathrm{End}(\mathscr{V}_h(\mathbf{C}))$$

satisfying a special case of (5.3.3(ii)). In this case the automorphy factor coincides with the respresentation τ_h, and in particular is rational with respect to the L_h-structure on $\mathscr{V}_{\hat{h}}$. The constructions in §4 naturally identify this L_h-structure on $\mathscr{V}_{\check{h}}$ with the rational structure on $[\mathscr{V}]|_{M(T,h)} \simeq \mathscr{V}_{\check{h}}(\mathbf{C}) \times M(T, h)$ provided by $H_B(\mathfrak{M}(\tau_h)/M(T, h))$; cf. also Proposition 4.1 of [13]. The following lemma is thus clear.

5.3.9 LEMMA: *Let $f \in \Gamma(M(G, X), [\mathscr{V}])$. Let L' be an extension of L_h, contained in $\mathbf{C}$. The following are equivalent:*

(*i*) *The restriction f_h of f to $M(T, h)$ is rational over L'.*

(*ii*) *The section $p(\mathscr{V}, h)^{-1}f_h$ of $H_{DR}(\mathfrak{M}(\tau_h)/M(T, h))(\mathbf{C})$ takes values in $H_B(\mathfrak{M}(\tau_h)/M(T, h)) \otimes_{L_h} L'$.*

(*iii*) *Let $\tilde{f}_h$ be the restriction of $\tilde{f}$ to $\{h\} \times T(\mathbf{A}^f)$. The function*

$$\tilde{p}(\mathscr{V}, h)^{-1} \cdot \tilde{f}_h : \{h\} \times T(\mathbf{A}^f) \to \mathscr{V}_{\check{h}}(\mathbf{C})$$

takes values in $\mathscr{V}_{\check{h}}(L')$.

We recall that for any CM pair $(T, h) \subset (G, X)$, the subset $M(T, h) \cdot G(\mathbf{A}^f) \subset M(G, X)$ is Zariski dense ([5], §5). In view of 1.2.4, Lemma 5.3.9 immediately implies the following proposition.

5.3.10 PROPOSITION: *Let* $f \in \Gamma(M(G, X), [\mathcal{V}])$, *and let* $L' \subset \mathbf{C}$ *be an extension of* L. *The following are equivalent.*

(*i*) *The section* f *is rational over* L'.

(*ii*) *For every CM pair* $(T, h) \subset (G, X)$, *and each* $\gamma \in G(\mathbf{A}^f)$, *the following condition holds: Let* β *be the composite of the inclusion* $M(T, h) \subset M(G, X)$ *with right translation by* γ. *The section* $p(\mathcal{V}, h)^{-1}\beta^*(f)$ *of* $H_{DR}(\mathfrak{M}(\tau_h)/M(T, h)) \otimes \mathbf{C}$ *takes values in* $H_B(\mathfrak{M}(\tau_h)/M(T, h)) \otimes_{L_h} L' \cdot E(T, h)$.

(*iii*) *For each* (T, h) *and* γ *as in* (*ii*), *the following condition holds: Let* $\gamma^* f$ *be the pullback of* f *under right translation by* γ, *and let* $\tilde{f}_{\gamma,h}$ *be the restriction of* $\gamma^* f$ *to* $\{h\} \times T(\mathbf{A}^f)$. *The function*

$$\tilde{p}(\mathcal{V}, h)^{-1} \cdot \tilde{f}_{\gamma,h} : \{h\} \times T(\mathbf{A}^f) \to \mathcal{V}_{\check{h}}(\mathbf{C})$$

takes values in $\mathcal{V}_{\check{h}}(L' \cdot E(T, h))$.

5.3.11 REMARK: If we are willing to extend the ground field so that the representation of T on $\mathcal{V}_{\check{h}}$ is diagonalizable, then we may take $\tilde{p}(\mathcal{V}, h)$ to be a function whose values are diagonal matrices. In this case, its entries are Shimura's period invariants [23]. As indicated in 4.1, these invariants, up to algebraic factors, are actually invariants of representations of the Serre group, and can therefore be transferred from one Shimura variety to another. This feature of the period invariants has been exploited to great effect by Shimura [24,56,58], who uses a variant of the rationality criterion 5.3.10 to prove theorems about the rationality of theta-liftings.

It should be stressed that $H_B(\mathfrak{M}(\tau_h)/M(T, h))$ consists of the restrictions to $M(T, h) \subset M(G, X)$ of sections of the canonical local system V_ρ^∇. If $(T', h') \subset (G, X)$ is another *CM* pair, we may evaluate $\tilde{f}$ at $G(\mathbf{A}^f)$-translates of $\{h'\} \times T'(\mathbf{A}^f)$. Since $\tilde{f}$ takes values in $\mathcal{V}_{\check{h}}(\mathbf{C})$, its restriction to $\{h'\} \times T'(\mathbf{A}^f)\gamma$ cannot be compared directly with periods of motives over $M(T', h')$. However, V_ρ^∇ does not depend on the *CM* pair (T, h), and we may use the relation (4.4.1) to identify $\mathcal{V}_{\check{h}}(\overline{\mathbf{Q}})$ with $\mathcal{V}_{\check{h}}(\overline{\mathbf{Q}})$. Thus, as in the work of Shimura, we obtain a criterion for $\overline{\mathbf{Q}}$-rationality based on evaluation of automorphic forms at arbitrary *CM* pairs.

The work of Blasius, which determines the period invariants up to multiplication by scalars in more precise number fields, should also be mentioned in this connection [28].

§6. Fourier-Jacobi Expansions

In this section we prove a general version of the classical q-expansion principle, along the lines developed by Shimura [53,54,20] and, more intrinsically, by Garrett [11,36]. The context of the principle, roughly speaking, is that arithmetic automorphic forms have arithmetic Fourier-Jacobi expansions along all rational proper boundary components, and that conversely, an automorphic form whose Fourier-Jacobi expansion is arithmetic at a single rational boundary component is *ipso facto* arithmetic. Of course this principle is vacuous unless the Shimura variety on which these forms are defined possesses a nontrivial boundary.

When (G, X) has a symplectic imbedding, the Fourier-Jacobi expansion along a rational boundary component of $M(G, X)$ has a modular interpretation, investigated by Brylinski in his thesis [31]. This interpretation is expressed in terms of the degeneration of a family of polarized abelian varieties, along the boundary of $M(G, X)$, into a family of (polarized) 1-motives. Each polarized abelian variety in the family is determined up to isogeny by the Hodge structure on its rational singular cohomology, and the degeneration along the boundary is faithfully reflected by the degeneration of the family of (polarized) Hodge structures into a family of (polarized) mixed Hodge structures. Although these geometric constructions are not available when (G, X) admits no symplectic imbedding, it turns out that the underlying linear algebraic data, in the form of the canonical local systems constructed in §4, suffice to prove an appropriate generalization of Brylinski's results.

6.1 Let (G, X), P, $\mathcal{V}$, and L be as in §5; we assume as before that $L \supset E(G, X)$. The following lemma will be proved in a moment.

6.1.1 LEMMA: *The reflex field* $E(G^{\text{even}}, \Delta(P)) = E(G, X)$.

First we derive a consequence from the lemma.

6.1.2 COROLLARY: *Let* $f \in \Gamma(M(G, X), [\mathcal{V}])$, *and let* $L' \subset \mathbf{C}$ *be an extension of* L. *The following are equivalent*:
(*i*) *The section* f *is rational over* L'.
(*ii*) *For every* $\gamma \in G(\mathbf{A}^f)$, *the following condition holds*: *Let* $i_\gamma : M(G^{\text{even}}, \Delta(P)) \to M(G, X)$ *be the composition of the natural inclusion with right translation by* γ. *Let* $[\mathcal{V}]^{even}$ *be the pullback of* $[\mathcal{V}]$ *to* $M(G^{\text{even}}, \Delta(P))$ *via the natural inclusion. Then* $i_\gamma^*(f)$ *is an* L'*-rational section of* $i_\gamma^*[\mathcal{V}] \simeq [\mathcal{V}]^{\text{even}}$.

PROOF: This is an immediate consequence of Lemma 6.11 and the fact that $M(G^{\text{even}}, \Delta(P)) \cdot G(\mathbf{A}^f)$ is Zariski dense in $M(G, X)$.

6.1.3 PROOF of Lemma 6.1.1: Let $h \in \Delta(P)$, and define $\mu = \mu_h : \mathbf{G}_m \to G_{\mathbf{C}}^{\text{even}} \subset G_{\mathbf{C}}$ as in 1.1. Let $\tilde{M}$ (resp. $\tilde{M}_P$) denote the G-conjugacy class (resp. G^{even}-conjugacy class) of μ. Let $\tau = w_P(-1)$ (notation 5.1.3), and let $\tilde{M}^\tau$ be the set of fixed points of τ in $\tilde{M}$. Since $\tau \in G(\mathbf{Q})$, $\tilde{M}^\tau$ is defined over $E(G, X)$. It thus suffices to prove

(6.1.3.1) $\quad \tilde{M}^\tau = \tilde{M}_P$.

We first construct a G^{even}-equivariant map $\epsilon: \tilde{M}^{\tau} \to \check{M}(G^{\text{even}}, \Delta(P)) \overset{\text{def}}{=} \check{M}^{\text{even}}$. Let $\nu \in \tilde{M}^{\tau}$, and let $\mathfrak{k}_{\nu}$, $\mathfrak{p}_{\nu}^{-}$, and $\mathfrak{p}_{\nu}^{+}$ be the eigenspaces in $\mathfrak{g}$, for $\nu(\mathbf{G}_m)$, of weight 0, 1, and -1 respectively. Let $\mathfrak{P}_{\nu}$ be the parabolic subgroup of G with Lie algebra $\mathfrak{k}_{\nu} \oplus \mathfrak{p}_{\nu}^{-}$. Then $\nu(\mathbf{G}_m) \subset \mathfrak{P}_{\nu}$. Let $\bar{\nu}: \mathbf{G}_m \to \mathfrak{P}_{\nu}/R_u\mathfrak{P}_{\nu}$ be the composite of ν with the natural projection. Then $(\mathfrak{P}_{\nu}, \bar{\nu}) \in \check{M}(G, X)$. But since ν commutes with τ, the eigenspace decomposition may be further refined:

$$\text{(6.1.3.2)} \quad \mathfrak{k}_{\nu} = \mathfrak{k}_{\nu}^{\tau} \oplus \mathfrak{k}_{\nu}^{-\tau}; \quad \mathfrak{p}_{\nu}^{-} = \mathfrak{p}_{\nu}^{\tau} \oplus \mathfrak{p}_{\nu}^{-\tau}$$

where $\mathfrak{k}_{\nu}^{\tau}$ (resp. $\mathfrak{k}_{\nu}^{-\tau}$) is the subspace of $\mathfrak{k}_{\nu}$ fixed by τ (resp. the -1 eigenspace of τ on $\mathfrak{k}_{\nu}$), and the second decomposition is defined analogously. Let $\mathfrak{P}_{\nu}^{\tau} \subset G^{\text{even}}$ be the subgroup with Lie algebra $\mathfrak{k}_{\nu}^{\tau} \oplus \mathfrak{p}_{\nu}^{\tau}$; then $\bar{\nu}(\mathbf{G}_m) \subset \mathfrak{P}_{\nu}^{\tau}/R_u\mathfrak{P}_{\nu}^{\tau}$, and one sees easily that

$$\text{(6.1.3.3)} \quad \epsilon(\nu) \overset{\text{def}}{=} (\mathfrak{P}_{\nu}^{\tau}, \bar{\nu}) \in \check{M}^{\text{even}},$$

and that ϵ is equivariant with respect to G^{even}.

Let $\nu \in \check{M}^{\tau}$; we want to show that ν is G^{even}-conjugate to μ. Now G^{even} acts transitively on $\check{M}^{\text{even}}$, so we may assume, after replacing ν by a G^{even}-conjugate, that $\epsilon(\nu) = \epsilon(\mu)$. It then follows that

$$\text{(6.1.3.4)} \quad \nu = \gamma\mu\gamma^{-1}, \text{ with some } \gamma \in P^{-} \overset{\text{def}}{=} R_u\mathfrak{P}_{\mu}.$$

Here we have used the fact that $\bar{\mu}(\mathbf{G}_m)$ is central in $\mathfrak{P}_{\mu}/R_u\mathfrak{P}_{\mu}$. Now the exponential map is an isomorphism of groups

$$\exp: \mathfrak{p}_{\mu}^{-} \simeq P^{-}.$$

By (6.1.3.2) we may write $\gamma = \gamma^{+} \cdot \gamma^{-}$, with $\gamma^{+} \in \exp \mathfrak{p}_{\mu}^{\tau}$, $\gamma^{-} \in \exp \mathfrak{p}_{\mu}^{-\tau}$. Since $\gamma^{+} \in G^{\text{even}}$, we may assume $\gamma = \gamma^{-}$; i.e., $\gamma^{\tau} = \gamma^{-1}$. But $\nu^{\tau} = \nu$. It follows from (6.1.3.4) that

$$\text{(6.1.3.5)} \quad \gamma^{-1}\mu\gamma = \gamma\mu\gamma^{-1}; \text{ i.e., } \gamma^2 \text{ commutes with } \mu(\mathbf{G}_m).$$

But (6.1.3.5) implies that γ^2 is in the subgroup K_{ν} of G whose Lie algebra is $\mathfrak{k}_{\nu}$. Since $P^{-} \cap K_{\nu} = \{1\}$, we have $\gamma^2 = 1$. Since γ is unipotent, we even have $\gamma = 1$. The lemma is proved.

Lemma 6.1.1 has another useful corollary:

6.1.4 COROLLARY: *The reflex field $E(G_P, F_P)$ is contained in $E(G, X)$.*

PROOF: Let (T, h) be a CM pair in $(G^{\text{even}}, \Delta(P))$. By 1.2.4 and Lemma 6.1.1, it suffices to show that there exists a CM pair $(T_P, h_P) \subset (G_P, F_P)$ such that

$E(T_P, h_P) \subset E(T, h)$. Now let $G' = G^{\text{even}}/G_l$, and let F' be the $G'(\mathbf{R})$-conjugacy class of homomorphisms $h: \underline{S} \to G'_{\mathbf{R}}$ generated by the image of $\Delta(P)$ under the natural map $G^{\text{even}} \to G'$. Let $(T', h') \subset (G', F')$ be the image of (T, h). Then $E(T', h') \subset E(T, h)$, by general principles of functoriality. On the other hand, G_P maps onto G', and its kernel is the $\mathbf{Q}$-split torus A_P. Let $T_P \subset G_P$ be the connected component of the inverse image of T' with respect to this map. Then $(T_P, \pi_P(h))$ is a CM pair in (G_P, F_P). Since the kernel of the map $T_P \to T'$ is $\mathbf{Q}$-split, we have $E(T_P, \pi_P(h)) = E(T', h')$, and we are done.

6.2 Our method for determining rationality of $f \in \Gamma(M(G, X), [\mathscr{V}])$ is based on the Fourier-Jacobi expansion of $i_\gamma^*(f)$ for any $\gamma \in G(\mathbf{A}^f)$, along the boundary components of $\Delta(P)^+$ of the form $F_P^+ \times \{pt\}$; where $\{pt\}$ refers to a rational point boundary component of D_P^+. The relation between this and the "normalized division-point values" of Shimura and Garrett will be explained below. Meanwhile, we will assume $(G, X) = (G^{\text{even}}, \Delta(P))$.

Let $P' = G_P \cdot U \subset G$. If $K \subset G(A^f)$ is an open compact subgroup, define

$$(6.2.1) \quad \Xi_K = \Xi_K(G, X, P) = P'(\mathbf{Q}) \backslash X \times P'(\mathbf{A}^f)/P'(\mathbf{A}^f) \cap K$$

$$= P'(\mathbf{Q}) \cap G(\mathbf{Q})_+ \backslash X^+ \times P'(\mathbf{A}^f)/P'(\mathbf{A}^f) \cap K$$

$$\Xi = \lim_{\leftarrow} \Xi_K.$$

The inclusion of $P'(\mathbf{A}^f)$ in $G(\mathbf{A}^f)$ defines a collection of natural maps

$$B_K : \Xi_K \to {}_K M(G, X)(\mathbf{C})$$

which piece together to define a $P'(\mathbf{A}^f)$-equivariant map

$$B: \Xi \to M(G, X)(\mathbf{C}).$$

On the other hand, the morphism $\pi_P: X^+ \to F_P^+$ induces a morphism

$$\pi: \Xi \to M(G_P, F_P)(\mathbf{C}).$$

Our first objective is to represent $B^*[\mathscr{V}]$ in terms of a bundle on $M(G_P, F_P)$. As before, we assume $\mathscr{V}$ satisfies hypothesis (5.3.6). It is convenient to introduce the homogeneous subbundle $\mathscr{V}_l \subset \tilde{V}_\rho$ such that, for each $\check{h} = (\mathfrak{P}_h, \mu) \in \check{M}(G, X)$ the fiber $\mathscr{V}_{l,\check{h}}$ is the subspace of $\tilde{V}_\rho$ fixed by $G_l \cap R_u\mathfrak{P}_h$. Then $\mathscr{V}_l$, and hence $[\mathscr{V}_l]$, is defined over L, and we have

$$[\mathscr{V}] \subset [\mathscr{V}_l] \subset [\tilde{V}_\rho].$$

On the other hand, let ρ_P^* be the representation of G_P on the subspace $V_{\rho_P}^* \subset V_\rho^*$ of vectors fixed under the action of U. (Recall that G_P normalizes U

in G^{even}). We may define the vector bundle $[V^*_{\rho_P}]$ over $M(G_P, F_P)$. Now the bundle $B^*[V^*_\rho]$ contains a local system $B^*V_\rho^{*,\nabla}$ in L-vector spaces. There is a complex analytic L-linear isomorphism

$$(6.2.2)\quad B^*V_\rho^{*,\nabla} \xrightarrow{\sim} \varprojlim_K P'(\mathbf{Q})\backslash V_\rho^*(L) \times X \times P'(\mathbf{A}^f)/P'(\mathbf{A}^f) \cap K,$$

and we let $W_\rho^\nabla \subset B^*V_\rho^{*,\nabla}$ be the $P'(\mathbf{A}^f)$-equivariant local subsystem which corresponds via (6.2.2) to

$$\varprojlim_K P'(\mathbf{Q})\backslash V^*_{\rho_P}(L) \times X \times P'(\mathbf{A}^f)/P'(\mathbf{A}^f) \cap K.$$

Then W_ρ^∇ does not depend on the choice of isomorphism (6.2.2). Let $\mathscr{W}_\rho$ be the $\mathcal{O}_\Xi$-subsheaf of $B^*[\tilde{V}_\rho^*]$ spanned by W_ρ^∇. The following lemma is clear by construction:

6.2.3 Lemma: *There is an isomorphism of $P'(\mathbf{A}^f)$-equivariant vector bundles*

$$j: \pi^*\tilde{V}^*_{\rho_P} \xrightarrow{\sim} \mathscr{W}_\rho$$

such that

$$(6.2.4)\quad j\left(V^\nabla_{\rho_P}\right) = W_\rho^\nabla.$$

We note that j is not uniquely determined by (6.2.4); in every case it can be multiplied by an element of $L^\times$ and still satisfy (6.2.4).

Let D_0 denote the composite morphism

$$B^*[\mathscr{V}_l] \to B^*\left[\tilde{V}_\rho\right] = B^*\left[\tilde{V}_\rho^*\right]^* \to \mathscr{W}_\rho^*$$

where the last arrow is the natural restriction map. Let

$$D = D_j = j^{-1} \circ D_0 : B^*[\mathscr{V}_l] \to \pi^*\left[\tilde{V}^*_{\rho_P}\right]^* = \pi^*\left[\tilde{V}_{\rho_P}\right].$$

This homomorphism, which is evidently $P'(\mathbf{A}^f)$-equivariant, is canonical up to automorphisms of W_ρ^∇. The main step in our study of Fourier-Jacobi expansions is the following proposition.

