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ON THE NÉRON MODEL OF JACOBIANS OF SHIMURA CURVES

Bruce W. Jordan and Ron A. Livné *

Let $\mathscr{B}$ be an indefinite rational quaternion algebra of discriminant Disc $\mathscr{B} > 1$ and denote by $V_{\mathscr{B}} = V_{\mathscr{B}}/\mathbb{Q}$ the corresponding Shimura curve. $V_{\mathscr{B}}$ has bad reduction exactly at the primes p dividing Disc $\mathscr{B}$; fix such a prime p. Let $\mathscr{J}/\mathbb{Z}_p$ be the Néron model of the jacobian of $V_{\mathscr{B}} \times_{\mathbb{Q}} \mathbb{Q}_p$. Denote by $\mathscr{J}_p^0$ the connected component of the special fiber $\mathscr{J}_p = \mathscr{J} \times_{\mathbb{Z}_p} \mathbb{F}_p$ and by $\Phi = \mathscr{J}_p/\mathscr{J}_p^0$ its group of connected components. The following problems are relevant to many arithmetic questions concerning $V_{\mathscr{B}}$:

1. Determine the structure of $\mathscr{J}_p^0/\mathbb{F}_p$.
2. Determine the group of connected components Φ.

It is the purpose of this paper to solve these problems.

To describe the answer we obtain, let $\hat{\mathscr{B}}$ be the rational definite quaternion algebra of discriminant $\dfrac{\text{Disc } \mathscr{B}}{p}$. Denote by $m(\hat{\mathscr{B}})$ the mass $\frac{1}{12} \prod_{q \mid \text{Disc } \hat{\mathscr{B}}} (q-1)$ of $\hat{\mathscr{B}}$. Let $B = B(p)$ be the Brandt matrix of degree p for $\hat{\mathscr{B}}$ relative to a fixed ordering of the ideal classes of $\hat{\mathscr{B}}$. B is an integral $h \times h$ matrix for which $p+1$ is an eigenvalue, where h is the class number of $\hat{\mathscr{B}}$. Hence we can write the characteristic polynomial $P_B(x)$ of B as

$$P_B(x) = (x - p - 1) \prod_{i=2}^{h} (x - \lambda_i).$$

In response to Problem 2 we establish the

THEOREM (2.3):
Let

$$e_2 = \prod_{q \mid \text{Disc } \mathscr{B}} \left(1 - \left(\frac{-4}{q}\right)\right), \qquad e_3 = \prod_{q \mid \text{Disc } \mathscr{B}} \left(1 - \left(\frac{-3}{q}\right)\right).$$

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Then

$$|\Phi| = \frac{p+1}{m(\hat{\mathscr{B}})c(\hat{\mathscr{B}})2^{e_2}3^{e_3}} \left| \prod_{i=2}^{h} (\lambda_i - (p+1))(\lambda_i + (p+1)) \right|,$$

where $c(\hat{\mathscr{B}}) = 8$ *if* Disc $\hat{\mathscr{B}} = 2$, $c(\hat{\mathscr{B}}) = 3$ *if* Disc $\hat{\mathscr{B}} = 3$, *and* $c(\hat{\mathscr{B}}) = 1$ *otherwise.*

In fact, we explain how to describe Φ in terms of the Brandt matrix B. In Theorem 3.1 we describe the connected component $\mathscr{J}_p^0$.

By the results of Raynaud [8] and Deligne-Rapoport [1], questions 1 and 2 are reduced to computations in linear algebra if one has a description of a regular model of $V_{\mathscr{B}}$ over $\mathbb{Z}_p$. In our case, Drinfeld [2] has constructed a scheme $M_{\mathscr{B}}/\mathbb{Z}$ whose fiber over $\mathbb{Q}$ is the Shimura curve $V_{\mathscr{B}}$. Moreover he has given a description of $M_{\mathscr{B}} \times \mathbb{Z}_p$ in terms of Mumford uniformization. By resolving singularities one obtains a regular scheme $\widetilde{M_{\mathscr{B}} \times \mathbb{Z}_p}$ over $\mathbb{Z}_p$. In Section 1 we give the intersection matrix of the special fiber $(\widetilde{M_{\mathscr{B}} \times \mathbb{Z}_p})_0$ in terms of the Brandt matrix B. Then in Sections 2 and 3 we carry out the computations necessary to answer our questions. The case where the interchanged algebra $\hat{\mathscr{B}}$ has discriminant 2 was treated by Ogg in [7].

The theorems we obtain are analogs of the results of Mazur and Rapoport [6] on elliptic modular jacobians. The arithmetic significance of Theorem 2.3, however, seems more involved. Suppose for simplicity that Disc $\mathscr{B} = pq$ with q prime. Then $P_B(x)$ is the characteristic polynomial of the Hecke operator $T(p)$ acting on the space $M_2(\Gamma_0(q))$ of modular forms of weight 2 for $\Gamma_0(q)$. What is remarkable is that the primes dividing $|\Phi|$ are essentially the primes of congruence between modular forms in $M_2(\Gamma_0(q))$ and newforms of weight 2 for $\Gamma_0(pq)$, cf. Ribet [9]. Hence Φ apparently detects fusion between newforms and old forms.

