COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 60, nº 2 (1986), p. 237-250
<http://www.numdam.org/item?id=CM_1986__60_2_237_0>

[1] Compositio Mathematica 60: 237–250 (1986)

BASE CHANGE FOR UNIT ELEMENTS OF HECKE ALGEBRAS

Robert E. Kottwitz

One of the ingredients in the comparison of trace formulas involves matching the orbital integrals of spherical functions; this is what Langlands [L] refers to as the "fundamental lemma". There is a special case of the fundamental lemma that has a simple local proof. Let G be a connected reductive group over a p-adic field F and assume that G is unramified (that is, quasi-split over F and split over an unramified extension of F). Let E be a finite unramified extension of F, let θ be a generator of $\mathrm{Gal}(E/F)$, and let $l=[E:F]$.

Consider a hyperspecial point x_0 in the building of G over F. We denote by K the stabilizer of x_0 in $G(F)$ and by $\mathscr{H}=\mathscr{H}(G(F), K)$ the corresponding Hecke algebra. Of course x_0 also gives rise to $K_E \subset G(E)$ and $\mathscr{H}_E=\mathscr{H}(G(E), K_E)$. There is a canonical homorphism $b: \mathscr{H}_E \to \mathscr{H}$, characterized by the following property:

$$\mathrm{tr}\ \pi_\varphi(b(f)) = \mathrm{tr}\ \pi_\psi(f)$$

for all $f \in \mathscr{H}_E$ and all unramified admissible homomorphisms $\varphi: W_F \to {}^L G$. Here ψ denotes the restriction of φ to W_E, and π_φ (resp. π_ψ) denotes the K-spherical (resp. K_E-spherical) representation of $G(F)$ (resp. $G(E)$) corresponding to φ (resp. ψ).

The fundamental lemma for the homomorphism $b: \mathscr{H}_E \to \mathscr{H}$ relates the stable orbital integrals of $b(f)$ to the "stable" twisted orbital integrals of f for any $f \in \mathscr{H}_E$. The precise statement requires definitions for stable conjugacy, stable orbital integrals, the twisted analogues, and the norm mapping $\mathscr{N}$. All of these are easier to define if the derived group G_{der} is simply connected. To keep the exposition simple we will now assume that G_{der} is simply connected, and then in the last section of the paper we will sketch a proof of the general case.

There are two forms of the norm mapping. The first is the mapping $N: G(E) \to G(E)$ defined by

$$N\delta = \delta\theta(\delta)\theta^2(\delta)\dots\theta^{l-1}(\delta).$$

The second is a mapping $\mathscr{N}$ from $G(E)$ to the set of stable conjugacy

Partially supported by NSF Grant DMS-8200785.

classes in $G(F)$. Since G_{der} is simply connected, stable conjugacy is the same as $G(\bar{F})$-conjugacy, where $\bar{F}$ is an algebraic closure of F. The conjugacy class of $N\delta$ in $G(\bar{F})$ is defined over F and therefore contains an element $\gamma \in G(F)$ (since G is quasi-split, G_{der} is simply connected, and char$(F)=0$ [K2]). By definition, $\mathscr{N}\delta$ is the stable conjugacy class of γ. The fiber of $\mathscr{N}$ through δ is the stable twisted conjugacy class of δ. The construction of $\mathscr{N}$ when G_{der} is not simply connected is given in [K2].

Let $\mathrm{d}g$ (resp. $\mathrm{d}g_E$) be the Haar measure on $G(F)$ (resp. $G(E)$) that gives K (resp. K_E) measure 1. For $\gamma \in G(F)$ and $f \in C_c^\infty(G(F))$ we denote by $O_\gamma(f)$ the orbital integral

$$\int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)\, \mathrm{d}g/\mathrm{d}t.$$

This requires a choice of Haar measure $\mathrm{d}t$ on $G_\gamma(F)$, but we leave the measure out of the notation.

Let $I = \mathrm{Res}_{E/F}G$. Then the automorphism θ of E/F induces an F-automorphism of I; this automorphism agrees with θ on $I(F)=G(E)$, and we will abuse notation slightly by using θ to denote both of them. For $\delta \in G(E)$ and $f \in C_c^\infty(G(E))$ we have the twisted orbital integral $O_{\delta\theta}(f)$, given by

$$\int_{I_{\delta\theta}(F)\backslash G(E)} f(g^{-1}\delta\theta(g))\, \mathrm{d}g_E/\mathrm{d}u,$$

where $I_{\delta\theta}$ denotes the fixed points of $\mathrm{Int}(\delta)\circ\theta$ on I. Of course $I_{\delta\theta}(F)$ is simply the twisted centralizer of δ in $G(E)$.