6.2.5 Proposition: *The homomorphism D is an isomorphism.*

Fix a point $h \in \Delta(P)$. Let $P^+ = P_h^+ \cap G_l$, $P^- = P_h^- \cap G_l$, in the notation of 5.2.1. Choose an element $c \in G_l(\mathbf{C})$ such that $cP^+c^{-1} = U$, as in 5.2.3, and let $U^- = cP^-c^{-1}$; then Lie $U = \mathfrak{g}^{-2}$, Lie $U^- = \mathfrak{g}^2$, in the notation of 5.1. We write $\mathfrak{p}^\pm =$ Lie $P^\pm$.

If H is a subgroup of G, $\mathfrak{h} = \text{Lie } H$, we let $V_\rho^H \subset V_\rho$ be the subspace of H-fixed vectors, and let $V_\rho^{H,1} \subset V_\rho$ be the subspace generated by $d\rho(X)v$, $\forall X \in \mathfrak{h}$, $v \in V_\rho$.

We note that $Ad(c^2)$ interchanges P^+ and P^-, and also interchanges U and U^- (cf. [38], §1).

6.2.5.1. Lemma: *Let H be one of the subgroups P^+, P^-, U, U^- of G_l, and let $\rho: G_l \to GL(V_\rho)$ be any representation. Let $H' = c^2 H c^{-2}$. We have*

$$V_\rho = V_\rho^H \oplus V_\rho^{H',1}.$$

Proof: Since all groups in question are conjugate, we may take $H = U$. It suffices to consider the case in which ρ is irreducible. Then V_ρ^U is an irreducible G_l^0-module, and is the lowest eigenspace for $w_P(\mathbf{G}_m) = A_P$. The lemma follows immediately.

Returning to the proof of the Proposition, we see that it suffices to prove that the composite map

$$V_\rho^{P^-} \hookrightarrow V_\rho = (V_\rho^*)^* \to \left((V_\rho^*)^U\right)^*$$

is an isomorphism. We drop the subscript ρ. First we establish that dim $V^{P^-} =$ dim$(V^*)^U$. Since P^- is conjugate to U, we know dim $V^{P^-} =$ dim V^U. We thus have to prove

$$(6.2.5.2) \quad \dim V^U = \dim(V^*)^U.$$

But $(V^*)^U$ is naturally isomorphic to $(V/V^{U,1})^*$. Moreover, dim $V^{U,1} =$ dim $V^{U^-,1}$. Thus (6.2.5.2) follows from Lemma 6.2.5.1.

The kernel of $V \to ((V^*)^U)^*$ is just $V^{U,1}$. The preceding paragraph thus reduces the proposition to the statement

$$(6.2.5.3) \quad V^P \cap V^{U,1} = \{0\}.$$

Write $c^{-1} = c^+ c_0 c^-$ as in 5.2.3, with $c^+ \in P^+(\mathbf{C})$, $c_0 \in K_h(\mathbf{C})$, $c^- \in P^-(\mathbf{C})$. We have to prove

$$(6.2.5.4) \quad c^{-1} V^{P^-} \cap V^{P^-,1} = \{0\}.$$

Evidently $c^{-1} V^{P^-} = c^+ V^{P^-}$. Say $c^+ = \exp(X^+)$, with $X^+ \in \mathfrak{p}^+(\mathbf{C})$. Then

$$(6.2.5.5) \quad \rho(c^+)v \equiv v (\text{mod } V^{P^-,1}) \ \forall v \in V.$$

Suppose $v \in V^P$ is such that $\rho(c^+)v \in V^{P^-,1}$. Then (6.2.5.5) implies that $v \in V^{P^-} \cap V^{P^-,1}$. Now Lemma 6.2.5.1 implies $v = 0$. The assertion (6.2.5.4), and therefore the Proposition, now follows immediately.

6.3 We continue to assume $(G, X) = (G^{\text{even}}, \Delta(P))$. In the last section we defined an isomorphism

$$D: B^*[\mathscr{V}_l] \tilde{\rightarrow} \pi^*[\tilde{V}_{\rho_P}].$$

Now any section of $\pi^*[\tilde{V}_{\rho_P}]$ over Ξ is assumed to be invariant with respect to $K \cap P'(\mathbf{A}^f)$, for some open compact subgroup $K \subset G(\mathbf{A}^f)$. Let $\Lambda^* = K \cap U(\mathbf{Q})$, and let $\Lambda = \text{Hom}(\Lambda^*, \mathbf{Z})$. We may view Λ^* as a lattice in $\mathfrak{g}^{-2}(\mathbf{Q})$. As in (5.1.7), we may identify D_P^+ $P(\mathbf{R})^0$-homogeneously with $\mathfrak{g}^{-2}(\mathbf{R}) \oplus iC$, where C is a self-adjoint $G_l^0(\mathbf{R})$-homogeneous cone in $\mathfrak{g}^{-2}(\mathbf{R})$. Such an identification is not unique, but we will see in 6.5.2, below, that there is a natural set of such identifications which is homogeneous under $P_l(\mathbf{Q}) \cap P_l(\mathbf{R})^0$; we admit this for now, and assume our given identification belongs to this set. Let $N = \dim \mathfrak{g}^{-2} = \dim D_P^+$, and let $\{a_1, \ldots, a_N\}$ be a $\mathbf{Z}$-basis for Λ^*, contained in C. Let $\{l_1, \ldots, l_N\}$ be the dual basis for Λ, and let

$$q_j = q_j(z) = e^{2\pi i l_j(z)} \quad z \in D_P^+, \ j = 1, \ldots, N.$$

More generally, if $\underline{\alpha} \in \Lambda$, $\underline{\alpha} = \sum_{j=1}^{N} \alpha_j l_j$, we define $q^{\underline{\alpha}} = \prod_{j=1}^{N} q_j^{\alpha_j}$. If we identify A_P with $\mathbf{G}_{\text{m}}$, then the function $q^{\underline{\alpha}}$ determines uniquely a function on $\Xi = \underset{\overleftarrow{K}}{\lim}\, P'(\mathbf{Q}) \cap G(\mathbf{Q})_+ \backslash F_P^+ \times D_P^+ \times P'(\mathbf{A}^f)/P'(\mathbf{A}^f) \cap K$ which does not depend on the variable in F_P^+ and is right invariant under $A_P(\hat{\mathbf{Z}}) \cdot G_P^{\text{der}}(\mathbf{A}^f)$. We denote this function $q^{\underline{\alpha}}$ as well. Let $K_P = K \cap P'(\mathbf{A}^f)/K \cap U(\mathbf{A}^f) \subset G_P(\mathbf{A}^f)$. The space Ξ_K is a fibration over ${}_{K_P}M(G_P, F_P)(\mathbf{C})$ whose fiber at each point is isomorphic to the domain D_P^+/Λ^*. Since $a_i \in C$, $i = 1, \ldots, N$, the domain D_P^+/Λ^* contains a product of punctured disks $D_\epsilon = \prod_{i=1}^{N} \{q_i \,|\, 0 < i < \epsilon\}$, for some $\epsilon < 0$. It follows that we have an imbedding (Laurent series)

$$(6.3.1) \quad \pi_* \pi^*[\tilde{V}_{\rho_P}]^{K \cap P'(\mathbf{A}^f)} \hookrightarrow \hat{\bigoplus_{\underline{\alpha} \in \Lambda}} [\tilde{V}_{\rho_P}]^{K_P} q^{\underline{\alpha}}$$

where the symbol $\hat{\oplus}$ refers to formal Laurent series. Alternatively, (6.3.1) may be viewed as the isotypic decomposition with respect to the natural right action of $U(\mathbf{R})$ on Ξ_K, which lifts naturally to $\pi^*[\tilde{V}_{\rho_P}]$. More generally, we have

$$(6.3.2) \quad \pi_* \pi^*[\tilde{V}_{\rho_P}] \hookrightarrow \hat{\bigoplus_{\underline{\alpha} \in \mathfrak{g}^{-2}(\mathbf{Q})^*}} [\tilde{V}_{\rho_P}] \cdot q^{\underline{\alpha}}.$$

Now let $f \in \Gamma(M(G, X), [\mathscr{V}]) \subset \Gamma(M(G, X), [\mathscr{V}_l])$. Then $D(B^*(f)) \in \Gamma(\Xi, \pi^*[\tilde{V}_{\rho_P}]) = \Gamma(M(G_P, F_P), \pi_*\pi^*[\tilde{V}_{\rho_P}]) \subset \hat{\bigoplus}_{\underline{\alpha} \in \mathfrak{g}^{-2}(\mathbf{Q})^*} \Gamma(M(G_P, F_P), [\tilde{V}_{\rho_P}]) \cdot q^{\underline{\alpha}}$.
Let

$$(6.3.3) \quad \text{F.J.}^P(f) = D(B^*(f)) = \sum_{\underline{\alpha} \in \mathfrak{g}^{-2}(\mathbf{Q})^*} f_{\underline{\alpha}} q^{\underline{\alpha}}$$

with $f_{\underline{\alpha}} \in \Gamma(M(G_P, F_P), [\tilde{V}_{\rho_P}])$. This is the invariant form of the Fourier-Jacobi series of f along F_P (up to the choice of imbedding $D_P^+ \hookrightarrow \mathfrak{g}^{-2}(C)$). We will explain in 6.8 what this has to do with the general Fourier-Jacobi expansion-i.e., the one involving theta-functions. However, it is clear that, at least in this case, the formula (6.3.3) coincides with the classical Fourier-Jacobi expansion of Piatetski-Shapiro [47]. Indeed, although no automorphy factor appears in the definition, a careful examination of the proof of Proposition 6.2.5 will reveal that the canonical automorphy factor $J^{h,P}$ of (5.2.3.7) was implicit in our construction. It is easy to check that there is a commutative diagram

$$(6.3.4) \quad \begin{array}{ccc} \Gamma(M(G, X), [\mathscr{V}]) & \xrightarrow{\sim} & M_{\mathscr{V}} \\ \downarrow{\scriptstyle \text{F.J.}^P} & & \downarrow{\scriptstyle \text{F.J.}^{\text{class}}} \\ \hat{\bigoplus}_{\underline{\alpha}} \Gamma(M(G_P, F_P), [\tilde{V}_{\rho_P}]) & \xrightarrow{\sim} & \hat{\bigoplus}_{\underline{\alpha}} \mathrm{M}_{\tilde{V}_{\rho_P}} \end{array}$$

where M and $M_{\tilde{V}_{\rho_P}}$ are defined as in 5.3 with respect to $J^{h,P}$, the horizontal arrows are given by (5.3.5), and F.J.$^{\text{class}}$ is the classical Fourier-Jacobi expansion, corresponding to the chosen automorphy factor.

6.3.5 If G has no rational quotient isomorphic to PGL(2), then the Koecher principle ([1], 10. 14) implies that $f_{\underline{\alpha}} \equiv 0$ for $\underline{\alpha} \notin C$ (we identify $\mathfrak{g}^{-2}(\mathbf{R})$ with its dual). If G has such a three-dimensional factor, we require that f satisfy this condition. Thus, by abuse of notation, we will read $\Gamma(M(G, X), [\mathscr{V}])$ as the space of sections "holomorphic at infinity."

The main theorem of this section is

6.4 THEOREM: *Assume $f \in \Gamma(M(G, X), [\mathscr{V}])$ is rational over the extension L' of L. Then each Fourier-Jacobi coefficient $f_{\underline{\alpha}} \in \Gamma(M(G_P, F_P), [\tilde{V}_{\rho_P}])$ is rational over L'. In other words F.J.^{P} is an L-rational homomorphism.*

When the pair (G, X) admits a symplectic imbedding, this theory can be derived from the results of Brylinski's thesis [31].

The formulation of the theorem presupposes that $M(G_P, F_P)$ and $[\tilde{V}_{\rho_P}]$ have L-rational structures, and thus makes implicit reference to Corollary 6.1.4.

The proof of this theorem occupies most of the remainder of this section. We begin by making a few reductions.

6.4.1 Lemma: *It suffices to treat the case in which G^{der} is simply connected.*

Proof: The techniques used in sections 3.14 and 4.6 apply in this case as well, so the proof will be omitted.

6.4.2 Lemma: *Suppose $(G, X) \subset (G', X')$, where G'/G is an abelian group. Let $P' = P \cdot Z_{G'} \subset G'$, and suppose Theorem 6.4 is valid for (G', X') and P'. Then it is valid for (G, X) and P.*

Proof: Let $G_{P'} = G_P \cdot Z_{G'} \subset G'$ and define $F_{P'}$ in the obvious way. Let $(\rho', V_{\rho'})$ be an L-rational representation of G′ whose restriction to G contains (ρ, V_ρ), and let $\mathscr{V}' \subset \tilde{V}_{\rho'}$ be a G'-homogeneous vector bundle over $\check{M}(G', X')$ which contains $\mathscr{V}$ as a G-homogeneous direct summand. It obviously suffices to check that Theorem 6.4 holds with $\mathscr{V}$ replaced by $\mathscr{V}'$. Let $\rho_{P'}$ be the representation of $G_{P'}$ defined as in 6.2. There is a commutative diagram

$$\begin{array}{ccc}
\Gamma(M(G', X'), [\mathscr{V}']) & \longrightarrow & \Gamma(M(G, X), [\mathscr{V}']) \\
\Big\downarrow \text{F.J.}^{P'} & & \Big\downarrow \text{F.J.}^{P} \\
\hat{\bigoplus\limits_{\underline{\alpha}}} \Gamma(M(G_{P'}, F_{P'}), [\tilde{V}_{\rho_{P'}}]) & \to & \hat{\bigoplus\limits_{\underline{\alpha}}} \Gamma(M(G_P, F_P), [\tilde{V}_{\rho_{P'}}])
\end{array}$$

where the horizontal arrows are L-rational. We need only check that the top arrow is surjective. But $M(G, X)$ is the inverse limit of a family of open closed subsets of ${}_{K'}M(G', X')$, as K' varies among the open compact subgroups of $G'(\mathbf{A}^f)$. Thus $\Gamma(M(G', X'), [\mathscr{V}']) \to \Gamma(M(G, X), [\mathscr{V}'])$ is surjective, and we are done.

6.4.3 Corollary: *It suffices to prove Theorem 6.4 in the special case that $G^{\mathrm{der}} = G_l$; i.e., that X^+ is equivalent to a rational tube domain and F_P^+ is a point.*

Proof: By Lemma 6.4.1, we may assume G^{der} is simply connected. Thus the natural homomorphism

$$\gamma : G \to G' \overset{\mathrm{def}}{=} G/G_h^{\mathrm{der}} \times G/G_l^{\mathrm{der}}$$

is injective. Let X' be the $G'(\mathbf{R})$-conjugacy class of homomorphisms $h' : \underline{S} \to G'_{\mathbf{R}}$ generated by $\gamma \circ h$, $h \in X$. Then the inclusion $(G, X) \subset (G', X')$ satisfies the hypotheses of Lemma 6.4.2. Thus we may replace (G, X) by (G', X').

But (G', X') is isomorphic to the product $(G/G_h^{\mathrm{der}}, X_l) \times (G/G_l^{\mathrm{der}}, X_h)$, for some X_l, X_h such that each connected component of X_l (resp. X_h) is isomorphic to D_P^+ (resp. F_P^+). Moreover, P' is the product of G/G_l^{der} with a parabolic subgroup P_l of G/G_h^{der}. Let $G^1 = G/G_h^{\mathrm{der}}$, $G^2 = G/G_l^{\mathrm{der}}$. Then $G'_{P'}$ is the product of G^2 with a subtorus $H \subset G^1$, and the corresponding boundary component $F_{P'}$ is the product of X_h with a point $\nu : \underline{S} \to H_{\mathbf{R}}$. Without loss of generality, we may assume that there exist homogeneous vector bundles $\mathscr{V}^1$, $\mathscr{V}^2$, over $M(G^1, X_l)$ and $M(G^2, X_h)$, respectively, such that $\mathscr{V}$ is the external tensor product $\mathscr{V}^1 \otimes \mathscr{V}^2$. Then we have a commutative diagram

$$\begin{array}{ccc}
\Gamma(M(G', X'), [\mathscr{V}]) & \xrightarrow{\ \sim\ } & \Gamma(M(G^1, X_l), [\mathscr{V}^1]) \otimes \Gamma(M(G^2, X_h), [\mathscr{V}^2]) \\
\Big\downarrow {\scriptstyle \mathrm{FJ}^{P'}} & & \Big\downarrow {\scriptstyle \mathrm{FJ.}^{P_l} \otimes Id} \\
\hat{\bigoplus}_{\underline{\alpha}} \Gamma(M(G'_{P'}, F_{P'}), [\tilde{V}_{\rho_{P'}}]) & \xrightarrow{\sim} & \hat{\bigoplus}_{\underline{\alpha}} \Gamma(M(H, \nu), [\tilde{W}_{\rho_H}])) \otimes \Gamma(M(G^2, X_h), [\mathscr{V}^2])
\end{array}$$

where $\rho_H = \rho_{P_l} : H \to GL(W_{\rho_H})$ is the representation of $H = G^1_{P_l}$ defined as in 6.2, and Id is the identity map. Theorem 6.4 for (G', X') and P' thus is a consequence of the corresponding assertion for (G^1, X_l) and P_l.

6.5 We henceforward assume that X is a rational tube domain and that F_P^+ is a point. Our technique, following Shimura and Garrett [53,36] will be to reduce Theorem 6.4 in this case to the case in which $G^{\mathrm{der}} = SL(2, \mathbf{Q})$. For this we need the following lemma.

6.5.1 Lemma: *Suppose (G, X) is a pair in which X is a tube domain over the rational zero-dimensional boundary component F. There exists a pair $(G', X') \subset (G, X)$ such that*

(*i*) $G' \supset Z_G$;

(*ii*) $G'^{,ad} \simeq PGL(2)_{\mathbf{Q}}$;

(*iii*) *Some union of connected components of F is a boundary component of X'; and*

(*iv*) $E(G', X') = E(G, X)$.

Proof: First suppose G is an adjoint group. We consider pairs $Y = (Y^+, Y^-)$ with $Y^+ \in \mathfrak{g}^{-2}(\mathbf{Q})$, $Y^- \in \mathfrak{g}^2(\mathbf{Q})$, such that $Y^+ \neq 0$, $Y^- \neq 0$, and $[Y^+, Y^-] \in \mathfrak{a}_P = \mathrm{Lie}\, A_P$. We know tht $\mathfrak{g}^2$ and $\mathfrak{g}^{-2}$ are linearly isomorphic to some rational Jordan algebra $J(X)$, for which the trace form $B(\ ,\ )$ is positive-definite ([26], p. 227; [49], V, §3). Then the condition that

$$(6.5.1.1) \quad B(Y^-, v) = 0 \ \ \forall v \in \mathfrak{g}^2(\mathbf{Q}) \text{ such that } [v, Y^+] = 0$$

is $\mathbf{Q}$-linear, and we assume that Y satisfies (6.5.1.1). Under this hypothesis. Y^- is determined by Y^+ up to a scalar multiple. Let

$$\mathfrak{g}'_Y = \mathfrak{a}_P \oplus \mathbf{Q}Y^+ \oplus \mathbf{Q}Y^-$$

and let G'_Y be the corresponding subgroup of G. Note that G'_Y depends only on Y^+. There is an isomorphism $\mathrm{PGL}(2)_{\mathbf{Q}} \simeq G'_Y$ and consequently an inclusion $i_Y : \mathrm{PGL}(2)_{\mathbf{Q}} \to G$. Let X'_Y be the union $\mathfrak{H}^{\pm}$ of the Poincare upper and lower half-planes, viewed as a $\mathrm{PGL}(2, \mathbf{R})$-conjugacy class of homomorphisms $h : \underline{S} \to \mathrm{PGL}(2)_{\mathbf{R}}$ (cf. 1.2). We have to find conditions under which $i_Y(X'_Y) \subset X$.

