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1. Introduction

Let l be a prime number, and let n be a positive integer. Let K be a cyclic extension field of the rational number field $\mathbf{Q}$ of degree l^n (i.e., $K/\mathbf{Q}$ is Galois with cyclic Galois group of order l^n.) Let σ be a generator of Gal$(K/\mathbf{Q})$, and let C_K be the ideal class group of K in the narrow sense. Let

$$C_K^{1-\sigma} = \{ a^{1-\sigma} : a \in C_K \}.$$

The following results are discussed in Section 2 of [2]. The group $C_K^{1-\sigma}$ is the narrow principal genus of K, and $C_K/C_K^{1-\sigma}$ is the narrow genus group of K. (All subsequent references in this paper to the principal genus and genus group mean narrow principal genus and narrow genus group.) The structure of the genus group is determined by the ramification indices of the ramified primes of $K/\mathbf{Q}$. More precisely, suppose exactly t finite primes of $\mathbf{Q}$ ramify in K, and suppose they have ramification indices

$$l^{n_1}, l^{n_2}, \dots, l^{n_t}, \quad \text{where } n = n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t \geqslant 1.$$

Then the genus group $C_K/C_K^{1-\sigma}$ is an abelian group of type $(l^{n_2}, \dots, l^{n_t})$; i.e., $C_K/C_K^{1-\sigma}$ is the direct product of cyclic groups of order l^{n_i} for $i = 2, \dots, t$.

We now focus our attention on the Sylow l-subgroup of the principal genus. We shall denote this Sylow l-subgroup by H_K. We let

$$H_K^{1-\sigma} = \{ a^{1-\sigma} : a \in H_K \} \quad \text{and} \quad H_K^l = \{ a^l : a \in H_K \}.$$

We note that $H_K/(H_K^{1-\sigma} \cdot H_K^l)$ is an elementary abelian l-group which may be viewed as a vector space over $\mathbf{F}_l$, the finite field with l elements. We let

$$R_K = \operatorname{rank}[H_K/(H_K^{1-\sigma} \cdot H_K^l)] = \dim_{\mathbf{F}_l}[H_K/(H_K^{1-\sigma} \cdot H_K^l)]. \tag{1.1}$$

We observe that R_K is the rank of $H_K/H_K^{1-\sigma}$ as an l-group, which is the same as the minimal number of generators of H_K as a module over Gal$(K/\mathbf{Q})$. (As a special case we observe that if $l = 2$ and $n = 1$, then $H_K^{1-\sigma} = H_K^2$, and hence R_K is the 2-rank of H_K, which is the same as the 4-rank of C_K.) In this paper we shall consider the following question: how likely is $R_K = 0$, $R_K = 1$,

$R_K = 2, \ldots$? When $n = 1$ (and thus $[K:\mathbf{Q}] = l$), we have already obtained answers to this question in [5] when $l = 2$ and in [4] and [6] when $l > 2$. Many of the techniques we used when $n = 1$ can also be used when $n > 1$, but the answers when $n > 1$ are quite different from the answers when $n = 1$.

In Section 2 we shall derive various results for the case where $l > 2$ and the genus group is a given finite abelian l-group. In Section 3 we consider the case where $l > 2$ and the genus group varies over all finite abelian l-groups with a given rank. We also examine what happens when the rank becomes arbitrarily large. In Section 4 we briefly describe the analogous results when $l = 2$. In the appendix we specify certain Markov processes that form parts of our calculations, and we present tables of values for certain densities.

2. Case $l > 2$ and given genus group

We assume that the prime $l > 2$ and that n is a positive integer. Both l and n will be fixed throughout this section. We let notations be the same as in Section 1, and we let $G_K = C_K/C_K^{1-\sigma}$, the genus group of K. Let t be a positive integer, and let $n_1, n_2, \ldots, n_t$ be integers such that $n = n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t \geqslant 1$. Let G be the abelian group of type $(l^{n_2}, \ldots, l^{n_t})$. (When $t = 1$, we let G be the trivial group.) For each nonnegative integer i and each positive real number x, we define

$$A(G) = \{\text{cyclic extensions } K \text{ of } \mathbf{Q} \text{ of degree } l^n \text{ with genus group } G_K \text{ isomorphic to } G\} \tag{2.1}$$

$$A(G)_x = \{K \in A(G) : \text{the conductor of } K \text{ is } \leqslant x\} \tag{2.2}$$

$$A_i(G) = \{K \in A(G) : R_K = i\} \text{ (cf. Equation 1.1)} \tag{2.3}$$

$$A_i(G)_x = \{K \in A_i(G) : \text{the conductor of } K \text{ is } \leqslant x\}. \tag{2.4}$$

Then we define the density $d_i(G)$ of $A_i(G)$ in $A(G)$ by

$$d_i(G) = \lim_{x \to \infty} \frac{|A_i(G)_x|}{|A(G)_x|} \tag{2.5}$$

where $|S|$ denotes the cardinality of a set S. The density $d_i(G)$ indicates how likely it is for a field K in $A(G)$ to have R_K equal to i, where $i = 0, 1, 2, \ldots$.

Next we recall some basic facts about the conductor f_K of a field K in $A(G)$. First f_K has the form $p_1 \ldots p_t$ or $p_1 \ldots p_{t-1} l^\alpha$, where $p_1 < p_2 < \cdots < p_t$ are rational primes different from l and $2 \leqslant \alpha \leqslant n + 1$. Since

$$|\{p_1 \ldots p_{t-1} l^\alpha \leqslant x\}| = o(|\{p_1 \ldots p_t \leqslant x\}|) \quad \text{as } x \to \infty,$$

it suffices to consider $f_K = p_1 \dots p_t$ when calculating $d_t(G)$ in Equation 2.5. Next the ramification indices impose certain restrictions on $p_1, \dots, p_t$. If the prime p_i has ramification index $l^{n_{j_i}}$ for $1 \leqslant i \leqslant t$, then $p_i \equiv 1 (\mathrm{mod}\ l^{n_{j_i}})$ for $1 \leqslant i \leqslant t$. Let K_{j_i} be the cyclic extension of $\mathbf{Q}$ of degree $l^{n_{j_i}}$ in which only the prime p_i ramifies. Then $K \subset K_{j_1} \dots K_{j_t}$, and an inductive argument shows that there are

$$\left[\prod_{i=1}^{t} (l^{n_{j_i}} - l^{n_{j_i}-1})\right] / (l^n - l^{n-1})$$

distinct fields K' in $A(G)$ that satisfy the following conditions: (i) K' is a subfield of $K_{j_1} \dots K_{j_t}$; (ii) the conductor of K' is $p_1 \dots p_t$; and (iii) the ramification index of p_i is $l^{n_{j_i}}$ for $1 \leqslant i \leqslant t$. Since $n_{j_1}, \dots, n_{j_t}$ is a reordering of $n_1, \dots, n_t$, then

$$\prod_{i=1}^{t} (l^{n_{j_i}} - l^{n_{j_i}-1}) = \prod_{i=1}^{t} (l^{n_i} - l^{n_i-1}).$$

Then we have

$$|A(G)_x| \sim \frac{\prod_{i=1}^{t}(l^{n_i} - l^{n_i-1})}{l^n - l^{n-1}} \cdot \sum_{(n_{j_i})} \sum_{\substack{p_1 \dots p_t \leqslant x \\ p_1 < \cdots < p_t}}^{(n_{j_i})} 1 \quad \text{as } x \to \infty. \tag{2.6}$$

Here $\sum_{(n_{j_i})}$ denotes a sum over all distinguishable orderings of the integers $n_1, \dots, n_t$, and $\sum^{(n_{j_i})}$ is a sum for a fixed ordering (n_{j_i}) in which $p_i \equiv 1 (\mathrm{mod}\ l^{n_{j_i}})$ for $1 \leqslant i \leqslant t$.

Now it is a standard fact in analytic number theory (cf. [7], Theorem 437) that

$$\sum_{\substack{p_1 \dots p_t \leqslant x \\ p_1 < \cdots < p_t}} 1 \sim \frac{x (\log \log x)^{t-1}}{(t-1)! \log x} \quad \text{as } x \to \infty.$$

However since we are summing only over primes in certain arithmetic progressions, we have to include a factor $(l^{n_{j_i}} - l^{n_{j_i}-1})^{-1}$ for each condition $p_i \equiv 1 (\mathrm{mod}\ l^{n_{j_i}})$. So

$$\sum_{\substack{p_1 \dots p_t \leqslant x \\ p_1 < \cdots < p_t}}^{(n_{j_i})} 1 \sim \frac{1}{\prod_{i=1}^{t}(l^{n_{j_i}} - l^{n_{j_i}-1})} \cdot \frac{x (\log \log x)^{t-1}}{(t-1)! \log x} \quad \text{as } x \to \infty. \tag{2.7}$$

Recall that

$$\prod_{i=1}^{t} (l^{n_{j_i}} - l^{n_{j_i}-1}) = \prod_{i=1}^{t} (l^{n_i} - l^{n_i-1}).$$

Hence the right side of Formula 2.7 does not depend on the particular ordering (n_{j_i}). Now let

$$v_w = |\{n_i : n_i = w\}|, \quad w = 1, 2, \ldots, n. \tag{2.8}$$

Since $1 \leqslant n_i \leqslant n$ for $1 \leqslant i \leqslant t$, then $v_1 + v_2 + \cdots + v_n = t$, and $(t!)/[(v_1!)\ldots(v_n!)]$ is the number of distinguishable orderings of the integers $n_1, n_2, \ldots, n_t$. Hence from Formulae 2.6, 2.7, and 2.8, we get

$$|A(G)_x| \sim \frac{1}{l^n - l^{n-1}} \cdot \frac{t!}{(v_1!)\ldots(v_n!)} \cdot \frac{x(\log\log x)^{t-1}}{(t-1)!\log x}$$

$$\text{as } x \to \infty. \tag{2.9}$$

Our next goal is to obtain an asymptotic formula for $A_t(G)_x$ (cf. Equation 2.4). To accomplish this task, we need a convenient method for calculating the quantity R_K defined by Equation 1.1. For this purpose we use Theorem 5.3 of [2]:

$$R_K = t - 1 - \operatorname{rank} M'_K, \tag{2.10}$$

where M'_K is a certain $t \times (t-1)$ matrix over $\mathbf{F}_l$. We can define the elements of M'_K as follows. Suppose K has conductor $p_1 \ldots p_t$ and p_i has ramification index $l^{n_{j_i}}$. We let the row indices of M'_K be $a = 1, 2, \ldots, t$ and the column indices be $b = 1, \ldots, i_0 - 1, \; i_0 + 1, \ldots, t$, where p_{i_0} is the smallest p_i with ramification index l^n. For primes $p \equiv 1 (\operatorname{mod} l)$, we let χ_p denote a Dirichlet character with conductor p and exponent l, and we let ζ be a primitive l^{th} root of unity. We define $[p_b, p_a] \in \mathbf{F}_l$ by

$$\zeta^{[p_b, p_a]} = \chi_{p_b}(p_a) \quad \text{for } 1 \leqslant a \leqslant t, \quad 1 \leqslant b \leqslant t. \tag{2.11}$$