For semisimple $\gamma \in G(F)$ we have the stable orbital integral SO_γ, given as a linear form on $C_c^\infty(G(F))$ by

$$SO_\gamma = \sum_{\gamma'} e(G_{\gamma'})O_{\gamma'},$$

where the sum is taken over a set of representatives γ' for the conjugacy classes within the stable conjugacy class of γ, and where $e(G_{\gamma'}) = \pm 1$ is the sign [K3] attached to the connected reductive F-group $G_{\gamma'}$. The distribution SO_γ depends on a choice of Haar measure on $G_\gamma(F)$. This measure is then transported to the inner twists $G_{\gamma'}$ and used to form $O_{\gamma'}$. One expects that SO_γ is a stable distribution for all semisimple γ. Of course this is true by definition if γ is regular semisimple.

For $\delta \in G(E)$ such that $N\delta$ is semisimple we have the "stable" twisted orbital integral

$$SO_{\delta\theta} = \sum_{\delta'} e(I_{\delta'\theta})O_{\delta'\theta},$$

where the sum is taken over a set of representatives δ' for the twisted conjugacy classes within the stable twisted conjugacy class of δ. In the same way as for SO_γ we use compatible measures on the groups $I_{\delta'\theta}(F)$.

Let $f_E \in C_c^\infty(G(E))$ and $f \in C_c^\infty(G(F))$. As usual we say that f_E, f have matching orbital integrals if for every semisimple $\gamma \in G(F)$ the stable orbital integral $SO_\gamma(f)$ vanishes unless the stable conjugacy class of γ is equal to $\mathscr{N}\delta$ for some $\delta \in G(E)$, in which case it is given by

$$SO_\gamma(f) = SO_{\delta\theta}(f_E).$$

Of course we are using compatible Haar measures on $G_\gamma(F)$, $I_{\delta\theta}(F)$ to form the orbital integrals; this has a meaning since $I_{\delta\theta}$ is an inner twist of G_γ [K2, Lemma 5.8].

The (conjectural) fundamental lemma for $b: \mathscr{H}_E \to \mathscr{H}$ asserts that f_E, $b(f_E)$ have matching orbital integrals for all $f_E \in \mathscr{H}_E$. The main result of this paper is that f_E, $b(f_E)$ have matching orbital integrals if f_E is the unit element of $\mathscr{H}_E$, namely, the characteristic function of K_E (recall that we normalized the measure on $G(E)$ so that K_E has measure 1). In this case $b(f_E)$ is the unit element of $\mathscr{H}$, namely, the characteristic function of K.

For $G = GL_n$ this result is not new – it follows immediately from Lemma 8.8 of [K1]. In fact that lemma shows that some other pairs of functions have matching orbital integrals (characteristic functions of corresponding parahoric subgroups of $GL_n(F)$, $GL_n(E)$, divided by the measures of the subgroups). Following a suggestion of J.-P. Labesse, this paper also proves a matching theorem for more general pairs of functions.

This more general matching theorem is the subject of §1. In §2 we make some remarks about twisted κ-orbital integrals of the functions considered in §1. In §3 we return to the unit elements of $\mathscr{H}_E$, $\mathscr{H}$ and follow a suggestion of J. Arthur by proving a matching theorem for some weighted orbital integrals. In §4 we sketch what to do when G_{der} is not simply connected.

1. Main result

In this section our situation will be somewhat more general than in the introduction. Let F, E, θ, l be as before. In particular we still insist that E/F be unramified. Let L denote the completion of the maximal unramified extension E^{un} of E. We have $E^{\text{un}} = F^{\text{un}}$ and we denote by σ the Frobenius automorphism of L over F.

Let G be a connected reductive group over F, no longer assumed to be unramified. We do assume, however, that G_{der} is simply connected. As before we write I for $\text{Res}_{E/F}G$ and θ for the F-automorphism of I

obtained from the field automorphism θ. Let K_L be an open bounded subgroup of $G(L)$ satisfying the following three conditions:

(a) $\sigma(K_L) = K_L$.

(b) The mapping $k \mapsto k^{-1}\sigma(k)$ from K_L to K_L is surjective.

(c) The mapping $k \mapsto k^{-1}\sigma^l(k)$ from K_L to K_L is surjective.

Let K (resp. K_E) be $G(F) \cap K_L$ (resp. $G(E) \cap K_L$). Note that the situation in the introduction can be recovered by taking K_L to be the stabilizer in $G(L)$ of the hyperspecial point x_0; then (a) is obvious and (b), (c) follow from a result of Greenberg [G] since the special fiber of $\boldsymbol{G}$ is connected, where $\boldsymbol{G}$ is the extension of G to a group scheme over $\mathfrak{o}$ determined by x_0 ($\mathfrak{o}$ denotes the valuation ring of F).

Let X, X_E, X_L denote $G(F)/K$, $G(E)/K_E$, $G(L)/K_L$ respectively. There are obvious inclusions $X \subset X_E \subset X_L$ and σ acts on X_L. Condition (b) (resp. (c)) implies that the fixed point set of σ (resp. σ^l) on X_L is equal to X (resp. X_E).

Choose Haar measures dg, dg_E on $G(F)$, $G(E)$ such that K, K_E have measure 1, and use these measures in forming orbital integrals. Let f (resp. f_E) denote the characteristic function of K (resp. K_E) in $G(F)$ (resp. $G(E)$).