Now G'_Y can be defined as a real Lie subgroup of $G(\mathbf{R})$, for any $Y^+ \in \mathfrak{g}^{-2}(\mathbf{R})$. We claim

(6.5.1.2) There is an open subset $\Gamma \subset \mathfrak{g}^{-2}(\mathbf{R})$ such that $\mathrm{i}_{\mathrm{Y}}(\mathrm{X}'_{\mathrm{Y}}) \subset \mathrm{X}$ for all $\mathrm{Y} \in \Gamma$.

In order to prove this claim, we may assume G to be an $\mathbf{R}$-simple group. In this case, $\mathrm{SL}(2, \mathbf{R})$-equivariant morphisms from the Poincare upper halfplane $\mathfrak{H}^+$ to X^+, and corresponding homomorphisms $\lambda : \mathrm{SL}(2, \mathbf{R}) \to G(\mathbf{R})$, are constructed in [26], Chapter III, §3. Let Y^+ be an element of $d\lambda(\mathfrak{sl}(2, \mathbf{R})) \cap \mathfrak{g}^{-2}(\mathbf{R})$; then it is easy to see that the given imbedding $\beta : \mathfrak{H}^+ \to X^+$ is of the form $i_Y|_{\mathfrak{H}^+}$, for $Y = (Y^+, Y^-)$ with an appropriate Y^-. It follows that for any $\gamma \in G_l^0(\mathbf{R})^0$, the translate $\gamma \circ \beta : \mathfrak{H}^+ \hookrightarrow X^+$ is of the form $i_{\gamma(Y)}$, where γ acts naturally by conjugation on $\mathfrak{g}^{-2}(\mathbf{R}) \times \mathfrak{g}^2(\mathbf{R})$. But the orbit of Y^+ under $G_l^0(\mathbf{R})^0$ is an open cone in $\mathfrak{g}^{-2}(\mathbf{R})$: in fact, it is isomorphic to the cone C of (5.1.7) ([26], p. 227). The assertion (6.5.1.2) now follows easily. But $\mathfrak{g}^{-2}(\mathbf{Q})$ is dense in $\mathfrak{g}^{-2}(\mathbf{R})$; thus if $Y^+ \in \Gamma \cap \mathfrak{g}^{-2}(\mathbf{Q}) \neq \emptyset$ we have a morphism of pairs $i_Y: (\mathrm{PGL}(2), X'_Y) \to (G, X)$.

We now drop the assumption that G be an adjoint group. Let X^{ad} be the $G^{\mathrm{ad}}(\mathbf{R})$-conjugacy class of homomorphisms $\underline{S} \to G^{\mathrm{ad}}_{\mathbf{R}}$ generated by the image of X in $G^{\mathrm{ad}}_{\mathbf{R}}$. The above argument provides us with a morphism of pairs i:$(\mathrm{PGL}(2), X'') \to (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. Since $G^{\mathrm{ad}}(\mathbf{Q})$ acts transitively on the set of connected components of X^{ad}, we may assume $i(X'')$ is contained in the image of X in X^{ad}, replacing i if necessary by its conjugate by some element in $G^{\mathrm{ad}}(\mathbf{Q})$. Now let G' be the inverse image of $i(\mathrm{PGL}(2))$ with respect to the natural map $G \to G^{\mathrm{ad}}$. Then the existence of a pair $(G', X') \subset (G, X)$ satisfying (i) and (ii) of the assertion of the lemma is obvious.

Next we verify (iii). We choose a connected component X^+ of X, such that $X^+ \cap X' \neq \emptyset$, and a connected component F^+ of F, which is a boundary component of X^+. We have to check that F^+ is in the closure of $X^+ \cap X'$ in the natural compactification of X^+ as a bounded domain ([26], §3). But the closure of any $A_P(\mathbf{R})$-orbit in X^+ contains F^+, and $A_P \subset G'$. This proves (iii).

It remains to verify (iv). First assume G is an adjoint group. Let $h \in X$, and let $c = c_{h,P}$ be the Cayley transform (5.2.3.1). Then $c\mu_h^2 c^{-1} = w_P : \mathbf{G}_m \to G$. It follows that the conjugacy class of μ_h contains a homomorphism defined over $\mathbf{Q}$, and is thus itself defined over $\mathbf{Q}$. Thus $E(G, X) = \mathbf{Q}$, provided G is an adjoint group (and X is a rational tube domain). Now in the general case, (iv) follows from Deligne's recipe for $E(G, X)$ ([5], 3.8). The lemma is proved.

6.5.2 Let $X^+ \subset X$ be as in the proof of Lemma 6.5.1. The lemma provides us with an imbedding of $G' = \mathrm{PGL}(2)_{\mathbf{Q}}$ in G^{ad}, and a $G'(\mathbf{R})^0$-equivariant map of the Poincaré upper half-plane $\mathfrak{H}^+$ into X^+. We say an isomorphism $X^+ \simeq \mathfrak{g}^{-2}(\mathbf{R}) \oplus iC$ (cf. 6.3) is *admissible* if for some such map of pairs $(G', \mathfrak{H}^+) \to (G^{\mathrm{ad}}, X^+)$ the induced map $X^+ \hookrightarrow \mathfrak{g}^{-2}(\mathbf{C})$ extends to a linear map $\mathbf{C} \to \mathfrak{g}^{-2}(\mathbf{C})$. Evidently conjugation by $P(\mathbf{Q}) \cap P(\mathbf{R})^0$ preserves the set of admissible isomorphisms. We assume that the identification $D_P^+ \simeq \mathfrak{g}^{-2}(\mathbf{R}) \oplus iC$ in 6.3 is admissible in this sense, relative to some imbedding $(G', \mathfrak{H}^+) \hookrightarrow (G_l^{\mathrm{ad}}, D_P^+)$. Our proof will show that $f \mapsto \mathrm{F.J.}^P(f)$ is L-rational, in the notation of Theorem 6.4, for any admissible isomorphism $D_P^+ \simeq \mathfrak{g}^{-2}(\mathbf{R}) \oplus iC$. This seems a little *ad hoc*, but it probably cannot be helped.

6.5.3 It will be convenient to make some further simplifications. Let $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ be as in the proof of Lemma 6.5.1. The proof of Lemma 2.5.5 of [6] shows that there exists a pair (G_*, X_*) mapping to $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ with $G_*^{\mathrm{ad}} = G^{\mathrm{ad}}$, G_*^{der} simply connected, $E(G_*, X_*) = \mathbf{Q}$, and $\mathbf{Z}_{G_*} \simeq \mathbf{G}_m$; the last two points follow from Deligne's construction and the fact that $E(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = \mathbf{Q}$, as we established in the proof of Lemma 6.5.1. Let (G_1, X_1) be the fiber product of (G, X) and (G_*, X_*) over $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, as in the proof of Corollary 4.6.4.

6.5.3.1 LEMMA: *It suffices to prove Theorem* 6.4 *for the pair* (G_1, X_1), *and the inverse image* P_1 *of* P *in* G_1.

PROOF: We know $E(G_1, X_1) = E(G, X)$. Let $\pi: M(G_1, X_1) \to M(G, X)$ be the natural map, and let $f \in \Gamma(M(G, X), [\mathscr{V}])$ be rational over the extension L of $E(G, X)$. If $\gamma \in G(\mathbf{A}^f)$, let $f_\gamma \in \Gamma(M(G, X), [\mathscr{V}])$ be the right translate of f by γ. Then $\pi^*(f_\gamma)$ is an L-rational element of $\Gamma(M(G_1, X_1), \pi^*[\mathscr{V}])$, for all $\gamma \in G(\mathbf{A}^f)$. But $\pi(M(G_1, X_1)) \cdot G(\mathbf{A}^f) = M(G, X)$. It follows that the $\mathrm{F.J.}^P(f)$ is determined by $\{\mathrm{F.H.}^{P_1}(\pi^*(f_\gamma)),\ \gamma \in G(\mathbf{A}^f)\}$. The lemma now follows easily.

6.5.3.2 LEMMA: *It suffices to prove Theorem* 6.4 *in the rational tube domain case with* $G^{\mathrm{ab}} = \mathbf{G}_m$, G^{der} *simply connected, and* $E(G, X) = \mathbf{Q}$.

PROOF: We may assume, by Lemma 6.5.3.1, that there is a map $(G, X) \to (G_*, X_*)$. Since G_* is simply connected, the diagonal map $G \to G_* \times G^{\mathrm{ab}}$ is injective. Thus there is an imbedding $(G, X) \hookrightarrow (G_*, \mathrm{X}_*) \times (G^{\mathrm{ab}}, Y)$, where Y is a point. By Lemma 6.4.2, we may replace (G, X) by the product $(G_*, X_*) \times (G^{\mathrm{ab}}, Y)$. But Theorem 6.4 for this product follows from the corresponding assertion for (G_*, X_*), as in the proof of Corollary 6.4.3.

6.6 LEMMA: *Theorem* 6.4 *is valid when* $G = GL(2)_{\mathbf{Q}}$, X *is the union* $\mathfrak{H}^+ \cup \mathfrak{H}^-$ *of the upper and lower half-planes in* $\mathbf{C}$, *and* P *is the upper triangular subgroup of* G.

PROOF: This is essentially the theory of the q-expansion and the Tate curve, as developed by Deligne and Rapoport in [35]. Since they only study q-expansions along connected components, we refer instead to the treatment of this subject in Brylinski's thesis [31], where it appears as a special case of the theory of families of 1-motives with absolute Hodge cycles. It should be clear to the reader that Brylinski's methods actually apply to all pairs (G, X) admitting symplectic imbeddings.

In this proof we develop only enough of the theory of the Tate curve to permit comparison between our Fourier-Jacobi expansion and the q-expansion of Deligne-Rapoport-Brylinski. The details omitted here can be found in [35,31].

We introducte the following notation. Let $N \in \mathbf{Z}$, $N \geqslant 3$, and let $K_N \subset GL(2, \hat{\mathbf{Z}})$ be the kernel of the map $GL(2, \hat{\mathbf{Z}}) \to GL(2, \mathbf{Z}/N\mathbf{Z})$. In our situation G_P can be identified with the diagonal subgroup $\mathbf{G}_m \times \mathbf{G}_m \subset P$, and $K_{N,P} = (1 + N\hat{\mathbf{Z}}) \times (1 + N\hat{\mathbf{Z}}) \subset \mathbf{G}_m(\mathbf{A}^f) \times \mathbf{G}_m(\mathbf{A}^f)$. We write $M(N, P) = {}_{K_{N,P}}M(G_P, F_P)$, $M(N) = {}_{K_N}M(G, X)$. We also have $P' = P$ in this case. We let $\Xi_N = \Xi_{K_N}$. Let $\mathscr{T}_N(\mathbf{C})$ be the group $U_P(\mathbf{C})/U_P(\mathbf{Q}) \cap K_N \simeq \mathbf{C}/N\mathbf{Z}$. There is an isomorphism

$$(6.6.1) \quad q^{1/N} : \mathscr{T}_N(\mathbf{C}) \xrightarrow{\sim} \mathbf{C}^\times; \quad z\big(\mathrm{mod}\ U_P(\mathbf{Q}) \cap K_N\big) \mapsto e^{2\pi i z/N}$$

which identifies $\mathscr{T}_N(\mathbf{C})$ with the group of $\mathbf{C}$-valued points of the torus $\mathscr{T}_N \simeq \mathbf{G}_m$, defined over $\mathbf{Q}$.

Let $\mathscr{F} : \{\mathbf{Q}\text{-schemes}\} \to \{\text{abelian groups}\}$ be the functor whose value at the test scheme S is $\mathrm{Hom}(\mathbf{Z}, \mathscr{T}_N(S))$. This functor is represented by $\mathscr{T}_N$, and there is thus a universal object

$$\begin{array}{ccc} \mathbf{Z}_{\mathscr{T}_N} & \xrightarrow{\tau_N} & \mathscr{T}_N \times \mathscr{T}_N \\ & \searrow \quad \swarrow & \\ & \mathscr{T}_N & \end{array}$$

where $\mathbf{Z}$ is regarded as a discrete algebraic group, the vertical arrows are the structure maps, and the triangle (6.6.2) is commutative.

Via the Borel imbedding β_X, we identify X with a subset of $\check{M}(G, X)(\mathbf{C}) \simeq P^1(\mathbf{C})$. Let $D(P)$ be the $U_P(\mathbf{C})$-orbit in $\check{M}(G, X)(\mathbf{C})$ containing X, and let $o \in D(P)$ denote the unique fixed piont of A_P. Let

$$\tilde{M}(N)(\mathbf{C}) = P(\mathbf{Q}) \backslash D(P) \times P(\mathbf{A}^f)/K_N.$$

Then $\tilde{M}(N)(\mathbf{C})$ fibers naturally over $M(N, P)(\mathbf{C})$, and the fibers are principal homogeneous spaces under $\mathscr{T}_N(\mathbf{C})$. Moreover, this fibration has a section: the subset

$$P(\mathbf{Q}) \backslash P(\mathbf{Q}) \cdot o \times P(\mathbf{A}^f)/K_N \subset \tilde{M}(N)(\mathbf{C})$$

maps isomorphically onto $M(N, P)(\mathbf{C})$. (We are using strong approximation for U_P). We thus have an isomorphism over $M(N, P)(\mathbf{C})$:

(6.6.3) $\quad \tilde{M}(N)(\mathbf{C}) \simeq M(N, P)(\mathbf{C}) \times \mathscr{T}_N(\mathbf{C}),$

which implies that $\tilde{M}(N)(\mathbf{C})$ is the set of $\mathbf{C}$-valued points of a $\mathscr{T}_N$-bundle $\tilde{M}(N) \simeq M(N, P) \times \mathscr{T}_N$ over $M(N, P)$, defined over $\mathbf{Q}$. Let $\tilde{\tau}_N : \mathbf{Z}_{\tilde{M}(N)} \to \mathscr{T}_{N,\tilde{M}(N)}$ be the pullback of (6.6.2) to $\tilde{M}(N)$ via the projection $\tilde{M}(N) \to \mathscr{T}_N$.

The imbedding $q^{1/N} : \mathscr{T}_N \xrightarrow{\sim} \mathbf{G}_m \hookrightarrow \mathbf{G}_a$ determines an action of $\mathscr{T}_N$ on $\mathbf{G}_a$, by multiplication. Let $\hat{\mathscr{T}}_N$ denote the complement of the origin in the formal completion at the origin of $\mathbf{G}_a$. Then there is a natural morphism of ringed spaces $\hat{\mathscr{T}}_N \to \mathscr{T}_N$; let

$$\hat{M}(N) = \tilde{M}(N) \times_{\mathscr{T}_N} \hat{\mathscr{T}}_N;$$

$$\hat{\tau}_N = \tilde{\tau}_N \times_{\mathscr{T}_N} Id : \mathbf{Z}_{\hat{M}(N)} \to \mathscr{T}_{N,\hat{M}(N)}.$$

Note that $\hat{\mathscr{T}}_N = \operatorname{Spec} \mathbf{Q}((q^{1/N}))$; thus $\hat{M}(N)$ is a scheme over $\mathbf{Q}((q^{1/N}))$. Let $\mathrm{Tate}_N(q)$ be the elliptic curve over $\hat{M}(N)$ whose group of points over $\mathbf{Q}((q^{1/N}))$ is isomorphic to

$$\hat{M}(N)\big(\mathbf{Q}((q^{1/N}))\big)/\hat{\tau}_N(N \cdot \mathbf{Z}).$$

This is obviously the pullback to $\hat{M}(N)$ of the Tate curve [35] over $\mathbf{Q}((q^{1/N}))$. The arguments of Brylinski [31], especially §2, imply that $\mathrm{Tate}_N(q)$ naturally has a level N structure over $\hat{M}(N)$. But $M(N)$ is the moduli space of elliptic curves with level N structure. Thus there is a classifying map, defined over $\mathbf{Q}$:

$$\beta_N : \hat{M}(N) \to M(N).$$

On the other hand, the inclusion $X \subset D(P)$ defines an imbedding $\Xi_N \to \tilde{M}(N)(\mathbf{C})$. The canonical morphism of ringed spaces $\hat{M}(N)(\mathbf{C}) \to \tilde{M}(N)(\mathbf{C})$ factors through a map of ringed spaces

(6.6.5) $\quad \hat{M}(N)(\mathbf{C}) \to \Xi_N.$

Recall the map $B_{K_N} : \Xi_N \to M(N)(\mathbf{C})$ of 6.2, and let $\mathscr{E}/\Xi_N$ denote the pullback via B_{K_N} of the universal elliptic curve over $M(N)(\mathbf{C})$. The homomorphism $\tilde{\tau}_N$ restricts to a homomorphism $j : \mathbf{Z}_{\Xi_N} \to \mathscr{T}_{N,\Xi_N}$, and the classical construction of the universal elliptic curve over $M(N)(\mathbf{C})$ shows that

(6.6.6) $\quad \mathscr{E} \simeq \mathscr{T}_N(\Xi_N)/j(N\mathbf{Z})$

(cf. [35], VII.4). Let $\beta_N^{\mp} : \hat{M}(N)(\mathbf{C}) \to M(N)(\mathbf{C})$ denote the composite of

(6.6.5) with B_{K_N}. It follows from (6.6.6) that $\beta_N = \beta_N^{\mp}$ on **C**-valued points.

Let $E(N)$ be the universal elliptic curve over $M(N)$; let $\omega(N) = \Omega^1_{E(N)/M(N)}$, $\mathscr{H}^1(N) = \mathscr{H}^1_{DR}(E(N)/M(N))$. Let $k \in \mathbf{Z}$, $k \geqslant 0$. By construction, $\beta_N^*(\omega(N)^{\otimes k}) = \Omega^1(\mathrm{Tate}_N(q)/\hat{M}(N))^{\otimes k}$ is canonically isomorphic to $((\Omega^1_{\mathscr{T}_N/\mathbf{Q}})^{\mathscr{T}_N})^{\otimes k}_{\hat{M}(N)}$, which in turn is isomorphic to $\hat{\pi}^*((\Omega^1_{\mathscr{T}_N/\mathbf{Q}})^{\mathscr{T}_N})^{\otimes k}_{M(N,P)}$, where $\hat{\pi}: \hat{M}(N) \to M(N, P)$ is the natural map, and $(\Omega^1_{\mathscr{T}_N/\mathbf{Q}})^{\mathscr{T}_N}$ is the space of invariant 1-forms on $\mathscr{T}_N$. For simplicity we denote this space Ω. We thus have an inclusion

$$(6.6.7)\quad \hat{\pi}^*\left(\Omega^{\otimes k}_{M(N,P)}\right) \xrightarrow{\sim} \beta_N^*\left(\omega(N)^{\otimes k}\right) \hookrightarrow \beta_N^*\left(\mathscr{H}^1(N)^{\otimes k}\right),$$

defined over **Q**. There is also an isomorphism

$$H^1(\mathscr{T}_N(\mathbf{C}), \mathbf{C}) \xrightarrow{\sim} \Omega(\mathbf{C})$$

given by integration of forms over cycles, and thus an inclusion

$$H^1(\mathscr{T}_N(\mathbf{C}), \mathbf{Z}) \hookrightarrow \Omega(\mathbf{C}).$$

The diligent reader will check that (6.6.7) restricts to a map

$$(6.6.8)\quad \hat{\pi}^*\left(H^1(\mathscr{T}_N(\mathbf{C}), \mathbf{Z})^{\otimes k}_{M(N,P)}\right) \hookrightarrow \beta_N^*\left(R^1 f_*(\mathbf{Q})^{\otimes k}\right)$$

where $f: E(N) \to M(N)$ is the natural map, and **Q** is the constant sheaf. It follows from (6.6.8) and Remark 4.5.3 that

(6.6.9) *The first arrow in* (6.6.7) *is the inverse of the isomorphism* $\mathrm{D}: \beta_{\mathrm{N}}^*(\omega(\mathrm{N})^{\otimes \mathrm{k}}) \xrightarrow{\sim} \pi^*(\Omega^{\otimes \mathrm{k}}_{\mathrm{M(N,P)}})$ *of Proposition* 6.2.5, *restricted to* $\hat{\mathrm{M}}(\mathrm{N})$.

