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Abstract Let P be a formally self-adjoint, conformally covariant differential operator on tensor fields, defined for Riemannian manifolds (M, g), and suppose that the leading symbol of P is always positive definite. For example, P could be the conformal Laplacian on functions, $D = \Delta + (n-2)K/4(n-1)$, where $n = \dim M$ and $K =$ scalar curvature. Then the t^0 coefficient in the Minakshisundaram-Pleijel (MP) asymptotic expansion of the L^2 trace of the heat operator $\exp(-tP)$ is a conformal invariant for compact, even dimensional M. Invariance and non-invariance results for the t^0 coefficient are obtained for some non-conformally covariant operators, and for all the operators studied, the first conformal variation of the other MP coefficients is given explicitly.

0. Introduction

Conformal geometry on a smooth manifold M with metric tensor g is a subject with a long history and a rapidly growing literature, on issues ranging from relativity and relativistic wave equations to curvature prescription, pseudoconvex comains in $\mathbb{C}^n$, and CR structure. Simply stated, conformal geometry deals with objects that deform nicely when g is replaced by $\Omega^2 g$, where Ω is a smooth positive function on M. At the center are always differential operators which depend only on the conformal class $\{\Omega^2 g\}$ of g; these are the *conformal covariants*. An example is the conformal Laplacian $D = \Delta + (n-2)K/4(n-1)$, where $n = \dim M$ and K is the scalar curvature, properly understood as acting between two bundles of densities. An important problem concerns *local invariants* $U = U(g)$; in local coordinates, invariant polynomials in derivatives of the metric tensor and its inverse: When does such a U or its integral deform as if Ω were just a constant? For the significance of this question in complex and CR geometry see Fefferman-Graham [1984], Burns-Diederich-Shnider [1977], and Jerison-Lee [1984].

In this paper we consider positive definite (Riemannian) metrics and associate to any formally self-adjoint conformal covariant P with positive definite leading symbol (necessary of even order $2l$) a *conformal index* $c(M, g, P)$, defined when M is compact and even-dimensional. By work of Branson [1982,1984], several such P exist. $c(M, g, P)$ is a number depending only on the conformal class of g, and is given by the integral of the local scalar

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invariant $U_{n/2}$ in the Minakshisundaram-Pleijel (MP) expansion of the kernel function $H(t, x, y)$ for $\exp(-tP)$:

$$\text{fibrewise trace } H(t, x, x) \sim \sum_{i=0}^{\infty} t^{(2i-n)/2l} U_i(x, P) \,\mathrm{d}\,\mathrm{vol}_g, \quad t \downarrow 0.$$

Otherwise stated, $c(M, g, P)$ is the t^0 term in the asymptotic expansion of the L^2 trace of $\exp(-tP)$. In the case of the conformal Laplacian, the conformal index is known to physicists as the *trace anomaly* (Hawking [1977], Dowker-Kennedy [1978]). A by-product of our proof of the invariance of $c(M, g, P)$ is a formula for the first conformal variation of the other $U_i(x, P)$ (n not necessarily even) which shows that for $i \neq n/2$, the Euler-Lagrange equation for conformal variation of $\int U_i$ reads $U_i = 0$; for volume-preserving conformal variation, $U_i = \text{constant}$.

The invariance of $c(M, g, P)$ is exactly the assertion that under $g \to \Omega^2 g$, the integral of $U_{n/2}$ deforms as if Ω were constant. Schimming [1981] (see also Wünch [1985], p. 186) has observed that in the case of the conformal Laplacian D, the MP coefficient $U_{(n-2)/2}$ behaves this way *before* integrating; specifically,

$$\bar{g} = \Omega^2 g \Rightarrow \overline{U}_{(n-2)/2} = \Omega^{2-n} U_{(n-2)/2}. \tag{0.1}$$

(The power of Ω involved depends on homogeneity considerations; $U_{n/2}$ is of exactly the right homogeneity to have a chance of producing a conformal index.) The proof of (0.1) is based on a formal connexion between the systems of transport equations giving: (1) the fundamental solution of the conformal d'Alembertian $\square + (n-2)K/4(n-1)$ on an n-dimensional Lorentz manifold (see, e.g., Friedlander [1975]), and (2) the fundamental solution of the heat equation (the kernel function for $\exp(-tD)$) based on the conformal Laplacian on an n-dimensional Riemannian manifold (see, e.g., Berger *et al.* [1971]). Presumably this proof extends to more general conformal covariants P whose Lorentz versions have appropriate hyperbolicity properties.

The natural extension of conformal index theory to non-positive definite metrics is not treated in this paper, nor are the analogous questions for CR structure; we believe, however, that these will be fruitful lines of enquiry. In fact, by analytic continuation in signature, one can assert the existence of conformal indices which are integrals of local invariants for compact M of any signature. In particular, these give biholomorphic invariants of pseudo-convex domains V in $\mathbb{C}^n$ through the construction of the Fefferman bundle (Fefferman [1976], Lee [1985]), which carries a Lorentz conformal structure reflecting the geometry of V. Within Riemannian geometry, an interesting question is that of a possible relationship between $c(M, g, D)$ and the critical values of the Yamabe functional

$$(D\varphi, \varphi)_{L^2} / \|\varphi\|^2_{L^{2n/(n-2)}},$$

and similarly for Yamabe functionals based on other conformal covariants (Branson [1984], Sec. 4). It should also be possible to relate conformal indices to Lefschetz fixed point formulas (see Gilkey [1979] for the relation of MP expansions to these formulas), and to treat manifolds with boundary.

This paper is organized as follows. In Section 1 we introduce the conformal Laplacian and define conformal covariants in general. Section 2 is an expanded account of the question raised above on conformal deformation of local invariants, and its relation to MP expansions. In Section 3, we prove the main theorems using the calculus of pseudo-differential operators and a generalization of a fundamental formula of Ray and Singer for the first variation of the L^2 trace of $\exp(-t\Delta)$. In Section 4.a, we look at the case of non-conformally covariant operators. In particular, we get a non-invariance result for $\int U_{n/2}(\Delta + aK$ on functions) for $a \neq (n-2)/4(n-1)$: a metric which is critical for conformal variations must satisfy a constant curvature condition. The square of the Dirac operator, though *not* conformally covariant, *does* produce a conformal index. In Section 4.b, we do some explicit illustrative calculations with U_0, U_1 and U_2 of the $\Delta + aK$ on functions. Section 4.c indicates how the second conformal variation can be calculated. In Section 4.d, we draw two curious consequences of the proofs in Section 3 in the case where (M, g) admits a conformal vector field T with infinitesimal conformal factor $\omega \in C^\infty(M, \mathbb{R})$: ω must be L^2-orthogonal to (1) $U_i(P)$ for $i \neq n/2$, and (2) $|\psi_1|^2 + \cdots + |\psi_m|^2$, where P is any formally self-adjoint conformal covariant with positive definite leading symbol, and $\psi_1, \ldots, \psi_m$ is an orthonormal basis of the real λ-eigensections of P, $\lambda \neq 0$. Section 4.e is about a connexion between the ideas of Section 3 and the theory of *Lax pairs*. In Section 5, we give our original argument for the invariance of the conformal index in the case of the conformal Laplacian D on functions; this was outlined in Branson-Ørsted [1984]. This argument has the advantage of involving only classical analysis through the transport equations, but is difficult to make in the general setting of Section 3. S. Rosenberg and T. Parker [1985] have also constructed a proof of invariance in this special case, based on an appropriate zeta function and properties of its analytic continuation. A zeta function argument for the invariance of $c(M, g, D)$ is also outlined in Dowker-Kennedy [1978], p. 906.

The authors are indebted to S. Rosenberg for pointing out the formula of Ray-Singer [1971], Sec. 6, a generalization of which is an important part of the argument in Section 3, and for pointing out that a proof of the invariance of $c(M, g, D)$ should also yield variational formulas for all the $\int U_i(D)$. Special acknowledgement is due to the referee for extensive education on the pseudo-differential operator calculus and its applications to Section 3.

1. The conformal Laplacian

Let (M, g) be a C^∞ compact Riemannian manifold of dimension $n \geqslant 2$. Let

$\Delta = -\nabla^{\lambda}\nabla_{\lambda}$ be the Laplacian on functions in M. It is well-known that the operator

$$D = \Delta + \frac{n-2}{4(n-1)}K \ (K = \text{scalar curvature})$$

is *conformally covariant* in the sense that if $0 < \Omega \in C^{\infty}(M)$, $\bar{g} = \Omega^2 g$, and $\bar{D}$ is the above operator calculated in the metric $\bar{g}$, then

$$D\left(\Omega^{(n-2)/2}\varphi\right) = \Omega^{(n+2)/2}\bar{D}\varphi, \quad \text{for all } \varphi \in C^{\infty}(M). \tag{1.1}$$

(That is, D determines an operator between two bundles of densities that depends only on the conformal class $\{\Omega^2 g\}$ of g.) The infinitesimal form of (1.1) may be written as follows: Let $\omega \in C^{\infty}(M)$, and let Ω run through the one-parameter family $e^{u\omega}$, $u \in \mathbb{R}$. Set

$$^{\cdot} = \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0}, \quad (\dot{D}\varphi)(x) = [D\varphi(x)]^{\cdot}.$$

Evaluating (1.1) at $x \in M$ and differentiating, we get

$$\dot{D} = -2\omega D + \frac{n-2}{2}[D, m_{\omega}], \tag{1.2}$$

where m_{ω} is multiplication by ω, and $[\cdot, \cdot]$ denotes the commutator.

Several other general conformally covariant differential operators P on tensor fields are known; a systematic account is given in Branson [1984]. By definition, such operators satisfy a covariance relation analogous to (1.1): if $\bar{g} = \Omega^2 g$,

$$P(\Omega^a\varphi) = \Omega^b\bar{P}\varphi \tag{1.3}$$

for all relevant tensor fields φ, and some a, $b \in \mathbb{R}$. The infinitesimal covariance law corresponding to (1.2) is then

$$\dot{P} = -(b-a)\omega P + a[P, m_{\omega}]. \tag{1.4}$$

2. The heat kernel and the Minakshisundaram-Pleijel coefficients

The objects of this paper is to calculate the conformal variation of the Minakshisundaram-Pleijel (MP) coefficients appearing in the small-time asymptotic expansion of the heat kernel based on a conformally covariant operator P. It will turn out that each such operator with strong enough ellipticity properties gives rise, through the MP expansion, to a *conformal index*, or numerical invariant of conformal structure.

Suppose M is a smooth n-dimensional compact manifold without boundary, let F be a Hermitian vector bundle over M, and let $P: C^\infty(F) \to C^\infty(F)$ be an elliptic differential operator of even order $2l$, with positive definite leading symbol. It is well-known (see, for example, Gilkey [1984,1980]) that the L^2 trace of the kernel function for $\exp(-tP)$ has an asymptotic expansion

$$\operatorname{Tr} \exp(-tP) \sim \sum_{i=0}^{\infty} t^{(2i-n)/2l} \int_M U_i \, \mathrm{d\,vol}$$

where the U_i are given by local polynomial expressions in derivatives of the total symbol of P.