The groups $G(F)$, $G(E)$, $G(L)$ act on X, X_E, X_L respectively. Furthermore θ acts on X_E (by some power of σ). Let $\delta \in G(E)$. Then

$$O_{\delta\theta}(f_E) = \sum_g \text{meas}(I_{\delta\theta}(F)\backslash I_{\delta\theta}(F)gK_E),$$

where g runs over a set of representatives for the elements of

$$I_{\delta\theta}(F)\backslash G(E)/K_E$$

such that $g^{-1}\delta\theta(g) \in K_E$. Writing x for $gK_E \in X_E$, we have $g^{-1}\delta\theta(g) \in K_E$ if and only if $\delta\theta x = x$. Let $I_{\delta\theta}(F)_x$ denote the stabilizer of x in $I_{\delta\theta}(F)$. Then

$$\text{meas}(I_{\delta\theta}(F)\backslash I_{\delta\theta}(F)gK_E) = \text{meas}(I_{\delta\theta}(F)_x)^{-1}.$$

Let $X_E^{\delta\theta}$ denote the set of fixed points of $\delta\theta$ on X_E (the product of δ and θ is taken in the semidirect product of $G(E)$ and $\text{Gal}(E/F)$). Then we have shown that

$$O_{\delta\theta}(f_E) = \sum_x \text{meas}(I_{\delta\theta}(F)_x)^{-1},$$

where x runs through a set of representatives for the orbits of $I_{\delta\theta}(F)$ on $X_E^{\delta\theta}$. Taking the special case $E = F$, we get a corollary that for $\gamma \in G(F)$

$$O_\gamma(f) = \sum_x \text{meas}(G_\gamma(F)_x)^{-1},$$

where x runs through a set of representatives for the orbits of $G_\gamma(F)$ on X^γ, the set of fixed points of γ on X.

Choose an integer j such that θ is equal to the restriction of σ^j to E. Of course j is relatively prime to l, and hence we can choose integers a, b such that $bl - aj = 1$. We are going to define a correspondence between $G(F)$ and $G(E)$. Let $\gamma \in G(F)$ and $\delta \in G(E)$. We write $\gamma \leftrightarrow \delta$ if there exists $c \in G(L)$ such that the following two conditions hold:

(A) $c\gamma^a\sigma^l c^{-1} = \sigma^l$,

(B) $c\gamma^b\sigma^j c^{-1} = \delta\sigma^j$.

In (A) and (B) the equalities are of elements in the semidirect product of $G(L)$ and the infinite cyclic group $\langle\sigma\rangle$ generated by σ. Let $\langle\gamma, \sigma\rangle$ be the subgroup generated by γ, σ. Then if γ, δ, c satisfy (A) and (B), it follows that $c\langle\gamma, \sigma\rangle c^{-1} = \langle\sigma^l, \delta\sigma^j\rangle$, the point being that $\gamma^a\sigma^l$, $\gamma^b\sigma^j$ generate the same subgroup as γ, σ. Let Y be any set on which the semidirect product acts. Then $y \mapsto cy$ induces a bijection from the fixed points of $\langle\gamma, \sigma\rangle$ on Y to the fixed points of $\langle\sigma^l, \delta\sigma^j\rangle$ on Y. Taking $Y = X_L$, we see that $x \mapsto cx$ induces a bijection from X^γ to $X_E^{\delta\theta}$. Taking $Y = G(L)$ with $G(L)$ acting by conjugation, we see that $g \mapsto cgc^{-1}$ induces an isomorphism from $G_\gamma(F)$ to $I_{\delta\theta}(F)$. It is then immediate from the expressions we obtained for $O_{\delta\theta}(f_E)$ and $O_\gamma(f)$ that

$$O_{\delta\theta}(f_E) = O_\gamma(f)$$

if the measures used on $G_\gamma(F)$, $I_{\delta\theta}(F)$ correspond under the isomorphism above.

What remains is to get a better understanding of the correspondence $\gamma \leftrightarrow \delta$. For which $\gamma \in G(F)$ do there exist $\delta \in G(E)$ such that $\gamma \leftrightarrow \delta$? Conditions (A), (B) can be rewritten as

(A′) $\gamma^a = c^{-1}\sigma^l(c)$,

(B′) $\delta = c\gamma^b\sigma^j(c^{-1})$.

If δ exists, then (A′) can be solved. Conversely, suppose that (A′) can be solved. Then we can use (B′) to define $\delta \in G(L)$ such that γ, δ, c satisfy (A), (B). But then $c\langle\gamma, \sigma\rangle c^{-1} = \langle\sigma^l, \delta\sigma^j\rangle$, which implies that σ^l, $\delta\sigma^j$ commute, and this in turn implies that $\delta \in G(E)$. We conclude that δ exists if and only if (A′) can be solved. The element $c \in G(L)$ appearing in (A′) is clearly determined up to left multiplication by an element of $G(E)$. Making such a change in c replaces δ by a θ-conjugate under $G(E)$. Thus if $\gamma \leftrightarrow \delta$, then $\gamma \leftrightarrow \delta'$ if and only if δ, δ' are θ-conjugate under $G(E)$.