If the ab entry of M'_K is m'_{ab} for $a = 1, 2, \ldots, t$ and $b = 1, \ldots, i_0 - 1$, $i_0 + 1, \ldots, t$, then

$$m'_{ab} = \begin{cases} -z_a[p_b, p_a] & \text{if } a \neq b \\ \sum\limits_{\substack{c=1 \\ c \neq a}}^{t} z_c[p_c, p_a] & \text{if } a = b \end{cases} \tag{2.12}$$

for some $z_a, z_c \in \mathbf{F}_l$. (Remark: Actually z_a is one of the integers $il^{n - n_{j_a}}$ such that $1 \leqslant i \leqslant l^{n_{j_a}}$ and $i \not\equiv 0 (\operatorname{mod} l)$, but in Equation 2.12 we are reducing $z_a (\operatorname{mod} l)$. A similar statement applies to z_c.) Note that if $l^{n_{j_a}} < l^n$, then all entries in row a of M'_K are zero, except perhaps for the aa entry. Also note that $\zeta^{m'_{ab}} = \chi_{p_b}^{-z_a}(p_a)$ if $a \neq b$.

Now if f_K is the conductor of K, then

$$|A_i(G)_x| = |\{K \in A(G) : R_K = i \text{ and } f_K \leqslant x\}|,$$

which can be estimated as follows by using Equation 2.10:

$$|A_i(G)_x| \sim \sum_{\substack{M' \\ \operatorname{rk} M' = t-1-i}} \sum_{(n_{j_i})} \sum^{(n_{j_i})}_{\substack{p_1 \cdots p_t \leqslant x \\ p_1 < \cdots < p_t}} \sum_{\substack{K \\ f_K = p_1 \cdots p_t}} \delta_{M'} \quad \text{as } x \to \infty, \tag{2.13}$$

where the first sum is over all $t \times (t-1)$ matrices M' over $\mathbf{F}_l$ with rank $M' = t - 1 - i$; the description of the second and third sums follows Formula 2.6; the fourth sum is over $K \in A(G)$ with conductor $f_K = p_1 \dots p_t$ such that each p_i has ramification index $l^{n_{j_i}}$; and $\delta_{M'} = 1$ if $M'_K = M'$ and $\delta_{M'} = 0$ if $M'_K \neq M'$.

To illustrate our method for evaluating Formula 2.13, we first suppose $G = G_1$, where G_1 is the abelian group of type $(l^{n_2}, \dots, l^{n_t})$ with $n_i = n$ for all i. Then each prime that ramifies in a field $K \in A(G_1)$ is totally ramified. Since $n_i = n$ for all i, there is only one distinguishable ordering (n_{j_i}), and each $p_i \equiv 1 (\text{mod } l^n)$. So Formula 2.13 becomes

$$|A_i(G_1)_x| \sim \sum_{\substack{M' \\ \operatorname{rk} M' = t-1-i}} \sum_{\substack{p_1 \cdots p_t \leqslant x \\ p_1 < \cdots < p_t \\ \text{each } p_i \equiv 1 (\text{mod } l^n)}} \sum_{\substack{K \\ f_K = p_1 \cdots p_t}} \delta_{M'} \quad \text{as } x \to \infty. \tag{2.14}$$

The calculations in Formula 2.14 are very similar to the calculations performed in [3], where the fields K are of degree l over $\mathbf{Q}$ instead of degree l^n over $\mathbf{Q}$. Because the notation will become somewhat complicated and because the calculations in [3] are quite lengthy, we shall first sketch the basic ideas used in evaluating Formula 2.14 and in computing Equation 2.5 when $G = G_1$. To each field K of degree l^n over $\mathbf{Q}$ with t ramified primes, each of which is totally ramified, we have associated a certain $t \times (t-1)$ matrix M'_K over $\mathbf{F}_l$. We shall perform certain row and column operations on M'_K and then create a new matrix M_K, which is a $t \times t$ matrix over $\mathbf{F}_l$, has rank $M_K =$ rank M'_K, and has the sum of the entries in each of its rows equal to zero. Then we shall derive an asymptotic formula which indicates that M_K is equally likely to be any of the $t \times t$ matrices Γ over $\mathbf{F}_l$ in which the sum of the entries in each row of Γ is zero. Since each such Γ is determined by its last $t-1$ columns, there are $l^{t(t-1)}$ such matrices Γ. If $N(t, r)$ is the number of $t \times (t-1)$ matrices over $\mathbf{F}_l$ with rank $= r$, then $N(t, r)/l^{t(t-1)}$ is the probability that a randomly selected $t \times (t-1)$ matrix over $\mathbf{F}_l$ has rank $= r$. We shall show that the density $d_i(G_1) = N(t, t-1-i)/l^{t(t-1)}$.

Now we begin the details. First note that $i_0 = 1$ when $G = G_1$. Then starting from the matrix M'_K whose entries are given by Equation 2.12, we obtain a matrix M''_K by multiplying row a of M'_K by z_a^{-1} for $1 \leqslant a \leqslant t$ and by multiplying column b of the resulting matrix by z_b for $2 \leqslant b \leqslant t$. Since all ramified primes in $K/\mathbf{Q}$ have ramification indices equal to l^n, then each z_a and z_b is a nonzero element of $\mathbf{F}_l$. So z_a^{-1} is a well-defined element of $\mathbf{F}_l$, and rank $M''_K =$ rank M'_K. To get our new matrix M_K, we insert a column 1 into M''_K whose $a1$ entry is $(-1) \times$ (the sum of the entries in row a of M''_K). Thus rank $M_K =$ rank $M''_K =$ rank M'_K, and we can describe the entries m_{ab} of M_K for $1 \leqslant a \leqslant t$ and $1 \leqslant b \leqslant t$ as follows:

$$m_{ab} = \begin{cases} -z_b[p_b, p_a] & \text{if } a \neq b \\ \sum_{\substack{c=1 \\ c \neq a}}^{t} z_c[p_c, p_a] & \text{if } a = b. \end{cases} \tag{2.15}$$

Note that $\zeta^{m_{ab}} = \chi_{p_b}^{-z_b}(p_a)$ for all $a \neq b$. Then except for the use of additive notation instead of multiplicative notation, our matrix M_K is just the transpose of the matrix M of [3]. (See, in particular, the discussion following Lemma 1 in [3].) Now we proceed as in Section 3 of [3]. We let $F = \mathbf{Q}(\zeta)$, and we let $\mathfrak{p}_i$ be a prime ideal of F above (p_i) for $1 \leqslant i \leqslant t$. We define the characters λ_i and ω_i as described on p. 198 of [3]. Then for $b < a$, we have $\lambda_b^{z_b}(\mathfrak{p}_a) = \zeta^{m_{ab}}$ and $\omega_b^{z_a}(\mathfrak{p}_a) = \zeta^{m_{ba}}$. (Recall that our M_K is the transpose of M in [3].) For $\alpha, \beta \in \mathbf{F}_l$, we define

$$\delta(z_b, z_a, \beta, \alpha) = \begin{cases} 1 & \text{if } (\lambda_b^{z_b}(\mathfrak{p}_a), \omega_b^{z_a}(\mathfrak{p}_a)) = (\zeta^{\beta}, \zeta^{\alpha}) \\ 0 & \text{otherwise} \end{cases} \tag{2.16}$$

Now suppose $\Gamma = [\gamma_{ab}]$ is an arbitrary $t \times t$ matrix over $\mathbf{F}_l$ such that the sum of the entries in each row of Γ is zero. Let

$$N(\Gamma) = \sum_{\substack{p_1 \cdots p_t \leqslant x \\ p_1 < \cdots < p_t \\ \text{each } p_i \equiv 1 (\text{mod } l^n)}} \sum_{\substack{K \\ f_K = p_1 \cdots p_t}} \delta_\Gamma \tag{2.17}$$

where $\delta_\Gamma = 1$ if $M_K = \Gamma$ and $\delta_\Gamma = 0$ if $M_K \neq \Gamma$. Then (cf. [3], Equation 4)

$$N(\Gamma) = \sum_{\substack{p_1 \leqslant x^{1/t} \\ p_1 \equiv 1 (\text{mod } l^n)}} \sum_{z_1 = 1}^{1} \sum_{\substack{p_1 < p_2 \leqslant (x/p_1)^{1/(t-1)} \\ p_2 \equiv 1 (\text{mod } l^n)}} Y_2 \cdots$$

$$\times \sum_{\substack{p_{t-1} < p_t \leqslant x/p_1 \cdots p_{t-1} \\ p_t \equiv 1 (\text{mod } l^n)}} Y_t \tag{2.18}$$

where

$$Y_a = \sum_{z_a} \prod_{b_a=1}^{a-1} \delta(z_{b_a}, z_a, \gamma_{ab_a}, \gamma_{b_a a}) \quad \text{for } a = 2, \dots, t \tag{2.19}$$

and $\sum_{z_a}$ is a sum over all $z_a = 1, \dots, l^n$ such that $z_a \neq 0 (\text{mod } l)$. Using the same arguments as those used in the proof of Lemma 3 in [3], we get

$$N(\Gamma) \sim \frac{1}{l^n - l^{n-1}} \cdot \frac{1}{l^{t(t-1)}} \cdot \frac{x(\log \log x)^{t-1}}{(t-1)! \log x} \quad \text{as } x \to \infty. \tag{2.20}$$

Note that this formula is valid for each $t \times t$ matrix Γ in which the sum of the entries in each row is zero. Next let $N(t, r)$ denote the number of $t \times t$ matrices Γ over $\mathbf{F}_l$ such that the sum of the entries in each row of Γ is zero and such that rank $\Gamma = r$. Note that $N(t, r)$ also represents the number of $t \times (t-1)$ matrices over $\mathbf{F}_l$ with rank $= r$. Then from Formulae 2.14, 2.17, and 2.20, we have

$$|A_i(G_1)_x| \sim \frac{N(t, t-1-i)}{l^{t(t-1)}} \cdot \frac{1}{l^n - l^{n-1}} \cdot \frac{x(\log \log x)^{t-1}}{(t-1)! \log x}$$
$$\text{as } x \to \infty. \tag{2.21}$$

Finally by combining Formulae 2.5, 2.9, and 2.21, and by observing that $v_1 = \cdots = v_{n-1} = 0$ and $v_n = t$ in Formula 2.9 when $G = G_1$, we get the following result.