Let $\tilde{\pi}: \tilde{M}(N) \to M(N, P)$ be the natural map. The action of $\mathscr{T}_N$ on $M(N)$ induces an isomorphism

$$\tilde{\pi}_*\left(\tilde{\pi}^*\left(\Omega^{\otimes k}_{M(N,P)}\right)\right) \simeq \hat{\bigoplus_{a \in \mathbf{Z}}} \Omega^{\otimes k}_{M(N,P)} \cdot q^{a/N}.$$

Upon passage to the formal completion, one obtains an isomorphism

$$(6.6.10)\quad \hat{\pi}_*\left(\hat{\pi}\left(\Omega^{\otimes k}_{M(N,P)}\right)\right) \simeq \hat{\bigoplus_{a \in \mathbf{Z}}} \Omega^{\otimes k}_{M(N,P)} \cdot q^{a/N}$$

where $\hat{\oplus}$ refers to *formal* Laurent series (cf. [31], 5.2). Let $f \in \Gamma(M(N), \omega^{\otimes k})$. Via (6.6.7) and (6.6.10), we obtain a Fourier expansion

$$\mathrm{F.J.}(f) \in \hat{\bigoplus_{a \in \mathbf{Z}}} \Gamma\left(M(N, P), \Omega^{\otimes k}_{M(N,P)}\right),$$

and the map $f \mapsto \mathrm{F.J.}(f)$ is defined over $\mathbf{Q}$. By (6.6.9), $\mathrm{F.J.}(f)$ as defined here coincides with $F.J.^P(f)$ as defined in (6.3.3). Since $\{K_N\}$ is cofinal in the set of open compact subgroups of $G(A^f)$, we have completed the proof of Lemma 6.6 for the vector bundles $\boldsymbol{\omega}(N)^{\otimes k}$, $k \geqslant 0$. But every vector bundle $[\mathscr{V}]/M(N)$ which satisfies hypothesis (5.3.6) is of the form $\boldsymbol{\omega}(N)^{\otimes k} \otimes [\mathbf{Q}(n)]$, where $G \to GL(\mathbf{Q}(n))$ is the nth power of the determinant representation. The validity of Theorem 6.4 for such a bundle is a trivial consequence of its validity for $\boldsymbol{\omega}(N)^{\otimes k}$. The lemma is proved.

6.7 We now return to the general case of a rational tube domain, under the hypotheses of Lemma 6.5.3.2. In this paragraph we prove a series of lemmas which reduces this case to the one-dimensional case treated in 6.6.

For the sake of brevity we write $V = V_{\rho_P}$, $M = M(G_P, F_P)$. The group G_P is isomorphic to $\mathbf{G}_m \times \mathbf{G}_m$, which we regard as a group scheme over $\mathbf{Z}$. We may thus assume that the representation $\rho_P : G_P \to GL(V)$ is defined over $\mathbf{Q} = E(G_P, F_P)$; the same is then true of the vector bundle $[\tilde{V}]$.

The formal series $\sum_{\underline{\alpha} \in \mathfrak{g}^{-2}(\mathbf{Q})^*} f_{\underline{\alpha}} q^{\underline{\alpha}}$, with $f_{\underline{\alpha}} \in \Gamma(M, [\tilde{V}])$, will be called rational over the extension L' of $\mathbf{Q}$ if $f_{\underline{\alpha}}$ is L'-rational for all $\underline{\alpha}$.

In the following lemmas, the action of $U_P(\mathbf{Q})$ on $\pi_* \pi^* [\tilde{V}]$ is the one induced by the right action of $U_P(\mathbf{Q}) \subset U_P(\mathbf{A}^f)$ on Ξ.

6.7.1 Lemma: *Suppose $f \in \Gamma(M, \pi_* \pi^* [\tilde{V}])$ has a Fourier-Jacobi expansion*

$$\mathrm{F.J.}^P(f) = \sum_{\underline{\alpha} \in \mathfrak{g}^{-2}(\mathbf{Q})^*} f_{\underline{\alpha}} q^{\underline{\alpha}}$$

which is rational over the extension L' of $\mathbf{Q}$. Then for any $u \in U_P(\mathbf{Q})$, the right translate $u^*(f) \in \Gamma(M, \pi_* \pi^* [\tilde{V}])$ also has an L'-rational Fourier expansion. In other words, $U_P(\mathbf{Q})$ acts $\mathbf{Q}$-rationally on $\bigoplus_{\underline{\alpha} \in \mathfrak{g}^{-2}(\mathbf{Q})^*} [\tilde{V}] \cdot q^{\underline{\alpha}}$.

Proof: Let x be a connected component – i.e., a point – of M. Then x is defined over the maximal cyclotomic extension $\mathbf{Q}_{\mathrm{ab}}$ of $\mathbf{Q}$, by hypothesis, and $M = x \cdot G_P(\hat{\mathbf{Z}})$. Moreover, we have the reciprocity law (1.2.2) for the action of $G_{\mathrm{ab}} \overset{\mathrm{def}}{=} \mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q})$ on the set of points of M. Namely, F_P is a point h; let $\mu = \mu_H : \mathbf{G}_m \to G_P$ be the morphism of 0.7. Define $r = \mu_{\mathbf{A}^f}^{-1} : \mathbf{G}_m(\mathbf{A}^f) \to G_P(\mathbf{A}^f)$. Let $\epsilon : \mathbf{G}_m(\hat{\mathbf{Z}}) \xrightarrow{\sim} G_{\mathrm{ab}}$ be the reciprocity homomorphism, normalized as in [6] to be the *inverse* of the usual Artin map. Then if $\sigma \in G_{\mathrm{ab}}$, $x \in M$, we have

$$(6.7.1.1) \quad x^{\sigma} = x \cdot \left(r \circ \epsilon^{-1}(\sigma)\right)$$

where the multiplication on the right hand side refers to right translation by $r \circ \epsilon^{-1}(\sigma) \in \mathbf{G}_P(\hat{\mathbf{Z}})$.

On the other hand, G_P is a Levi component of P. Let $\gamma: G_P \to \mathbf{G}_m$ be the root defining the adjoint action of G_P on U_P. As in the proof of Lemma 5.1.10, $\mathfrak{g}^{-2} = \mathrm{Lie}(U_P)$ is of Hodge type $(-1, -1)$ with respect to $h: \underline{S} \to G_{P,\mathbf{R}}$. It follows that $\gamma \circ \mu: \mathbf{G}_m \to \mathbf{G}_m$ is the map $t \mapsto t^{-1}$. Consequently $\gamma_{\mathbf{A}^f} \circ r: \mathbf{G}_m(\mathbf{A}^f) \to \mathrm{Aut}(U_P(\mathbf{A}^f)) \simeq \mathrm{Aut}(\mathbf{A}^f)$ takes $t \in \mathbf{G}_m(\mathbf{A}^f)$ to multiplication by t in $\mathrm{Aut}(U_P(\mathbf{A}^f))$. We write $u^*(f) = \sum_{\underline{\alpha}} \underline{\alpha}(u^*f)_{\underline{\alpha}} \cdot q^{\underline{\alpha}}$ for the Fourier expansion of $u^*(f)$. It suffices to consider $L' = \mathbf{Q}$. We may then consider $f_{\underline{\alpha}}$ as a collection $\{f_{\underline{\alpha}}^x\}_{x \in M}$ of elements of $V(\mathbf{Q}_{\mathrm{ab}})$ such that

$$(6.7.1.2) \quad (f_{\underline{\alpha}}^x)^\sigma = f_{\underline{\alpha}}^{\sigma(x)} = f_{\underline{\alpha}}^{x \cdot (r \circ \epsilon^{-1}(\sigma))} \quad \sigma \in G_{\mathrm{ab}}.$$

Likewise we may write $(u^*f)_{\underline{\alpha}} = \{(u^*f)_{\underline{\alpha}}^x\}_{x \in M}$. We have to check that these quantities also satisfy (6.7.1.2).

Suppose $\underline{\alpha}$ belongs to the lattice $\Lambda = \mathrm{Hom}(K \cap U(\mathbf{Q}), \mathbf{Z}) \subset \mathfrak{g}^{-2}(\mathbf{Q})^*$, for some K as in 6.3. We may think of $q^{\underline{\alpha}}$ as a left $P'(\mathbf{Q})$-invariant function on $X^+ \times P'(\mathbf{A}^f)/K \cap P'(\mathbf{A}^f)$, determined by its restriction to $X^+ \times G_P(\hat{\mathbf{Z}}) \times U_P(\mathbf{A}^f)$:

$$q^{\underline{\alpha}}(z, g, v) = \mathrm{e}(\underline{\alpha}(z)) \quad z \in X^+,\ g \in G_P(\hat{\mathbf{Z}}),\ v \in U_P(\mathbf{A}^f)$$

where $\mathrm{e}(x) = \mathrm{e}^{2\pi i x}$. Then

$$(6.7.1.3) \quad q^{\alpha}((z, g, v) \cdot u) = q^{\underline{\alpha}}(u_g^{-1} z, g, v) = \mathrm{e}(-\underline{\alpha}(u_g)) q^{\underline{\alpha}}(z, g, v)$$
$$z \in X^+,\ g \in G_P(\hat{\mathbf{Z}}),\ v \in U_P(\mathbf{A}^f),\ u \in U_P(\mathbf{Q})$$

where $u_g \in U(\mathbf{Q})$, $u_g \equiv gug^{-1} (\mathrm{mod}\ K \cap U(\mathbf{Q}))$. Note that by definition, $u_g \equiv \gamma_{\mathbf{A}^f}(g) \cdot u (\mathrm{mod}\ K \cap U(\mathbf{Q}))$. Thus, suppose $g = r \circ \epsilon^{-1}(\sigma)$, $\sigma \in G_{\mathrm{ab}}$. It follows from our previous remarks that

$$(6.7.1.4) \quad q^{\underline{\alpha}}((z, r \circ \epsilon^{-1}(\sigma), v) \cdot u)$$
$$= \mathrm{e}(-\epsilon^{-1}(\sigma) \underline{\alpha}(u)) q^{\underline{\alpha}}(z, r \circ \epsilon^{-1}(\sigma), v)$$
$$= \mathrm{e}(\underline{\alpha}(u))^\sigma q^{\underline{\alpha}}(z, r \circ \epsilon^{-1}(\sigma), v)$$

by our normalization of ϵ.

Suppose x_0 is the image of $F_P \times \{1\}$ in M. It follows from the above that

$$(6.7.1.5) \quad (u^*f)_{\underline{\alpha}}^x = \zeta_{u,\underline{\alpha}}^x \cdot f_{\underline{\alpha}}^x$$

for some root of unity $\zeta_{u,\underline{\alpha}}^x$ which satisfies

$$(6.7.1.6) \quad \zeta_{u,\underline{\alpha}}^{x_0} = \mathrm{e}(-\underline{\alpha}(u))$$

and more importantly

$$(6.7.1.7) \quad \zeta_{u,\underline{\alpha}}^{x\cdot(r\circ\epsilon^{-1}(\sigma))} = \left(\zeta_{u,\underline{\alpha}}^{x}\right)^{\sigma} \quad x \in M,\ \sigma \in G_{\mathrm{ab}}.$$

Now it follows from (6.7.1.5), (6.7.1.7), and (6.7.1.2) that

$$(u^*f)_{\underline{\alpha}}^{x} = (u^*f)_{\underline{\alpha}}^{x\cdot(r\circ\epsilon^{-1}(\sigma))},$$

which proves the lemma.

Let $G' = GL(2)_{\mathbf{Q}}$, $X' = \mathfrak{H}^+ \cup \mathfrak{H}^-$ as in Lemma 6.6, and let $s:(G', X') \to (G, X)$ be an imbedding as in Lemma 6.5.1. For any $u \in U_P(\mathbf{Q})$, let s^u be the conjugate of s by u^{-1}. Note that the boundary group $G'_{P\cap G'}$ and boundary component $F_{P\cap G'}$ attached to $P \cap G' \subset G'$ are equivalent to the pair $(G_P, F_P) \approx (\mathbf{G}_m \times \mathbf{G}_m, \{\text{point}\})$. We assume, as in the proof of Lemma 6.5.1, that $G' \supset A_P$; then we have $G'_{P\cap G'} = G_P$. To the pair (G', X') we associate a fibration $\pi' : \Xi' \to M$, and we have an imbedding $\Xi' \hookrightarrow \Xi$ over M.

Assume the isomorphism $X^+ \simeq \mathfrak{g}^{-2}(\mathbf{R}) \oplus iC$ is chosen so that the corresponding morphism $\mathfrak{H}^+ \overset{s}{\hookrightarrow} X^+ \hookrightarrow \mathfrak{g}^{-2}(\mathbf{C})$ extends to a linear map, as in 6.5.2. Then for any $\underline{\alpha} \in \mathfrak{g}^{-2}(\mathbf{Q})^*$, we obtain an element $s^*(\underline{\alpha}) \in (\mathfrak{g}')^{-2}(\mathbf{Q})^*$. We identify $(\mathfrak{g}')^{-2}(\mathbf{Q})^*$ with $\mathbf{Q}$ in such a way that the positive elements of the two spaces coincide. If $\rho_P : G_P \to GL(V)$ is as above, the inclusion $\Xi' \to \Xi$ induces a homomorphism

$$\Gamma(\Xi, \pi^*[\tilde{V}]) \to \Gamma(\Xi', (\pi')^*[\tilde{V}]),$$

and a corresponding map on Fourier series

$$T_s : \hat{\bigoplus_{\underline{\alpha}\in\mathfrak{g}^{-2}(\mathbf{Q})^*}} \Gamma(M, [\tilde{V}]) \cdot q^{\underline{\alpha}} \to \hat{\bigoplus_{n\in\mathbf{Q}}} \Gamma(M, [\tilde{V}]) \cdot q^n$$

which is given by

$$(6.7.2) \quad \sum_{\underline{\alpha}} f_{\underline{\alpha}} q^{\underline{\alpha}} \mapsto \sum_{n}{}' \left(\sum_{s'(\underline{\alpha})=n} f_{\underline{\alpha}} \right) q^n.$$

Similarly, for any $u \in U_P(\mathbf{Q})$, s^u induces a map T_{s^u} on Fourier series given by

$$(6.7.2.1) \quad \sum_{\underline{\alpha}} f_{\underline{\alpha}} \cdot q^{\underline{\alpha}} \mapsto \sum_{n} \left(\sum_{s^*(\underline{\alpha})=n} (u^*f)_{\underline{\alpha}} \right) q^n.$$

Note that $f_{\underline{\alpha}} = 0$ unless $\underline{\alpha} \in C \cap \mathfrak{g}^{-2}(\mathbf{Q})^* \cap K$, for some open compact subgroup $K \subset G(\mathbf{A}^f)$, where we identify $\mathfrak{g}^{-2}(\mathbf{Q})^*$ with $U_P(\mathbf{Q})$ via the exponential map. It follows as in [36], Lemma 3.1 that the inner sum in the right hand side of (6.7.2), (6.7.2.u) is *finite*.

6.7.3 LEMMA: *Let* $F = \sum_{\underline{\alpha}} f_{\underline{\alpha}} q^{\underline{\alpha}} \in \hat{\bigoplus}_{\underline{\alpha} \in \mathfrak{g}^{-2}(\mathbf{Q})^* \cap C \cap K} \Gamma(M, [\tilde{V}]) \cdot q^{\underline{\alpha}}$ *with K as above, where* $\hat{\oplus}$ *denotes* formal *Laurent series. Suppose* $T_{s^u}(F) \in \hat{\bigoplus}_{n \in \mathbf{Q}} \Gamma(M, [\tilde{V}]) \cdot q^n$ *is rational over the extension L′ of* $\mathbf{Q}$ *for all* $u \in U_P(\mathbf{Q})$. *Then F is L′-rational.*

PROOF: It follows from (6.7.2) and Lemma 6.2.1 that the map $F \mapsto T_{s^u}(F)$ is rational over $\mathbf{Q}$ for all $u \in U_P(\mathbf{Q})$. The lemma is thus a consequence of the following lemma.

6.7.4 LEMMA: *Let F be as in Lemma* 6.7.3. *Suppose* $T_{s^u}(F) = 0$ *for all* $u \in U_P(\mathbf{Q})$. *Then* $f \equiv 0$.

PROOF: The proof is based on an idea of Shimura ([53], p. 502). Let $x \in M$, and let $F^x = \sum_{\underline{\alpha}} f^x_{\underline{\alpha}} \cdot q^{\underline{\alpha}}$, in the notation of the proof of Lemma 6.7.1. We have to check that $F^x = 0$ for all $x \in M$; thus we fix $x = x_0 \cdot (r \circ \epsilon^{-1}(\sigma))$, say. We are given that

$$\sum_{s^*(\underline{\alpha}) = n} (u^* f)^x_{\underline{\alpha}} = 0 \quad \forall n \in \mathbf{Q},\ u \in U_P(\mathbf{Q}).$$

By (6.7.1.5)–(6.7.1.7), this implies that

$$\sum_{s^*(\underline{\alpha}) = n} \mathrm{e}(\underline{\alpha}(u))^{\sigma} f^x_{\underline{\alpha}} = 0 \quad \forall n \in \mathbf{Q},\ u \in U_P(\mathbf{Q}). \tag{6.7.4.1}$$

Now in analogy with the Lemma on p. 502 of [53], we can find a subset Y of $U_P(\mathbf{Q})$, with the same cardinality as $Y' = \{\underline{\alpha} \in \mathfrak{g}^{-2}(\mathbf{Q})^* \cap K \cap C \mid s^*(\underline{\alpha}) = n\}$, such that

$$\det\big(\mathrm{e}(\underline{\alpha}(u))^{\sigma}\big)_{u \in Y, \underline{\alpha} \in Y'} = \det\big(\mathrm{e}(\underline{\alpha}(u))\big)^{\sigma}_{u \in Y, \underline{\alpha} \in Y'} \neq 0; \tag{6.7.4.2}$$

cf. also [36], Proposition 6.2. It follows from (6.7.4.1) and (6.7.4.2) that $F^x \equiv 0$.

Now the proof of Theorem 6.4 is obvious. It follows from properties of canonical local systems that, up to a multiplicative constant in $\mathbf{Q}^{\times}$, the following diagram commutes:

(6.7.5)

$$\begin{array}{ccc}
\Gamma(M(G, X), [\mathcal{V}]) & \xrightarrow{M(s^u)^*} & \Gamma(M(G', X'), M(s^u)^*[\mathcal{V}]) \\
\Big\downarrow F.J.^P & & \Big\downarrow F.J.^{P \cap G'} \\
\hat{\bigoplus}'_{\underline{\alpha} \in \mathfrak{g}^{-2}(\mathbf{Q})^*} \Gamma(M, [\tilde{V}]) \cdot q^{\underline{\alpha}} & \underset{T_{s^u}}{\rightarrow} & \hat{\bigoplus}_{n \in \mathbf{Q}} \Gamma(M, [\tilde{V}]) \cdot q^n
\end{array} \qquad u \in U_P(\mathbf{Q})$$

where $M(s^u) : M(G', X') \to M(G, X)$ is the imbedding induced by s^u and $\hat{\oplus}'$ refers to the subspace of $\sum_{\underline{\alpha}} f_{\underline{\alpha}} \cdot q^{\underline{\alpha}}$ for which there exists some K as above such that $f_{\underline{\alpha}} \neq 0 \Rightarrow \underline{\alpha} \in \mathfrak{g}^{-2}(\mathbf{Q})^* \cap C \cap K$. Let $f \in \Gamma(M(G, X), [\mathscr{V}])$ be rational over L'. Then $M(s^u)^*(f)$ is L'-rational for all $u \in U_P(\mathbf{Q})$. It follows from Lemma 6.6 that $\text{F.J.}^{P \cap G'}(M(s^u)^*(f))$ is L'-rational for all $u \in U_P(\mathbf{Q})$. Since (6.7.5) is commutative, the rationality of $\text{F.J.}^P(f)$ is a consequence of Lemma 6.7.3. Theorem 6.4 is proved.