Next we consider $\delta \in G(E)$ and ask whether there exists $\gamma \in G(F)$ such that $\gamma \leftrightarrow \delta$. Inverting the matrix

$$\begin{bmatrix} a & l \\ b & j \end{bmatrix},$$

we see that (A), (B) are equivalent to

(C) $(\delta\sigma^j)^l \sigma^{-jl} = c\gamma c^{-1}$,

(D) $(\delta\sigma^j)^{-a} \sigma^{bl} = c\sigma c^{-1}$

(of course we are using that γ, σ commute and that σ^l, $\delta\sigma^j$ commute). We can rewrite (C), (D) as

(C′) $N\delta = c\gamma c^{-1}$,

(D′) $(\delta\sigma^j)^{-a} \sigma^{aj} = c\sigma(c^{-1})$.

If γ exists, then (D′) can be solved. Conversely, suppose that (D′) can be solved. Then we can use (C′) to define $\gamma \in G(L)$ such that γ, δ, c satisfy (C), (D). But (C) and (D) imply that γ, σ commute and hence that $\gamma \in G(F)$. We conclude that γ exists if and only if (D′) can be solved. Furthermore (D′) determines c up to right multiplication by $G(F)$, and changing c by an element of $G(F)$ replaces γ by a conjugate under $G(F)$. Thus if $\gamma \leftrightarrow \delta$, then $\gamma' \leftrightarrow \delta$ if and only if γ, γ' are conjugate in $G(F)$.

What we now know about the correspondence $\gamma \leftrightarrow \delta$ can be summarized as follows. The correspondence sets up a bijection from the set of conjugacy classes in $G(F)$ of elements $\gamma \in G(F)$ such that (A′) can be solved to the set of θ-conjugacy classes in $G(E)$ of elements $\delta \in G(E)$ such that (D′) can be solved. Furthermore (C′) tells us that if $\gamma \leftrightarrow \delta$, then $\mathcal{N}\delta = \gamma$.

To complete the picture we need to know that there are enough corresponding elements of $G(F)$, $G(E)$. First we show that if $\gamma \in G(F)$ and X^γ is non-empty, then there exists $\delta \in G(E)$ such that $\gamma \leftrightarrow \delta$. Indeed, replacing γ by a conjugate, we may assume that $\gamma \in K$. Then our assumption (c) on K_L implies that (A′) can be solved.

Next we show that if $\delta \in G(E)$ and $X_E^{\delta\theta}$ is non-empty, then there exists $\gamma \in G(F)$ such that $\gamma \leftrightarrow \delta$. Indeed, replacing δ by a θ-conjugate in $G(E)$, we may assume that $\delta\theta$ fixes the base point of $X_E = G(E)/K_E$. Then $(\delta\sigma^j)^{-a}$ and σ^{aj} both fix the base point of X_L, as does their product $(\delta\sigma^j)^{-a}\sigma^{aj} \in G(L)$. Therefore $(\delta\sigma^j)^{-a}\sigma^{aj} \in K_L$ and assumption (b) on K_L implies that (D′) can be solved.

There is one further remark that we need to make before stating the main result of the paper. Suppose that $\gamma \leftrightarrow \delta$. Choose $c \in G(L)$ such that γ, δ, c satisfy (C), (D). We have already seen that $g \mapsto cgc^{-1}$ induces an isomorphism from $G_\gamma(F)$ to $I_{\delta\theta}(F)$. Since (C), (D) determine c up to right multiplication by an element of $G_\gamma(F)$, the isomorphism is canonical up to inner automorphisms of $G_\gamma(F)$.

THEOREM: *The correspondence* $\gamma \leftrightarrow \delta$ *induces a bijection from the set of conjugacy classes of* $\gamma \in G(F)$ *such that* $O_\gamma(f) \neq 0$ *to the set of* θ*-conjugacy classes of* $\delta \in G(E)$ *such that* $O_{\delta\theta}(f_E) \neq 0$. *Moreover if* $\gamma \leftrightarrow \delta$, *then* $\gamma = \mathcal{N}\delta$, $G_\gamma(F)$ *is isomorphic to* $I_{\delta\theta}(F)$, *and* $O_\gamma(f) = O_{\delta\theta}(f_E)$.

Since $O_\gamma(f) \neq 0$ (resp. $O_{\delta\theta}(f_E) \neq 0$) if and only if X^γ (resp. $X_E^{\delta\theta}$) is non-empty, the theorem follows from the remarks made above.