Suppose now that F is a tensor bundle and the symbol of P is given by a universal polynomial expression in the derivatives of the metric tensor, so that the U_i are also given by such expressions. By homogeneity considerations, if P deforms according to

$$\bar{g} = A^2 g \Rightarrow \bar{P} = A^{-2l} P, \quad 0 < A \in \mathbb{R}$$

under uniform dilation of the metric, the U_i will deform similarly:

$$\bar{U}_i = A^{-2i} U_i.$$

By Weyl's invariant theory, this means that U_i is a *level 2i local scalar invariant*, i.e., an $\mathbb{R}$-linear combination of universal expressions

$$\operatorname{trace}\left(R_{jklm|\alpha} \cdots R_{stuv|\beta}\right), \quad (2 + |\alpha|) + \cdots + (2 + |\beta|) = 2i,$$

where R is the Riemann curvature tensor, α and β are multi-indices, indices after the bar denote covariant derivatives, and "trace" represents some partitioning of all the indices into pairs, the raising of one index in each pair, and contraction to a scalar. In particular, the *MP coefficients*

$$a_i = \int_M U_i \, \mathrm{d\,vol}$$

will be C^∞ functions of the real parameter u, as the metric ranges through the $\mathrm{e}^{2u\omega} g$, for a fixed $\omega \in C^\infty(M)$. Recalling the notation of Section 1, we shall show that if P satisfies all the above assumptions and is formally self-adjoint and conformally covariant, then

$$\dot{a}_i = (n - 2i) \int_M \omega U_i \, \mathrm{d\,vol}. \tag{2.1}$$

In particular, $a_{n/2}$ is a conformal invariant for even n.

One could also ask for the conformal variation of the U_i themselves. Local conformal invariants, i.e. level m local scalar invariants L for which

$$\bar{g} = \Omega^2 g \Rightarrow \bar{L} = \Omega^{-m} L, \quad 0 < \Omega \in C^\infty(M),$$

or, infinitesimally,

$$\dot{g} = 2\omega g \Rightarrow \dot{L} = -m\omega L,$$

are rare (see Fefferman-Graham [1984], Günther-Wünsch [1985]). The simplest nontrivial local invariant, K, does not have this property: applying (1.2) to $\varphi = 1$, we get

$$\dot{K} = -2\omega K + 2(n-1)\Delta\omega. \tag{2.2}$$

But it was remarked by Schimming [1981] (see also Wünsch [1985], p. 186) that for even n, the quantity $U_{(n-2)/2}$ obtained from the conformal Laplacian D is a local conformal invariant. Schimming's argument exploits a formal connexion between the classical system of transport equations defining the U_i (see Berger et al. [1971]) and Hadamard's transport system for the fundamental solution of the conformal d'Alembertian $\square + (n-2)K/4(n-1)$ on n-dimensional Lorentz manifolds (see, e.g., Friedlander [1975]). It is too much, however, to expect that all the U_i generated by D will be local conformal invariants, even though this would be consistent with (2.1). For example,

$$(4\pi)^{n/2} U_1 = -(n-4)K/12(n-1) \tag{2.3}$$

(see Section 4.b below) is invariant only for $n = 4$, where Schimming's remark applies.

From the point of view of conformal variation of the U_i, (2.1) says that

$$\dot{U}_i \cong -2iU_i \ (\text{mod range } \delta),$$

where $\delta = d^*$ is the divergence carrying one-forms to functions: $\delta\eta = -\nabla^\lambda \eta_\lambda$. Indeed, the metric variation $\dot{g} = 2\omega g$ has the effect $(d\,\text{vol})^\cdot = n\omega\, d\,\text{vol}$ on the invariant measure, and it is shown in Branson [1984] that $\dot{U}_i$ is a linear combination of universal expressions of the form

$$\text{trace}\left(R_{jklm|\alpha} \cdots R_{stuv|\beta}\,\omega_{|\gamma}\right),$$

$$(2 + |\alpha|) + \cdots + (2 + |\beta|) + |\gamma| = 2i.$$

After repeated integration by parts, this reduces modulo the range of δ to ωL, where L is some level $2i$ local scalar invariant. Exact divergences integrate to zero, and since ω is arbitrary in $C^\infty(M)$, (2.1) forces $L = U_i$.

3. Variation of a general heat kernel

In this section we prove a general theorem on the variation of the heat kernel of a formally self-adjoint differential operator D with positive definite leading symbol. This will allow us to compute the first conformal variation of the L^2 trace of $\exp(-tD)$ for conformally covariant D with this strong ellipticity property. The main technique is that of pseudo-differential operators as developed in, e.g., Seeley [1967] and Gilkey [1974,1984]. The main point is that the standard manipulations with symbols and operators between Sobolev spaces can be made differentiable in an external real parameter u. As a result, the MP expansion can be differentiated term by term in u.

In the following, fix a smooth Hermitian vector bundle F over a smooth compact manifold M, of dimension n and without boundary, with smooth positive measure μ. Let $D: C^\infty(F) \to C^\infty(F)$ be a formally self-adjoint differential operator of even order $2l$ with positive definite leading symbol. (In particular, D is elliptic.) Actually we want to consider a smoothly varying one-parameter family of such setups: let u run through a real interval $(-\epsilon, \epsilon)$. If

$$\mu(u) = f(x, u)\, \mathrm{d}x^1 \dots \mathrm{d}x^n$$

in local coordinates, $\langle \cdot, \cdot \rangle(u)$ is the Hermitian structure on F, and $D(u)$ is our one-parameter family of differential operators (formally self-adjoint in $\mu(u)$ and $\langle \cdot, \cdot \rangle(u)$, with leading symbol positive definite in $\langle \cdot, \cdot \rangle(u)$), and if φ and ψ are C^∞ sections of F, then

$$f(x, u), \quad \langle \varphi(x), \psi(x) \rangle_x(u), \quad \text{and } [D(u)\varphi](x)$$

should be jointly C^∞ in x and u. In our applications, F will be a tensor bundle with Hermitian structure given by the choice of a Riemannian metric on M; this choice will also determine μ. D will also be built naturally from the metric, which will run through a one-parameter family $g(u)$ within a conformal class.

REMARK 3.1: $D(u)$ has real discrete spectrum $\lambda_j(u) \uparrow +\infty$, $j = 0, 1, 2, \dots$. Following through the estimates in, e.g., Lemma 1.6.4 of Gilkey [1984], we can get a lower bound Λ on the bottom eigenvalue $\lambda_0(u)$ which is *uniform* for u in some compact interval $[-\delta, \delta]$. This allows us to fix a cone C of small slope about the ray $\{z \in \mathbb{R},\ z \geqslant \min(\Lambda, 0)\}$ in the complex plane as in Gilkey [1984], p. 48, which encloses the spectrum of $D(u)$ for all $u \in [-\delta, \delta]$.

REMARK 3.2: The $L^2(F)$ norm varies with u, since it depends on $\langle \cdot, \cdot \rangle(u)$ and $\mu(u)$, but the different norms are all equivalent. The same can be said of the Sobolev class $L_s^2(F)$ norms

$$\| \varphi \|_{L_s^2}(u) = \| (D(u) - \Lambda + 1)^{s/2l} \varphi \|_{L^2}(u).$$

In fact, there are positive constants $c(s)$ and $C(s)$ for which

$$c(s)\,\|\varphi\|_{L_s^2}(0) \leqslant \|\varphi\|_{L_s^2}(u) \leqslant C(s)\,\|\varphi\|_{L_s^2}(0), \quad \text{all } u \in [-\delta, \delta].$$

Thus it will not be important for the estimates below whether we think of these norms as varying, or just fix the $u = 0$ norms.

We denote by S_k the class of symbols $p = p(x, \xi, \lambda, u)$ defining endomorphisms of F_x $(x \in M)$ satisfying (in local coordinates)

(3.1.a) p is smooth in the cotangent variables (x, ξ) and in the parameter u;

(3.1.b) p is holomorphic in the complex parameter λ on the complement of the fixed cone C;

(3.1.c) $|\partial_x^\alpha \partial_\xi^\beta \partial_\lambda^\gamma \partial_u^\delta p|$

$$\leqslant \text{const}(\alpha, \beta, \gamma, \delta)(1 + |\xi|^{2l} + |\lambda|)^{(k-|\beta|-2l|\gamma|)/2l}$$

for all multi-indices α, β and all γ, δ.

This symbol class is invariant under coordinate changes, and is transferred from $\mathbb{R}^n$ to M using partitions of unity in the usual way. Composition in the algebra of pseudo-differential operators satisfies

$$\sigma(PQ) \sim \sum_\alpha \partial_\xi^\alpha \sigma(P) \cdot \left(\frac{1}{\sqrt{-1}}\partial_x\right)^\alpha \sigma(Q)/\alpha!, \tag{3.2}$$

where α runs over multi-indices, and " $\sim$ " means modulo lower order terms, as in Gilkey [1984], Lemma 1.7.1. Let $|\cdot|_{s,s'}$ denote the norm in the spaces $\mathscr{B}(L_s^2, L_{s'}^2)$ of bounded operators from L_s^2 to $L_{s'}^2$. If $|P|_{-m,m}$ is finite for $m > k + n/2$, then P has a C^k kernel function $L(x, y, P)$ (Seeley [1969], p. 181), and the sup norm of the k-th derivatives of L can be estimated:

$$|L(x, y, P)|_{L_k^\infty} \leqslant \text{const}(k)\,|P|_{-m,m}. \tag{3.3}$$

(See Seeley [1969], p. 181, or Gilkey [1984] Lemma 1.2.9.) Furthermore, (3.3) can be made uniform for u in some interval $[-\delta, \delta]$. If $\sigma(P) \in S_{-k}$, then 1.7.1(b) of Gilkey [1984] together with the uniformity in u implies

$$|P(\lambda, u)|_{-m,m} \leqslant \text{const}(m)(1 + |\lambda|)^{-m} \tag{3.4}$$

for k greater than or equal to some constant $k(m)$. For the resolvent $R(\lambda) = (D - \lambda)^{-1}$, Theorem 1 (page 269) of Seeley [1969] or Lemma 1.6.6 of Gilkey [1984] implies

$$|R(\lambda)|_{s,s} \leqslant \text{const}(s)(1 + |\lambda|)^{a(s)} \tag{3.5}$$

uniformly in u, for some positive power $a(s)$.