6.8 We now return to the general case of a pair (G, X) and a maximal rational parabolic subgroup $P \subset G$. We define the pair $(G^{\text{even}}, \Delta(P))$ and, for any $\gamma \in G(\mathbf{A}^f)$, the map $i_\gamma : M(G^{\text{even}}, \Delta(P)) \to M(G, X)$ as in 6.1. Fix a homogeneous bundle $\mathscr{V}$ on $\check{M}(G, X)$, and let $[\mathscr{V}]^{\text{even}}$ be the pullback of $[\mathscr{V}]$ to $M(G^{\text{even}}, \Delta(P))$ via the natural inclusion in $M(G, X)$. We assume henceforward that $[\mathscr{V}]^{\text{even}}$ satisfies hypothesis (5.3.6) for $(G^{\text{even}}, \Delta(P))$.

Let $w \in W_P(\mathbf{Q})$, regarded as a subgroup of $G(\mathbf{A}^f)$. We let $P^{\text{even}} = P \cap G^{\text{even}}$. Define $\rho_P : G_P \to GL(V_{\rho_P})$ as in 6.2, and let

$$\text{F.J.}^{P,w} : \Gamma(M(G, X), [\mathscr{V}]) \to \hat{\bigoplus_{\underline{\alpha} \in \mathfrak{g}^{-2}(\mathbf{Q})^*}} \Gamma\left(M(G_P, F_P), \left[\tilde{V}_{\rho_P}\right]\right) \cdot q^{\underline{\alpha}}$$

be defined by the formula

$$\text{F.J.}^{P,w}(f) = \text{F.J.}^{P^{\text{even}}}(i_w^*(f)).$$

Our results imply that the map $\text{F.J.}^{P,w}$ is rational over the field of definition of $\mathscr{V}$ as a homogeneous vector bundle. On the other hand, an easy calculation shows that, on a connected component of $M(G, X)$, the coefficients of $\text{F.J.}^{P,w}(f)$ are what Garrett refers to as "normalized division-point values" (ND-PV's) of the theta functions in the classical Fourier-Jacobi series of f [11]. Such ND-PV's had previously been used by Shimura, in a number of cases, to define arithmetic automorphic forms [20,21]; however, he did not give them any special name.

For those who are concerned with such things, then, our results imply that rationality in terms of values at CM points as in Proposition 5.3.10, implies rationaltiy in terms of ND-PV's of Fourier-Jacobi coefficients. It seems pointless to make this equivalence more precise without a specific application in mind.

One can also ask for a version of the *q-expansion principle*, namely, is it true that a form all of whose ND-PV's are rational over the field L, say, is itself L-rational? The following is the (essentially tautological) answer:

6.9 THEOREM (*q-expansion principle*). *Let $\mathscr{V}$ be a homogeneous vector bundle on $\check{M}(G, X)$, rational over the extension L of $E(G, X)$. Let $f \in \Gamma(M(G, X), [\mathscr{V}])$, and let L' be an extension of L. Assume $[\mathscr{V}]^{even}$ satisfies hypothesis* (5.3.6) *for* $(G^{\text{even}}, \Delta(P))$.

(*i*) *Suppose* $F.J.^{P,w}(f)$ *is rational over* L', *for all* $w \in W_P(\mathbf{Q})$. *Then the pullback of* f *to the closure* $\overline{M}^P$ *in* $M(G, X)$ *of* $M(G^{even}, \Delta(P))\ W_P(\mathbf{Q})$ *is rational over* L'. *The subset* $\overline{M}^P \subset M(G, X)$ *is a union of connected components of* $M(G, X)$.

(*ii*) *More generally, let* $\Sigma \subset G(\mathbf{A}^f)$ *be a subset such that* $\overline{M}^P \cdot \Sigma = M(G, X)$. *Suppose* $F.J.^{P^{even}}(i_\gamma^*(f))$ *is rational over* L', *for all* $\gamma \in W_P(\mathbf{Q}) \cdot \Sigma$. *Then* f *is rational over* L'.

PROOF: This is a trivial consequence of the continuity of f. The only point requiring elaboration is the fact that $\overline{M}^P$ is a union of connected components of $M(G, X)$. In fact, let M^0 be a connected component of $M(G^{\text{even}}, \Delta(P))$, say $M^0 =$ the image in $M(G^{\text{even}}, \Delta(P))$ of $\Delta(P)^+ \times \{\gamma\}$, $\gamma \in G^{\text{even}}(\mathbf{A}^f)$. The closure of $M^0 \cdot W_P(\mathbf{Q})$ in $M(G, X)$ is the same as the closure of $M^0 \cdot W_P(\mathbf{A}^f)$ in $M(G, X)$. Since $W_P(\mathbf{R}) \cdot \Delta(P)^+$ is a connected component X^+ of X, it folows from strong approximation for W_P that the closure of $M^0 \cdot W_P(\mathbf{Q})$ in $M(G, X)$ is the image of $X^+ \times \{\gamma\}$ in $M(G, X)$.

6.10 Let E^{ab} be the maximal abelian extension of $E(G, X)$, and let M^0 be a connected component of $\overline{M}^P$; then M^0 is rational over E^{ab}. Let $[\mathscr{V}]^0$ be the restriction of $[\mathscr{V}]$ to M^0. Let L' be an extension of $L \cdot E^{\text{ab}}$. It follows from Theorem 6.9 that if $f \in \Gamma(M^0, [\mathscr{V}]^0)$ has L'-rational ND-PV's along F_P^+, then f is L'-rational. Theorem 6.4 then implies that f has L'-rational ND-PV's along every other rational boundary component. I include this remark in response to a question of R. Indik.

REMARK 6.11: We have defined the Fourier-Jacobi series as a homomorphism of global sections of certain vector bundles. However, our method extends without much difficulty to define the Fourier-Jacobi series of meromorphic sections as well. This can be applied in particular to the structure sheaf $\mathcal{O}_{M(G,X)}$. As in Brylinski's thesis [31], 5.2.3, we can then conclude that certain of Mumford's toroidal compactifications of $M(G, X)$-namely, the smooth projective compactifications studied by Tai in [26], IV-are defined over $E(G, X)$.

REMARK 6.12: In order to define the map F.J.^{P} in 6.3, we assumed in 6.2 that $\mathscr{V}$ satisfies hypothesis (5.3.6). In fact, one sees easily, upon tracing through the subsequent steps in the proof of Theorem 6.4, that one only needs a hypothesis relative to the subgroup $G_l \subset G$. For example, the following less strict hypothesis suffices: Assume $(G, X) = (G^{\text{even}}, \Delta(P))$, and let $h \in \Delta(P)$; let K_h be the centralizer of h in G. Let $\mu: K_h \to GL(\mathscr{V}_h)$ be the isotropy representation. Then there is a finite-dimensional representation $\rho_h: G_l \to GL(\mathscr{V}_{\rho_h})$, and an imbedding $\mathscr{V}_h \to \mathscr{V}_{\rho_h}$ of $K_h \cdot (P_h^- \cap G_l)$-modules, such that the corresponding morphism of homogeneous vector bundles $\mathscr{V} \to \mathscr{V}_\rho$ is defined over L.

§7. Applications: examples of differential operators

7.0. This section generalizes some of the results of §3 and §6 of [37], and is included with a view to future applications to special values of the L-functions of Piatetski-Shapiro and Rallis [48]. It will also be clear that our methods include a construction of the non-holomorphic differential operators studied by Shimura in [57]; these operators are also known as Maass operators. Our methods are based on the work of Jakobsen and Vergne on the decomposition of representations of holomorphic type [40]. The presentation was strongly influenced by suggestions of Deligne.

7.1. Let $(G_1, X_1) \subset (G_2, X_2)$ be an inclusion of pairs satisfying (1.1.1)-(1.1.3). Let $\check{M}_i = \check{M}(G_i, X_i)$, and let $\mathscr{V}_i$ be a homogeneous vector bundle over $\check{M}_i$, $i = 1, 2$. Our goal is to classify homogeneous differential operators $\delta : u^*(\text{jet}^n(\mathscr{V}_2)) \to \mathscr{V}_1$, where $u: \check{M}_1 \hookrightarrow \check{M}_2$ is the natural map, in terms of representation theory (cf. 3.1). Let $F^p\mathscr{V}_i \subset \mathscr{V}_i$ be as in 3.1.3, $i = 1, 2$, $p \in \mathbf{Z}$. In order to obtain a nice answer, we assume in what follows that there exist integers p_i, $i = 1, 2$, such that

$$(7.1.1) \quad \mathscr{V}_i = F^{p_i}\mathscr{V}_i \quad \text{and} \quad F^{p_i+1}\mathscr{V}_i = 0,\ i = 1, 2.$$

Thus for each $x = (\mathfrak{P}_x, \mu_x) \in \check{M}_i$, $R_u\mathfrak{P}_x$ acts trivially on $\mathscr{V}_{i,x}$, and $\mu_x(\mathbf{G}_m) \subset \mathfrak{P}_x/R_u\mathfrak{P}_x$ acts through the character $z \mapsto z^{-p_i}$ on $\mathscr{V}_{i,x}$.

In what follows, $\mathscr{V}_2$ will be thought of as fixed, $\mathscr{V}_1$ as variable; $\mathscr{V}_2$ is rational over the field $L \supset E(G_1, X_1)$. Any differential operator $\delta : u^*(\text{jet}^n\mathscr{V}_2) \to \mathscr{V}_1$ is equivalent, by duality, to a homomorphism $\delta^* : \mathscr{V}_1^* \to u^*(\text{jet}^n\mathscr{V}_2)^*$. As in (3.9.1), there is a natural isomorphism

$$(7.1.2) \quad \mathscr{D}_{\check{M}_2} \otimes \mathscr{V}_2^* \xrightarrow{\sim} (\text{jet}^\infty\mathscr{V}_2)^* \overset{\text{def}}{=} \varinjlim_n (\text{jet}^n\mathscr{V}_2)^*$$

where the limit is taken with respect to the duals to the natural projection maps $\text{jet}^n\mathscr{V}_2 \to \text{jet}^m\mathscr{V}_2$, $n \geqslant m$. If $x = (\mathfrak{P}_x, \mu)$ is a point in $\check{M}_2$, then by (3.9.2) the isomorphism (7.1.2) specializes at x to

$$(7.1.3) \quad U\mathfrak{g}_2 \otimes_{U(\text{Lie}\,\mathfrak{P}_x)} \mathscr{V}_{2,x}^* \xrightarrow{\sim} (\text{jet}^\infty\mathscr{V}_2)_x^*$$

where the action of Lie $\mathfrak{P}_x$ on $\mathscr{V}_{2,x}^*$ is as described in 3.9. Recall that the left-hand side of (7.1.3) is denoted $\mathbf{D}(\mathscr{V}_{2,x}^*, x)$. If we assume $x = u(h)$, $h \in \check{M}_1$, then we see that δ^* is given, at h, by a homomorphism also denoted δ^*

$$\delta^* : \mathscr{V}_{1,h}^* \to \mathbf{D}(\mathscr{V}_{2,x}^*, x).$$

The hypothesis (7.1.1) implies that the image of δ^* is contained in

$$\mathbf{D}(\mathscr{V}_{2,x}^*, x)^{R_u\mathfrak{P}_x}.$$

Suppose h is rational over the extension L' of $E(G_1, X_1)$. The association $\mathscr{V} \mapsto \mathscr{V}_h$ defines an equivalence of categories

$$\left\{\begin{matrix}\text{Homogeneous vector}\\ \text{bundles over } \check{M}_1,\\ \text{rational over } L'\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{Representations}\\ \rho: \mathfrak{P}_h \to GL(V_\rho),\\ \text{rational over } L'\end{matrix}\right\}$$

here $h = (\mathfrak{P}_h, \mu) \in \check{M}_1(L')$. If we assume $L = L'$, we thus have the following lemma:

7.2 LEMMA: *Let $h = (\mathfrak{P}_1, \mu_1) \in \check{M}_1$ be defined over L; let $u(h) = (\mathfrak{P}_2, \mu_2) \in \check{M}_2$. Then each $\mathfrak{P}_1/R_u\mathfrak{P}_1$-homomorphism from $\mathscr{V}_{1,h}^*$ to $\mathbf{D}(\mathscr{V}_{2,u(h)}^*, u(h))^{R_u\mathfrak{P}_1}$, rational over L, gives rise to an L-rational differential operator from $\mathscr{V}_2$ to $\mathscr{V}_1$, and every such differential operator arises in this way.*

In the statement of the above lemma, we are using implicitly the fact that the action of Lie $\mathfrak{P}_2$ on $\mathbf{D}(\mathscr{V}_{2,u(h)}^*, u(h))$ integrates to an action of the group $\mathfrak{P}_2$.

7.3. Henceforward, for brevity, we write $\mathbf{D}(\mathscr{V}_2^*, u(h))$ for $\mathbf{D}(\mathscr{V}_{2,u(h)}^*, u(h))$. Suppose in Lemma 7.2 that h is a special point; i.e., there is a torus $H/\mathbf{Q}$, a $\mathbf{Q}$-homomorphism $\gamma: H \to G_1$ and a homomorphism $h'': \underline{S} \to H_{\mathbf{R}}$, such that $h = \beta_{X_1}(\gamma \circ h'') \in \beta_{X_1}(X_1) \subset \check{M}_1$ (β_{X_1} = the Borel imbedding of 3.1). Then $\mathfrak{P}_1 = K_h \cdot P_h^-$, in the notation of 5.2. We write $K_1 = K_h$, $P_1^- = P_h^-$. Likewise, we write $\mathfrak{P}_2 = K_2 \cdot P_2^-$. Let $\tilde{\mu}_h: \mathbf{G}_m \to K_1 \subset \mathfrak{P}_1$ be a lifting of $\mu_1: \mathbf{G}_m \to \mathfrak{P}_1/R_u\mathfrak{P}_1$. Then

$$(7.3.1)\quad \mathbf{D}(\mathscr{V}_2^*, u(h))^{R_u\mathfrak{P}_1} = \bigoplus_{i \in \mathbf{Z}} \left(\mathbf{D}(\mathscr{V}_2^*, u(h))^{P_1^-}\right)^i,$$

where $\tilde{\mu}_h(z)$ acts on $(\mathbf{D}(\mathscr{V}_2^*, u(h))^{P_1^-})^i$ as z^{-i}, $i \in \mathbf{Z}$, $z \in \mathbf{G}_m$. Since $\tilde{\mu}_h$ is defined over $E(H, h)$, the decomposition (7.3.1) is rational over $L_h = L \cdot E(H, h)$.

Now define $\mathfrak{n}$ to be the normal bundle of $\check{M}_1$ in $\check{M}_2$. Let $M_i = U\mathfrak{g}_1 \cdot U^i\mathfrak{g}_2(1 \otimes \mathscr{V}_{2,u(h)}^*) \subset \mathbf{D}(\mathscr{V}_2^*, u(h))$ $i = 0, 2, \ldots$. Here $U^0\mathfrak{g}_2 \subset u^1\mathfrak{g}_2 \subset \ldots \subset u^i\mathfrak{g}_2 \subset \ldots \subset u\mathfrak{g}_2)$ is the Poincaré-Birkhoff-witt filtration. Then $M_0 \subset M_1 \subset \ldots \subset M_i \subset \ldots$ is a filtration of $\mathbf{D}(\mathscr{V}_2^*, u(h))$ by $U\mathfrak{g}_1$-submodules and M_i/M_{i-1} is canonically isomorphic, for $i = 0, 1, \ldots,$ to $A_i \stackrel{\text{def}}{=} U\mathfrak{g}_1 \otimes_{U(\text{Lie}\,\mathfrak{P}_1)} (\text{Sym}^i(\mathfrak{n}_h) \otimes \mathscr{V}_{2,u(h)}^*)$, where we let $M_{-1} = \{0\}$ (cf. [40], p. 31). Here P_1^- acts trivially on $\text{Sym}^i(\mathfrak{n}_h)$, and K_1 acts via the isotropy representation.

7.3.2. LEMMA: *Suppose we have a decomposition over $\mathbf{C}$ as $U\mathfrak{g}_1$-modules.*

$$(7.3.3)\quad j: \bigoplus_{i=0}^{\infty} A_i \xrightarrow{\sim} \mathbf{D}(\mathscr{V}_2^*, u(h))$$

such that each of the summands on the left is completely reducible over $U\mathfrak{g}_1$. *Then the decomposition* (7.3.3) *is defined over* L_h *and* $j\left(\bigoplus_{\iota=0}^{k} A_\iota\right) = M_k$. *Moreover, suppose j is normalized so that the composition* $A_\iota \overset{j}{\hookrightarrow} M_\iota \to M_\iota/M_{\iota-1} \simeq A_\iota$ *is the identity. Then j is uniquely determined, and is itself defined over* L_h.

PROOF: We first claim that the image of $\bigoplus_\iota 1 \otimes (\mathrm{Sym}^\iota(\mathfrak{n}_h) \otimes \mathscr{V}^*_{2,u(h)})$ under j generates $\mathbf{D}(\mathscr{V}_2^*, u(h))^{P_1^-}$. In fact, it is known that, for any irreducible $\mathfrak{P}_1$-module V, $U\mathfrak{g}_1 \otimes_{U(\mathrm{Lie}\,\mathfrak{P}_1)} V$ has a unique absolutely irreducible quotient ([40], p. 35). But it is clear that any element $v \in (U\mathfrak{g}_1 \otimes_{U(\mathrm{Lie}\,\mathfrak{P}_1)} V)^{P_1^-}$ generates a $U\mathfrak{g}_1$-submodule of $U\mathfrak{g}_1 \otimes_{U(\mathrm{Lie}\,\mathfrak{P}_1)} V$, which is *proper* if $v \notin 1 \otimes V$. Thus if $U\mathfrak{g}_1 \otimes_{U(\mathrm{Lie}\,\mathfrak{P}_1)} V$ is irreducible, then $(U\mathfrak{g}_1 \otimes_{U(\mathrm{Lie}\,\mathfrak{P}_1)} V)^{P_1^-} = 1 \otimes V$. The claim now follows easily from the hypotheses of the lemma, upon taking V to be any irreducible $\mathfrak{P}_1$-submodule of $\mathrm{Sym}^\iota(\mathfrak{n}_h) \otimes \mathscr{V}^*_{2,u(h)}$.

But $\tilde{\mu}_h(z)$ acts as $z^{\iota - p_2}$ on $1 \otimes \mathrm{Sym}^\iota(\mathfrak{n}_h) \otimes \mathscr{V}^*_{2,u(h)}$. In other words, $j(1 \otimes \mathrm{Sym}^\iota(\mathfrak{n}_h) \otimes \mathscr{V}^*_{2,u(h)}) = \mathbf{D}(\mathscr{V}_2^*, u(h))^{P_1^-})^{\iota - p_2}$. Since 7.3.1 is defined over L_h, $j(A_\iota) = N_\iota$ is also defined over L_h. Since the $\mathbf{D}(\mathscr{V}_2^*, u(h))^{P_1^-})^{\iota - p_2}$ are disjoint for different i, and since the $U\mathfrak{g}_1$- modules they generate have no P_1^--fixed vectors in common for different i, it follows that $M_k = \bigoplus_{\iota=0}^{k} N_k$, and the uniqueness is clear. The rationality of j follows from its uniqueness.