PROPOSITION 2.1: *Let l be an odd prime; let n and t be positive integers; and let G_1 be the abelian group of type $(l^{n_2}, \dots, l^{n_t})$ with $n_2 = \cdots = n_t = n$. (If $t = 1$, we let G_1 be the trivial group.) Let $N(t, r)$ be the number of $t \times t$ matrices Γ over $\mathbf{F}_l$ such that the sum of the entries in each row of Γ is zero and such that* rank $\Gamma = r$, *where $0 \leqslant r \leqslant t-1$. Let $d_i(G_1)$ be the density defined by Equation 2.5. Then*

$$d_i(G_1) = N(t, t-1-i)/l^{t(t-1)} \quad \textit{for } 0 \leqslant i \leqslant t-1.$$

Since $N(t, r)$ also represents the number of $t \times (t-1)$ matrices over $\mathbf{F}_l$ with rank $= r$, we can use [8] to get

$$N(t, t-1-i) = \prod_{j=0}^{t-2-i} \frac{(l^t - l^j)(l^{t-1-j} - 1)}{(l^{j+1} - 1)}.$$

Then

$$d_i(G_1) = \left[\prod_{j=0}^{t-2-i} \frac{(l^t - l^j)(l^{t-1-j} - 1)}{(l^{j+1} - 1)} \right] \Big/ l^{t(t-1)}$$

$$= \frac{l^{t(t-1-i)} l^{(t-1)(t-1-i)-(t-2-i)(t-1-i)/2}}{l^{t(t-1)} l^{(t-1-i)(t-i)/2}}$$

$$\times \prod_{j=0}^{t-2-i} \frac{(1 - l^{j-t})(1 - l^{j-t+1})}{(1 - l^{-j-1})}$$

$$= l^{-i(i+1)} \sum_{j=0}^{t-2-i} \frac{(1 - l^{j-t})(1 - l^{j-t+1})}{(1 - l^{-j-1})}.$$

If we let $k = t - 1 - j$, we get the following result.

COROLLARY 2.2: *Let notations be as in Proposition* 2.1. *Then*

$$d_i(G_1) = \frac{l^{-i(i+1)} \left[\prod_{k=i+1}^{t-1} (1 - l^{-k-1})(1 - l^{-k}) \right]}{\left[\prod_{j=1}^{t-1-i} (1 - l^{-j}) \right]} \quad \textit{for } 0 \leqslant i \leqslant t - 1.$$

We now return to the general case of an abelian group G of type $(l^{n_2}, \ldots, l^{n_t})$ with $n = n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t \geqslant 1$. We want to evaluate the right side of Formula 2.13. We let $u = v_n$ (cf. Equation 2.8), and to simplify subsequent notation, we assume that the ordering (n_{j_i}) has $n_{j_i} = n_i$ for $1 \leqslant i \leqslant t$. (All of our arguments can be appropriately modified to handle other orderings (n_{j_i}).) Next we replace the matrix M_K' whose entries are given by Equation 2.12 with a $t \times t$ matrix M_K created as follows. First note that $i_0 = 1$ since we have assumed that $n_{j_1} = n_1 = n$. We multiply row a of M_K' by z_a^{-1} for $1 \leqslant a \leqslant u$, and we multiply column b of the resulting matrix by z_b for $2 \leqslant b \leqslant u$. We denote this new matrix by M_K''. To get M_K, we insert a column 1 into M_K'' whose $a1$ entry is $(-1) \times$ (the sum of the entries in columns $2, \ldots, u$ of row a of M_K''). Thus rank M_K = rank M_K'' = rank M_K', and M_K has the following form:

$$M_K = \begin{pmatrix} M_1 & M_2 \\ O & D \end{pmatrix}, \tag{2.22}$$

where M_1 is a $u \times u$ matrix over $\mathbf{F}_l$ in which the sum of the entries in each row is zero; M_2 is a $u \times (t-u)$ matrix over $\mathbf{F}_l$; O is the $(t-u)\times u$ zero matrix; and D is a $(t-u)\times(t-u)$ diagonal matrix over $\mathbf{F}_l$. If m_{ab} is the ab entry of M_K for $1 \leqslant a \leqslant t$, $1 \leqslant b \leqslant t$, then

$$m_{ab} = \begin{cases} -z_b[p_b, p_a] & \text{if } 1 \leqslant a \leqslant u,\ 1 \leqslant b \leqslant u,\ a \neq b \\ \sum_{\substack{c=1 \\ c\neq a}}^{u} z_c[p_c, p_a] & \text{if } 1 \leqslant a \leqslant u,\ a = b \\ -[p_b, p_a] & \text{if } 1 \leqslant a \leqslant u,\ u+1 \leqslant b \leqslant t \\ \sum_{c=1}^{u} z_c[p_c, p_a] & \text{if } u+1 \leqslant a \leqslant t,\ a = b \\ 0 & \text{otherwise.} \end{cases} \tag{2.23}$$

We note that M_1 is the same kind of matrix as the matrix M_K in our analysis of the case $G = G_1$. We shall use some of the same ideas we used in that case. Let $\Gamma = [\gamma_{ab}]$ be a $t \times t$ matrix over $\mathbf{F}_l$ such that Γ has the same form as the matrix on the right side of Equation 2.22. Let

$$N(\Gamma) = \sum_{\substack{p_1 \cdots p_t \leqslant x \\ p_1 < \cdots < p_t}}^{(n_{j_i})} \sum_{\substack{K \\ f_K = p_1 \cdots p_t}} \delta_\Gamma \tag{2.24}$$

where (n_{j_i}) is the ordering with $n_{j_i} = n_i$ for $1 \leqslant i \leqslant t$, and where $\delta_\Gamma = 1$ if $M_K = \Gamma$ and $\delta_\Gamma = 0$ if $M_K \neq \Gamma$. Let $\delta(z_b, z_a, \beta, \alpha)$ be defined by Equation 2.16. Next let

$$Y_a = \sum_{z_a} \prod_{b_a=1}^{a-1} \delta(z_{b_a}, z_a, \gamma_{ab_a}, \gamma_{b_a a}) \quad \text{for } a = 2, \ldots, u \tag{2.25}$$

and

$$Y_a'({}_a\beta_1, \ldots, {}_a\beta_u) = \sum_{z_a} \prod_{b_a=1}^{u} \delta(z_{b_a}, 1, -{}_a\beta_{b_a}, \gamma_{b_a a})$$

$$\text{for } a = u+1, \ldots, t, \tag{2.26}$$

where $\sum_{z_a}$ denotes a sum over all $z_a = il^{n-n_a}$ with $1 \leqslant i \leqslant l^{n_a}$ and $i \not\equiv 0 \pmod{l}$, and ${}_a\beta_{b_a}$ is an arbitrary element of $\mathbf{F}_l$ for $b_a = 1, \ldots, u$. Now let

$$Y_a'' = \sum_{{}_a\beta_1 + \cdots + {}_a\beta_u = \gamma_{aa}} Y_a'({}_a\beta_1, \ldots, {}_a\beta_u) \quad \text{for } a = u+1, \ldots, t. \tag{2.27}$$

Then

$$N(\Gamma) = \sum_{\substack{p_1 \leqslant x^{1/t} \\ p_1 \equiv 1 (\text{mod } l^n)}} \sum_{z_1=1}^{1} \sum_{\substack{p_1 < p_2 \leqslant (x/p_1)^{1/(t-1)} \\ p_2 \equiv 1 (\text{mod } l^n)}} Y_2 \dots$$

$$\times \sum_{\substack{p_{u-1} < p_u \leqslant (x/p_1 \cdots p_{u-1})^{1/(t-u+1)} \\ p_u \equiv 1 (\text{mod } l^n)}} Y_u$$

$$\times \sum_{\substack{p_u < p_{u+1} \leqslant (x/p_1 \cdots p_u)^{1/(t-u)} \\ p_{u+1} \equiv 1 (\text{mod } l^{n_{u+1}})}} Y''_{u+1} \cdots \sum_{\substack{p_{t-1} < p_t \leqslant x/p_1 \cdots p_{t-1} \\ p_t \equiv 1 (\text{mod } l^{n_t})}} Y''_t. \quad (2.28)$$

By using techniques similar to those used in the proof of Lemma 3 in [3], we can then obtain

$$N(\Gamma) \sim \frac{1}{l^n - l^{n-1}} \cdot \frac{1}{l^{u(t-1)+t-u}} \cdot \frac{x(\log \log x)^{t-1}}{(t-1)! \log x} \quad \text{as } x \to \infty. \quad (2.29)$$

The coefficients in Formula 2.29 can be explained intuitively as follows. Each condition $p_a \equiv 1(\text{mod } l^{n_a})$ contributes a factor $1/(l^{n_a} - l^{n_a - 1})$ for $1 \leqslant a \leqslant t$, but each $\sum_{z_a}$ in Y_a or Y'_a contributes a factor $(l^{n_a} - l^{n_a-1})$ for $2 \leqslant a \leqslant t$. So we have a net factor of $1/(l^n - l^{n-1})$. Next we consider the factor $1/l^{u(t-1)+t-u}$. First we note that the product in each Y_a introduces a factor $1/l^{2(a-1)}$ for $a = 2, \dots, u$. The product in each Y'_a introduces a factor $1/l^{2u}$, but each sum $\sum_{{}_a\beta_1 + \cdots + {}_a\beta_u = \gamma_{aa}}$ introduces a factor l^{u-1}. So each Y''_a contributes a factor $1/l^{u+1}$. Finally

$$\left[\prod_{a=2}^{u} 1/l^{2(a-1)}\right] \cdot \left[\prod_{a=u+1}^{t} 1/l^{u+1}\right] = 1/l^{u(u-1)+(t-u)(u+1)}$$

$$= 1/l^{u(t-1)+t-u}.$$

We also note that there are $l^{u(t-1)+t-u}$ matrices Γ of the same form as the matrix on the right side of Equation 2.22. So Formula 2.29, which is valid for each Γ of the prescribed form, indicates that each such Γ is equally likely to occur.