We can find approximate resolvents $R_k(\lambda)$ by constructing, in each coordinate patch, the symbol of an operator which approximately inverts $D-\lambda$, and patching these operators together with a partition of unity. If $\sigma(D)=p_{2l}+p_{2l-1}+\cdots+p_0$ with p_j j-homogeneous in ξ, and our trial resolvent symbol is $r_{-2l}+\cdots+r_{-2l-k}$ with r_i jointly i-homogeneous in ξ and $\lambda^{1/2l}$, we define

$$r_{-2l}=(p_{2l}-\lambda)^{-1},$$

$$r_{-2l-m}=-r_{-2l}\left(\sum_{\substack{i+j-|\alpha|=-m\\ i>-2l-m}}\partial_\xi^\alpha p_j\cdot\left(\frac{1}{\sqrt{-1}}\partial_x\right)^\alpha r_i/\alpha!\right),$$

$m=1,2,\ldots.$

By (3.2), this will assure that

$$\sigma(I-(D-\lambda)R_k(\lambda))\in S_{-k-1}. \tag{3.6}$$

Fix m and let $a(m)$ be as in estimate (3.5). Let $q=m+a(m)$, and let $k>k(q)$ be chosen to apply estimate (3.4) to $I-(D-\lambda)R_k(\lambda)$:

$$\begin{aligned}|R(\lambda)-R_k(\lambda)|_{-m,m}&\\ &=|R(\lambda)(I-(D-\lambda)R_k(\lambda))|_{-m,m}\\ &\leqslant|R(\lambda)|_{m,m}|I-(D-\lambda)R_k(\lambda)|_{-m,m}\\ &\leqslant\mathrm{const}(m)(1+|\lambda|)^{a(m)}|I-(D-\lambda)R_k(\lambda)|_{-q,q}\\ &\leqslant\mathrm{const}(m)(1+|\lambda|)^{-m}.\end{aligned} \tag{3.7}$$

Let $H(t,x,y)\in F_x\otimes F_y^*\otimes|\Lambda^n|_y$ be the kernel function of $\exp(-tD)$ ($|\Lambda^n|$ is the bundle of densities, sections of which are measures), and $H_k(t,x,y)$ the kernel of the approximate heat semigroup

$$E_k(t)=\frac{1}{2\pi i}\int_{\partial C}\mathrm{e}^{-t\lambda}R_k(\lambda)\,\mathrm{d}\lambda,$$

defined in analogy with

$$\exp(-tD)=\frac{1}{2\pi i}\int_{\partial C}\mathrm{e}^{-t\lambda}R(\lambda)\,\mathrm{d}\lambda.$$

Estimate (3.7) gives rise to the operator estimate

$$|\exp(-tD)-E_k(t)|_{-m,m}\leqslant\mathrm{const}(m)t^m,$$

uniformly in u for large k and small t (see Gilkey [1984], p. 53). Estimates (3.3) and (3.7) now imply

$$|H(t, x, y) - H_k(t, x, y)|_{L^\infty_m} \leqslant \mathrm{const}(m) t^m, \tag{3.8}$$

uniformly in u for large k and small t.

Now the $H_k(t, x, x)$, and thus by (3.8) $H(t, x, x)$, are easily seen to admit asymptotic expansions in t. The same will be true of the kernel functions of the u-derivatives of $\exp(-tD)$ and $E_k(t)$; this is the important point of the main result of this section:

THEOREM 3.3: *Let M, F, $\mu(u)$, $D(u)$, and C be as above. Let $H(t, x, y) \in F_x \otimes F_y^* \otimes |\Lambda^n|_y$ be the kernel function of* $\exp(-tD)$. *Then*

(*a*) *The fibrewise trace of $H(t, x, x)$ has an asymptotic expansion*

$$\mathrm{Tr}_{F_x} H(t, x, x) \sim \sum_{\iota=0}^{\infty} t^{(2\iota-n)/2l} V_\iota(x, D, u), \quad t \downarrow 0, \tag{3.9}$$

where $V_\iota(x, D, u)$ is a smooth measure given in local coordinates by a polynomial in the jets of the total symbol of $D(u)$. As a result,

$$\mathrm{Tr}_{L^2} \exp(-tD) \sim \sum_{i=0}^{\infty} t^{(2\iota-n)/2l} \int_M V_\iota(x, D, u), \quad t \downarrow 0. \tag{3.10}$$

(*b*) $\mathrm{Tr}_{L^2} \exp(-tD)$ *is smooth in u, and one can differentiate* (3.10) *term by term to get*

$$\frac{\mathrm{d}}{\mathrm{d}u} \mathrm{Tr}_{L^2} \exp(-tD) \sim \sum_{\iota=0}^{\infty} t^{(2\iota-n)/2l} \frac{\mathrm{d}}{\mathrm{d}u} \int_M V_\iota(x, D, u), \quad t \downarrow 0. \tag{3.11}$$

PROOF: First note that the L^2 trace does not depend on which of the equivalent norms $\|\cdot\|_{L^2(u)}$ we use. The existence of the asymptotic expansions (3.9), (3.10) was already indicated above, following, e.g., Gilkey [1984]. For (b), consider the operators involved as $\mathscr{B}(L^2_s, L^2_{s'})$-valued functions of u for appropriate s and s', and differentiate:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}(R(\lambda) - R_k(\lambda)) &= \frac{\mathrm{d}}{\mathrm{d}u}(R(\lambda)(I - (D - \lambda)R_k(\lambda)))\\ &= R(\lambda)\frac{\mathrm{d}}{\mathrm{d}u}(I - (D - \lambda)R_k(\lambda))\\ &\quad - R(\lambda)\frac{\mathrm{d}D}{\mathrm{d}u}R(\lambda)(I - (D - \lambda)R_k(\lambda)).\end{aligned} \tag{3.12}$$

The two terms on the right in (3.12) can now be estimated separately. For the second, set $q = m + 2l + a(m) + a(m+2l)$ and let $k > k(q)$:

$$
\begin{aligned}
&|R(\lambda)\frac{\mathrm{d}D}{\mathrm{d}u}R(\lambda)(I-(D-\lambda)R_k(\lambda))|_{-m,m} \\
&\leqslant |R(\lambda)|_{m,m}\left|\frac{\mathrm{d}D}{\mathrm{d}u}\right|_{m+2l,m}|R(\lambda)|_{m+2l,m+2l} \\
&\quad\times|I-(D-\lambda)R_k(\lambda)|_{-m,m+2l} \\
&\leqslant \mathrm{const}(m)(1+|\lambda|)^{a(m)+a(m+2l)}|I-(D-\lambda)R_k(\lambda)|_{-q,q} \\
&\leqslant \mathrm{const}(m)(1+|\lambda|)^{-m}.
\end{aligned} \tag{3.13}
$$

Similarly, for the first term,

$$
\begin{aligned}
&|R(\lambda)\frac{\mathrm{d}}{\mathrm{d}u}(I-(D-\lambda)R_k(\lambda))|_{-m,m} \\
&\leqslant |R(\lambda)|_{m,m}\left|\frac{\mathrm{d}}{\mathrm{d}u}(I-(D-\lambda)R_k(\lambda))\right|_{-m,m} \\
&\leqslant \mathrm{const}(m)(1+|\lambda|)^{a(m)}\left|\frac{\mathrm{d}}{\mathrm{d}u}(I-(D-\lambda)R_k(\lambda))\right|_{-q,q} \\
&\leqslant \mathrm{const}(m)(1+|\lambda|)^{-m},
\end{aligned} \tag{3.14}
$$

since $(\mathrm{d}/\mathrm{d}u)(I-(D-\lambda)R_k(\lambda))$ has symbol in S_{-k}. Now we can perform a contour integration and argue as in the proof of Lemma 1.7.3 of Gilkey [1984] to get norm estimates

$$\left|\frac{\mathrm{d}}{\mathrm{d}u}(\exp(-tD)-E_k(t))\right|_{-m,m}\leqslant \mathrm{const}(m)t^m$$

for large k and small t. Estimate (3.3) and (3.12)–(3.14) then show that

$$\left|\frac{\mathrm{d}}{\mathrm{d}u}(H(t,x,y)-H_k(t,x,y))\right|_{L^\infty_m}\leqslant \mathrm{const}(m)t^m$$

for large k and small t. There are similar estimates for the higher u-derivatives. Let $f(t)$ and $f_k(t)$ be the L^2 traces of $\exp(-tD)$ and $E_k(t)$ respectively. Setting $x=y$, taking the fibrewise trace, and integrating over M, we get

$$\left|\frac{\mathrm{d}}{\mathrm{d}u}(f(t)-f_k(t))\right|\leqslant \mathrm{const}(m)t^m, \tag{3.15}$$

for large k and small t.

This reduces the proof of (b) to that of the corresponding statement for the approximate traces $f_k(t)$. Now the operator $E_k(t)$ is built from the

$$e_j(t) = \frac{1}{2\pi i}\int_{\partial C} e^{-t\lambda} r_j(\lambda)\,\mathrm{d}\lambda$$

using a partition of unity, in exactly the way that $R_k(\lambda)$ is built from the $r_j(\lambda)$. The e_j are infinitely smoothing symbols, and

$$H_k(t, x, x) = \sum_{j=-2l-k}^{-2l} \left(\int_{\mathbf{R}^n} e_j(t, x, \xi)\,\mathrm{d}\xi \right) \mathrm{d}x_{\mathbf{R}^n},$$

where $\mathrm{d}x_{\mathbf{R}^n}$ is the coordinate-dependent measure on a chart containing x and trivializing F. By the joint homogeneity of r_j in ξ and $\lambda^{1/2l}$,

$$\mathrm{Tr}_{F_x} e_j(t, x, \xi)\,\mathrm{d}\xi = t^{-(j+2l+n)/2l}\frac{1}{2\pi i}\int_{t\partial C} \mathrm{Tr}_{F_x} r_j(\lambda, x, \xi)\,\mathrm{d}\lambda\,\mathrm{d}\xi.$$

Defining

$$V_{-(j+2l)/2}(x) = \left(\int_{\mathbf{R}^n} \frac{1}{2\pi i}\int_{\partial C} \mathrm{Tr}_{F_x} r_j(\lambda, x, \xi)\,\mathrm{d}\lambda\,\mathrm{d}\xi \right) \mathrm{d}x_{\mathbf{R}^n}$$

and using Cauchy's theorem to shift the path of integration from $t\partial C$ to ∂C, we get

$$f_k(t) = \sum_{m=0}^{k} t^{(m-n)/2l}\int_M V_{m/2}(x).$$

As in Gilkey [1984], p. 54, the V_i vanish for half-integral i. Since all integrals above converge absolutely, we can differentiate with respect to u under the integral sign to obtain

$$\frac{\mathrm{d}}{\mathrm{d}u} f_k(t) = \sum_{m=0}^{k} t^{(m-n)/2l}\int_M \frac{\mathrm{d}}{\mathrm{d}u} V_{m/2}(x),$$

which, in view of (3.15), completes the proof of the theorem. □

For future purposes, we record a generalization of Theorem 3.3. The arguments above suffice to prove:

THEOREM 3.4: *In Theorem 3.3,* $\exp(-tD(u))$ *may be replaced by* $B(u)\exp(-tD(u))$, *where* $B(u)$ *is a smooth one-parameter auxiliary family of* *b-th order differential operators, provided we replace* $t^{(2\iota-n)/2l}$ *with* $t^{(2\iota-n-c)2l}$, *where* $c=2[b/2]$. (3.11) *also holds for higher u-derivatives*:

$$\frac{\mathrm{d}^k}{\mathrm{d}u^k}\mathrm{Tr}_{L^2} B\exp(-tD) \sim \sum_{\iota=0}^{\infty} t^{(2\iota-n-c)/2l}\frac{\mathrm{d}^k}{\mathrm{d}u^k}\int_M V_\iota(x, D, B, u),$$

$t\downarrow 0$. □

The left-hand side of (3.11) can now be computed using a formula of Ray and Singer [1971]. For the sake of completeness, we give the proof of this in a general setting. (Ray and Singer treat the case of metric deformations of the Laplacian.)