We may express this in another way:

7.3.4. COROLLARY: *Under the hypotheses of* 7.3.2, *there is, for* $i = 0, 1, \ldots,$ *a unique differential operator* δ_ι *from* $\mathscr{V}_2$ *to* $u^*\mathscr{V}_2 \otimes \mathrm{Sym}^\iota(\ker(u^*\Omega^1_{\check{M}_2} \to \Omega^1_{\check{M}_1}))$, *defined over* L_h, *and such that the composite map*

$$(7.3.4.1)\quad \begin{aligned} &u^*\mathscr{V}_2 \otimes \mathrm{Sym}^\iota\left(\ker\left(u^*\Omega^1_{\check{M}_2} \to \Omega^1_{\check{M}_1}\right)\right) \\ &\overset{\alpha}{\to} u^*\left(\ker\left(\mathrm{jet}^\iota(\mathscr{V}_2) \to \mathrm{jet}^{\iota-1}(\mathscr{V}_2)\right)\right) \\ &\hookrightarrow u^*\mathrm{jet}^\iota\mathscr{V}_2 \overset{\delta_\iota}{\to} u^*\mathscr{V}_2 \otimes \mathrm{Sym}^\iota\left(\ker\left(u^*\Omega^1_{\check{M}_2} \to \Omega^1_{\check{M}_1}\right)\right), \end{aligned}$$

where α *arises from multiplication of jets, is the identity.*

PROOF: This is just a reformulation of 7.3.2, taking into account Lemma 7.2.

Of course, the existence of the decomposition 7.3.3 over $\mathbf{C}$, and the assumption of complete reducibility, is independent of the choice of h, since G_1 acts transitively on $\check{M}_1$; h need not even be a CM point. By (1.2.4), the uniqueness of δ_ι implies the following theorem:

7.4. THEOREM: *Assume, for some* (*and thus for every*) *point* $h = (\mathfrak{P}_1, \mu_1) \in \check{M}_1$, *with* $u(h) = (\mathfrak{P}_2, \mu_2) \in \check{M}_2$, *the module* $\mathbf{D}(\mathscr{V}_2^*, u(h))$ *is completely reducible* *over* $U\mathfrak{g}_1 \otimes \mathbf{C}$, *and has the decomposition* 7.3.3 *over* $\mathbf{C}$. *Then, for each* i, *there exists a unique differential operator* δ_i *from* $\mathscr{V}_2$ *to* $u^*\mathscr{V}_2 \otimes \mathrm{Sym}^i(\ker(u^*\Omega^1_{\check{M}_2} \to \Omega^1_{\check{M}_1}))$, *defined over* L, *and such that* (7.3.4.1) *is the identity.*

7.5. Corollary: *Let* $\tilde{u}: M(G_1, X_1) \to M(G_2, X_2)$ *be the canonical map. Under the hypotheses of Theorem* 7.4, *there is a differential operator* $[\delta_i]$ *from* $[\mathscr{V}_2]$ *to* $[u^*\mathscr{V}_2] \otimes \mathrm{Sym}^i(\ker(\tilde{u}^*\Omega^1_{M(G_2,X_2)} \to \Omega^1_{M(G_1,X_1)}))$, $i = 0, 1, \ldots,$ *such that the analogue of* 7.3.4.1 *is the identity.*

Proof: This follows immediately from Theorem 4.8 and the relation $[\Omega^1_{\check{M}(G,X)}] \simeq \Omega^1_{M(G,X)}$, which is a consequence of 3.2.2.(b).

7.6. From the proof of Lemma 7.3.2, it is clear that, if $\mathbf{D}(\mathscr{V}_2^*, h)$ is completely reducible over $U\mathfrak{g}_1$, then it has the decomposition 7.3.3. The question of when these hypotheses are satisfied was taken up in [40] and, in the symplectic case, in [37]. In practice we only consider the case in which $U\mathfrak{g}_2 \otimes_{U(\mathrm{Lie}\,\mathfrak{P}_2)} \mathscr{V}^*_{2,h}$ integrates to a *unitary* representation of $G_2(\mathbf{R})^0$, in which case the hypothesis of 7.4 is evident.

One important special case is when $(G_2, X_2) = (G \times G, X \times X)$, in which $(G, X) = (G_1, X_1)$ is imbedded diagonally.

7.7. Corollary: *Let* $\mathscr{V}$ *and* $\mathscr{W}$ *be homogeneous vector bundles on* $\check{M}$, *satisfying the hypothesis* (7.1.1), *such that for some (or any) point* $h = (\mathfrak{P}, \mu) \in M$, $U\mathfrak{g} \otimes_{U(\mathrm{Lie}\,\mathfrak{P})} \mathscr{V}_h^*$ *and* $U\mathfrak{g} \otimes_{U(\mathrm{Lie}\,\mathfrak{P})} \mathscr{W}_h^*$ *are associated to unitary representations of* $G(\mathbf{R})^0$. *Then for* $i = 0, 1, \ldots,$ *we have a differential operator* δ^i *from the external tensor product* $[\mathscr{V}] \otimes [\mathscr{W}]$ *over* $M(G, X) \times M(G, X)$ *to* $[\mathscr{V} \otimes \mathscr{W}] \otimes \mathrm{Sym}^i\Omega^1_{M(G,X)}$ *over* $M(G, X)$, *imbedded diagonally in* $M(G, X) \otimes M(G, X)$. *The* δ^i *are functorial with respect to imbeddings* $(G_1, X_1) \subset (G_2, X_2)$, *and with respect to homomorphisms of vector bundles. If* $\mathscr{V}$ *and* $\mathscr{W}$ *are rational over an extension* $L \supset E(G, X)$, *then the* δ^i *are defined over* L.

7.8. Let $\mathscr{V}$ be any homogeneous vector bundle over $\check{M}$. Let $h: \underline{S} \to G_{\mathbf{R}}$ be an element of X, and let $x = \beta_X(h) \in \check{M}$. Then $h(\underline{S})$ acts on the fiber $\mathscr{V}_x$, and we have the decomposition $\mathscr{V}_{x,\mathbf{C}} = \bigoplus_{p,q} \mathscr{V}_x^{p,q}$, as in 0.7. It is easy to see that as x varies, the above decomposition traces out a canonical $G(\mathbf{R})$-homogeneous decomposition of C^∞ *vector bundles*:

$$(7.8.1) \quad \beta_X^*(\mathscr{V})^\infty \simeq \bigoplus_{p,q} \mathscr{V}^{p,q}$$

Let

$$[\mathscr{V}]^{p,q} = \varprojlim_K G(\mathbf{Q}) \backslash \mathscr{V}^{p,q} \times G(\mathbf{A}^f)/K \subset [\mathscr{V}]^\infty$$

Then corresponding to (7.8.1), we have a canonical isomorphism

(7.8.2) $$[\mathcal{V}]^\infty \xrightarrow{\sim} \bigoplus_{p,q} [\mathcal{V}]^{p,q}$$

The restriction of (7.8.2) to $F^p[\mathcal{V}]^\infty$ determines a canonical isomorphism of C^∞ vector bundles

$$\left(F^p[\mathcal{V}]/F^{p+1}[\mathcal{V}]\right)^\infty \xrightarrow{\sim} \bigoplus_q [\mathcal{V}]^{p,q}.$$

We may thus interpret (7.8.2) as a canonical C^∞ splitting of the Hodge filtration on $[\mathcal{V}]$:

(7.8.3) $$\underline{\mathrm{Split}}_{[\mathcal{V}]} : [\mathcal{V}]^\infty \xrightarrow{\sim} \bigoplus_p \left(F^p[\mathcal{V}]/F^{p+1}[\mathcal{V}]\right)^\infty.$$

7.9. The algebraic differential operators constructed in 7.7 fit into a collection of non-holomorphic differential operators of the type considered by Maass, Shimura, Katz, and the author [43,57,41,38]. These are constructed as follows: Let $\mathcal{V}$ be a homogeneous vector bundle over $\check{M}$, as in 7.1.1, and let $n \geqslant 0$ be an integer. Then the Hodge filtration (3.1.3) on $\mathrm{jet}^n[\mathcal{V}]$ coincides with the filtration by order of jets; consequently,

$$\bigoplus_p F^p\mathrm{jet}^n[\mathcal{V}]/F^{p+1}\mathrm{jet}^n[\mathcal{V}] = \bigoplus_{i=0}^{n} [\mathcal{V}] \otimes \mathrm{Sym}^i\Omega^1_{M(G,X)}.$$

Let

$$\underline{\mathrm{Split}} = \underline{\mathrm{Split}}_{\mathrm{jet}^n[\mathcal{V}]} : \mathrm{jet}^n[\mathcal{V}]^\infty \xrightarrow{\sim} \bigoplus_{i=0}^{n} \left([\mathcal{V}] \otimes \mathrm{Sym}^i\left(\Omega^1_{M(G,X)}\right)\right)^\infty$$

be the isomorphism (7.8.3) of C^∞-vector bundles; let $\underline{\mathrm{Split}}(i)$ denote $\underline{\mathrm{Split}}$ followed by projection on the ith factor, $i = 0, \ldots, n$. If $\mathrm{jet}^n : \Gamma(M(G, X), [\mathcal{V}]) \to \Gamma(M(G, X), \mathrm{jet}^n[\mathcal{V}])$ is the jet map, then the map $f \mapsto \underline{\mathrm{Split}}(i)(\mathrm{jet}^n(f))$ defines a C^∞-differential operator depending only on i, and not on n:

(7.9.1) $$D^i : [\mathcal{V}]^\infty \to [\mathcal{V}]^\infty \otimes \mathrm{Sym}^i\left(\Omega^1_{M(G,X)}\right)^\infty.$$

Now let $\mathcal{V}_1$ and $\mathcal{V}_2$ be L-rational homogeneous vector bundles on $\check{M}$, as in 7.7. Let $(\mathrm{jet}^\infty[\mathcal{V}_1] \otimes \mathrm{jet}^\infty[\mathcal{V}_2])^i$ be the quotient of $\mathrm{jet}^\infty[\mathcal{V}_1] \otimes \mathrm{jet}^\infty[\mathcal{V}_2]$ whose sections are jets of total order at most i in the two variables, $i = 0, 1, \ldots$. Then the differential operators δ^i of 7.7 are equivalent to homomorphisms, also denoted δ^i

$$\delta^i : \left(\mathrm{jet}^\infty[\mathcal{V}_1] \otimes \mathrm{jet}^\infty[\mathcal{V}_2]\right)^i \to [\mathcal{V}_1 \otimes \mathcal{V}_2] \otimes \mathrm{Sym}^i\left(\Omega^1_{M(G,X)}\right).$$

We write Ω^1 for $\Omega^1_{M(G,X)}$. The decomposition (7.3.3) may be interpreted as a set of isomorphisms, for every $n \geqslant 0$

$$(7.9.2) \quad (\mathrm{jet}^\infty[\mathscr{V}_1] \otimes \mathrm{jet}^\infty[\mathscr{V}_2])^n \xrightarrow{\sim} \bigoplus_{i=0}^{n} \mathrm{jet}^{n-i}([\mathscr{V}_1 \otimes \mathscr{V}_2] \otimes \mathrm{Sym}^i(\Omega^1)),$$

arising from the jet maps associated to the maps δ^i. Thus suppose $f_i \in \Gamma(M(G, X), [\mathscr{V}_i])$, $i = 1, 2$, and (7.9.2) identifies

$$\mathrm{jet}^a f_1 \otimes \mathrm{jet}^{n-a} f_2 \text{ with } \sum_{i=0}^{n} A_i(\mathrm{jet}^{n-i}\delta^i(f_1 \otimes f_2))$$

where $A_i \in \mathrm{Hom}(\mathrm{jet}^{n-i}([\mathscr{V}_1 \otimes \mathscr{V}_2] \otimes \mathrm{Sym}^i(\Omega^1)), (\mathrm{jet}^\infty[\mathscr{V}_1] \otimes \mathrm{jet}^\infty[\mathscr{V}_2])^n)$ is an L-rational injective homomorphism of vector bundles, $i = 0, \dots, n$. Let A_i' be the restriction of A_i to the subbundle $\mathrm{Sym}^{n-i}(\Omega^1) \otimes ([\mathscr{V}_1 \otimes \mathscr{V}_2] \otimes \mathrm{Sym}^i(\Omega^1))$ of $\mathrm{jet}^{n-i}([\mathscr{V}_1 \otimes \mathscr{V}_2] \otimes \mathrm{Sym}^i(\Omega^1))$. If we now split the Hodge filtration, we see that

$$(7.9.3) \quad \underline{\mathrm{Split}}(a) \otimes \underline{\mathrm{Split}}(n-a)\left(\sum_{i=0}^{n} A_i' D^{n-i}\delta^i(f_1 \otimes f_2)\right) = D^a f_1 \otimes D^{n-a} f_2.$$

We may interpret (7.9.3) as a statement about the rationality of the "holomorphic part" of $D^a f_1 \otimes D^{n-a} f_2$, since $A_n'\delta^n(f_1 \otimes f_2)$ can be recovered from (7.9.3) in a rational way. This statement may also be viewed in another way.

7.10. Let $\hat{X} = X \times G(\mathbf{A}^f)$, as in 4.2. Suppose $f \in \Gamma(M(G, X), [\mathscr{V}])$, for some $\mathscr{V}$ as above. If $\mathscr{E}$ is a vector bundle over $M(G, X)$, let $\tilde{\mathscr{E}}$ be its lifting to $\hat{X}$, and denote the lifts of sections and homomorphisms of vector bundles analogously. Then $[\tilde{\mathscr{V}}]$ is homogeneous under $G(\mathbf{R})$ (even under $G(\mathbf{A})$). Thus if $h \in \hat{X}$ is any point, $[\tilde{\mathscr{V}}]$ lifts to the bundle $G(\mathbf{A}) \times [\tilde{\mathscr{V}}]_h$ over $G(\mathbf{A})$, and f lifts to a function $\phi = \phi_f \in \varinjlim_K C^\infty(G(\mathbf{Q}) \backslash G(\mathbf{A})/K, [\tilde{\mathscr{V}}]_h(\mathbf{C}))$. We write V_h instead of $[\tilde{\mathscr{V}}]_h(\mathbf{C})$. The enveloping algebra $U(\mathfrak{g}_{\mathbf{C}})$ acts by right differentiation on $\varinjlim_K C^\infty(G(\mathbf{Q}) \backslash G(\mathbf{A})/K, V_h)$; if $\phi \in C^\infty(G(\mathbf{Q}) \backslash G(\mathbf{A})/K, V_h)$ for some K and $Y \in U(\mathfrak{g}_{\mathbf{C}})$, we denote this action $\phi \mapsto Y * \phi$. It is well known that the fact that f is holomorphic implies that $Y * \phi = 0 \ \forall Y \in \mathfrak{p}^-(\mathbf{C})$, where $\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_h \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ as in 5.2; cf. [37], Prop. 2.3 for a proof in a special case. Define $\mathscr{V}_f$ to be the $U(\mathfrak{g}_{\mathbf{C}})$-submodule of $\varinjlim_K C^\infty(G(\mathbf{Q}) \backslash G(\mathbf{A})/K, \mathbf{C})$ generated by $\{L(\phi) \mid L \in V_h^*\}$. Let $\hat{\beta}: \hat{X} \to \check{M}(\mathbf{C})$ denote projection on X followed by β_X. If $\hat{\beta}(h) = (\mathfrak{P}, \mu) \in \check{M}(\mathbf{C})$, then by the hypothesis of 7.7 we have the isomorphism

$$(7.10.1) \quad U(\mathfrak{g}_{\mathbf{C}}) \otimes_{U(\mathrm{Lie}\,\mathfrak{P})} \mathscr{V}_h^* \xrightarrow{\sim} \mathscr{V}_f.$$

On the other hand, as in the remarks preceding Lemma 7.2, $U(\mathfrak{g}_{\mathbf{C}}) \otimes_{U(\mathrm{Lie}\,\mathfrak{P})} \mathscr{V}_h^*$ is the fiber of h of the bundle $(\mathrm{jet}^\infty[\tilde{\mathscr{V}}])^*$. Define $\mathfrak{p}^+$ as above: of course $\mathfrak{p}^+(\mathbf{C})$ is canonically isomorphic to the holomorphic tangent spece $T_{X,h}$ to $\hat{X}$ at h. We write $\mathfrak{p}_h^+ = \mathfrak{p}^+$ to emphasize this dependence on h. The definition of $\mathfrak{p}_h^+$ implies immediately that

(7.10.2) The splitting (7.8.3) of the Hodge filtration on $\mathrm{jet}^\infty[\mathscr{V}]$ corresponds by duality to the decomposition, over each $h \in X$,

$$\mathrm{U}(\mathfrak{g}_{\mathbf{C}}) \otimes_{\mathrm{U}(Lie\,\mathfrak{P})} \mathrm{V}_{\mathrm{h}}^* = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(\mathfrak{p}_h^+) \otimes V_h^*.$$

Now $D^i f \in \Gamma({}_K M(G, X), ([\mathscr{V}] \otimes \mathrm{Sym}^i \Omega^1)^\infty)$ can be lifted, in analogy with the above procedure, to a function

$$\Delta^i f \in \mathbf{C}^\infty\big(G(\mathbf{Q}) \backslash G(\mathbf{A})/K,\ V_h \otimes \mathrm{Sym}^i(T_{X,h}^*)\big).$$

It follows from (7.10.2) that, for any $L \in (V_h \otimes \mathrm{Sym}^i(T_{X,h}^*))^*$,

$$L(\Delta^i f) \in \mathscr{V}_f \quad i = 0, 1, \ldots .$$

(This argument replaces and generalizes the computations in §6 of [37].) In fact $L(\Delta^i f)$ is in the subspace corresponding, by (7.9.3) and (7.9.4), to $\mathrm{Sym}^i(\mathfrak{p}_h^+) \otimes V_h^*$. We have thus canonically lifted $D^i f$ to an element of $\mathrm{Hom}(V_h^* \otimes \mathrm{Sym}^i(T_{X,h}), \mathscr{V}_f) = \mathrm{Hom}(V_h^* \otimes \mathrm{Sym}^i(\mathfrak{p}_h^+), \mathscr{V}_f)$.

It is thus apparent that the differential operators D^i are the operators discussed in special cases by Maass, Selberg, Shimura, and Katz [43,53,57,41], and in general by the author in [38], where, in their incarnation as homogeneous differential operators on X, they were called *Maass operators*.

7.10.3 Remark: Suppose $h \in \hat{X}$ is of the form (h_0, γ) $h_0 \in X$, $\gamma \in G(\mathbf{A}^f)$, where (T, h_0) is a CM pair in (G, X). Let L be the field of definition of $\mathscr{V}$, and let $L_h = L \cdot E(T, h_0)$. It follows from (3.5.1) (a) that the splitting (7.10.2) is rational over L_h. Let $J = J^{h_0, 0}$ be the canonical automorphy factor of 5.2. Using J, one can associated to $\Delta^i f$ a C^∞-function $\widetilde{D^i f}$ on $\hat{X}$, with values in $V_h \otimes \mathrm{Sym}^i(T_{X,h}^*)$, satisfying 5.3.3(ii), where τ is the representation of K_{h_0} on $V_h \otimes \mathrm{Sym}^i(T_{X,h}^*)$. Precisely,

$$\widetilde{D^i f}(z, \gamma) = J_\tau(g_z, h_0) \cdot f(g_z, \gamma) \quad z \in X, \gamma \in G(\mathbf{A}^f)$$

where g_z is any element of $G(\mathbf{R})$ such that $g_z(h_0) = z$. The rationality of (7.10.2) implies, by the argument of Katz in [41], that $\widetilde{D^i f}$ has the same rationality properties at h as f does. Suppose that f is rational over L', for example. Then $\tilde{p}(V \otimes \mathrm{Sym}^i \Omega^1_{\check{M}(G,X)}, h)$ $\widetilde{D^i f}(h)$ takes values in

$V_{\check{h}}(L' \cdot L_h) \otimes \mathrm{Sym}^r(\Omega^1_{\check{M}(G,X),\check{h}})(L' \cdot L_h)$, as in Proposition 5.3.10, where $\check{h} = \beta_X(h_0)$. These ideas were first systematically investigated by Shimura [53], and were developed by him in [57] in great generality.

7.11. We now work out in detail a specific example which is directly related to the L-functions studied by Piatetski-Shapiro and Rallis [48]. Applications will be discussed in a future work.