We now return to Formula 2.13. Our calculations have determined that

$$|A_\iota(G)_x| \sim \sum_{\substack{\Gamma \\ \text{rk}\,\Gamma = t-1-\iota}} \sum_{(n_{j_t})} N(\Gamma) \quad \text{as } x \to \infty, \quad (2.30)$$

where Γ ranges over all matrices of the form specified on the right side of Equation 2.22; (n_{j_t}) ranges over all distinguishable orderings of $n_1, \dots, n_t$; and $N(\Gamma)$ is given by Formula 2.29. Let $N(t, u, r)$ denote the number of Γ's with rank $\Gamma = r$, where $0 \leqslant r \leqslant t-1$. Since the number of distinguishable orderings of $n_1, \dots, n_t$ is $(t!)/[(v_1!)\dots(v_n!)]$ (cf. Equation 2.8) and since $u = v_n$, then

$$|A_t(G)_x| \sim \frac{N(t, u, t-1-i)}{l^{u(t-1)+t-u}} \cdot \frac{1}{l^n - l^{n-1}} \cdot \frac{t!}{(v_1!)\dots(v_n!)} \cdot \frac{x(\log\log x)^{t-1}}{(t-1)!\,\log x} \quad \text{as } x \to \infty. \tag{2.31}$$

Finally from Formulae 2.5, 2.9, and 2.31, we obtain the following result.

PROPOSITION 2.3: *Let l be an odd prime number; let n and t be positive integers; and let G be the abelian group of type $(l^{n_2}, \dots, l^{n_t})$ with $n = n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t \geqslant 1$. (If $t = 1$, we let G be the trivial group.) Let u be the largest integer such that $n_u = n$, and let r be an integer with $0 \leqslant r \leqslant t-1$. Let $N(t, u, r)$ be the number of $t \times t$ matrices Γ over $\mathbf{F}_l$ of the form specified on the right side of Equation* 2.22 *such that* rank $\Gamma = r$. *Let $d_i(G)$ be the density defined by Equation* 2.5. *Then*

$$d_i(G) = \frac{N(t, u, t-1-i)}{l^{u(t-1)+t-u}} \quad \textit{for } 0 \leqslant i \leqslant t-1.$$

We conclude this section by describing an algorithm for computing

$$N(t, u, t-1-i)/l^{u(t-1)+t-u}.$$

LEMMA 2.4: *Let $\Gamma = [\gamma_{ab}]$ be a $t \times t$ matrix over $\mathbf{F}_l$ of the form specified on the right side of Equation* 2.22. *Let $r =$* rank Γ *and $s =$* rank D. *Let $\Gamma_1 = [\gamma'_{ab}]$ be a $(t+1) \times (t+1)$ matrix over $\mathbf{F}_l$ such that $\gamma'_{ab} = \gamma_{ab}$ for $1 \leqslant a \leqslant t$ and $1 \leqslant b \leqslant t$. Also suppose that Γ_1 has the form specified on the right side of Equation* 2.22, *except with t replaced by $t+1$. Let D_1 denote the $(t+1-u) \times (t+1-u)$ diagonal submatrix in the lower right corner of Γ_1. Then of all possible Γ_1,*

(*i*) $l^{u+1} - l^u$ *have* rank $\Gamma_1 = r+1$ *and* rank $D_1 = s+1$;
(*ii*) $l^u - l^{r-s}$ *have* rank $\Gamma_1 = r+1$ *and* rank $D_1 = s$;
(*iii*) l^{r-s} *have* rank $\Gamma_1 = r$ *and* rank $D_1 = s$.

PROOF: Rank $D_1 = s+1 \Leftrightarrow \gamma'_{t+1,t+1} \neq 0$. So there are $l-1$ choices for $\gamma'_{t+1,t+1}$ that give rank $D_1 = s+1$. Since rank $D_1 = s+1$ implies rank $\Gamma_1 = r+1$, then the entries $\gamma'_{a,t+1}$ can be any elements of $\mathbf{F}_l$ for $1 \leqslant a \leqslant u$. So there are $(l-1)l^u$ matrices Γ_1 with rank $\Gamma_1 = r+1$ and rank $D_1 = s+1$. So (i) is proved. Now suppose rank $D_1 = s$; then $\gamma'_{t+1,t+1} = 0$. So there is one choice for $\gamma'_{t+1,t+1}$ in this case. Then rank $\Gamma_1 = r \Leftrightarrow$ the column vector

$$\begin{pmatrix} \gamma'_{1,t+1} \\ \vdots \\ \gamma'_{t,t+1} \end{pmatrix}$$

is a linear combination of the columns of $\Gamma \Leftrightarrow$ that column vector is a linear combination of the columns of Γ that do not contain the nonzero elements of D. Note that the columns of Γ that do not contain the nonzero elements of D span a space of dimension $r - s$ over $\mathbf{F}_l$. So (iii) is proved, and then (ii) follows easily. □

REMARK: Let $N(t, u, r, s)$ be the number of matrices Γ of the form specified on the right side of Equation 2.22 such that rank $\Gamma = r$ and rank $D = s$. Note that $N(u, u, r, 0)$ is the same as $N(u, u, r)$ in Proposition 2.3, which is the same as $N(u, r)$ in Proposition 2.1. Hence we know how to calculate $N(u, u, r, 0)$ and $N(u, u, r)$ for $u = 1, 2, 3, \ldots$ and $0 \leqslant r \leqslant u - 1$. Then we can use Lemma 2.4 inductively to compute $N(t, u, r, s)$ for $t = u + 1, u + 2, \ldots$ and each possible r and s. Then we can calculate $N(t, u, r)$ in Proposition 2.3 as follows:

$$N(t, u, r) = \sum_s N(t, u, r, s).$$

We can formulate the calculation of $d_l(G)$ in terms of a denumerable Markov process (cf. [5], Section 4). We let u be a fixed positive integer, and we let

$$g_{t,u,i} = N(t, u, t-1-i)/l^{u(t-1)+t-u} \tag{2.32}$$

for $t = u, u+1, u+2, \ldots$ and $0 \leqslant i \leqslant t-1$. From the above remark $N(u, u, u-1-i) = N(u, u-1-i)$, and hence from Proposition 2.1 and Corollary 2.2, we get

$$g_{u,u,i} = \frac{l^{-i(i+1)}\left[\prod_{k=i+1}^{u-1}(1-l^{-k-1})(1-l^{-k})\right]}{\left[\prod_{j=1}^{u-1-i}(1-l^{-j})\right]} \tag{2.33}$$

for $u = 1, 2, 3, \ldots$ and $0 \leqslant i \leqslant u-1$. Next we let

$$x_{t,u,(i,w_i)} = N(t, u, t-1-i, t-u-w_i)/l^{u(t-1)+t-u} \tag{2.34}$$

for $t = u, u+1, u+2, \ldots$; $i = 0, 1, 2, \ldots$; and $0 \leqslant w_i \leqslant i+1$. We note that $x_{t,u,(i,w_i)} = 0$ if $i > t-1$ or $w_i > t-u$. Also

$$g_{t,u,i} = \sum_{w_i} x_{t,u,(i,w_i)}. \tag{2.35}$$

If we divide each term in Lemma 2.4 by l^{u+1}, and if we let $r = t-1-i$, $s = t-u-w_i$, $j = (t+1) - 1 - \text{rank } \Gamma_1$, and $w_j = (t+1) - u - \text{rank } D_1$, then we get Markov Process E_u that appears in the appendix. Markov Process E_u can be used to calculate the quantities $x_{t,u,(i,w_i)}$, and then Proposition 2.3 and Equations 2.32 and 2.35 can be used to calculate $d_i(G)$.

3. Case $l > 2$ and varying genus groups

We let notations be the same as in Section 2. We define (cf. Equations 2.1 through 2.5)

$$B_t = \{\text{cyclic extensions } K \text{ of } \mathbf{Q} \text{ of degree } l^n$$

$$\text{with exactly } t \text{ primes of } \mathbf{Q} \text{ ramified in } K/\mathbf{Q}\} \tag{3.1}$$

$$B_{t;x} = \{K \in B_t : \text{the conductor of } K \text{ is } \leqslant x\} \tag{3.2}$$

$$B_{t,i} = \{K \in B_t : R_K = i\} \text{ (cf. Equation 1.1)} \tag{3.3}$$

$$B_{t,i;x} = \{K \in B_{t,i} : \text{the conductor of } K \text{ is } \leqslant x\} \tag{3.4}$$

$$d_{t,i} = \lim_{x \to \infty} \frac{|B_{t,i;x}|}{|B_{t;x}|}. \tag{3.5}$$

So $d_{t,i}$ is a density involving the cyclic extensions of $\mathbf{Q}$ of degree l^n with a given number of ramified primes (rather than a given genus group). We note that

$$|B_{t,i;x}| = \sum_G |A_i(G)_x| \tag{3.6}$$

and

$$|B_{t;x}| = \sum_G |A(G)_x|, \tag{3.7}$$

where $\sum_G$ denotes a sum over all finite abelian groups of type $(l^{n_2}, \ldots, l^{n_t})$, where $n_2, \ldots, n_t$ range over all integers such that $n = n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t \geqslant 1$. From Formulae 2.8, 2.9, 2.31, 3.5, 3.6, and 3.7, we get

$$d_{t,l} = \frac{\sum'_{v_1, \ldots, v_n} \frac{N(t, u, t-1-i)}{l^{u(t-1)+t-u}} \cdot \frac{t!}{(v_1!) \ldots (v_n!)}}{\sum'_{v_1, \ldots, v_n} \frac{t!}{(v_1!) \ldots (v_n!)}} \tag{3.8}$$

where $\Sigma'_{v_1, \ldots, v_n}$ is a sum over all nonnegative integers $v_1, \ldots, v_n$ such that $v_1 + \cdots + v_n = t$ and $v_n \geqslant 1$. We also recall that $u = v_n$. If $n = 1$, then $u = v_n = t$, and hence

$$d_{t,l} = \frac{N(t, t, t-1-i)}{l^{t(t-1)}}. \tag{3.9}$$

Since $N(t, t, t-1-i) = N(t, t-1-i)$, then we can use Proposition 2.1 and Corollary 2.2 to calculate $d_{t,l}$ in Equation 3.9. So when $n = 1$, our calculation is complete.