In order to use the theorem to prove that f, f_E have matching orbital integrals, there is a technical point to check. Suppose that γ, δ, c satisfy (A), (B). Then Lemma 5.8 of [K2] gives us an inner twisting $\beta : I_{\delta\theta} \to G_\gamma$, canonical up to inner automorphisms of $G_\gamma(\bar{F})$. Assume now that γ is semisimple. We want to check that there exists an F-isomorphism $\alpha : I_{\delta\theta} \xrightarrow{\sim} G_\gamma$ whose restriction to $I_{\delta\theta}(F)$ is given by $g \mapsto c^{-1}gc$ and which differs from β by an inner automorphism of $G_\gamma(\bar{L})$. This will show that if we use $g \mapsto c^{-1}gc$ to transport a Haar measure on $I_{\delta\theta}(F)$ over to $G_\gamma(F)$, the two measures will be compatible in the sense that arises in the definition of matching orbital integrals. It will also show that the signs $e(G_\gamma)$ and $e(I_{\delta\theta})$ are equal. We see from [K2] that if $d \in G(\bar{F})$ and $N\delta = d\gamma d^{-1}$, then we can take β to be $\mathrm{Int}(d)^{-1} \circ p$, where $p : I_{\delta\theta} \to G_{N\delta}$ (over E) is the restriction to $I_{\delta\theta}$ of the projection of $I_E = G_E \times \cdots \times G_E$ onto the factor indexed by the identity element of $\mathrm{Gal}(E/F)$ (the l factors are indexed by the elements of $\mathrm{Gal}(E/F)$). Let $\alpha = \mathrm{Int}(c)^{-1} \circ p$. Then α, β differ by an inner automorphism of $G_\gamma(\bar{L})$ (use (C′) to see this), and what remains is to show that α is defined over F. It is obvious that α is defined over L. Since the functor $A \mapsto \mathrm{Isom}_A(I_{\delta\theta}, G_\gamma)$ from (F-algebras) to (sets) is representable by a scheme over F (here we use that γ is semisimple and that G_{der} is simply connected in order to conclude that the groups G_γ, $I_{\delta\theta}$ are connected and reductive), it is enough to show that α commutes with σ. We will do this by showing that α commutes with σ^j and σ^l; this is enough since j, l are relatively prime. Direct calculation shows that

$$\sigma^j(\alpha) = \mathrm{Int}\big(\sigma^j(c^{-1})\delta^{-1}c\big) \circ \alpha,$$

$$\sigma^l(\alpha) = \mathrm{Int}\big(\sigma^l(c^{-1})c\big) \circ \alpha,$$

and (A′), (B′) imply that

$$\sigma^j(c^{-1})\delta^{-1}c = \gamma^{-b},$$

$$\sigma^l(c^{-1})c = \gamma^{-a}.$$

Since γ is central in G_γ, this proves that $\sigma^j(\alpha) = \sigma^l(\alpha) = \alpha$.

COROLLARY: *The functions f, f_E have matching orbital integrals.*

This follows immediately from the theorem and the technical point that we just checked. However, we need to say a few more words about the corollary. If G is not quasi-split, the most natural notion of matching orbital integrals would involve "stable' twisted orbital integrals on $G(E)$ and stable orbital integrals on a quasi-split inner form of G. In fact, if G is not quasi-split, the stable conjugacy class of $N\delta$ need not contain any F-rational elements, hence the stable norm $\mathcal{N}\delta$ does not always exist in $G(F)$. Nevertheless the corollary is true and even has the following supplement: if $\mathcal{N}\delta$ does not exist in $G(F)$, then $SO_{\delta\theta}(f_E) = 0$. To prove the supplement, note that if $SO_{\delta\theta}(f_E) \neq 0$, then there exists a stable θ-conjugate δ' of δ such that $O_{\delta'\theta}(f_E) \neq 0$; therefore there exists $\gamma \in G(F)$ such that $\gamma \leftrightarrow \delta'$, and then it follows that $\gamma = \mathcal{N}\delta' = \mathcal{N}\delta$.

2. κ-orbital integrals and the dependence of $\gamma \leftrightarrow \delta$ on j, a, b

We keep the notation and assumptions of §1. We have not yet used the full strength of the theorem in §1, which proved a matching result for orbital integrals, not just stable orbital integrals. Consider an element $\delta \in G(E)$ such that $N\delta$ is regular and semisimple. Then $I_{\delta\theta}$ is a torus. For any stable θ-conjugate $\delta' \in G(E)$ of δ there is an invariant

$$\mathrm{inv}(\delta, \delta') \in H^1(F, I_{\delta\theta})$$

measuring the difference between δ, δ'. This invariant sets up a bijection from the set of θ-conjugacy classes in the stable θ-conjugacy class of δ to the set

$$\ker\left[H^1(F, I_{\delta\theta}) \to H^1(F, I)\right].$$

As usual we can define twisted κ-orbital integrals $O^\kappa_{\delta\theta}$ for any character κ on the group $H^1(F, I_{\delta\theta})$ by putting

$$O^\kappa_{\delta\theta} = \sum_{\delta'} \langle \mathrm{inv}(\delta, \delta'), \kappa \rangle O_{\delta'\theta},$$

where δ' runs over a set of representatives for the θ-conjugacy classes in the stable θ-conjugacy class of δ. Suppose that $O_{\delta'\theta}(f_E) \neq 0$ for some stable θ-conjugate δ' of δ. It does no harm to replace δ by δ', and so we may as well assume that $O_{\delta\theta}(f_E) \neq 0$. Then there exists $\gamma \in G(F)$ such that $\gamma \leftrightarrow \delta$. Of course γ is regular and semisimple, and G_γ is a torus T.