PROPOSITION 3.5: *With assumptions as in Theorem* 3.3, *we have*

$$\frac{\mathrm{d}}{\mathrm{d}u}(\mathrm{Tr}\exp(-tD(u))) = -t\ \mathrm{Tr}\left(\frac{\mathrm{d}}{\mathrm{d}u}(D(u))\exp(-tD(u))\right). \tag{3.16}$$

PROOF: $\exp(-tD(u))$ is infinitely smoothing and forms a semigroup (see, e.g., Seeley [1967], p. 301). Without loss of generality, we can prove (3.16) at $u=0$. Let $D=D(0)$ and

$$\dot{D}=\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0}D(u).$$

For a fixed $t>0$, consider

$$\begin{aligned}\exp(-2tD(u)) &= \exp(-tD(u))\exp(-tD(u))\\ &= \frac{1}{2\pi i}\int_{\partial C}\mathrm{e}^{-t\lambda}(D(u)-\lambda)^{-1}\exp(-tD(u))\,\mathrm{d}\lambda,\end{aligned}$$

which is convergent in the trace norm and u-differentiable. We then get

$$\begin{aligned}&\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0}(\mathrm{Tr}\exp(-2tD(u)))\\ &= 2\ \mathrm{Tr}\left(\frac{\mathrm{d}}{\mathrm{d}u}\exp(-tD(u))\mid_{u=0}\exp(-tD)\right)\\ &= 2\cdot\frac{1}{2\pi i}\int_{\partial C}\mathrm{e}^{-t\lambda}\ \mathrm{Tr}\left(-(D-\lambda)^{-1}\dot{D}(D-\lambda)^{-1}\exp(-tD)\right)\mathrm{d}\lambda\\ &= -2\ \mathrm{Tr}\left(\dot{D}\frac{1}{2\pi i}\int_{\partial C}\mathrm{e}^{-t\lambda}(D-\lambda)^{-2}\exp(-tD)\,\mathrm{d}\lambda\right),\end{aligned}$$

since the fact that $\exp(-tD)$ is infinitely smoothing justifies cyclic permutation of operators under the trace. Integrating by parts, we get

$$\begin{aligned}\frac{d}{du}\bigg|_{u=0}&(\operatorname{Tr}\exp(-2t\,D(u)))\\ &=2\operatorname{Tr}(\dot{D}(-t\exp(-tD)\exp(-tD)))\\ &=-2t\operatorname{Tr}(\dot{D}\exp(-2tD)).\qquad\square\end{aligned}$$

We are now in a position to associate global invariants to differential operators satisfying a very general type of covariance law. Our main application will be to the case in which u is a parameter of conformal deformation for the metric tensor, and $D(u)$ is a conformally covariant differential operator on tensor fields which is naturally constructed from the metric.

THEOREM 3.6: *Let M, F, $\mu(u)$, $D(u)$, C, and V_i be as in Theorem 3.3. Assume that*

$$\frac{d}{du}\bigg|_{u=0}D(u)=-(b-a)\omega D(0)+a[D(0),m_\omega]\tag{3.17}$$

for some $a, b\in\mathbb{R}$ and $\omega\in C^\infty(M,\mathbb{R})$, where m_ω is multiplication by ω and $[\cdot,\cdot]$ is the commutator. Then for the coefficients in (3.9),

$$\frac{d}{du}\bigg|_{u=0}\int_M V_i(x,D,u)=\frac{b-a}{2l}(n-2i)\int_M\omega(x)V_i(x,D,0).\tag{3.18}$$

PROOF: The $P(u)=m_{\exp(-u\omega)}D(0)m_{\exp(u\omega)}$ are an *isospectral* family of operators satisfying the assumptions of Proposition 3.5, $P(u)$ being self-adjoint in the Hermitian structure $\langle\cdot,\cdot\rangle(0)$ and measure $e^{2u\omega}\mu(0)$. Thus

$$\frac{d}{du}\operatorname{Tr}_{L^2}\exp(-tP(u))=\frac{d}{du}\sum_{j=0}^{\infty}e^{-t\lambda_j(0)}=0.$$

Since

$$\dot{P}=[D(0),m_\omega],$$

(3.16) with $P(u)$ in place of $D(u)$ gives

$$\operatorname{Tr}_{L^2}([D(0),m_\omega]\exp(-tD(0)))=0.$$

This, along with (3.17) and (3.16) (now applied to $D(u)$), gives

$$(\mathrm{Tr}_{L^2}\exp(-tD))^{\cdot} = (b-a)t\,\mathrm{Tr}_{L^2}(\omega D(0)\exp(-tD(0)))$$

$$= -(b-a)t\,\mathrm{Tr}_{L^2}\left(\omega\cdot\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}\exp(-t(1+\epsilon)D(0))\right).$$

Now the trace and the ϵ-derivative may be interchanged:

$$(\mathrm{Tr}_{L^2}\exp(-tD))^{\cdot} = -(b-a)\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}\mathrm{Tr}_{L^2}(\omega\exp(-t(1+\epsilon)D(0))). \tag{3.19}$$

But the operator $\omega\exp(-t(1+\epsilon)D)$ has kernel function $\omega(x)H((1+\epsilon)t, x, y)$, so that by (3.9),

$$\mathrm{Tr}_{L^2}\,\omega\exp(-t(1+\epsilon)D(0))$$

$$\sim\sum_{i=0}^{\infty}t^{(2i-n)/2l}(1+\epsilon)^{(2i-n)/2l}\int_M\omega(x)V_i(x, D, 0),\quad t\downarrow 0. \tag{3.20}$$

Theorem 3.4 and (3.19), (3.20) now give two asymptotic expansions for $(\mathrm{Tr}\exp(-tD))^{\cdot}$, and comparing terms, we get (3.18). □

COROLLARY 3.7: *Let M be a smooth compact manifold of dimension $n>1$ and without boundary. Let g be a Riemannian metric on M, and consider the one-parameter conformal family of metrics $g(u)=\mathrm{e}^{2u\omega}g$, where ω is fixed in $C^\infty(M,\mathbb{R})$. Let $\mathrm{d}\,\mathrm{vol}(u)$ be the Riemannian measure and $D(u)=\Delta(u)+(n-2)K(u)/4(n-1)$ the conformal Laplacian determined by $g(u)$. Let $U_i(x, D, u)$ be the level $2i$ local scalar invariants given by* (3.9):

$$V_i(x, D, u) = U_i(x, D, u)\,\mathrm{d}\,\mathrm{vol}(u).$$

(*Recall Section* 2.) *Then*

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\int_M U_i\,\mathrm{d}\,\mathrm{vol}\right) = (n-2i)\int_M \omega U_i\,\mathrm{d}\,\mathrm{vol}, \tag{3.21}$$

and in fact

$$\frac{\mathrm{d}U_i}{\mathrm{d}u}\cong -2i\omega U_i\ (\mathrm{mod\ range}\ \delta), \tag{3.22}$$

where $\delta=d^*$ is the divergence. In particular, $c(M, g, D)=\int_M U_{n/2}\,\mathrm{d}\,vol$ is a conformal invariant for even n.

PROOF: We only need to prove (3.21) at $u=0$, since the statement about $(\mathrm{d}/\mathrm{d}u)|_{u=u_0}$ will follow upon replacing g by $e^{2u_0\omega}g$. But this is just an application of Theorem 3.6 with $a=(n-2)/2$, $b-a=2$, and $l=1$ (recall (1.2)). The argument for (3.22) was given in Section 2. (3.21) shows that $c(M, g, D)$ is constant on one-parameter conformal families of metrics, and thus on the conformal class of g. □

It should be noted that Corollary 3.7 is directly accessible by elementary methods using transport equations. This was our first approach (Branson-Ørsted [1984]), and it is still of some potential interest, especially for the case of noncompact M. Therefore, we have given an elementary treatment of the case of the conformal Laplacian D is an appendix (Section 5). We also do some explicit calculations with U_0, U_1 and U_2 of D in Section 4.

There are other known formally self-adjoint, conformally covariant operators whose leading symbols are positive definite. Branson [1982] introduced a general second-order conformally covariant operator $D_{2,k}$ on differential forms of arbitrary order k, with leading term $(n-2k+2)\delta\mathrm{d}+(n-2k-2)\,\mathrm{d}\delta$, for $n\neq 1, 2$. Now the symbol of d is exterior multiplication $\epsilon(\xi)$, and that of δ interior multiplication $\iota(\xi)$. $\iota(\xi)\epsilon(\xi)$ and $\epsilon(\xi)\iota(\xi)$ are positive semidefinite for $\xi\neq 0$, and $\iota(\xi)\epsilon(\xi)+\epsilon(\xi)\iota(\xi)=|\xi|^2$. Thus $D_{2,k}$ is grist for our mill if $k<(n-2)/2$. Paneitz [1983] gave a general fourth-order conformally covariant operator D_4 on functions, with leading term Δ^2, for $n\neq 1, 2$. Branson [1984] generalized this to a fourth-order $D_{4,k}$ on k-forms, $n\neq 1, 2, 4$, with leading term $(n-2k+4)(\delta\mathrm{d})^2+(n-2k-4)(\mathrm{d}\delta)^2=\Delta\{(n-2k+4)\delta\mathrm{d}+(n-2k-4)\,\mathrm{d}\delta\}$. This has positive definite leading symbol for $k<(n-4)/2$. In each case the number $b-a$ appearing in (3.17) (recall also (1.4)) is the order of the operator $2l$, so we have:

COROLLARY 3.8: *For* $D=D_{2,k}$ $(k<(n-2)/2)$, D_4 $(n>2)$, *or* $D_{4,k}$ $(k<(n-4)/2)$, *the local scalar invariants* U_i *given by* (3.9) *through* $V_i=U_i\,\mathrm{d}\,\mathrm{vol}$ *satisfy* (3.21) *and* (3.22), *and* $c(M, g, D)=\int_M U_{n/2}\,\mathrm{d}\,\mathrm{vol}$ *is a conformal invariant for even* n. □

More generally, we have Corollary 3.10 below.

DEFINITION 3.9: The following tensor fields on a Riemannian manifold (M, g) are said to be *natural*: (1) g, (2) $g^{\#}=(g^{\alpha\beta})$, (3) the Riemann tensor R; and if S and T are natural tensors, (4) CT, where C is any contraction, (5) ∇T, where ∇ is the Riemannian covariant derivative, (6) $S\otimes T$. The following differential operators on tensor fields are said to be natural: (1) $T\otimes$, where T is a natural tensor, (2) any transposition of tensor indices, (3) ∇, (4) contractions, (5) compositions of (1)–(4).