Now suppose $n > 1$. Let $\binom{t}{u} = (t!)/(u!)((t-u)!)$. Then

$$d_{t,l} = \frac{\sum_{u=1}^{t} \frac{N(t, u, t-1-i)}{l^{u(t-1)+t-u}} \cdot \binom{t}{u} \cdot \sum'_{v_1, \ldots, v_{n-1}} \frac{(t-u)!}{(v_1!) \ldots (v_{n-1}!)}}{\sum_{u=1}^{t} \binom{t}{u} \sum'_{v_1, \ldots, v_{n-1}} \frac{(t-u)!}{(v_1!) \ldots (v_{n-1}!)}}$$

where $\sum'_{v_1, \ldots, v_{n-1}}$ is a sum over all nonnegative integers $v_1, \ldots, v_{n-1}$ such that $v_1 + \cdots + v_{n-1} = t - u$. Now the terms $(t-u)!/[(v_1!) \ldots (v_{n-1}!)]$ are multinomial coefficients, and hence

$$\sum'_{v_1, \ldots, v_{n-1}} \frac{(t-u)!}{(v_1!) \ldots (v_{n-1}!)} = (n-1)^{t-u}.$$

So

$$d_{t,l} = \frac{\sum_{u=1}^{t} \frac{N(t, u, t-1-i)}{l^{u(t-1)+t-u}} \cdot \binom{t}{u} \cdot (n-1)^{t-u}}{\sum_{u=1}^{t} \binom{t}{u} (n-1)^{t-u}}. \tag{3.10}$$

Now $\sum_{u=0}^{t}\binom{t}{u}(n-1)^{t-u}=n^t$; hence

$$\sum_{u=1}^{t}\binom{t}{u}(n-1)^{t-u}=n^t-(n-1)^t.$$

If we divide both numerator and denominator of Equation 3.10 by n^t, we get

$$d_{t,i}=\frac{\sum_{u=1}^{t}\frac{N(t,u,t-1-i)}{l^{u(t-1)+t-u}}\cdot\binom{t}{u}\cdot\frac{(n-1)^{t-u}}{n^t}}{1-\left(\frac{n-1}{n}\right)^t} \tag{3.11}$$

Thus we have the following result.

THEOREM 3.1: *Let l be an odd prime number, and let n and t be positive integers. If u and r are integers with $1 \leqslant u \leqslant t$ and $0 \leqslant r \leqslant t-1$, let $N(t, u, r)$ be the number of $t \times t$ matrices Γ over $\mathbf{F}_l$ of the form specified on the right side of Equation* 2.22 *such that* rank $\Gamma = r$. *Let $d_{t,i}$ be the density defined by Equation* 3.5 *for $0 \leqslant i \leqslant t-1$. Then $d_{t,i}$ is given by the formula in Corollary* 2.2 *if $n = 1$, and $d_{t,i}$ is given by Equation* 3.11 *if $n > 1$.*

REMARK: The quantities $N(t, u, t-1-i)/l^{u(t-1)+t-u}$ can be calculated using Markov Process E_u and Equations 2.32 and 2.35 for $1 \leqslant u \leqslant t$. Tables 4 and 5 in the appendix contain values of $d_{t,i}$ for $l = 3$ and 5.

We now examine what happens to $d_{t,i}$ as $t \to \infty$ (cf. [6] when $n = 1$).

THEOREM 3.2: *Let notations be as in Theorem* 3.1. *If $n = 1$, then*

$$\lim_{t\to\infty} d_{t,i}=\frac{l^{-i(i+1)}\prod_{k=1}^{\infty}(1-l^{-k})}{\left[\prod_{k=1}^{i}(1-l^{-k})\right]\left[\prod_{k=1}^{i+1}(1-l^{-k})\right]} \quad \textit{for } i=0,1,2,\ldots.$$

If $n > 1$, then $\lim_{t\to\infty} d_{t,i}=0$ *for $i = 0, 1, 2, \ldots$.*

PROOF: The proof when $n = 1$ follows easily from the formula in Corollary 2.2. So suppose $n > 1$. We rewrite Equation 3.11 as follows:

$$d_{t,i}=\frac{1}{a}\sum_{u=1}^{t} g_{t,u,i}h_{t,u} \tag{3.12}$$

where $a = 1 - [(n-1)/n]^t$; $g_{t,u,i}$ is given by Equation 2.32; and

$$h_{t,u} = \binom{t}{u}\left(\frac{1}{n}\right)^u \left(\frac{n-1}{n}\right)^{t-u}. \tag{3.13}$$

Note that $g_{t,u,i}$ has been defined in Equation 2.32 for $u \geqslant 1$; we now extend its definition in Equation 2.32 to include the case $u = 0$. (The matrices are diagonal $t \times t$ matrices when $u = 0$.) So

$$d_{t,i} \leqslant \frac{1}{a} \sum_{u=0}^{t} g_{t,u,i} h_{t,u}. \tag{3.14}$$

Let $\epsilon > 0$ be arbitrary. We want to show $d_{t,i} < \epsilon$ for all sufficiently large t. Now there exists $T_1 > 0$ such that for $t \geqslant T_1$, $a > 1/2$. So for $t \geqslant T_1$, we have

$$d_{t,i} < 2 \sum_{u=0}^{t} g_{t,u,i} h_{t,u}. \tag{3.15}$$

Next we let U_t be the random variable with possible values $u = 0, 1, \ldots, t$ and with $\text{prob}(U_t = u) = h_{t,u}$. So U_t is a binomial random variable with expected value $t(1/n)$ and standard deviation $\sqrt{t(1/n)((n-1)/n)}$. For sufficiently large t, U_t is approximately normally distributed, and the standard deviation is much smaller than the expected value. So there exists $T_2 > 0$ such that for $t \geqslant T_2$, $\text{prob}(U_t > 3t/4) < \epsilon/4$. (Note that the expected value is $\leqslant t/2$ since $n \geqslant 2$.) So for $t \geqslant \max(T_1, T_2)$,

$$\begin{aligned} d_{t,i} &< 2 \sum_{u \leqslant 3t/4} g_{t,u,i} h_{t,u} + 2 \sum_{u > 3t/4} g_{t,u,i} h_{t,u} \\ &\leqslant 2 \sum_{u \leqslant 3t/4} g_{t,u,i} h_{t,u} + 2 \sum_{u > 3t/4} h_{t,u} \quad (\text{since } g_{t,u,i} \leqslant 1) \\ &< 2 \sum_{u \leqslant 3t/4} g_{t,u,i} h_{t,u} + \epsilon/2. \end{aligned}$$

Thus it suffices to show that

$$\sum_{u \leqslant 3t/4} g_{t,u,i} h_{t,u} < \epsilon/4$$

for all sufficiently large t. Now recall that

$$g_{t,u,i} = N(t, u, t-1-i)/l^{u(t-1)+t-u} \tag{3.16}$$

and

$$N(t, u, t-1-i) = \sum_{s=0}^{t-u} N(t, u, t-1-i, s). \tag{3.17}$$

From the form of the matrix on the right side of Equation 2.22, we see that $N(t, u, t-1-i, s) = 0$ if $s < t-1-i-u$. So we need $s \geqslant t-1-i-u$. Also note that

$$\frac{N(t, u, t-1-i, s)}{l^{u(t-1)+t-u}} \leqslant \frac{N(\text{rk } D = s)}{l^{t-u}} \tag{3.18}$$

where $N(\text{rk } D = s)$ denotes the number of $(t-u) \times (t-u)$ diagonal matrices D over $\mathbf{F}_l$ with rank $D = s$. Also note that

$$N(\text{rk } D = s) = \binom{t-u}{s}(l-1)^s(1)^{t-u-s} = \binom{t-u}{s}(l-1)^s. \tag{3.19}$$

Let S_{t-u} be the random variable with possible values $s = 0, 1, \ldots, t-u$ and with $\text{prob}(S_{t-u} = s) = q_{t-u,s}$, where

$$q_{t-u,s} = \binom{t-u}{s}\left(\frac{l-1}{l}\right)^s\left(\frac{1}{l}\right)^{t-u-s}. \tag{3.20}$$

Then the expected value of S_{t-u} is $(t-u)(l-1)/l$, and the standard deviation of S_{t-u} is $\sqrt{(t-u)(l-1)/l^2}$. For large $t-u$, the standard deviation is much smaller than the expected value. So there exists $T_3 > 0$ such that for $t-u \geqslant T_3$, we have

$$\sum_{s \geqslant t-u-1-i} q_{t-u,s} < \epsilon/4.$$

Now if $u \leqslant 3t/4$ and $t \geqslant 4T_3$, then $t-u \geqslant T_3$. So for $t \geqslant 4T_3$, we have from Formulae 3.16 through 3.20.