Lemma 5.8 of [K2] gives us a canonical isomorphism $T \xrightarrow{\sim} I_{\delta\theta}$, allowing us to view κ as a character on $H^1(F, T)$ and to form κ-orbital integrals

$$O_\gamma^\kappa = \sum_{\gamma'} \langle \text{inv}(\gamma, \gamma'), \kappa \rangle O_{\gamma'},$$

where γ' runs over a set of representatives for the conjugacy classes in the stable conjugacy class of γ.

PROPOSITION 1: $O_{\delta\theta}^\kappa(f_E) = O_\gamma^\kappa(f)$.

Of course the significance of the proposition is that whenever one is able to express the κ-orbital integrals of f in terms of stable orbital integrals of a function on an endoscopic group H of G, the proposition will then express $O_{\delta\theta}^\kappa(f_E)$ in terms of stable orbital integrals on H, which may also be regarded as an endoscopic group for the pair (I, θ) [S].

To prove the proposition it is enough to show that if γ' is stably conjugate to γ, if δ' is stably θ-conjugate to δ, and if $\gamma' \leftrightarrow \delta'$, then $\text{inv}(\gamma, \gamma') = \text{inv}(\delta, \delta')$. This is sufficient since the elements γ', δ' that do not take part in the correspondence contribute zero to $O_\gamma^\kappa(f)$, $O_{\delta\theta}^\kappa(f_E)$. In order to prove that $\text{inv}(\gamma, \gamma') = \text{inv}(\delta, \delta')$ it is convenient to use the injection

$$H^1(F, T) \to B(T)$$

defined in [K4, §1], where $B(T)$ denotes $H^1(\langle\sigma\rangle, T(L))$. Choose c, $c' \in G(L)$ such that γ, δ, c and γ', δ', c' satisfy (A), (B). Since $H^1(L, T)$ is trivial, we can also choose $g \in G(L)$ such that $\gamma' = g\gamma g^{-1}$. The image of $\text{inv}(\gamma, \gamma')$ in $B(T)$ is represented by the 1-cocycle

$$\sigma^k \mapsto g^{-1}\sigma^k(g)$$

of $\langle\sigma\rangle$ in $T(L)$.

As in §1 we write $p: I_{\delta\theta} \to G_{N\delta}$ (over E) for the restriction to $I_{\delta\theta}$ of the projection of $I_E = G_E \times \cdots \times G_E$ on the factor indexed by the identity element of $\text{Gal}(E/F)$. The canonical isomorphism from $I_{\delta\theta}$ to T is given by $\text{Int}(c)^{-1} \circ p$. It is easy to see that there exists a unique element $h \in I(L)$ such that

(a) the image of h under the projection of $I(L) = G(L) \times \cdots \times G(L)$ onto the factor indexed by the identity element of $\text{Gal}(E/F)$ is equal to dgc^{-1} (note that dgc^{-1} conjugates $N\delta$ into $N\delta'$),

(b) $\delta' = h\delta\theta(h)^{-1}$.

The image of $\text{inv}(\delta, \delta')$ in $B(T)$ is represented by the 1-cocycle

$$\sigma^k \mapsto \big(\text{Int}(c)^{-1} \circ p\big)\big(h^{-1}\sigma^k(h)\big)$$

of $\langle\sigma\rangle$ in $T(L)$.

We will now show that with the choices we have made the two 1-cocycles of $\langle\sigma\rangle$ in $T(L)$ are equal (not just cohomologous). Since j, l are relatively prime, it is enough to show that

$$\left(\mathrm{Int}(c)^{-1}\circ p\right)\left(h^{-1}\sigma^k(h)\right)=g^{-1}\sigma^k(g)$$

for $k=j$, l. First we take $k=j$. The equality $\delta'=h\delta\theta(h)^{-1}$ implies that $p(h^{-1}\sigma^j(h))$ is equal to

$$\left(dgc^{-1}\right)^{-1}\cdot\delta'\cdot\sigma^j\left(dgc^{-1}\right)\cdot\delta^{-1}.$$

Therefore $(\mathrm{Int}(c)^{-1}\circ p)(h^{-1}\sigma^j(h))$ is equal to

$$g^{-1}d^{-1}\delta'\sigma^j dgc^{-1}\sigma^{-j}\delta^{-1}c.$$

Using (B) for δ and δ', we can simplify this expression, obtaining

$$g^{-1}(\gamma')^b\sigma^j(g)\gamma^{-b}.$$

Using $\gamma'=g\gamma g^{-1}$, we can simplify it further, obtaining $g^{-1}\sigma^j(g)$.

Next we take $k=l$. Then $(\mathrm{Int}(c)^{-1}\circ p)(h^{-1}\sigma^l(h)$ is equal to

$$c^{-1}\cdot\left(dgc^{-1}\right)^{-1}\cdot\sigma^l\left(dgc^{-1}\right)\cdot c.$$

Using (A′) for c and d we can simplify this expression, obtaining

$$g^{-1}(\gamma')^a\sigma^l(g)\gamma^{-a}.$$

Using $\gamma'=g\gamma g^{-1}$, we can simplify it further, obtaining $g^{-1}\sigma^l(g)$. This completes the proof of the proposition.