COROLLARY 3.10: *Let* D *be a natural differential operator of order* $2l$, *on tensor fields of a certain degree and symmetry type in Riemannian manifolds. Suppose* *that D is always formally self-adjoint, always has positive definite leading symbol, is $(-2l)$-homogeneous under uniform dilations,*

$$\bar{g} = A^2 g \Rightarrow \bar{D} = A^{-2l} D, \quad 0 < A \in \mathbb{R}, \tag{3.23}$$

and is conformally covariant:

$$\bar{g} = \Omega^2 g \Rightarrow Dm_{\Omega^a} = \Omega^b \bar{D}. \tag{3.24}$$

Then the local scalar invariants U_l given by (3.9) *through $\mathcal{U}_l = U_l$* d vol *satisfy* (3.21) *and* (3.22), *and $c(M, g, D) = \int_M U_{n/2}$* d vol *is a conformal invariant for even n.*

PROOF: We need only observe that (3.23) and (3.24) together force $b - a = 2l$. □

4. Remarks, further results, and some explicit calculations

a. The proof of Theorem 3.6 has content even for operators which are not conformally covariant. For example, let $P = \Delta + aK = D + bK$ on functions, where $a \in \mathbb{R}$ and D is the conformal Laplacian (so that $b = a - (n-2)/4(n-1)$). Then

$$\dot{P} = -2\omega P + \frac{n-2}{2}[P, m_\omega] + 2(n-1)b\,\Delta\omega$$

by (1.2) and (2.2), and the proof of Theorem 3.6 shows that

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\int_M U_i \,\mathrm{d}\,\mathrm{vol}\right) = (n-2i)\int_M \omega U_i \,\mathrm{d}\,\mathrm{vol} - 2(n-1)b \times \int_M (\Delta\omega) U_{i-1} \,\mathrm{d}\,\mathrm{vol}. \tag{4.1}$$

This gives a non-invariance result: if the "extra" term in (4.1) is to be zero for all $\omega \in C^\infty(M, \mathbb{R})$, we must have $U_{i-1} \perp$ range Δ in L^2:

THEOREM 4.1: *Let n be even, and let P be as above with $b \neq 0$. (In particular, P could be the ordinary Laplacian on functions if $n > 2$.) Then a metric g is critical for the functional $\int_M U_{n/2}$* d vol *within its conformal class if and only if $U_{(n-2)/2}(g)$ is constant. In particular, if $n = 4$, g is critical for $\int_M U_2$* d vol *within its conformal class if and only if g has constant scalar curvature.* □

The assertion about $n=4$ comes from (2.3). When $n=2$, $U_{(n-2)/2}=U_0=(4\pi)^{-1}$ so $\int_M U_1\,\mathrm{d}\,\mathrm{vol}$ is always a conformal invariant, as we know it should be: U_1 must be a multiple of K, and $\int_M K\,\mathrm{d}\,\mathrm{vol}$ is a *topological* invariant in dimension 2.

In another direction, let (M, g) be a compact Riemannian manifold with spin structure, and consider the Dirac operator P on the spin bundle S. With the inner product given by the adjunction $A_x: S_x \to S_x^*$ (see, e.g., Kosmann [1972] Sec. I.3.2), S is a Hermitian vector bundle and P is self-adjoint. Though P^2 is not conformally covariant, it still produces a global conformal invariant. Recall (Kosmann [1975]) that P is covariant: if $0<\Omega\in C^\infty(M)$,

$$\bar{g}=\Omega^2 g, \quad \bar{\gamma}=\Omega^{-1}\gamma \Rightarrow P\big(\Omega^{(n-1)/2}\psi\big)=\Omega^{(n+1)/2}\bar{P}\psi,$$

where γ is the fundamental section of $TM\otimes S\otimes S^*$. (The convention for identifying spinors produced by conformally related metrics is as in Choquet-Bruhat and Christodoulou [1981].) But for $\Omega=\exp(u\omega)$ and ${}^{\cdot}=\mathrm{d}/\mathrm{d}u$,

$$\begin{aligned}(P^2)^{\cdot} &= P\dot{P}+\dot{P}P\\ &= P\Big(-\omega P+\frac{n-1}{2}[P, m_\omega]\Big)+\Big(-\omega P+\frac{n-1}{2}[P, m_\omega]\Big)P\\ &= -\omega P^2-Pm_\omega P+\frac{n-1}{2}\big[P^2, m_\omega\big].\end{aligned}$$

To follow the proof of Theorem 3.6, we need to calculate $\mathrm{Tr}_{L^2}((P^2)^{\cdot}\exp(-tP^2))$. From the unfamiliar term in $(P^2)^{\cdot}$, we get

$$\begin{aligned}\mathrm{Tr}\big(Pm_\omega P\exp(-tP^2)\big) &= \mathrm{Tr}\big(\omega P\exp(-tP^2)P\big)\\ &= \mathrm{Tr}\big(\omega^2\exp(-tP^2)\big),\end{aligned}$$

since the fact that $\exp(-tP^2)$ is infinitely smoothing justifies cyclic permutation of operators under the trace. Thus

$$\mathrm{Tr}\big((P^2)^{\cdot}\exp(-tP^2)\big)=-2\,\mathrm{Tr}\big(\omega P^2\exp(-tP^2)\big),$$

and we have the conclusion of Corollary 3.7 for the local invariants U_i produced by P^2:

$$\Big(\int_M U_i\,\mathrm{d}\,\mathrm{vol}\Big)^{\cdot}=(n-2i)\int_M \omega U_i\,\mathrm{d}\,\mathrm{vol},$$

and $c(M, g, \gamma, P^2) = \int_M U_{n/2} \, d \operatorname{vol}$ is a conformal invariant for even n.

b. Let R, r, and K be the Riemann, Ricci, and scalar curvatures, with sign conventions that make r and K positive on standard spheres. Let

$$J = \frac{1}{2(n-1)} K, \quad V^\alpha{}_\beta = \frac{1}{n-2}\left(r^\alpha{}_\beta - J\delta^\alpha{}_\beta\right), \quad n > 2,$$

$$C^{\alpha\beta}{}_{\lambda\mu} = R^{\alpha\beta}{}_{\lambda\mu} + V_\lambda{}^\beta \delta^\alpha{}_\mu - V_\mu{}^\beta \delta^\alpha{}_\lambda + V_\mu{}^\alpha \delta^\beta{}_\lambda - V_\lambda{}^\alpha \delta^\beta{}_\mu, \quad n > 2.$$

C is the *Weyl conformal curvature tensor*, and V and J carry the information in r and K in a way that is convenient for conformal deformation theory.

Consider the operators $P = \Delta + aK = D + bK$ on functions (b not necessarily nonzero), where D is the conformal Laplacian. For convenience, in this section only, we write U_i for what was formerly $(4\pi)^{n/2} U_i$. By Gilkey [1975],

$$U_0 = 1,$$

$$U_1 = \left(\tfrac{1}{6} - a\right) K = 2\left(\tfrac{1}{6} - a\right)(n-1) J,$$

$$\begin{aligned} 180 U_2 &= \| R \|^2 - \| r \|^2 + 90\left(\tfrac{1}{6} - a\right)^2 K^2 - 30\left(\tfrac{1}{5} - a\right) \Delta K \\ &= \| C \|^2 - (n-2)(n-6) \| V \|^2 \\ &\quad + \left[8 - 3n + 360\left(\tfrac{1}{6} - a\right)^2 (n-1)^2\right] J^2 \\ &\quad - 60\left(\tfrac{1}{5} - a\right)(n-1) \Delta J, \end{aligned} \tag{4.2}$$

using $V^\alpha{}_\alpha = J$, where $\| R \|^2 = R^{\alpha\beta\lambda\mu} R_{\alpha\beta\lambda\mu}$, $\| r \|^2 = r^{\alpha\beta} r_{\alpha\beta}$, etc. In the special case $b = 0$,

$$\tfrac{1}{6} - a = \frac{4-n}{12(n-1)}, \quad \tfrac{1}{5} - a = \frac{6-n}{20(n-1)}, \tag{4.3}$$

so

$$U_1(D) = \frac{4-n}{6} J,$$

$$\begin{aligned} 180 U_2(D) &= \| C \|^2 - (n-2)(n-6) \| V \|^2 \\ &\quad + \tfrac{1}{2}(5n-16)(n-6) J^2 + 3(n-6) \Delta J. \end{aligned}$$

(3.22) predicts that

$$U_i(D)^{\bullet} = -2 i \omega U_i(D) + (\text{exact divergence}).$$

By Branson [1984], Secs. 1d and 2a,

$$
\begin{aligned}
\dot{J} &= -2\omega J + \Delta\omega \\
(J^2)^{\cdot} &= -4\omega J^2 + 2J\Delta\omega \\
(\|V\|^2)^{\cdot} &= -4\omega\|V\|^2 - 2V^{\alpha}{}_{\beta}\omega_{|\alpha}{}^{\beta} \\
(\|C\|^2)^{\cdot} &= -4\omega\|C\|^2 \\
(\Delta J)^{\cdot} &= -4\omega\Delta J + \Delta^2\omega - (n-6)J_{|\alpha}\omega_{|}{}^{\alpha} - 2J\Delta\omega.
\end{aligned}
\tag{4.4}
$$

This and the Bianchi identity $V^{\alpha}{}_{\beta|\alpha} = J_{|\beta}$ give an explicit expression for $U_2(D)^{\cdot} + 4\omega U_2(D)$ as an exact divergence:

$$
\begin{aligned}
&180\{U_2(D)^{\cdot} + 4\omega U_2(D)\} \\
&\quad = (n-6)\nabla^{\beta}\{2(n-2)V^{\alpha}{}_{\beta}\omega_{|\alpha} - (5n-22)J\omega_{|\beta} + 3\omega_{|\alpha}{}^{\alpha}{}_{\beta}\}.
\end{aligned}
$$

When $n = 6$, $U_2(D)^{\cdot} + 4\omega U_2(D) = 0$, as we know it should be by Schimming's remarks on $U_{(n-2)/2}$. When $n = 4$, $U_1(D)$ is a linear combination of $\|C\|^2$, an exact divergence, and the Pfaffian (Euler characteristic density)

$$
\begin{aligned}
\mathrm{Pff} &= \frac{1}{32\pi^2}(\|R\|^2 - 4\|r\|^2 + K^2) \\
&= \frac{1}{32\pi^2}(\|C\|^2 - 8\|V\|^2 + 8J^2),
\end{aligned}
$$

since

$$
n = 4 \Rightarrow 180U_2(D) = \|C\|^2 + 4\|V\|^2 - 4J^2 - 6\Delta J.
$$

Of course, $U_1 + 2\omega U_1 = 2(\frac{1}{6} - a)(n-1)\Delta\omega$ is an exact divergence for all n and any $P = \Delta + aK$.