$$\begin{aligned}
\sum_{u \leqslant 3t/4} g_{t,u,i} h_{t,u} &= \sum_{u \leqslant 3t/4} \left(\sum_{s=0}^{t-u} \frac{N(t, u, t-1-i, s)}{l^{u(t-1)+t-u}} \right) h_{t,u} \\
&\leqslant \sum_{u \leqslant 3t/4} \left(\sum_{s \geqslant t-u-1-i} \frac{N(\text{rk } D = s)}{l^{t-u}} \right) h_{t,u} \\
&= \sum_{u \leqslant 3t/4} \left(\sum_{s \geqslant t-u-1-i} q_{t-u,s} \right) h_{t,u} \\
&< \frac{\epsilon}{4} \sum_{u \leqslant 3t/4} h_{t,u} < \frac{\epsilon}{4}
\end{aligned}$$

since $\sum_{u=0}^{t} h_{t,u} = 1$. So for $t \geqslant \max(T_1, T_2, 4T_3)$, we have $d_{t,i} < \epsilon$. □

The results in Theorem 3.2 are interesting for several reasons. First the difference between the cases $n = 1$ and $n > 1$ is quite substantial. Second, the case $n = 1$ has an interesting relationship to conjectures of Cohen and Lenstra. For cyclic extensions K of $\mathbf{Q}$ of degree l^n, Cohen and Lenstra have made conjectures concerning the prime-to-l-part of the ideal class group of K. Since we have been dealing with the l-part of the class group, the Cohen-Lenstra Conjectures do not apply directly to our analysis. However when $n = 1$, our formula for $\lim_{t\to\infty} d_{t,i}$ is the same type of formula that appears in Theorem 6.3 of [1]. This suggests that the Cohen–Lenstra Conjectures could be extended to include the l-part of the principal genus when $n = 1$. When $n > 1$, the situation is more complicated. For the prime-to-l-part of the ideal class group, the Cohen-Lenstra Conjectures deal with finite modules over the product of rings of integers of $\mathbf{Q}(\zeta_{l^i})$ for $1 \leqslant i \leqslant n$, where ζ_{l^i} is a primitive l^i-th root of unity. However for the l-part of the principal genus, it would seem appropriate to consider finite modules over the ring

$$\Lambda = \mathbf{Z}[x]/\left(1 + x + \cdots + x^{l^n - 1}\right)\mathbf{Z}[x]$$

where $\mathbf{Z}[x]$ is the polynomial ring over $\mathbf{Z}$ in one indeterminate x. One might then make a fundamental assumption for the l-part of the principal genus analogous to the second part of Fundamental Assumptions 8.1 on p. 54 of [1], except with Λ modules of order a power of l. We are not sufficiently familiar with the module theory for the ring Λ to know whether the resulting conjectures are consistent with our Theorem 3.2 when $n > 1$. However our results should provide a starting point for future research.

We conclude this section with another limit formula. We start from Equation 3.10, and we note that $d_{t,i}$ is a function of n, although in our previous calculations we have assumed that n was fixed. We shall write $d_{t,i}(n)$ for $d_{t,i}$, and we shall calculate $\lim_{n\to\infty} d_{t,i}(n)$ for fixed t and i. From Equation 3.10 it is clear that as n becomes large, the dominant terms in both the numerator and denominator are the terms with $u = 1$. If we divide both numerator and denominator by $(n-1)^{t-1}$ and let $n \to \infty$, we get

$$\lim_{n\to\infty} d_{t,i}(n) = \frac{N(t, 1, t-1-i)}{l^{2(t-1)}}. \tag{3.21}$$

To evaluate $N(t, 1, t-1-i)$, we note that the matrices we are considering have the form specified on the right side of Equation 2.22 with $u = 1$. So M_1 is the 1×1 zero matrix, and hence it suffices to consider matrices

$$\Gamma = \begin{pmatrix} M_2 \\ D \end{pmatrix} \tag{3.22}$$

where M_2 is a $1 \times (t-1)$ matrix, and D is a $(t-1) \times (t-1)$ diagonal matrix.

We let $s = t - 1 - i$. From the form of the matrices in Equation 3.22, it is easy to see that

$$N(t, 1, s) = N(\text{rk } D = s) \cdot l^s \cdot 1^{t-1-s} + N(\text{rk } D = s - 1) \cdot l^{s-1} \cdot (l^{t-s} - 1) \tag{3.23}$$

where $N(\text{rk } D = s)$ is given by Equation 3.19 with $u = 1$. For $s = 0$ (and hence $i = t - 1$), we have $N(\text{rk } D = s - 1) = 0$, and then $N(t, 1, 0) = 1$. For $s = t - 1 - i > 0$, we can use Equations 3.19, 3.21, and 3.23 to obtain the following result.

PROPOSITION 3.3: *Let notations be the same as in Theorem* 3.1, *except that we write* $d_{t,i}(n)$ *instead of* $d_{t,i}$. *Then*
(*i*) *if* $i = t - 1$, *then*

$$\lim_{n \to \infty} d_{t,i}(n) = l^{-2t+2};$$

(*ii*) *if* $0 \leqslant i \leqslant t - 2$, *then*

$$\lim_{n \to \infty} d_{t,i}(n) = \binom{t-1}{i}(1 - l^{-1})^{t-1-i} l^{-2i} + \binom{t-1}{i+1}(1 - l^{-1})^{t-2-i}(1 - l^{-i-1}) l^{-i-1}.$$

REMARK: The quantity $\lim_{n \to \infty} d_{t,i}(n)$ is not related to the behavior of the l-rank of the ideal class group in a $\mathbf{Z}_l$-extension. (Recall that a $\mathbf{Z}_l$-extension of $\mathbf{Q}$ is an infinite Galois extension of $\mathbf{Q}$ whose Galois group is isomorphic to the additive group of the l-adic integers $\mathbf{Z}_l$.) For fixed l, there is a unique $\mathbf{Z}_l$-extension of $\mathbf{Q}$; only the prime l ramifies in this $\mathbf{Z}_l$-extension; and the l-rank of the ideal class group is zero for every subextension of the $\mathbf{Z}_l$-extension.

4. Case $l = 2$

In this section we consider Galois extensions $K/\mathbf{Q}$ of degree 2^n with cyclic Galois groups. We shall sketch the results for these extensions that are analogous to results which appear in Sections 2 and 3 for the case $l > 2$. Most of the details will be omitted. For quadratic fields (i.e., for $n = 1$) we have already obtained the appropriate analogs in [5]. So we assume $n \geqslant 2$. We shall consider separately the complex extensions of degree 2^n and the real extensions of degree 2^n. Although the density results in [5] are different for complex quadratic fields and real quadratic fields, we shall see that the density results are the same for complex fields of degree 2^n and real fields of degree 2^n if $n \geqslant 2$. We first focus on complex fields of degree 2^n. We let t be a positive integer, and we let $n_1, n_2, \ldots, n_t$ be integers such that $n = n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t \geqslant 1$. We let G be the abelian group of type $(2^{n_2}, \ldots, 2^{n_t})$. (When $t = 1$, we let G be the trivial group.) We define

$$A(G) = \{\text{complex cyclic extensions } K \text{ of } \mathbf{Q} \text{ of degree } 2^n \text{ with genus group isomorphic to } G\}. \tag{4.1}$$

For each nonnegative integer i and each positive real number x, and with $A(G)$ given by Equation 4.1, we define $A(G)_x$, $A_i(G)$, and $A_i(G)_x$ exactly as in Equations 2.2, 2.3, and 2.4. With this $A(G)_x$ and $A_i(G)_x$, we define

$$d_i(G) = \lim_{x \to \infty} \frac{|A_i(G)_x|}{|A(G)_x|}. \tag{4.2}$$

We then proceed to obtain the following analog of Formula 2.9.

$$|A(G)_x| \sim \frac{1}{2} \cdot \frac{1}{2^n - 2^{n-1}} \cdot \frac{t!}{(v_1!) \ldots (v_n!)} \cdot \frac{x(\log\log x)^{t-1}}{(t-1)! \log x} \quad \text{as } x \to \infty. \tag{4.3}$$

Here the v_i's are defined by Equation 2.8, and the factor $1/2$ is introduced since asymptotically only one-half of the cyclic extensions K of $\mathbf{Q}$ of degree 2^n are complex.

Before evaluating $|A_i(G)_x|$ for arbitrary G, we first consider $G = G_1$, where G_1 is the abelian group of type $(2^{n_2}, \ldots, 2^{n_t})$ with $n_2 = \cdots = n_t = n$. Then we can obtain the $t \times t$ matrix M_K over $\mathbf{F}_2$ whose entries m_{ab} are given by Equation 2.15, and as in the case $l > 2$, the sum of the entries in each row of M_K is zero in $\mathbf{F}_2$. We now have reached a point where the analysis of the case $l = 2$ differs significantly from the analysis of the case $l > 2$. The matrix M_K has certain special properties when $l = 2$ and $n \geqslant 2$. First each $z_b \equiv 1 (\text{mod } 2)$. Furthermore, since $n \geqslant 2$ and each $p_a \equiv 1 (\text{mod } 2^n)$, then the quadratic reciprocity law implies that $[p_b, p_a] = [p_a, p_b]$ for all a and b. So the matrix M_K is symmetric. Since the sum of the entries in each row of M_K is zero and since M_K is symmetric, then the sum of the entries in each column of M_K is zero. So M_K is uniquely determined by its $(t-1) \times (t-1)$ submatrix whose entries are m_{ab} for $2 \leqslant a \leqslant t$ and $2 \leqslant b \leqslant t$. Furthermore, since this submatrix is symmetric, M_K is actually determined by its entries m_{ab} with $2 \leqslant a \leqslant b \leqslant t$. Note that there are $t(t-1)/2$ such elements. Let $N'(t, r)$ denote the number of symmetric $t \times t$ matrices Γ over $\mathbf{F}_2$ such that the sum of the entries in each row of Γ is zero and such that rank $\Gamma = r$. Note that

$N'(t, r)$ also represents the number of $(t-1)\times(t-1)$ symmetric matrices over $\mathbf{F}_2$ with rank $= r$. Then we can obtain the following analog of Formula 2.21.

$$|A_i(G_1)_x| \sim \frac{N'(t, t-1-i)}{2^{t(t-1)/2}}\cdot\frac{1}{2}\cdot\frac{1}{2^n-2^{n-1}}\cdot\frac{x(\log\log x)^{t-1}}{(t-1)!\log x}$$

$$\text{as } x\to\infty. \tag{4.4}$$

Then combining Formula 4.2, 4.3, and 4.4, we get the following result.