Let E be a CM field, f a totally real subfield, $[E:F] = 2$; let $\sigma \in \mathrm{Gal}(E/F)$ be the non-trivial automorphism. Let V^1 be an n-dimensional vector space over E, equipped with a non-degenerate σ-Hermitian form $<,>^1$, and let G^1 be the group of unitary similitudes of $<,>$; i.e., the group which preserves $<,>^1$ up to a scalar in $F^\times$. Let $G = R_{F/\mathbf{Q}}G^1$, $V = R_{E/\mathbf{Q}}V^1$; then G acts naturally on V, and preserves the symmetric form $<,> = R_{F/\mathbf{Q}} \mathrm{Tr}_{E/F} <,>^1$ up to a multiple. We diagonalize $<,>$ on $V_{\mathbf{C}}$; let $V^{-1,0}$ (resp. $V^{0,-1}$) be the subspace of $V_{\mathbf{C}}$ on which $<,>$ is positive-definite (resp. negative-definite). Then $V^{-1,0}$ and $V^{0,-1}$ are conjugate over $\mathbf{R}$. Thus V has a Hodge structure for which $V^{-1,0}$ (resp. $V^{0,-1}$) is of type $(-1,0)$ (resp. $(0,-1)$). The corresponding homomorphism $h: \underline{S} \to GL(V_{\mathbf{R}})$ factors through $G_{\mathbf{R}}$, and we let X be the $G(\mathbf{R})$-conjugacy class of h. The pair (G, X) satisfies axioms (1.1.1)-(1.1.4).

If we start with $-<,>^1$ instead of $<,>^1$, the construction provides us with a different conjugacy class $X^\#$ of homomorphisms $h: \underline{S} \to G_{\mathbf{R}}$; it is easy to see that $X^\#$ is isomorphic to X with the conjugate complex structure. Now let $W^1 = V^1 \oplus V^1$, with σ-Hermitian form $(\ ,\ )^1 = <,>^1 \oplus (-<,>^1)$. Let H^1 be the group of unitary similitudes of $(\ ,\ )^1$, $H = R_{F/\mathbf{Q}}H^1$, and let D be the $H(\mathbf{R})$-conjugacy class of homomorphisms $h: \underline{S} \to H_{\mathbf{R}}$, constructed as above. Let $\nu: G \to R_{F/\mathbf{Q}}\mathbf{G}_m$ be the restriction of scalars from F to $\mathbf{Q}$ of the morphism which sends $g \in G^1$ to the factor by which it multiplies $<,>^1$. Let $S(G \times G) = \{(g_1, g_2) \in G \times G \mid \nu(g_1) = \nu(g_2)\}$. Evidently there is an imbedding $(S(G \times G), X \times X^\#) \overset{\phi}{\hookrightarrow} (H, D)$.

Let $GU(p, q)$ be the group of similitudes of the standard Hermitian form on $\mathbf{C}^{p+q}$ with signature (p, q). Let $r = [F:\mathbf{Q}]$. Then $G_{\mathbf{R}} \simeq \prod_{i=1}^{r} GU(p_i, q_i)$ $p_i + q_i = n$, and $H_{\mathbf{R}} \simeq \prod_{i=1}^{r} GU(n, n)$. Correspondingly, we have decompositions

$$X \approx \prod_{i=1}^{r} X_{p_i, q_i}, \quad X^\# \approx \prod_{i=1}^{r} X^\#_{q_i, p_i}, \quad D \approx \prod_{i=1}^{r} D_{n,n}.$$

A connected component of X_{p_i,q_i} (resp. $X^\#_{q_i,p_i}$) is equivalent to the Hermitian symmetric space $SU(p_i, q_i)/S(U(p_i) \times U(q_i))$, which may be represented as a bounded symmetric domain in the space of $p_i \times q_i$-dimensional (resp. $q_i \times p_i$-dimensional)) complex matrices. Similarly, a connected component of $D_{n,n}$ is a bounded symmetric domain in the space of $n \times n$ complex matrices. On connected components, the imbedding $\phi: X \times X^{\#} \hookrightarrow D$ can be represented on the ith factor by the imbedding

$$(7.11.1) \quad p_i[\underbrace{(M_1)}_{q_i} \times \underbrace{(M_2)}_{p_i}]\, q_i \mapsto \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

We let G_i be the factor $GU(p_i, q_i)$ of $G_{\mathbf{R}}$. Let X_i^+, $X_i^{\#,+}$, D_i^+ be connected components of X_{p_i,q_i}, $X^{\#}_{q_i,p_i}$, and $D_{n,n}$ (in the ith place), respectively, regarded as above as bounded symmetric domains in complex matrix spaces. Let $K_i \subset G_i$ be the stabilizer of the origin in X_i^+, and define $K_i^{\#} \subset G_i$ and $\tilde{K}_i \subset H_i$ likewise. Then $K = \prod_{i=1}^{r} K_i$ (resp. $K^{\#} = \prod_{i=1}^{r} K_i^{\#}$, resp. $\tilde{K} = \prod_{i=1}^{r} \tilde{K}_i$) is the stabilizer of a point h (resp. $h^{\#}$, $\tilde{h}$) in X (resp. $X^{\#}$, resp. D) and $\phi(h \times h^{\#}) = \tilde{h}$. Let $\mu = \mu_h$, $\mu^{\#} = \mu_{h^{\#}}$, $\tilde{\mu} = \mu_{\tilde{h}}$ be the corresponding homomorphisms of $\mathbf{G}_m$ into $K_{\mathbf{C}}$, $K^{\#}_{\mathbf{C}}$, and $K_{\mathbf{C}}$, respectively, and let μ_i, $\mu_i^{\#}$, $\tilde{\mu}_i$ denote their projections on the ith factors of the respective complex groups.

Define h_i, $h_i^{\#}$, and $\tilde{h}_i$ likewise. We may identify K_i with the subgroup

(7.11.2)

$$\left\{ t \cdot \begin{pmatrix} g_{p_i} & 0 \\ 0 & g_{q_i} \end{pmatrix} \Big|\, t \in \mathbf{R}^{\times},\ g_{p_i} \in U(p_i),\ g_{q_i} \in U(q_i) \right\} \subset GU(p_i, q_i)$$

where $U(p)$ is the compact unitary group of degree p. Similarly, and in the obvious notation, we have

$(7.11.2)^{\#}$

$$K_i^{\#} = \mathbf{R}^{\times} \cdot \begin{pmatrix} U(q_i) & 0 \\ 0 & U(p_i) \end{pmatrix}; \quad \tilde{K}_i = \mathbf{R}^{\times} \cdot \begin{pmatrix} U(n) & 0 \\ 0 & U(n) \end{pmatrix}.$$

With this identification, we have

$$(7.11.3) \quad h_i(z) = \begin{pmatrix} z^{-1} I_{p_i} & 0 \\ 0 & \bar{z}^{-1} I_{q_i} \end{pmatrix}$$

$$h_i^{\#}(z) = \begin{pmatrix} z^{-1} I_{q_i} & 0 \\ 0 & \bar{z}^{-1} I_{p_i} \end{pmatrix}$$

$$\tilde{h}_i(z) = \begin{pmatrix} z^{-1} I_n & 0 \\ 0 & \bar{z}^{-1} I_n \end{pmatrix} \quad z \in \underline{S}(\mathbf{R}) \simeq \mathbf{C}^{\times}.$$

The tangent space to X_i^+ (resp. $X_i^{+,\#}$, resp. D_i^+) at the origin is naturally isomorphic to the space of complex matrices of degree $p_i \times q_i$ (resp. $q_i \times p_i$, resp. $n \times n$). If we denote these tangent spaces T_i, $T_i^{\#}$, and $\tilde{T}_i$ respectively, then the inclusion map $d\phi: T_i \oplus T_i^{\#} \hookrightarrow \tilde{T}_i$ is given by (7.11.1). Let $\mathfrak{n}_i = \mathfrak{n}_{h,h^{\#},i} = \tilde{T}_i / d\phi(T_i \oplus T_i^{\#})$ be the normal space to $\phi(X_i^+ \times X_i^{+,\#})$ in D_i^+ at the origin. Then $\mathfrak{n}_i$ is obviously isomorphic to the space $M_{p_i}(\mathbf{C}) \times M_{q_i}(\mathbf{C})$; here we write M_a for the space of $a \times a$ matrices. Let ρ_i denote the isotropy representation of $K_i \times K_i^{\#}$ on $\mathfrak{n}_i$. It is evidently determined by its restriction to $(K_i \cap U(p_i, q_i)) \times (K_i^{\#} \cap U(q_i, p_i))$, and we compute it as follows:

For simplicity, we drop the subscript i in this paragraph. Let $g^1 = (g_p^1, g_q^1) \in U(p) \times U(q) = K \cap U(p, q)$; let $g^2 = (g_q^2, g_p^2) \in U(q) \times U(p) = K^{\#} \cap U(q, p)$. Let $(A_p, B_q) \in M_p(\mathbf{C}) \times M_q(\mathbf{C}) \simeq \mathfrak{n}$. Then a trivial computation shows

$$(7.11.4) \quad \rho(g^1, g^2)(A_p, B_q) = \left(g_p^1 A_p (g_p^2)^{-1}, \; g_q^2 B_q (g_q^1)^{-1}\right).$$

For $i \in \mathbf{Z}$, $i \geqslant 0$, let $\mathrm{St}_i: GL(i) \to GL(i)$ be the identity representation. The matrix representations (7.11.2), (7.11.2)$^{\#}$ identify $(K \cap U(p, q))_{\mathbf{C}} \times (K^{\#} \cap U(q, p))_{\mathbf{C}}$ with $GL(p)_{\mathbf{C}} \times GL(q)_{\mathbf{C}} \times GL(q)_{\mathbf{C}} \times GL(p)_{\mathbf{C}}$. From (7.11.4) we see that

(7.11.5) The restriction of ρ to $(K \cap U(p, q))_{\mathbf{C}} \times (K^{\#} \cap U(q, p))_{\mathbf{C}}$ is equivalent to $\mathrm{St}_p \boxtimes \mathrm{St}_q^* \boxtimes \mathrm{St}_q \boxtimes \mathrm{St}_p^*$

where $\boxtimes$ denotes the tensor product of representations of distinct groups, and St_i^* is the contragredient to St_i.

By a *polynomial* representation of $GL(i)$ we mean an irreducible representation which occurs in $\mathrm{St}_i^{\otimes n}$, for some integer $n \geqslant 0$. Then n is uniquely determined, and is called the *degree* of ρ. The following theorem is classical:

7.11.6. THEOREM: (*Schur*, [51]). *Let ρ be any polynomial representation of $GL(i)$, $i \geqslant 0$, of degree n. Then the tensor product representation $\rho \boxtimes \rho^*$ of $GL(i) \times GL(i)$ occurs with multiplicity one in* $\mathrm{Sym}^n(\mathrm{St}_i \boxtimes \mathrm{St}_i^*)$.

Now we return to the general case; i.e., $K = \prod_{i=1}^{r} K_i$, etc. An irreducible representation ρ of $K_i \times K_i^{\#}$ is called *positive* if

(i) The restriction of ρ to $K_i \cap Z_G \times K_i^{\#} \cap Z_G$ is trivial

(ii) We identify $(K_i \cap U(p_i, q_i))_{\mathbf{C}} \times (K_i^{\#} \cap U(q_i, p_i))_{\mathbf{C}}$ as in (7.11.2) with $GL(p_i)_{\mathbf{C}} \times GL(q_i)_{\mathbf{C}} \times GL(q_i)_{\mathbf{C}} \times GL(p_i)_{\mathbf{C}}$. Suppose that ρ decomposes correspondingly as the tensor product $\rho_1 \boxtimes \rho_2 \boxtimes \rho_3 \boxtimes \rho_4$. Then $\rho_1^* \simeq \rho_4$, $\rho_2^* \simeq \rho_3$, and ρ_1 and ρ_3 are polynomial representations.

An irreducible representation ρ of $K \times K^{\#}$ is called positive if $\rho = \boxtimes_{i=1}^{r} \rho^i$, where ρ^i is a positive representation of $K_i \times K_i^{\#}$. As a corollary to (7.11.5) and Theorem 7.11.6, we have

7.11.7. COROLLARY: *Let* $\mathfrak{n}_{(h,h^{\#})}$ *denote the fiber at* $\phi(h \times h^{\#}) = \tilde{h}$ *of the normal bundle of* $\phi(X \times X^{\#})$ *in* D. *We identify* $(K_{\mathbf{C}} \cap \operatorname{Ker} \nu) \times (K_{\mathbf{C}}^{\#} \cap \operatorname{Ker} \nu)$ *with* $\prod_{i=1}^{r} GL(p_i)_{\mathbf{C}} \times GL(q_i)_{\mathbf{C}} \times GL(q_i)_{\mathbf{C}} \times GL(p_i)_{\mathbf{C}}$ *via* (7.11.2). *Then the isotropy representation of* $K \times K^{\#}$ *on* $\bigoplus_{i=0}^{\infty} \operatorname{Sym}^i \mathfrak{n}_{(h,h^{\#})}$ *decomposes as the sum* $\oplus \rho$ *over all positive representations of* $K \times K^{\#}$, *each taken with multiplicity one.*

This also has a global formulation. Let $\check{M}_1 = \check{M}(S(G \times G), X \times X^{\#})$, $\check{M}_2 = \check{M}(H, D)$. Let $\beta_{X \times X^{\#}}(h, h^{\#}) = x \in \check{M}_1$. Let $\rho : (K \times K^{\#}) \cap S(G \times G) \to GL(V_\rho)$ be an algebraic representation. If $x = (\mathfrak{P}, \mu)$, then ρ may be extended to a representation of $\mathfrak{P}$ trivial on $R_u \mathfrak{P}$; in this way we associate to ρ a homogeneous vector bundle $\mathscr{V}_\rho$ on $\check{M}_1$. The field of definition E_ρ of $\mathscr{V}_\rho$ is a finite extension of $E(S(G \times G), X \times X^{\#}) = E(G, X)$, as one sees by choosing h and $h^{\#}$ to be special points.

7.11.8. COROLLARY: *We have the decomposition of homogeneous vector bundles*

$$\bigoplus_{i=0}^{\infty} \operatorname{Sym}^i\left(\operatorname{Ker}\left(\Omega^1_{\check{M}_2} \to \Omega^1_{\check{M}_1}\right)\right) \mid_{\check{M}_1} \simeq \bigoplus_{\rho} \mathscr{V}_\rho^*$$

Here ρ *runs through the set of positive representations of the stabilizer* $(K \times K^{\#}) \cap S(G \times G)$ *of an arbitrary point* $x \in \check{M}_1$. *The projection map*

$$\bigoplus_{i=0}^{\infty} \operatorname{Sym}^i\left(\operatorname{Ker}\left(\Omega^1_{\check{M}_2} \to \Omega^1_{\check{M}_1}\right)\right) \to \mathscr{V}_\rho^*$$

is defined over E_ρ.

PROOF: The first part follows from the fact that our points h, $h^{\#}$ were chosen arbitrarily. The second part follows from the fact that $\mathscr{V}_\rho^*$ occurs in the sum with multiplicity one.

It now follows from Corollary 7.5 that, for a sufficiently "positive" [2] homogeneous vector bundle $\mathscr{V}$ on $\check{M}_2$, rational over L, say, there are $L \cdot E_\rho$-rational differential operators $[\mathscr{V}] \to [\mathscr{V}] \mid_{M(S(G \times G), X \times X^{\#})} \otimes \mathscr{V}_\rho^*$, for every positive ρ as above. For future applications, it is worth-while to work out the case in which $\mathscr{V}$ is a line bundle.

We know that a homogeneous line bundle on $\check{M}_2$ is determined by a one-dimensional representation $\tau : \tilde{K} \to GL(V_\tau)$. Composing τ with $\tilde{h}_i : \underline{S} \to \tilde{K}_{i,\mathbf{R}}$ allows us to write $V_\tau = V_\tau^{r_i, s_i}$ as in 0.7: here r_i and s_i depend on τ. The

[2] That is, *negative* in the sense of algebraic geometry.

hypotheses of Theorem 7.4, applied to the corresponding bundle $\mathscr{V}_\tau$ on $\check{M}_2$, are satisfied when the anti-holomorphic type module $\mathbf{D}(\mathscr{V}_\tau^*, \beta_D(\tilde{h}))$ integrates to a unitary representation of $H(\mathbf{R})^0$. This depends on the corresponding assertion for each factor. Thus, suppose $\beta_D(\tilde{h}) = (\mathfrak{P}, \mu)$, and $\mathfrak{P}_\mathbf{C} = \prod_{i=1}^{r} \mathfrak{P}_{i,\mathbf{C}} \subset H_\mathbf{C} = \prod_{i=1}^{r} GU(n, n)_\mathbf{C}$. Suppose moreover that $\tau = \tau_1 \boxtimes \ldots \boxtimes \tau_r$, where $\tau_i : \tilde{K}_i \to GL(V_{\tau_i})$ is a representation; note that r_i and s_i depend only on τ_i. Then we need to know that

$$(7.11.9) \quad \mathbf{D}_{\tau_i} = U(\mathfrak{gu}(n, n)_\mathbf{C}) \otimes_{U(\mathrm{Lie}\, \mathfrak{P}_{i,\mathbf{C}})} V_{\tau_i}^*$$

integrates to a unitary representation of $GU(n, n)$.

The unitarity of (7.11.9) depends only on the restriction of τ_i to $S(U(n) \times U(n)) = \{(g_1, g_2) \in U(n) \times U(n) \mid \det g_1 = \det g_2^{-1}\} \subset \tilde{K}_i$. Let $d_i = r_i - s_i$. It follows from results of [62] (see also [61]) that

$$(7.11.10) \quad \mathbf{D}_{\tau_i} \text{ is unitarizable if and only if } d_i \geqslant n$$

The following proposition is an immediate consequence of (7.11.10) and the preceding argument.

7.11.11. Proposition: *Let $\mathscr{V}$ be a line bundle on $\check{M}_2 = \check{M}(H, D)$. Suppose at one (hence at every) point $\beta_D(h) = (\mathfrak{P}, \mu)$ of $\beta_D(D) \subset \check{M}_2$, the isotropy representation $\tau : \mathfrak{P} \to GL(V_{\beta_D(h)})$ has the form $\tau_1 \boxtimes \ldots \boxtimes \tau_r$ as above, $\tau_i : \tilde{K}_i \to GL(V_{\beta_i})$. Define r_i, s_i as above, and suppose that $r_i - s_i \geqslant n$, $i = 1, \ldots, r$. Let L be the field of definition of $\mathscr{V}$, and let ρ be any positive representation of the stabilizer of an arbitrary point $(h, h^\#) \in X \times X\#$. There is an $L \cdot E_\rho$ rational differential operator from $[\mathscr{V}]$ to $[\mathscr{V}] \mid_{M(S(G \times G), X \times X^\#)} \otimes [\mathscr{V}_\rho]$.*

We remark that if $r_1 = r_2 = \ldots r_r$, $s_1 = \ldots = s_r$, then $L = \mathbf{Q}$.

§8. Applications: Holomorphic Eisenstein Series

Let (G, X) and P be as in §6. Let f be a cusp form on the "boundary component" $M(G_P, F_P)$ of $M(G, X)$, with coefficients in the vector bundle $[\mathscr{V}_P]$. Under certain hypotheses on $[\mathscr{V}_P]$, we can attach to f an absolutely convergent holomorphic Eisenstein series $E(f)$ on $M(G, X)$; this is a section of a vector bundle $[\mathscr{V}]$ over $M(G, X)$, which is not generally determined uniquely by $[\mathscr{V}_P]$. In [12] we proved that, under the assumption that (G, X) admits a symplectic imbedding, the arithmeticity of f implies that of $E(f)$. In this section, we use the results of §6 to eliminate the assumption that (G, X) admits a symplectic imbedding. When X^+ is an exceptional tube domain and F_P^+ is a point, results of this type were proved by Baily and Tsao [27,60] and Karel (to appear). Recently Shimura has found explicit formulas for the

Fourier coefficients in the case of certain classical domains, and Indik has done the same for the tube domain over the positive light cone in n dimensions [59,39]; their methods apply notably to Eisenstein series beyond the range of absolute convergence.