We can also make a computational test of (4.1); here we expect

$$
U_l(P)^{\cdot} + 2l\omega U_l(P) \cong -2(n-1)b(\Delta\omega)U_{l-1} \ (\text{mod range } \delta). \tag{4.5}
$$

The U_0 and U_1 cases are clear; for U_2,

$$
\begin{aligned}
&180(U_2(P) - U_2(D)) \\
&\quad = 360\left(b^2 + 2b\cdot\frac{n-4}{12(n-1)}\right)(n-1)^2J^2 + 60b(n-1)\Delta J
\end{aligned}
$$

by (4.2) and (4.3). Thus by (4.4),

$$\begin{aligned}
&U_2(P)^{\cdot} + 4\omega U_2(P)\\
&\quad\cong (n-1)b\Big\{\big[4(n-1)b + \tfrac{2}{3}(n-4)\big]\,J\Delta\omega\\
&\qquad\qquad - \tfrac{1}{3}\big[(n-6)J_{|\alpha}\omega_{|}{}^{\alpha} + 2J\Delta\omega\big]\Big\}\\
&\quad\cong (n-1)b\{4(n-1)b + \tfrac{2}{3}(n-4) - \tfrac{1}{3}(n-6) - \tfrac{2}{3}\}J\Delta\omega\\
&\quad\cong (n-1)b\{4(n-1)b + \tfrac{1}{3}(n-4)\}J\Delta\omega \quad (\text{mod range } \delta),
\end{aligned}$$

since $J_{|\alpha}\omega_{|}{}^{\alpha} - J\Delta\omega = \nabla^{\alpha}(J\omega_{|\alpha}) \cong 0$. But by (4.2),

$$U_1(P) = U_1(D) - 2b(n-1)J = \left\{\frac{4-n}{6} - 2b(n-1)\right\}J,$$

so that

$$U_2(P)^{\cdot} + 4\omega U_2(P) \cong -2(n-1)bU_1\Delta\omega \quad (\text{mod range } \delta),$$

as predicted by (4.5).

c. To treat the question of conformal stability for the functionals $\int_M U_{n/2}(D_2 + bK)\mathrm{d}\,\mathrm{vol}$ of Theorem 4.1, one must calculate their *second* conformal variation. By (4.1), this is equivalent to calculating the first variation of $U_{(n-2)/2}$. One can do this by adapting the argument of Schimming mentioned in Sec. 2. Details of this calculation will appear separately.

d. The isospectrality argument in the proof of Theorem 3.6 could be replaced by the observation (made in the proof of Proposition 3.5) that the presence of an infinitely smoothing operator justifies cyclic permutation of operators under the trace:

$$\begin{aligned}
\mathrm{Tr}_{L^2}(Dm_{\omega}\exp(-tD)) &= \mathrm{Tr}_{L^2}(m_{\omega}\exp(-tD)D)\\
&= \mathrm{Tr}_{L^2}(m_{\omega}D\exp(-tD)),
\end{aligned}$$

so that $\mathrm{Tr}_{L^2}([D, m_{\omega}]\exp(-tD)) = 0$. This calculation goes through if m_{ω} is replaced by any finite-order pseudo-differential operator. Two curious consequences of this are obtained of we consider a conformal vector field T on (M, g),

$$L_T g = 2\omega g, \tag{4.6}$$

where L_T is the Lie derivative and $\omega \in C^\infty(M, \mathbb{R})$. (In classical notation, (4.6) reads $\nabla_\alpha T_\beta + \nabla_\beta T_\alpha = 2\omega g_{\alpha\beta}$, so necessarily $n\omega = \nabla^\lambda T_\lambda$.) If D is a conformally covariant operator satisfying (1.4), then

$$[D, L_T] + a[D, m_\omega] = (b-a)\omega D; \tag{4.7}$$

see, e.g., Branson [1984]. If D is as in Section 3, the left-hand side of (4.7) contributes nothing upon multiplying by $\exp(-tD)$ and tracing: if $b \neq a$,

$$\begin{aligned} 0 &= \operatorname{Tr}(\omega D \exp(-tD)) \\ &\sim -\sum_{i=0}^{\infty} \left(\frac{\mathrm{d}}{\mathrm{d}t} t^{(2i-n)/2l}\right) \int_M \omega U_i(D)\, \mathrm{d}\,\mathrm{vol}. \end{aligned} \tag{4.8}$$

(Recall the argument giving (3.19), (3.20).)
This yields:

COROLLARY 4.2: *Let D be a natural differential operator* (*in the sense of* Definition 3.9) *which is conformally covariant in the sense of* (1.4), *with $b \neq a$. Suppose that D is always formally self-adjoint with positive definite leading symbol, and let U_i be the local scalar invariants given by* (3.9) *through $V_i = U_i$* d vol. *If ω is an infinitesimal conformal factor as in* (4.6), *then $\omega \perp U_i$ in L^2 for $i \neq n/2$.* □

If we knew that the conformal vector field T integrated to a local one-parameter group $\{h_u\}$ of *global* transformations (necessarily conformal), we could argue alternatively that the $[D, L_T]$ term in (4.7) contributes nothing because

$$[D, L_T] = -\frac{\mathrm{d}}{\mathrm{d}u}\left((h_u^{-1}\cdot)Dh_u\cdot\right),$$

and the $(h_u^{-1}\cdot)Dh_u\cdot$ are isospectral. Here $h_u\cdot$ is the natural action of the diffeomorphism h_u on tensors (Helgason [1978], p. 90).

Using (4.8) in another way, we get

$$\begin{aligned} 0 &= \operatorname{Tr}_{L^2}(\omega D \exp(-tD)) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_M \omega(x) \operatorname{Tr}_{F_x} H(t, x, x) \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=0}^{\infty} \mathrm{e}^{-\lambda_j t}\int_M \omega |\varphi_j|^2\, \mathrm{d}\,\mathrm{vol} = \sum_{j=0}^{\infty} \lambda_j\, \mathrm{e}^{-\lambda_j t}\int_M \omega |\varphi_j|^2\, \mathrm{d}\,\mathrm{vol} \end{aligned}$$

for all $t > 0$, where H is the kernel function for $\exp(-tD)$, and $\{\varphi_j\}$ is an orthonormal basis of $L^2(F)$ with $D\varphi_j = \lambda_j \varphi_j$. $|\varphi_j|^2$ is calculated in the natural Riemannian structure carried by tensor bundles over a Riemannian manifold. But we cannot have $\sum a_k e^{-\mu_k t} = 0$ for all $t > 0$ for distinct μ_k unless all the a_k are zero. Thus we have:

COROLLARY 4.3: *Let D be as in Corollary 4.2, and suppose that ω is an infinitesimal conformal factor as in (4.6). Let $\psi_{j,1}, \dots, \psi_{j,m_j}$ be an orthonormal basis of the λ_j eigenspace of D, $\lambda_j \neq 0$. Then*

$$\omega \perp |\psi_{j,1}|^2 + \cdots + |\psi_{j,m_j}|^2 \text{ in } L^2. \quad \square \tag{4.9}$$

On the sphere S^n with the standard metric, a basis of the infinitesimal conformal factors ω is given by the homogeneous coordinate functions $x^0, \dots, x^n$. Since S^n is a conformally flat Einstein manifold, all local scalar invariants are constant. Since $\int_{S^n} x^\alpha \, \mathrm{d\,vol} = 0$, Corollary 4.2 checks in this example. For Corollary 4.3, at least in the case where D is the conformal Laplacian on functions $\Delta + (n-2)K/4(n-1) = \Delta + (n-2)n/4$, note that each eigenspace consists of polynomials in the x^α of a certain homogeneity. Thus if Ψ_j^2 is the function on the right in (4.9), $x^\alpha \Psi_j^2$ is odd in x^α, so that $\int_{S^n} x^\alpha \Psi_j^2 \, \mathrm{d\,vol} = 0$.

e. The ideas in this paper are naturally connected with the theory of *Lax pairs* (Kubawara [1982]). Take, for example, one of the operators $P = \Delta + aK$ on functions and vary the metric $g = g_u$. If we can find a curve B_u of skew-adjoint (unbounded) operators on L^2 with

$$\frac{\mathrm{d}P_u}{\mathrm{d}u} = [P_u, B_u], \tag{4.10}$$

then the operator family P_u will lie within a unitary equivalence class, and thus be *isospectral*. (This setup requires that the L^2 inner product, and thus the Riemannian measure $\mathrm{d\,vol}_u$, be independent of u. Kubawara [1982] shows that this can be arranged by absorbing the change in $\mathrm{d\,vol}_u$ into the action of the diffeomorphism group, which acts trivially from the point of view of isospectrality.) Because of problems with domains of unbounded operators, one cannot necessarily make sense of (4.10). But if we assume that the B_u are finite order pseudo-differential operators, (4.10) makes sense on $C^\infty(M)$, a dense subspace of $L^2(M, g)$. The Ray-Singer formula (Proposition 3.5) and the observation at the beginning of Section 4.d then show that the spectral distribution $\sum e^{-t\lambda_j}$ is independent of u, even if the B_u are not skew:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\left(\sum e^{-t\lambda_j}\right) &= \frac{\mathrm{d}}{\mathrm{d}u}\mathrm{Tr}\exp(-tP) = -t\,\mathrm{Tr}\left(\frac{\mathrm{d}P}{\mathrm{d}u}\exp(-tP)\right)\\ &= -t\,\mathrm{Tr}([P, B]\exp(-tP)) = 0.\end{aligned} \tag{4.11}$$

The point of view of the present paper is that if

$$\frac{\mathrm{d}P_u}{\mathrm{d}u} = [P_u, B_u] + Q_u$$

for some nicely behaved Q_u, only Q_u affects the first variation in the spectral distribution. Analysis of $\mathrm{Tr}_{L^2}Q \exp(-tP)$ then reveals which aspects of the spectral distribution are stationary under the deformation; in our examples, these are the conformal indices.

5. Appendix: The transport equations

In this section we give the original "classical" proof of Corollary 3.7 outlined in Branson-Ørsted [1984]. The u-differentiability implied in (3.21) is no problem, since universal local scalar invariants and the Riemannian measure are u-differentiable. The strategy for proving (3.21) is to look at the initial value problem which defines the heat kernel,

$$\begin{cases} (\partial_t + D_x)H(t, x, y) = 0 \\ H(t, \cdot, y) \to \delta_y(\cdot), \quad t \downarrow 0. \end{cases} \tag{5.1}$$

where $D = \Delta + (n-2)K/4(n-1)$ on functions and δ is the Dirac distribution; and its formal conformal variation

$$\begin{cases} (\partial_t + D_x)h(t, x, y) + \dot{D}_x H(t, x, y) = 0 \\ h(t, \cdot, y) \to -n\omega(y)\delta_y(\cdot), \quad t \downarrow 0. \end{cases} \tag{5.2}$$

Here $\dot{D}$ is $(\mathrm{d}/\mathrm{d}u)|_{u=0}D$ as the metric runs through a one-parameter family $\mathrm{e}^{2u\omega}g$, $\omega \in C^\infty(M, \mathbb{R})$. The initial condition in (5.2) is motivated by the fact that $(\mathrm{d\ vol})^\cdot = n\omega \,\mathrm{d\ vol}$, and we now think of our kernels as functions, not measures. Calculating the unique solution to (5.2) in two different ways will give us two asymptotic expansions of $\int_M h(t, y, y)\,\mathrm{d\ vol}(y)$ as $t \downarrow 0$, and termwise comparison will give (3.21). We will not need to prove that $h = \dot{H}$.