PROPOSITION 4.1 (Complex fields): *Let $n\geqslant 2$ and $t\geqslant 1$ be integers, and let G_1 be the abelian group of type $(2^{n_2}, 2^{n_3},\dots,2^{n_t})$ with $n_2=n_3=\cdots=n_t=n$. (If $t=1$, we let G_1 be the trivial group.) Let $N'(t, r)$ be the number of $t\times t$ symmetric matrices Γ over $\mathbf{F}_2$ such that the sum of the entries in each row of Γ is zero and such that* rank $\Gamma=r$, *where $0\leqslant r\leqslant t-1$. Let $d_t(G_1)$ be the density defined by Equation* 4.2. *Then*

$$d_t(G_1)=N'(t, t-1-i)/2^{t(t-1)/2}\quad \textit{for } 0\leqslant i\leqslant t-1.$$

Since $N'(t, r)$ also represents the number of $(t-1)\times(t-1)$ symmetric matrices over $\mathbf{F}_2$ with rank = r, we can use Lemma 18 of [9] to compute $N'(t, t-1-i)$. Then after several intermediate calculations, we get the following corollary of Proposition 4.1.

COROLLARY 4.2: *Let notations be the same as in Proposition* 4.1. *Then*

$$d_t(G_1)=\frac{2^{-i(i+1)/2}\displaystyle\prod_{k=i+1}^{t-1}(1-2^{-k})}{\displaystyle\prod_{j=1}^{[(t-1-i)/2]}(1-2^{-2j})}$$

where $[y]$ denotes the greatest integer $\leqslant$ the real number y.

We now return to the general case of an abelian group G of type $(2^{n_2},\dots,2^{n_t})$ with $n=n_1\geqslant n_2\geqslant\cdots\geqslant n_t\geqslant 1$. We let $u=v_n$ (cf. Equation 2.8). Now the analog of Equation 2.22 is

$$M_K=\begin{pmatrix} M_1 & M_2\\ O & D\end{pmatrix} \tag{4.5}$$

where M_1 is a symmetric $u\times u$ matrix over $\mathbf{F}_2$ in which the sum of the entries in each row is zero; M_2 is a $u\times(t-u)$ matrix over $\mathbf{F}_2$; O is the $(t-u)\times u$

zero matrix; and D is the $(t-u)\times(t-u)$ diagonal matrix over $\mathbf{F}_2$ with each diagonal entry equal to the sum of the entries in the corresponding column of M_2. Note that $M_K=[m_{ab}]$ is uniquely determined by its $(t-1)\times(t-1)$ submatrix whose entries are m_{ab} for $2\leqslant a\leqslant t$ and $2\leqslant b\leqslant t$. We note that there are $2^{u(u-1)/2+u(t-u)}$ matrices of the form specified on the right side of Equation 4.5. We let $N'(t, u, r)$ denote the number of matrices Γ of the form specified on the right side of Equation 4.5 such that rank $\Gamma=r$, where $0\leqslant r\leqslant t-1$. Then the analog of Formula 2.31 is

$$|A_t(G)_x| \sim \frac{N'(t, u, t-1-i)}{2^{u(u-1)/2+u(t-u)}}\cdot\frac{1}{2}\cdot\frac{1}{2^n-2^{n-1}}\cdot\frac{t!}{(v_1!)\dots(v_n!)}$$

$$\cdot\frac{x(\log\log x)^{t-1}}{(t-1)!\log x}\quad\text{as } x\to\infty. \tag{4.6}$$

Then from Formulae 4.2, 4.3, and 4.6, we obtain the following result.

PROPOSITION 4.3 (Complex fields): *Let $n\geqslant 2$ and $t\geqslant 1$ be integers, and let G be the abelian group of type $(2^{n_2},\dots,2^{n_t})$ with $n=n_1\geqslant n_2\geqslant\cdots\geqslant n_t\geqslant 1$. (If $t=1$, we let G be the trivial group.) Let u be the largest integer such that $n_u=n$, and let r be an integer with $0\leqslant r\leqslant t-1$. Let $N'(t, u, r)$ be the number of $t\times t$ matrices Γ over $\mathbf{F}_2$ of the form specified on the right side of Equation 4.5 such that* rank $\Gamma=r$. *Let $d_i(G)$ be the density defined by Equation 4.2. Then*

$$d_i(G)=\frac{N'(t, u, t-1-i)}{2^{u(u-1)/2+u(t-u)}}\quad \textit{for } 0\leqslant i\leqslant t-1.$$

To use Proposition 4.3, we need an algorithm for computing $N'(t, u, t-1-i)/2^{u(u-1)/2+u(t-u)}$. The analog of Lemma 2.4 is the following lemma.

LEMMA 4.4: *Let $\Gamma=[\gamma_{ab}]$ be a $t\times t$ matrix over $\mathbf{F}_2$ of the form specified on the right side of Equation 4.5. Let $r=$* rank Γ *and $s=$* rank D. *Let $\Gamma_1=[\gamma'_{ab}]$ be a $(t+1)\times(t+1)$ matrix over $\mathbf{F}_2$ such that $\gamma'_{ab}=\gamma_{ab}$ for $1\leqslant a\leqslant t$ and $1\leqslant b\leqslant t$. Also suppose that Γ_1 has the form specified on the right side of Equation 4.5, except with t replaced by $t+1$. Let D_1 denote the $(t+1-u)\times(t+1-u)$ diagonal submatrix in the lower right corner of Γ_1. Then of all possible Γ_1,*

(*i*) 2^{u-1} *have* rank $\Gamma_1=r+1$ *and* rank $D_1=s+1$;
(*ii*) $2^{u-1}-2^{r-s}$ *have* rank $\Gamma_1=r+1$ *and* rank $D_1=s$;
(*iii*) 2^{r-s} *have* rank $\Gamma_1=r$ *and* rank $D_1=s$.

Next let

$$g'_{t,u,i}=N'(t, u, t-1-i)/2^{u(u-1)/2+u(t-u)}. \tag{4.7}$$

Since $N'(u, u, u-1-i) = N'(u, u-1-i)$, then Proposition 4.1 and Corollary 4.2 imply

$$g'_{u,u,i} = \frac{2^{-i(i+1)/2} \prod_{k=i+1}^{u-1} (1-2^{-k})}{\prod_{j=1}^{[(u-1-i)/2]} (1-2^{-2j})} \tag{4.8}$$

for $u = 1, 2, 3, \ldots$ and $0 \leqslant i \leqslant u-1$. Now analogous to Markov Process E_u, we can create Markov Process E'_u. (See the appendix for details.) Then

$$g'_{t,u,i} = \sum_{w_i} x'_{t,u,(i,w_i)}. \tag{4.9}$$

So we can use Markov Process E'_u to calculate the quantities $x'_{t,u,(i,w_i)}$, and then Proposition 4.3 and Equations 4.7 through 4.9 can be used to calculate $d_i(G)$.

Our next goal is to present the appropriate analogs of the results in Section 3. We let

$$B_t = \{\text{complex cyclic extensions } K \text{ of } \mathbf{Q} \text{ of degree } 2^n \text{ with exactly } t \text{ finite primes of } \mathbf{Q} \text{ ramified in } K/\mathbf{Q}\}. \tag{4.10}$$

For each nonnegative integer i and each positive real number x, and with B_t given by Equation 4.10, we define $B_{t;x}$, $B_{t,i}$, and $B_{t,i;x}$ exactly as in Equations 3.2, 3.3, and 3.4. With this $B_{t;x}$ and $B_{t,i;x}$, we define

$$d_{t,i} = \lim_{x\to\infty} \frac{|B_{t,i;x}|}{|B_{t;x}|}. \tag{4.11}$$

Then we can proceed to obtain the following analog of Theorem 3.1 and Theorem 3.2.

THEOREM 4.5 (Complex fields): *Let $n \geqslant 2$ and $t \geqslant 1$ be integers. If u is a positive integer such that $u \leqslant t$ and if $0 \leqslant r \leqslant t-1$, we let $N'(t, u, r)$ be the number of $t \times t$ matrices Γ over $\mathbf{F}_2$ of the form specified on the right side of Equation* 4.5 *such that* rank $\Gamma = r$. *Let $d_{t,i}$ be the density defined by Equation* 4.11 *for $0 \leqslant i \leqslant t-1$. Then*

$$d_{t,i} = \frac{\sum_{u=1}^{t} \frac{N'(t, u, t-1-i)}{2^{u(u-1)/2+u(t-u)}} \cdot \binom{t}{u} \cdot \frac{(n-1)^{t-u}}{n^t}}{1-\left(\frac{n-1}{n}\right)^t}.$$

Furthermore $\lim_{t\to\infty} d_{t,i} = 0$ *for* $i = 0, 1, 2, \ldots$.

REMARK: The quantities $N'(t, u, t-1-i)/2^{u(u-1)/2+u(t-u)}$ can be calculated by using Markov Process E'_u and Equations 4.7 and 4.9 for $1 \leqslant u \leqslant t$.

Finally for the complex fields of degree 2^n, we can obtain the following analog of Proposition 3.3.