In order to define the correspondence $\gamma\leftrightarrow\delta$ we had to choose integers j, a, b such that the restriction of σ^j to E was θ and such that $bl-aj=1$. This raises an obvious question: How does the correspondence depend on the choice of j, a, b? It turns out that the correspondence is independent of j, b, but is dependent on a. To see how the correspondence changes when j, a, b are replaced by j', a', b', we suppose that we have γ, $\gamma'\in G(F)$, $\delta\in G(E)$, c, $c'\in G(L)$ such that γ, δ, c satisfy (A), (B) for j, a, b and γ', δ, c' satisfy (A), (B) for j', a', b'. Then γ, γ' are stably conjugate and we can measure the difference between the two correspondences by calculating $\mathrm{inv}(\gamma,\gamma')\in H^1(F, G_\gamma)$. At this point we assume that γ is semisimple, so that G_γ is connected and we can embed $H^1(F, G_\gamma)$ in $B(G_\gamma)$. The set $B(G_\gamma)$ can be identified with the set of σ-conjugacy classes in $G_\gamma(L)$.

PROPOSITION 2: *The image of* $\mathrm{inv}(\gamma,\gamma')$ *in* $B(G_\gamma)$ *is equal to the*

σ-conjugacy class of γ^{-n} in $G_\gamma(L)$, where n is defined by the equality $a' = a + nl$.

We also write $j' = j + ml$; then $b' = b + nj + ma + mnl$. We have $c\gamma c^{-1} = N\delta = c'\gamma'(c')^{-1}$, and hence $\gamma' = g\gamma g^{-1}$, where $g = (c')^{-1}c$. Therefore the image of inv(γ, γ') in $B(G_\gamma)$ is equal to the σ-conjugacy class of x, where $x = g^{-1}\sigma(g)$.

We will now show that $x = \gamma^{-n}$. We have

$$x = c^{-1}c' \cdot \sigma \cdot (c')^{-1}c \cdot \sigma^{-1},$$

and using (D) for c' and then replacing j' by $j + ml$, we find that

$$x = c^{-1}(\delta\sigma^j)^{-a'}\sigma^{l(b+nj)}c\sigma^{-1}.$$

Finally, replacing a' by $a + nl$ and then using (C) and (D) for c, we find that $x = \gamma^{-n}$. In carrying out these steps we must remember that σ^l commutes with δ. This finishes the proof of the proposition.

3. Weighted orbital integrals

We return to the situation in the introduction, so that G is again unramified. The hyperspecial point x_0 determines an extension of G to a connected reductive group over the valuation ring $\mathfrak{o}$ of F, and we have $K_L = G(\mathfrak{o}_L)$. Let M be a Levi subgroup of G over $\mathfrak{o}$. We write $\mathfrak{a}_M$ for the real vector space

$$\mathrm{Hom}_{\mathbf{Z}}(\mathrm{Hom}_F(M, \mathbb{G}_m), \mathbb{R})$$

and define a homomorphism

$$H_M : M(L) \to \mathfrak{a}_M$$

by requiring that for $x \in M(L)$

$$\exp\langle H_M(x), \lambda\rangle = |\lambda(x)|$$

for all $\lambda \in \mathrm{Hom}_F(M, \mathbb{G}_m)$. Here we have extended the normalized absolute value on $F^\times$ to an absolute value on $L^\times$. Let P be a parabolic subgroup of G having M as Levi component and write N for the unipotent radical of P. We define a function

$$H_P : G(L) \to \mathfrak{a}_M$$

by putting $H_P(g) = H_M(m)$, where g has been written as mnk for

$m \in M(L)$, $n \in N(L)$, $k \in K_L$. For $g \in G(L)$ and $\lambda \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_M, \mathbb{C})$ we set

$$v_P(\lambda, g) = \mathrm{e}^{-\lambda(HP(g))}.$$

For fixed g the functions $\lambda \mapsto v_P(\lambda, g)$ form a (G, M) family [A], and this (G, M) family determines a number $v_M(g)$. In this way we have constructed a weight function v_M on $G(L)$; it is left invariant under $M(L)$ and right invariant under K_L.

It is obvious that the restriction of v_M to $G(F)$ is the weight function on $G(F)$ that Arthur uses to define weighted orbital integrals. Let γ be a regular semisimple element of $M(F)$. The weighted orbital integral that we are referring to is

$$WO_\gamma(\phi) = \int_{G_\gamma(F)\backslash G(F)} \phi(g^{-1}\gamma g) v_M(g)\, \mathrm{d}g/\mathrm{d}t$$

for $\phi \in C_c^\infty(G(F))$.