The first expansion comes from the integral formula for the solution of (5.2):

$$h(t, x, y) = -n\omega(y)H(t, x, y) \\ - \int_0^t \mathrm{d}\tau \int_M H(t-\tau, x, z)\dot{D}_z H(\tau, z, y)\,\mathrm{d}z.$$

(Convergence of the integral near $\tau = t$ is clear; for convergence near $\tau = 0$ rewrite the integral as $\int_M ((\dot{D}_z)^* H(t-\tau, x, z)) H(\tau, z, y)\, \mathrm{d}z$. Here and below, we write $\mathrm{d}z$ for $\mathrm{d}\,\mathrm{vol}(z)$.) This gives

$$\int_M h(t, y, y)\, \mathrm{d}y = -n \int_M \omega(y) H(t, y, y)\, \mathrm{d}y$$
$$- \int_M \mathrm{d}y \int_0^t \mathrm{d}\tau \int_M H(t-\tau, y, z) \dot{D}_z H(\tau, z, y)\, \mathrm{d}z. \tag{5.3}$$

The second term on the right in (5.3) can be rewritten as

$$- \int_0^t \mathrm{d}\tau \int_M \mathrm{d}z \dot{D}_z \left\{ \int_M \mathrm{d}y H(t-\tau, y, w) H(\tau, z, y) \right\} \Big|_{w=z}$$
$$= - \int_0^t \mathrm{d}\tau \int_M \mathrm{d}z \dot{D}_z H(t, z, w) \,|_{w=z}$$
$$= -t \int_M \mathrm{d}z \dot{D}_z H(t, z, w) \,|_{w=z}$$

by the semigroup law for the heat kernel. Writing $\dot{D}_z H(t, z, w)\,|_{w=z} = \dot{D}_{(1)} H(t, z, z)$ to indicate that $\dot{D}$ acts in the first space variable, we have

$$\int_M h(t, y, y)\, \mathrm{d}y = -n \int_M \omega(y) H(t, y, y)\, \mathrm{d}y$$
$$- t \int_M \dot{D}_{(1)} H(t, y, y)\, \mathrm{d}y.$$

(This can be regarded as a "classical" version of the Ray-Singer formula (3.16).) Using our formula (1.2) for $\dot{D}$ and the fact that H solves the heat equation, we get

$$\int_M h(t, y, y)\, \mathrm{d}y = -n \int_M \omega(y) H(t, y, y)\, \mathrm{d}y$$
$$- 2t \int_M \omega(y)\, \partial_t H(t, y, y)\, \mathrm{d}y$$
$$- \frac{n-2}{2} t \int_M \left[D_{(1)}, m_{\omega(y)} \right] H(t, y, y)\, \mathrm{d}y. \tag{5.4}$$

We would now like to show that the last term on the right in (5.4) vanishes. Consider the problem

$$\begin{cases} (\partial_t + D_x)k(t, x, y) + [D_x, m_{\omega(x)}]H(t, x, y) = 0 \\ k(t, \cdot, y) \to -2\omega(y)\delta_y(\cdot), \quad t \downarrow 0. \end{cases} \tag{5.5}$$

The (unique) solution of (5.5) is clearly $k(t, x, y) = -(\omega(x) + \omega(y))H(t, x, y)$; yet the argument carried out above in the case of h gives

$$\int_M k(t, y, y)\, \mathrm{d}y = -2\int_M \omega(y)H(t, y, y)\, \mathrm{d}y$$
$$-t\int_M [D_{(1)}, m_{\omega(y)}]H(t, y, y)\, \mathrm{d}y.$$

This shows that $\int_M [D_{(1)}, m_{\omega(y)}]H(t, y, y)\, \mathrm{d}y = 0$, so (5.4) implies

$$\int_M h(t, y, y)\, \mathrm{d}y + n\int_M \omega(y)H(t, y, y)\, \mathrm{d}y$$
$$\sim \sum_{i=0}^{\infty} (n-2i)\left(\int_M \omega(y)U_i(y)\, \mathrm{d}y\right)t^{i-n/2}, \quad t \downarrow 0, \tag{5.6}$$

where the U_i are the local invariants of Corollary 3.7. (That the asymptotic expansion of $H(t, x, y)$ can be differentiated term by term with respect to t can be shown by carefully following the argument in Berger et al. [1971]. Specifically, it is easy to show that one can take D_x term by term; the result then follows from the transport and heat equations. The argument with k is a "classical" version of the isospectrality argument in the proof of Theorem 3.6. The operator $m_{\Omega^{-1}}Dm_{\Omega}$ is formally self-adjoint in the smooth measure Ω^2 d vol$_g$, and the initial condition in (5.5) just keeps track of this change of measure.)

To get another asymptotic expansion for $\int_M h(t, y, y)\, \mathrm{d}y$, we examine the classical construction of H via the transport equations as in Berger et al. [1971]. Let ϵ be a uniform injectivity radius for the exponential map on M, and let $\eta(x, y) \in C^\infty(M \times M, [0, 1])$ be a cutoff function which is 1 in a neighborhood of the diagonal $\{x = y\}$ and 0 on $\{\mathrm{dist}(x, y) \geqslant \epsilon\}$. Then H has an asymptotic expansion

$$H(t, x, y) \sim \mathrm{e}^{-\sigma/4t}\eta \sum_{l=0}^{\infty} \varphi_l(x, y)t^{l-n/2}, \quad t \downarrow 0, \tag{5.7}$$

where $\sigma = \sigma(x, y) = (\text{dist}(x, y))^2$, and the two-point functions $\varphi_i \in C^\infty(\{\sigma < \epsilon^2\}, \mathbb{R})$ are uniquely defined by the system

$$\begin{cases} 2\langle \mathrm{d}\sigma, \mathrm{d}\varphi_0\rangle - (\Delta\sigma + 2n)\varphi_0 = 0, & \varphi_0(y, y) = (4\pi)^{-n/2} \\ 2\langle \mathrm{d}\sigma, \mathrm{d}\varphi_i\rangle - (\Delta\sigma + 2n - 4i)\varphi_i = -4D\varphi_{i-1}, & \\ \varphi_i \text{ bounded as } x \to y, \quad i > 0. & \end{cases}$$

Here d, Δ, and D act in the x variable, and $\langle\ ,\ \rangle$ is the pairing of one-forms determined by g.

Letting everything depend on the conformal parameter u, we can find an injectivity radius which is uniform over M, and uniform for u in some $[-\delta, \delta]$, by the standard theorem on smooth dependence on coefficients for solutions of initial value ODE problems (see, e.g., Miller and Murray [1954], Sec. 4.5). Thus we can pick a cutoff η which is independent of u. Formal term-by-term differentiation of the right-hand side of (5.7) then gives

$$\mathrm{e}^{-\sigma/4t}\eta \sum_{i=0}^{\infty}\left(-\frac{\dot\sigma}{4t}\varphi_i + \dot\varphi_i\right)t^{i-n/2}. \tag{5.8}$$

We pause to calculate $\dot\sigma$.

LEMMA 5.1: *Let ϵ be as above. Then σ is C^∞ in u on $\{\sigma < \epsilon^2\}$, and $\dot\sigma = 2\tilde\omega\sigma$, where $\tilde\omega = \tilde\omega(x, y)$ is the average of ω along the g-geodesic $z(r)$, $0 \leqslant r \leqslant 1$, joining y to x:*

$$\tilde\omega = \int_0^1 \omega(z(r))\,\mathrm{d}r.$$

PROOF: Fix $y \in M$, and let $v_1, \dots, v_n$ be a g-orthonormal basis of the tangent space M_y, so that $v_\alpha(u) = \mathrm{e}^{-u\omega(y)}v_\alpha$ is an $\mathrm{e}^{2u\omega}g$-orthonormal basis. Let $x^\alpha(u)$ be normal coordinates adapted to these bases: the exponential at y of $x^\alpha(u)v_\alpha(u)$ in the metric $\mathrm{e}^{2u\omega}g$ is x. Then by the above smooth dependence theorem and the smooth dependence of inverse functions on parameters, x^α is C^∞ in u on $\{\sigma < \epsilon^2\}$. Since $\sigma = g_{\alpha\beta}(y)x^\alpha x^\beta$ (see, e.g., Friedlander [1975], Theorem 1.2.3), σ is C^∞ in u.

Working now in a fixed coordinate system on $\{x \mid \sigma(x, y) < \epsilon^2\}$, if $z(r)$, $0 \leqslant r \leqslant 1$ is the geodesic joining y to x, then

$$\sigma = g_{\alpha\beta}(z(r))\frac{\mathrm{d}z^\alpha}{\mathrm{d}r}\frac{\mathrm{d}z^\beta}{\mathrm{d}r},$$

independent of r (i.e., $z(r)$ has constant speed). Thus

$$\dot{\sigma} = \left(\int_0^1 g_{\alpha\beta}(z(r)) \frac{\mathrm{d}z^\alpha}{\mathrm{d}r} \frac{\mathrm{d}z^\beta}{\mathrm{d}r} \mathrm{d}r \right)^{\cdot}$$

$$= \int_0^1 \left\{ 2g_{\alpha\beta} \frac{\mathrm{d}z^\alpha}{\mathrm{d}r} \frac{\mathrm{d}\dot{z}^\beta}{\mathrm{d}r} + 2\omega(z(r))\sigma + (\partial_\mu g_{\alpha\beta}) \frac{\mathrm{d}z^\alpha}{\mathrm{d}r} \frac{\mathrm{d}z^\beta}{\mathrm{d}r} \dot{z}^\mu \right\} \mathrm{d}r. \tag{5.9}$$

Since $\dot{z}^\mu(0) = \dot{z}^\mu(1) = 0$,

$$\int_0^1 g_{\alpha\beta} \frac{\mathrm{d}z^\alpha}{\mathrm{d}r} \frac{\mathrm{d}\dot{z}^\beta}{\mathrm{d}r} \mathrm{d}r = -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} \left(g_{\alpha\beta} \frac{\mathrm{d}z^\alpha}{\mathrm{d}r} \right) \dot{z}^\beta \, \mathrm{d}r$$

$$= -\int_0^1 \left\{ g_{\alpha\beta} \frac{\mathrm{d}^2 z^\alpha}{\mathrm{d}r^2} + (\partial_\mu g_{\alpha\beta}) \frac{\mathrm{d}z^\mu}{\mathrm{d}r} \frac{\mathrm{d}z^\alpha}{\mathrm{d}r} \right\} \dot{z}^\beta \, \mathrm{d}r$$

$$= \int_0^1 (\Gamma_{\mu\lambda\beta} - \partial_\mu g_{\lambda\beta}) \frac{\mathrm{d}z^\mu}{\mathrm{d}r} \frac{\mathrm{d}z^\lambda}{\mathrm{d}r} \dot{z}^\beta \, \mathrm{d}r$$

$$= -\tfrac{1}{2} \int_0^1 (\partial_\beta g_{\mu\lambda}) \frac{\mathrm{d}z^\mu}{\mathrm{d}r} \frac{\mathrm{d}z^\lambda}{\mathrm{d}r} \dot{z}^\beta \, \mathrm{d}r,$$

by the geodesic equation

$$\frac{\mathrm{d}^2 z^\lambda}{\mathrm{d}r^2} = -\Gamma_{\alpha\beta}{}^\lambda(z(r)) \frac{\mathrm{d}z^\alpha}{\mathrm{d}r} \frac{\mathrm{d}z^\beta}{\mathrm{d}r},$$