PROPOSITION 4.6: *Let notations be the same as in Theorem 4.5, except that we write $d_{t,i}(n)$ instead of $d_{t,i}$. Then*

$$\lim_{n\to\infty} d_{t,i}(n) = \binom{t-1}{i} \cdot 2^{-(t-1)} \text{ for } 0 \leqslant i \leqslant t-1.$$

Table 1. Values of $d_{t,i}$ for imaginary quadratic fields.

t \ i	0	1	2	3	4
1	1.0				
2	0.5	0.5			
3	0.4375	0.46875	0.09375		
4	0.375	0.515625	0.101563	0.007813	
5	0.350586	0.523682	0.117188	0.008240	0.000305
⋮					
10	0.300759	0.565305	0.126574	0.007182	0.000178
⋮					
20	0.289408	0.576950	0.128222	0.005365	0.000055
⋮					
∞	0.288788	0.577576	0.128350	0.005239	0.000047

Table 2. Values of $d_{t,i}$ for real quadratic fields.

t \ i	0	1	2	3	4
1	1.0				
2	0.75	0.25			
3	0.6875	0.28125	0.03125		
4	0.648438	0.3125	0.037109	0.001953	
5	0.627930	0.328369	0.041504	0.002136	6.1×10^{-5}
⋮					
10	0.588735	0.370372	0.039456	0.001410	2.7×10^{-5}
⋮					
20	0.578191	0.384230	0.036831	0.000745	4.2×10^{-6}
⋮					
∞	0.577576	0.385051	0.036672	0.000699	3.0×10^{-6}

Table 3. Values of $d_{t,i}$ for fields of degree 2^n, $n \geqslant 2$.

n	t \ i	0	1	2	3	4
$\geqslant 2$	1	1.0				
$\geqslant 2$	2	0.5	0.5			
2	3	0.285714	0.535714	0.178571		
2	4	0.179167	0.470833	0.298958	0.051042	
2	5	0.119960	0.393775	0.356288	0.116558	0.013420
2	10	0.024077	0.144000	0.290773	0.287104	0.168142
2	20	0.001330	0.014635	0.059755	0.132808	0.194300
3	3	0.263158	0.532895	0.203947		
3	4	0.145192	0.441346	0.338221	0.075240	
3	5	0.083531	0.335919	0.384331	0.169140	0.027080
3	10	0.007894	0.065905	0.190206	0.276559	0.243540
3	20	0.000127	0.002024	0.012248	0.040940	0.090652
4	3	0.256757	0.527027	0.216216		
4	4	0.135357	0.424643	0.352054	0.087946	
4	5	0.073183	0.309924	0.388087	0.193449	0.035356
4	10	0.004590	0.043727	0.145917	0.249445	0.260704
4	20	0.000035	0.000668	0.004818	0.019308	0.051395
5	3	0.254098	0.522541	0.223361		
5	4	0.131267	0.414465	0.358698	0.095571	
5	5	0.068926	0.295739	0.387845	0.206875	0.040615
5	10	0.003428	0.034550	0.123795	0.230500	0.264137
5	20	0.000016	0.000333	0.002650	0.011762	0.034760
$\geqslant 2$	∞	0	0	0	0	0

We now consider the real fields of degree 2^n. If we replace the word "complex" by "real" in Equation 4.1, we see that all subsequent results are still valid. (Recall that in this paper all references to the "genus group" actually refer to the "narrow genus group".) So we have the following theorem.

THEOREM 4.7 (Real fields): *Let $n \geqslant 2$ be an integer. Then the statements of Proposition* 4.1, *Corollary* 4.2, *Proposition* 4.3, *Theorem* 4.5, *and Proposition* 4.6 *are valid if we use real cyclic extensions of* $\mathbf{Q}$ *of degree* 2^n *instead of complex cyclic extensions of* $\mathbf{Q}$ *of degree* 2^n.

REMARK: For quadratic extensions of $\mathbf{Q}$ (i.e., the case $n = 1$), the density results are different in the complex and real cases because the properties of the associated matrices are different. (See [5] for details.) Some numerical values for $d_{t,i}$ appear in Tables 1, 2, and 3. The results in Table 3 apply to both the complex and real cases.

Appendix

Markov Process E_u ($u = 1, 2, 3, \ldots$):

States $x_{t,u,(i,w_i)}$ with $t = u, u+1, u+2, \ldots$; $i = 0, 1, 2, \ldots$; and $0 \leqslant w_i \leqslant i + 1$. Let $X_{t,u} = (x_{t,u,(0,0)}, \ldots, x_{t,u,(i,w_i)}, \ldots)$, where the component $x_{t,u,(i,w_i)}$ precedes $x_{t,u,(j,w_j)}$ if $i < j$, or if $i = j$ and $w_i < w_j$. Then $X_{t+1,u} = X_{t,u}Q_{E_u}$, where

$$Q_{E_u} = \left[q_{(i,w_i),(j,w_j)}\right]$$

with $i = 0, 1, 2, \ldots$; $j = 0, 1, 2, \ldots$; $0 \leqslant w_i \leqslant i+1$; $0 \leqslant w_j \leqslant j+1$;

$$q_{(i,w_i),(j,w_j)} = \begin{cases} 1 - l^{-1} & \text{if } j = i,\ w_j = w_i \\ l^{-1} - l^{-i+w_i-2} & \text{if } j = i,\ w_j = w_i + 1 \\ l^{-i+w_i-2} & \text{if } j = i+1,\ w_j = w_i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Table 4. Values of $d_{t,i}$ for fields of degree 3^n.

n	t \ i	0	1	2	3	4
$\geqslant 1$	1	1.0				
$\geqslant 1$	2	0.888889	0.111111			
1	3	0.855967	0.142661	0.001372		
1	4	0.845400	0.152642	0.001957	1.9×10^{-6}	
1	5	0.841921	0.155911	0.002165	2.8×10^{-6}	2.9×10^{-10}
1	10	0.840196	0.157529	0.002272	3.3×10^{-6}	5.1×10^{-10}
1	20	0.840189	0.157536	0.002272	3.3×10^{-6}	5.1×10^{-10}
1	∞	0.840189	0.157535	0.002272	3.3×10^{-6}	5.1×10^{-10}
2	3	0.778366	0.214384	0.007251		
2	4	0.680655	0.292382	0.026531	0.000431	
2	5	0.597154	0.346068	0.053432	0.003319	0.000026
2	10	0.318202	0.405158	0.198353	0.062027	0.013819
2	20	0.080359	0.214500	0.258196	0.211907	0.132395
3	3	0.762400	0.228431	0.009169		
3	4	0.641503	0.319323	0.038440	0.000734	
3	5	0.534270	0.377668	0.081145	0.006857	0.000060
3	10	0.204177	0.361859	0.263645	0.121503	0.038835
3	20	0.026549	0.102360	0.181771	0.217770	0.196493
4	3	0.755867	0.234086	0.010047		
4	4	0.625893	0.328913	0.044299	0.000895	
4	5	0.509689	0.386773	0.094479	0.008978	0.000081
4	10	0.164244	0.330199	0.281828	0.151395	0.055723
4	20	0.014604	0.065847	0.137820	0.193930	0.204762
$\geqslant 2$	∞	0	0	0	0	0

Initial vector:

$$X_{u,u} \text{ has } \begin{cases} x_{u,u,(i,0)} = g_{u,u,i} & \text{for } 0 \leqslant i \leqslant u-1 \text{ (cf. Eq. 2.33)} \\ x_{u,u,(i,w_i)} = 0 & \text{otherwise.} \end{cases}$$

Markov Process E'_u ($u = 1, 2, 3, \ldots$):

States $x'_{t,u,(i,w_i)}$ with $t = u, u+1, u+2, \ldots$; $i = 0, 1, 2, \ldots$; and $0 \leqslant w_i \leqslant i$. Let $X'_{t,u} = (x'_{t,u,(0,0)}, \ldots, x'_{t,u,(i,w_i)}, \ldots)$, where the component $x'_{t,u,(i,w_i)}$ precedes $x'_{t,u,(j,w_j)}$ if $i < j$, or if $i = j$ and $w_i < w_j$. Then $X'_{t+1,u} = X'_{t,u} Q_{E'_u}$, where

$$Q_{E'_u} = \left[q'_{(i,w_i),(j,w_j)} \right]$$

with $i = 0, 1, 2, \ldots$; $j = 0, 1, 2, \ldots$; $0 \leqslant w_i \leqslant i$; $0 \leqslant w_j \leqslant j$;

$$q'_{(i,w_i),(j,w_j)} = \begin{cases} 2^{-1} & \text{if } j = i,\ w_j = w_i \\ 2^{-1} - 2^{-i+w_i-1} & \text{if } j = i,\ w_j = w_i + 1 \\ 2^{-i+w_i-1} & \text{if } j = i+1,\ w_j = w_i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Table 5. Values of $d_{t,i}$ for fields of degree 5^n.

n	t \ i	0	1	2	3	4
$\geqslant 1$	1	1.0				
$\geqslant 1$	2	0.96	0.04			
1	3	0.95232	0.047616	0.000064		
1	4	0.950796	0.049124	0.000079	4.1×10^{-9}	
1	5	0.950492	0.049426	0.000082	5.1×10^{-9}	1.0×10^{-14}
1	10	0.950416	0.049501	0.000083	5.4×10^{-9}	1.4×10^{-14}
1	20	0.950416	0.049501	0.000083	5.4×10^{-9}	1.4×10^{-14}
1	∞	0.950416	0.049501	0.000083	5.4×10^{-9}	1.4×10^{-14}
2	3	0.915017	0.084151	0.000832		
2	4	0.869698	0.126053	0.004231	0.000018	
2	5	0.825736	0.163700	0.010245	0.000318	4.2×10^{-7}
2	10	0.626809	0.292909	0.067013	0.011643	0.001485
2	20	0.320969	0.341251	0.202906	0.091520	0.032089
3	3	0.907048	0.091837	0.001115		
3	4	0.848044	0.145178	0.006746	0.000032	
3	5	0.787233	0.194600	0.017451	0.000715	9.8×10^{-7}
3	10	0.513190	0.338679	0.114621	0.028014	0.004867
3	20	0.188105	0.288914	0.247971	0.157182	0.076619
4	3	0.903749	0.095004	0.001247		
4	4	0.839107	0.152808	0.008044	0.000040	
4	5	0.771283	0.206552	0.021200	0.000964	1.3×10^{-6}
4	10	0.465177	0.350133	0.137772	0.038344	0.007486
4	20	0.141286	0.252645	0.252121	0.184071	0.102938
$\geqslant 2$	∞	0	0	0	0	0

Initial vector:

$$X'_{u,u} \text{ has } \begin{cases} x'_{u,u,(i,0)} = g'_{u,u,i} & \text{for } 0 \leqslant i \leqslant u-1 \text{ (cf. Eq. 4.8)} \\ x'_{u,u,(i,w_i)} = 0 & \text{otherwise.} \end{cases}$$

References

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