Since most of the argument is contained in [12], we content ourselves with a formulation of the main result and a brief indication of its proof. We assume throughout that (G, X) satisfies (4.0.1) and (4.0.2), so that the methods of §6 apply.

8.1. As in [12], in order to set up the Eisenstein series, it is convenient to begin with the Φ_P operator. Choose a homogeneous vector bundle $\mathscr{V}$ over $\check{M}(G, X)$ satisfying hypothesis 7.1.1 (with "i" omitted) and an open compact subgroup $K \subset G(\mathbf{A}^f)$. In [12], 2.2, we associate to K and P, in a non-canonical way, a collection $\{K_{s,t}\}$ of open compact subgroups of $G_P(\mathbf{A}^f)$, indexes by a finite set $\{s, t\}$ of elements of $G(\mathbf{A}^f)$, whose exact nature need not concern us. We choose a point $h \in X^+$; it is convenient to assume $h \in \Delta(P)^+$, in the notation of §5. Let K_h be the centralizer of h in G, and let μ be the isotropy representation of K_h on the fiber $\mathscr{V}_h = \mathscr{V}_{\beta_X(h)}$ of $\mathscr{V}$. Let $J = J^{h,P}$ be the canonical automorphy factor of 5.2, and let $\psi = \psi_h : N_P \to K_h$ be the homomorphism of algebraic groups defined in (5.2.2.1). Let Z_P be the identity component of $\bigcap_{\chi \in X_{\mathbf{Q}}(N_P)} \operatorname{Ker} \chi$. Let μ_P be the representation of $K_{h,P} = K_h \cap G_P$ on $\mathscr{V}_{h,P} = \mathscr{V}_h^{\psi(Z_P)}$. Evidently $\mathscr{V}_{h,P}$ is the fiber at $\beta_{F_P}(\pi_P(h))$ of a homogeneous vector bundle $\mathscr{V}_P$ over $\check{M}(G_P, F_P)$.

8.1.1. Lemma: *Suppose $\mathscr{V}$ is defined over the extension L of $E(G, X)$. Then so is $\mathscr{V}_P$.*

Proof: Let $(H, h) \subset (G^{\text{even}}, \Delta(P))$ be a CM pair, let $L_h = L \cdot E(H, h)$. It suffices, by (1.2.4) and Lemma 6.1.1, to prove that for any (H, h), $\mathscr{V}_{h,P}$ is an L_h-rational subspace of $\mathscr{V}_h$. But it follows from the remarks following (5.2.3.9) that the homomorphism ψ is defined over $E(H, h)$. Thus $\mathscr{V}_{h,P}$ is obviously L_h-rational.

In 2.2 of [12], we defined a map

$$\Phi_P : \Gamma({}_K M(G, X), [\mathscr{V}]) \to \bigoplus_{s,t} \Gamma({}_{K^{s,t}} M(G_P, F_P), [\mathscr{V}_P]).$$

The definition is as follows: Let $h \in \Delta(P)^+$ as above, and let

$$\operatorname{Triv}_h : \Gamma({}_K M(G, X), [\mathscr{V}]) \xrightarrow{\sim} M_{\mathscr{V}}(K)$$

be the morphism (5.3.4) associated to $J^{h,P}$. Let $h_P = \pi_P(h)$, and let

$$\operatorname{Triv}_{h_P} : \Gamma({}_{K^{s,t}} M(G_P, F_P), [\mathscr{V}_P]) \xrightarrow{\sim} M_{\mathscr{V}_P}(K^{s,t})$$

be defined similarly. Let

$$\Phi^{\text{class}}_{P,1,1}(K): M_{\mathscr{V}}(K) \to M_{\mathscr{V}_P}(K^{1,1});$$

$$K^{1,1} = \left(K \cap P(\mathbf{A}^f)/K \cap W_P(\mathbf{A}^f)\right) \cap G_P(\mathbf{A}^f)$$

be the constant term of the classical Fourier-Jacobi expansion along F_P, as in 2.2 of [12]. Similarly, let $K_{ts} = tsK(ts)^{-1}$, $K^{s,t} = K^{1,1}_{ts}$, and let $\Phi^{\text{class}}_{P,s,t}$ be the composition

$$M_{\mathscr{V}}(K) \overset{(ts)^{-1}}{\to} M_{\mathscr{V}}(K_{ts}) \overset{\Phi^{\text{class}}_{P,1,1}(K_{ts})}{\to} M_{\mathscr{V}_P}(K^{s,t})$$

where the first arrow is right translation by $(ts)^{-1}$. Finally, let

$$\Phi^{\text{class}}_P : M_{\mathscr{V}}(K) \to \bigoplus_{s,t} M_{\mathscr{V}_P}(K^{s,t})$$

be the direct sum of the $\Phi^{\text{class}}_{P,s,t}$. Then there is a commutative diagram

$$(8.1.2)\quad \begin{array}{ccc} \Gamma({}_K M(G, X), [\mathscr{V}]) & \overset{\Phi_P}{\to} & \bigoplus_{s,t} \Gamma({}_{K^{s,t}} M(G_P, F_P), [\mathscr{V}_P]) \\ \text{Triv}_h \downarrow \wr & & \wr \downarrow \text{Triv}_{h_P} \\ M_{\mathscr{V}}(K) & \underset{\Phi^{\text{class}}_P}{\longrightarrow} & \bigoplus_{s,t} M_{\mathscr{V}_P}(K^{s,t}) \end{array}$$

On the other hand, let $\mathscr{V}^{\text{even}}$ be the pullback of $\mathscr{V}$ to $M(G^{\text{even}}, \Delta(P))$. Suppose, as in (5.3.6), that $\mathscr{V} \subset \tilde{V}_\rho$, and let $\tilde{V}_{\rho_P}$ be as defined in 6.2. Then $\mathscr{V}_P$ is naturally a subbundle of $\tilde{V}_{\rho_P}$. Let

$$\text{F.J.}^P_{0,s,t} : \Gamma(M(G, X), [\mathscr{V}]) \to \Gamma\left({}_{K^{s,t}} M(G_P, F_P), \left[\tilde{V}_{\rho_P}\right]\right)$$

be the composition of right translation by $(ts)^{-1}$:

$$\Gamma\left({}_K M(G, X), [\mathscr{V}]\right) \overset{(ts)^{-1}}{\to} \Gamma\left({}_{K_{ts}} M(G, X), [\mathscr{V}]\right)$$

with the coefficient of F.J.^P corresponding to $\alpha = 0$:

$$\Gamma\left({}_{K_{ts}} M(G, X), [\mathscr{V}]\right) \to \Gamma\left({}_{K^{s,t}} M(G_P, F_P), \left[\tilde{V}_{\rho_P}\right]\right).$$

Let

$$\text{F.J.}^P_0 = \bigoplus_{s,t} \text{F.J.}^P_{0,s,t} : \Gamma\left({}_K M(G, X), [\mathscr{V}]\right) \to \bigoplus_{s,t} \Gamma\left({}_{K^{s,t}} M(G_P, F_P), \left[\tilde{V}_{\rho_P}\right]\right).$$

Finally, let

$$j: \bigoplus_{s,t} \Gamma\left({}_{K^{s,t}}M(G_P, F_P), [\mathscr{V}_P]\right) \hookrightarrow \bigoplus_{s,t} \Gamma\left({}_{K^{s,t}}M(G_P, F_P), \left[\tilde{V}_{\rho_P}\right]\right)$$

be the natural inclusion. It follows from (6.3.4) and (8.1.2) that

$$\mathrm{F.J.}_0^P = j \circ \Phi_P : \Gamma\left({}_K M(G, X), [\mathscr{V}]\right)$$

$$\to \bigoplus_{s,t} \Gamma\left({}_{K^{s,t}}M(G_P, F_P), \left[\tilde{V}_{\rho_P}\right]\right).$$

As a Corollary of Theorem 6.4 and Lemma 8.1.1, we thus have

8.2. COROLLARY: *Suppose* $\mathscr{V}$ *satisfies* (5.3.6). *Then the map*

$$\Phi_P : \Gamma\left({}_K M(G, X), [\mathscr{V}]\right) \to \bigoplus_{s,t} \Gamma\left({}_{K^{s,t}}M(G_P, F_P), [\mathscr{V}_P]\right)$$

is defined over L if $\mathscr{V}$ *is rational over L.*

Let $\Gamma^0({}_K M(G, X), [\mathscr{V}]) = \bigcap_P \mathrm{Ker}(\Phi_P) \subset \Gamma({}_K M(G, X), [\mathscr{V}])$, where P runs through the set of maximal rational proper parabolic subgroups of G; this is the space of *cusp forms* of level K, with coefficients in $[\mathscr{V}]$. The analogous definition makes sense when (G, X) is replaced with (G_P, F_P), of course. The following is an immediate consequence of Corollary 8.2.

8.3. COROLLARY: *Assume* $\mathscr{V}$ *is rational over L. Assume the pullback of* $\mathscr{V}$ *to* $\check{M}(G^{even}, \Delta(P))$ *satisfies hypothesis* (5.3.6). *Then* $\Gamma^0({}_K M(G, X), [\mathscr{V}])$ *is an L-rational subspace of* $\Gamma({}_K M(G, X), [\mathscr{V}])$.

Let $\Gamma^0 = \Gamma^0(M(G, X), [\mathscr{V}]) = \varinjlim_K \Gamma^0({}_K M(G, X), [\mathscr{V}])$. Then Γ^0 is a $G(\mathbf{A}^f)$-stable L-rational subspace of $\Gamma(M(G, X), [\mathscr{V}])$, if L is as above.

8.4. We now assume that $\mathscr{V}$ satisfies the *convergence conditions* (2.5.1.3) of [12]. Let $\Gamma(\mathscr{V}, K) = \Gamma({}_K M(G, X), [\mathscr{V}])$, $\Gamma^0(\mathscr{V}_P, K^{s,t}) = \Gamma^0({}_{K^{s,t}}M(G_P, F_P), [\mathscr{V}_P])$. Then formation of Eisenstein series, as described in [12], 2.4, defines a homomorphism

$$E: \bigoplus_{s,t} \Gamma^0\left(\mathscr{V}_P, K^{s,t}\right) \to \Gamma(\mathscr{V}, K)$$

such that

$$\Phi_P \circ E: \bigoplus_{s,t} \Gamma^0\left(\mathscr{V}_P, K^{s,t}\right) \to \bigoplus_{s,t} \Gamma^0\left(\mathscr{V}_P, K^{s,t}\right)$$

is the *identity* ([12], Proposition 2.4.4).

Now we can state the main result of this section, generalizing Theorem 3.2.1 of [12].

8.5. THEOREM: *Assume $\mathscr{V}$ is a homogeneous vector bundle over $\check{M}(G, X)$, which is rational over the field $L \supset E(G, X)$. Suppose $\mathscr{V}$ satisfies the convergence conditions* (2.5.1.3) *of* [12], *and suppose $\mathscr{V}^{even}$, defined as above, satisfies hypothesis* (5.3.6). *Then the map*

$$E: \bigoplus_{s,t} \Gamma^0(\mathscr{V}_P, K^{s,t}) \to \Gamma(\mathscr{V}, K)$$

is rational over L.

REMARK: The hypothesis (5.3.6) is really not a restriction, as we explain below.

PROOF: We lose no generality by assuming $K = \prod_l K_l$, where K_l is a compact open subgroup of $G(\mathbf{Q}_l)$ and l runs over the set of rational primes. Under the hypothesis, the morphism

$$\Phi_P: \Gamma(\mathscr{V}, K) \to \bigoplus_{s,t} \Gamma^0(\mathscr{V}_P, K^{s,t})$$

is defined over L. Let l be a rational prime, and let $H_l(\mathbf{Q})$ be the algebra of $\mathbf{Q}$-valued K_l-biinvariant functions on $G(\mathbf{Q}_l)$. Then $H_l(\mathbf{Q})$ acts naturally on $\Gamma(\mathscr{V}, K)$, for each l, via [12], 2.3. If we know that the action of $H_l(\mathbf{Q})$ preserves the L-rational structure of $\Gamma(\mathscr{V}, K)$, then the proof of Theorem 3.2.1 of [12] goes over word for word to give us our result.

But let $\gamma \in G(\mathbf{Q}_l)$, and define $T(\gamma) \in \mathrm{End}(\Gamma(\mathscr{V}, K))$ as follows: Let $K_l\gamma K_l = \coprod_{\delta \in \Delta} K_l\delta$ be a left coset decomposition of $K_l\gamma K_l$. For any $\alpha \in G(\mathbf{A}^f)$, let r_α denote right translation by α on $\Gamma(M(G, X), [\mathscr{V}])$. Let $T(\gamma): \Gamma(M(G, X), [\mathscr{V}]) \to \Gamma(M(G, X), [\mathscr{V}])$ be $\sum_{\delta \in \Delta} r_\delta$. Then $T(\gamma)$ evidently leaves stable the subspace $\Gamma(\mathscr{V}, K)$ of $\Gamma(M(G, X), [\mathscr{V}])$, and preserves its L-rational structure. It is clear that the $T(\gamma)$ span the image of $H_l(\mathbf{Q})$ in $\mathrm{End}(\Gamma(\mathscr{V}, K))$. The proof is complete.

8.6. REMARK: The assumption that $\mathscr{V}^{\text{even}}$ satisfies hypothesis (5.3.6) is only made in order to define the Fourier-Jacobi series along P. As was remarked in 6.12, this is actually stronger than what we need. In fact, let $h \in \Delta(P)$, $K_h^{\text{even}} = K_h \cap G^{\text{even}}$ its stabilizer in G^{even}; let $\mu: K_h^{\text{even}} \to GL(\mathscr{V}_h)$ be the isotropy representation. Then it suffices to assume that there is a finite-dimensional representation $\rho_h: K_h^{\text{even}} \cdot G_l \to GL(\mathscr{V}_{\rho_h})$, giving rise to a homogeneous vector bundle $\mathscr{V}_\rho$ defined over L, and an imbedding $\mathscr{V}_h \to \mathscr{V}_{\rho_h}$ of $(K_h P_h^- \cap K_h^{\text{even}} \cdot G_l)$-modules.

But the convergence conditions ([12], 2.5.1.3) imply that such a ρ_h exists. In fact, one sees easily that it suffices to find ρ_h when G_l is the derived subgroup of G^{even}. Moreover, it suffices to assume μ is an irreducible representation. One sees easily, furthermore, that the existence of ρ_h depends only on the real structure of G^{even}. We thus assume $G = G^{\text{even}}$ is a real algebraic group such that G^{ad} is simple, that P is a maximal (real) parabolic subgroup of G which fixes a point boundary component F of X^+, and that X^+ is thereby a tube domain over F. Then the map

$$J : J^{h,P} : A_{P,\mathbf{C}} \to K_{h,\mathbf{C}}$$

defines a $\mathbf{C}$-rational isomorphism of A_ρ with the center of K_h. The convergence conditions [12], 2.5.1.3 translate into a condition on the restriction of μ to Z_{K_h} from which the existence of ρ_h follows easily.

§9. Further questions

In this section we mention some problems related to the arithmeticity of automorphic forms, but which have not been treated in this article.

9.1. Shimura has studied the arithmetic properties of forms of half-integral weight on the Siegel modular variety [20], and more generally has investigated the arithmetic properties of theta-liftings [24,56]. Among other treatments of this subject, we may mention those of Shintani, Niwa, Zagier, Oda, Kudla, and Rallis and Schiffman; cf. [42,45] and references in [24,56]. The results of Waldspurger [63,64] are well-known and should also be mentioned in this context. Perhaps the most striking consequences of this theory are the relations among periods of automorphic forms on different groups, discovered by Shimura in [58]; interesting results along the same line have been announced by Oda [46], who interprets them in the language of motives.

The bundles whose sections are forms of half-integral weight do not arise from the functor $\mathscr{V} \mapsto [\mathscr{V}]$ of Theorem 4.8. This is scarcely surprising since in general forms of half-integral weight generate representations of a double cover of $G(\mathbf{A}^f)$, whereas our automorphic vector bundles are equipped with $G(\mathbf{A}^f)$-action. However, Mumford's theory of algebraic thetafunctions [44] contains the arithmetic theory of forms of half-integral weight on the Siegel modular variety cf. [71]. Needless to say, this approach to the arithmetic of automorphic forms is essentially the same as the one based on the Fourier expansion of the classical theta-nullwerte, which were already present in Shimura's early work on the subject [52]. An *a priori* proof of the existence over $\mathbf{Q}$ of the bundle of forms of half-integral weight, with an action of a double cover of $G(\mathbf{A}^f)$, but not relying on the theory of theta-functions, would be of some interest.

9.2. If (G, X) has a symplectic imbedding, then there is naturally a family $\mathcal{A}(G, X)$ of abelian varieties over $M(G, X)$. Let $f: \mathcal{A}(G, X) \to M(G, X)$ be the natural map. Associated to $\mathcal{A}(G, X)$ is not only the flat vector bundle $\mathcal{H}^1_{DR}(\mathcal{A}(G, X)/M(G, X))$, with its canonical local system $R^1 f_* \mathbf{Q}$, but also the corresponding l-adic sheaves $R^1 f_* \mathbf{Q}_l$ which have rational structures over $E(G, X)$. Is it possible to construct analogous l-adic local systems in the general case? A partial answer to this question, in the case of certain (G, X) of abelian type, is provided by Shimura in [19], §7.

9.3. The attempt to extend our results to characteristic $p > 0$ runs up against a number of obstacles, most prominently the absence in general of good models for $M(G, X)$ in characteristic p. The recent success of Chai and Faltings in constructing smooth compactifications in characteristic p of the variety of moduli of principally polarized abelian varieties ([32], [71] leads one to hope that the resolution of this problem is not so distant as had been believed.

9.4. We have proved that the vector bundles on $M(G, X)$ defined by classical factors of automorphy (cf. (5.3.2) ff.) have rational structures over specific number fields. This implies that their spaces of sections, which are naturally identified with spaces of automorphic forms, also have rational structures; i.e., they have bases of *arithmetic* automorphic forms. Shimura introduced two criteria for deciding when an automorphic form, given as an analytic function of several complex variables, is arithmetic; these criteria are generalized in 9.3 and 6.9. Shimura has used these criteria to powerful effect in his investigations of theta-liftings and special values of L-functions; cf. [53,55,58] for some typical applications of Shimura's methods.

In general, it is believed that the bundles $[\mathcal{V}]$ satisfying $F^p[\mathcal{V}] = [\mathcal{V}]$, $F^{p+1}[\mathcal{V}] = \{0\}$, for some $p \in \mathbf{Z}$, have cohomology mostly in one preferred dimension $i(\mathcal{V})$. At least when $\mathcal{V}$ is sufficiently regular and the $\mathbf{Q}$-rank of G^{ad} is zero, this is a theorem [33]; partial results are available in other cases (cf. e.g. [50], [70]). In principle, most of this cohomology (all of it, when $\mathbf{Q}$-rank $G^{\mathrm{ad}} = 0$) is represented by L^2-harmonic forms on X. On the other hand, the spaces $H^{\cdot}(M(G, X), [\mathcal{V}])$ have natural structures over number fields. T. Oda has asked the following question: how can one recognize the L^2-harmonic representatives of arithmetic cohomology classes in degree $i > 0$? The Zucker conjecture [67], and its recent resolution in many cases by Borel [29] and Zucker [68] suggests that it is at least as natural to ask the analogous question for the intersection cohomology groups of the Baily-Borel compactification of $M(G, X)$, with coefficients in the flat bundles $[\tilde{V}]$.

In certain cases, necessary and sufficient criteria for arithmeticity can be obtained by considering the restriction to imbedded products of modular curves; this is be treated at length in [73]. However, this is clearly a very special technique.

References

This is a continuation of the list of references at the end of Part I. For the reader's convenience we reproduce those sources numbered [1] through [25] which are cited in Part II, preserving the original numbers

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