where $\Gamma_{\alpha\beta}{}^\lambda = \tfrac{1}{2} g^{\mu\lambda}(\partial_\beta g_{\alpha\mu} + \partial_\alpha g_{\beta\mu} - \partial_\mu g_{\alpha\beta})$ are the Christoffel symbols. This reduces (5.9) to

$$\dot{\sigma} = 2\sigma \int_0^1 \omega(z(r)) \, \mathrm{d}r,$$

as desired. □

Returning to the main argument, the lemma and (5.8) suggest asymptotically expanding $h + \tilde{\omega}\sigma H/2t$ in the hope that the $\dot{\varphi}_l$ will appear. First we try to obtain two-point functions $\varphi_l(x, y)$ for which

$$\mathrm{e}^{-\sigma/4t} \sum_{l=0}^{\infty} \left(\psi_l - \frac{\tilde{\omega}\sigma}{2t} \varphi_l \right) t^{l-n/2} \tag{5.10}$$

formally solves (5.2). Setting $R = t^{-n/2}\,\mathrm{e}^{-\sigma/4t}$, we have

$$(\partial_t + D_x)\big(R\psi_i t^i\big)$$

$$= R\left\{\frac{1}{2t}\langle \mathrm{d}\sigma, \mathrm{d}\psi_i\rangle - \frac{1}{4t}(\Delta\sigma + 2n - 4i)\psi_i + D\psi_i\right\}t^i,$$

$$(\partial_t + D_x)\left(-\frac{\tilde{\omega}\sigma}{2t}R\varphi_i t^i\right)$$

$$= R\left\{\frac{\tilde{\omega}\sigma}{2t^2}\varphi_i - \frac{\Delta(\tilde{\omega}\sigma)}{2t}\varphi_i + \frac{1}{t}\langle \mathrm{d}(\tilde{\omega}\sigma), \mathrm{d}\varphi_i\rangle\right.$$

$$\left. - \frac{1}{4t^2}\langle \mathrm{d}(\tilde{\omega}\sigma), \mathrm{d}\sigma\rangle\varphi_i\right\}t^i,$$

$$\dot{D}\big(R\varphi_i t^i\big) = R\left\{\dot{D}\varphi_i + \frac{\omega\Delta\sigma}{2t}\varphi_i + \frac{\omega\sigma}{2t^2}\varphi_i - \frac{1}{t}\omega\langle \mathrm{d}\sigma, \mathrm{d}\varphi_i\rangle\right.$$

$$\left. + \frac{n-2}{4}\langle \mathrm{d}\sigma, \mathrm{d}\omega\rangle\varphi_i\right\}t^i,$$

by the product rule $D(fg) = fDg + (\Delta f)g - 2\langle \mathrm{d}f, \mathrm{d}g\rangle$, (1.2), and $\langle \mathrm{d}\sigma, \mathrm{d}\sigma\rangle = 4\sigma$. Here all differential operators act in the x variable, $\omega = \omega(x)$, and $\tilde{\omega} = \tilde{\omega}(x, y)$. Now if $r = \sqrt{\sigma}$ is the radial geodesic polar coordinate at y, $\partial_r\tilde{\omega} = (\omega - \tilde{\omega})/r$, or $\langle \mathrm{d}\sigma, \mathrm{d}\tilde{\omega}\rangle = 2(\omega - \tilde{\omega})$. Thus the series (5.10) will formally solve the differential equation for h if

$$2\langle \mathrm{d}\sigma, \mathrm{d}\psi_i\rangle - (\Delta\sigma + 2n - 4i)\psi_i - 2\Delta(\tilde{\omega}\sigma)\varphi_i + 4\langle \mathrm{d}(\tilde{\omega}\sigma), \mathrm{d}\varphi_i\rangle$$

$$+ 2\omega(\Delta\sigma)\varphi_i - 4\omega\langle \mathrm{d}\sigma, \mathrm{d}\varphi_i\rangle + (n-2)\langle \mathrm{d}\sigma, \mathrm{d}\omega\rangle\varphi_i$$

$$= -4D\psi_{i-1} - 4\dot{D}\varphi_{i-1}, \tag{5.11}$$

where we put $\psi_{-1} = \varphi_{-1} = 0$.

We claim that one solution of (5.11) is $\psi_i = \dot{\varphi}_i$. The explicit formulas

$$\varphi_0 = (4\pi)^{-n/2}$$

$$\varphi_i = -(4\pi)^{-n/2}\int_0^1 \left.\frac{D\varphi_{i-1}}{\varphi_0}\right|_{x=z(s)} s^{i-1}\,\mathrm{d}s, \quad i > 0$$

show that the φ_i are differentiable in u. The $\dot{\varphi}_i$ must satisfy the conformal variation of the transport equations for φ_i, viz.

$$
\begin{aligned}
&-2\langle \mathrm{d}\sigma, \mathrm{d}\dot{\varphi}_i\rangle + 4\langle \mathrm{d}(\tilde{\omega}\sigma), \mathrm{d}\varphi_i\rangle - 4\omega\langle \mathrm{d}\sigma, \mathrm{d}\varphi_i\rangle \\
&\quad -(\Delta\sigma + 2n - 4i)\dot{\varphi}_i - (\Delta(2\tilde{\omega}\sigma) + \dot{\Delta}\sigma)\varphi_i \\
&\quad = -4D\dot{\varphi}_{i-1} - 4\dot{D}\varphi_{i-1},
\end{aligned}
\tag{5.12}
$$

by Lemma 5.1 and $\langle\cdot,\cdot\rangle^{\cdot} = -2\omega\langle\cdot,\cdot\rangle$. But $\dot{\Delta} = -2\omega\Delta - (n-2)\langle \mathrm{d}\omega, \mathrm{d}\cdot\rangle$ by (1.2) and (2.2), so (5.12) is just (5.11) for $\psi_i = \dot{\varphi}_i$.

The above calculations show that if $\mathscr{H} = \partial_t + D_x$,

$$
H_k = R\eta \sum_{i=0}^{k} \varphi_i t^i, \quad \text{and} \quad h_k = R\eta \sum_{i=0}^{k} \left(\dot{\varphi}_i - \frac{\tilde{\omega}\sigma}{2t}\varphi_i\right) t^i,
$$

then

$$
\mathscr{H}h_k + \dot{D}H_k = t^{k-(n/2)} b_k, \quad b_k \in C^\infty([0,\infty)\times M\times M),
\tag{5.13}
$$

in analogy with

$$
\mathscr{H}H_k = t^{k-(n/2)} B_k, \quad B_k \in C^\infty([0,\infty)\times M\times M).
$$

This was the point of getting a formal solution of the h-equation. Since $\dot{\varphi}_0(y,y) = 0$ and $\sigma R/t \to 2n\delta_y(x)$ as $t\downarrow 0$, h_k has the right initial condition:

$$
h_k(t,\cdot,y) \to -n\omega(y)\delta_y(\cdot) \text{ as } t\downarrow 0.
\tag{5.14}
$$

Now recall the method by which the exact heat kernel H is constructed from the parametrix H_k (Berger et al. [1971], Sec. III.E). Define the *convolution* $F*G$ for, say, $F, G \in C^0([0,\infty)\times M\times M)$ by

$$
(F*G)(t,x,y) = \int_0^t \mathrm{d}\tau \int_M F(\tau,x,z)G(t-\tau,z,y)\,\mathrm{d}z.
$$

Then if $F \in C^0([0,\infty)\times M\times M)$ and $k > n/2$, $H_k * F$ exists, and

$$
\mathscr{H}(H_k * F) = F + (\mathscr{H}H_k) * F.
\tag{5.15}
$$

This means $H = H_k + H_k * F_k$ will satisfy the heat equation if $\mathscr{H}H_k + F_k + (\mathscr{H}H_k)*F_k = 0$, or

$$
F_k = \sum_{l=1}^{\infty} (-\mathscr{H}H_k)^{*l};
$$

for large k, $H - H_k = H_k * F_k = 0(\mathrm{t}^{k+1-(n/2)})$.

Because of the initial condition (5.14), the identity for h_k analogous to (5.15) is

$$\mathscr{H}(h_k{*}F)+\dot{D}(H_k{*}F)=-n\omega(y)F+(\mathscr{H}h_k+\dot{D}H_k)*F.$$

Thus a trial solution $h=h_k+h_k{*}F_k+H_k{*}f_k$ will satisfy $\mathscr{H}h+\dot{D}H=0$ if

$$\mathscr{H}h_k+\dot{D}H-n\omega(y)F_k+(\mathscr{H}h_k+\dot{D}H_k)*F_k-\dot{D}(H_k{*}F_k)+f_k$$
$$+(\mathscr{H}H_k)*f_k=0,$$

or, since $H-H_k=H_k{*}F_k$,

$$\mathscr{H}h_k+\dot{D}H_k-n\omega(y)F_k+(\mathscr{H}h_k+\dot{D}H_k)*F_k+f_k+(\mathscr{H}H_k)*f_k$$
$$=0. \tag{5.16}$$

Formally, this says that $(I+(\mathscr{H}H_k)^*)f_k=-L$, where

$$L=\mathscr{H}h_k+\dot{D}H_k-n\omega(y)F_k+(\mathscr{H}h_k+\dot{D}H_k)*F_k.$$

This leads us to look at

$$f_k=-L-F_k{*}L,$$

since $I+F_k{}^*$ formally inverts $I+(\mathscr{H}H_k)^*$, and it is straightforward to show that this f_k does indeed satisfy (5.16). Since F_k and $\mathscr{H}h_k+\dot{D}H_k$ are $0(t^{k-(n/2)})$ for large k, L and f_k are $0(t^{k-(n/2)})$ also, giving $h-h_k=h_k{*}F_k+H_k{*}f_k=0(t^{(k-(n/2)})$. Thus

$$h(t,x,y)\sim \mathrm{e}^{-\sigma/4t}\eta\sum_{\iota=0}^{\infty}\left(\dot{\varphi}_\iota(x,y)-\frac{\tilde{\omega}\sigma}{2t}\varphi_\iota(x,y)\right)t^{\iota-n/2},$$

so that

$$h(t,y,y)\sim\sum_{\iota=0}^{\infty}\dot{U}_\iota(y)t^{\iota-n/2}.$$

Since $(\mathrm{d}y)^{\cdot}=n\omega\,\mathrm{d}y$,

$$\int_M h(t,y,y)\,\mathrm{d}y$$
$$\sim\sum_{\iota=0}^{\infty}\left\{\left(\int_M U_\iota(y)\,\mathrm{d}y\right)^{\cdot}-n\int_M\omega(y)U_\iota(y)\,\mathrm{d}y\right\}t^{\iota-n/2}. \tag{5.17}$$

Comparing (5.17) and (5.6), we get

$$\left(\int_M U_i \,\mathrm{d}\,\mathrm{vol}\right)^{\bullet} = (n-2i)\int_M \omega U_i \,\mathrm{d}\,\mathrm{vol},$$

as desired for (3.21).

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