After working through Arthur's definition is twisted weighted orbital integrals, one finds that the necessary weight function on $G(E)$ is none other than the restriction of v_M to $G(E)$ (up to a scalar which will be 1 in a suitable normalization). Let $\delta \in M(E)$ and assume that $N\delta$ is regular and semisimple. Then the twisted weighted orbital integral that we are referring to is

$$WO_{\delta\theta}(\phi) = \int_{I_{\delta\theta}(F)\backslash G(E)} \phi(g^{-1}\delta\theta(g)) v_M(g)\, \mathrm{d}g_E/\mathrm{d}u$$

for $\phi \in C_c^\infty(G(E))$.

As before we let f, f_E denote the characteristic functions of K, K_E. Suppose that our elements $\gamma \in M(F)$ and $\delta \in M(E)$ are related by the correspondence $\gamma \leftrightarrow \delta$ for the group M, so that there exists $c \in M(L)$ such that γ, δ, c satisfy (A) and (B).

Proposition: $WO_{\delta\theta}(f_E) = WO_\gamma(f)$.

The proof is a slight variant of the proof that $O_{\delta\theta}(f_E) = O_\gamma(f)$. Since v_M is right invariant under K_L it descends to a function w_M on $X_L = G(L)/K_L$. We have

$$WO_\gamma(f) = \sum_x \mathrm{meas}\big(G_\gamma(F)_x\big)^{-1} w_M(x),$$

where x runs through a set of representatives for the orbits of $G_\gamma(F)$ on X^γ. There is a similar formula for $WO_{\delta\theta}(f_E)$. The bijection $x \mapsto cx$ from

X^γ to $X_E^{\delta\theta}$ matches up the terms in the two formulas, and to finish the proof of the proposition we have only to note that the left invariance of v_M under $M(L)$ implies that $w_M(x) = w_M(cx)$.

Before finishing this section we should observe that enough γ, δ are related by the correspondence $\gamma \leftrightarrow \delta$ for M. Suppose that γ is a regular semisimple element of $M(F)$ such that $WO_\gamma(f) \neq 0$. Then there exists $g \in G(F)$ such that $g^{-1}\gamma g \in K$. Choose a parabolic subgroup P of G with Levi component M and unipotent radical N. Writing $g = mnk$ with $m \in M(F)$, $n \in N(F)$, $k \in K$ and using that $P(\mathfrak{o}) = M(\mathfrak{o})N(\mathfrak{o})$, we see that $m^{-1}\gamma m \in M(\mathfrak{o})$. The discussion in §1 then shows that there exists $\delta \in M(E)$ such that $\gamma \leftrightarrow \delta$ in the group M. Similarly, if $WO_{\delta\theta}(f_E) \neq 0$, then there exists $\gamma \in M(F)$ such that $\gamma \leftrightarrow \delta$ in the group M.

4. Groups G for which G_{der} is not simply connected

In proving our special case of the fundamental lemma we assumed that G_{der} was simply connected. We will now show that this assumption can be dropped. Choose a finite unramified extension F' of F that splits G and contains E; then there exists an extension H of G by a central torus Z such that

(a) H_{der} is simply connected,

(b) Z is a product of copies of $\text{Res}_{F'/F}\mathbb{G}_m$.

In the terminology of [K2, §5] H is an unramified z-extension of G adapted to E. Note that $H(F)$ maps onto $G(F)$.

It is not hard to see that the fundamental lemma for G, E, θ follows from the fundamental lemma for H, E, θ. The point is that there is a surjective homomorphism from the Hecke algebra of H (for the hyperspecial maximal compact subgroup of $H(F)$ corresponding to K) to the Hecke algebra of G, obtained by mapping f_H to f_G, where

$$f_G(x) = \int_{Z(F)} f_H(x_0 z)\, \mathrm{d}z.$$

Here x_0 is an element of $H(F)$ that maps to x and $\mathrm{d}z$ is the Haar measure on $Z(F)$ that gives measure 1 to the maximal compact subgroup of $Z(F)$. The mapping $f_H \mapsto f_G$ gives us (by means of the Satake isomorphism) a mapping

$$\mathbb{C}[X_*(S_H)]^{\Omega(F)} \to \mathbb{C}[X_*(S_G)]^{\Omega(F)},$$

where S_G is a maximal F-split torus of G, S_H is the corresponding maximal F-split torus of H, and $\Omega(F)$ is the relative Weyl group of S_G in G. The mapping is simply the homomorphism induced by $X_*(S_H) \to X_*(S_G)$, which is surjective since $H^1(F, X_*(Z))$ is trivial. From this it is also clear that $f_H \mapsto f_G$ is compatible with the base change homomorphisms b for H and G. Furthermore, the orbital integrals of f_G can be obtained from the orbital integrals of f_H by integrating over $Z(F)$. There is an analogous statement for twisted orbital integrals, in which the integration is over $Z(E)/(\theta - id)Z(E)$. Finally, the assumption that F' contains E implies that the norm map induces an isomorphism

$$Z(E)/(\theta - id)Z(E) \xrightarrow{\sim} Z(F).$$

Putting all this together, one can now check that the fundamental lemma for H implies the fundamental lemma for G.

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(Oblatum 28-X-1985)

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