§1. The intersection matrix

We first recall the description of the special fiber $M_{\mathscr{B}} \times \mathbb{F}_p$ provided by Drinfeld [2]. For details see [4] and Kurihara [5]. Fix a maximal order $\hat{\mathscr{M}} \subset \hat{\mathscr{B}}$ and set

$$\Gamma_0 = \left(\hat{\mathscr{M}} \otimes \mathbb{Z}\left[\frac{1}{p}\right]\right)^{\times} \Big/ \mathbb{Z}\left[\frac{1}{p}\right]^{\times}$$

$$\Gamma_+ = \left\{ x \in \left(\hat{\mathscr{M}} \otimes \mathbb{Z}\left[\frac{1}{p}\right]\right)^{\times} | \mathrm{Norm}(x) \in p^{2\mathbb{Z}} \right\} \Big/ \mathbb{Z}\left[\frac{1}{p}\right]^{\times},$$

where Norm: $\hat{\mathscr{B}} \to \mathbb{Q}$ is the reduced norm. Identify $\hat{\mathscr{B}} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ with the algebra of 2×2 matrices over $\mathbb{Q}_p$. Then Γ_0 and Γ_+ are discrete cocompact subgroups of $\mathrm{PGL}_2(\mathbb{Q}_p)$. Let Δ be the Bruhat-Tits building of $\mathrm{SL}_2(\mathbb{Q}_p)$ with vertices Ver Δ and edges Ed Δ. The groups Γ_0 and Γ_+ act on Δ and the quotients are finite oriented graphs with lengths in the sense of Kurihara [5]. The vertices $\mathrm{Ver}(\Gamma_0 \backslash \Delta)$ correspond to the ideal classes of $\hat{\mathscr{B}}$ and we denote them by $v_1, \dots, v_h$ with the same ordering used to write B. The weight $f(v)$ of a vertex $v \in \mathrm{Ver}(\Gamma_0 \backslash \Delta)$ and the length $\ell(y)$ of an edge $y \in \mathrm{Ed}(\Gamma_0 \backslash \Delta)$ are defined as the orders of their stabilizers in Γ_0. The integer $\ell(y)$ is always 1, 2, or 3. Define $h \times h$ matrices $N^k = (N_{ij}^k)_{1 \leqslant i,j \leqslant h}$ for $1 \leqslant k \leqslant 3$ by

$$N_{ij}^k = \text{number of } y \in \mathrm{Ed}(\Gamma_0 \backslash \Delta) \text{ with } v_i = o(y),\ v_j = t(y)$$

where $o(y)$ is the initial vertex of y and $t(y)$ the terminal vertex. Set F equal to the $h \times h$ diagonal matrix with $F_{ii} = f(v_i)$, $1 \leqslant i \leqslant h$. Then

$$B = \left(N^1 + \tfrac{1}{2}N^2 + \tfrac{1}{3}N^3\right)F; \tag{1.1}$$

see Kurihara [5], (4-4). Let St v_i denote $\{ y \in \mathrm{Ed}(\Gamma_0 \backslash \Delta) \mid o(y) = v_i \}$. As $\#\{ \tilde{y} \in \mathrm{Ed}\ \Delta \mid o(\tilde{y}) = \tilde{v} \} = p + 1$ for any $\tilde{v} \in \mathrm{Ver}\ \Delta$ we have

$$p + 1 = \sum_{y \in \mathrm{St}\ v_i} \frac{f(v_i)}{f(y)} = f(v_i) \sum_{j=1}^{h} \left(N_{ij}^1 + \tfrac{1}{2}N_{ij}^2 + \tfrac{1}{3}N_{ij}^3\right). \tag{1.2}$$

We can write $\Gamma_0 = \Gamma_+ \amalg \Gamma_+ \gamma_p$ where $\gamma_p \in \hat{\mathscr{M}}$ has norm p. γ_p induces an involution w_p of $\Gamma_+ \backslash \Delta$ which fixes no vertex and no (oriented) edge. In fact we may write $\mathrm{Ver}(\Gamma_+ \backslash \Delta) = \{ v_{i\ell} \}$ with $1 \leqslant i \leqslant h$; $1 \leqslant \ell \leqslant 2$, where v_{i1} and v_{i2} lie above $v_i \in \mathrm{Ver}(\Gamma_0 \backslash \Delta)$ and $w_p v_{i\ell} = v_{i,3-\ell}$. Moreover, we may suppose that liftings $\tilde{v}_{i\ell}, \tilde{v}_{jm} \in \mathrm{Ver}\ \Delta$ of $v_{i\ell}, v_{jm} \in \mathrm{Ver}(\Gamma_+ \backslash \Delta)$ are at a distance congruent to $\ell - m$ modulo 2. Hence no edge connects $v_{i\ell}$ and $v_{j\ell}$ ($\ell = 1, 2$; $1 \leqslant i, j \leqslant h$). By Drinfeld [2] $\Gamma_+ \backslash \Delta$ is canonically identified with the dual graph $G = G(M_{\mathscr{B}} \times \mathbb{Z}_p/\mathbb{Z}_p)$ of the special fiber $\mathscr{M}_{\mathscr{B}} \times \mathbb{F}_p$, and Frobenius acts on G as w_p (for this "Geometric Eichler-Shimura Relation" see also [4]). Let $\tilde{G}$ be the dual graph of the special fiber of the resolution of singularities $\widetilde{M_{\mathscr{B}} \times \mathbb{Z}_p/\mathbb{Z}_p}$ of $M_{\mathscr{B}} \times \mathbb{Z}_p/\mathbb{Z}_p$. For an edge $y \in \mathrm{Ed}(\Gamma_0 \backslash \Delta)$ let $\hat{y}$ be the edge above it in $G = \Gamma_+ \backslash \Delta$ such that $\mathrm{o}(\hat{y}) \in \{ v_{i1} \}_{i=1}^{h}$. Then $\tilde{G}$ is obtained from G by replacing $\hat{y}$ together with its opposite edge by a chain

$$\mathrm{o}(\hat{y}) - w_{y1} - \cdots - w_{y,\ell(y)-1} - t(\hat{y})$$

whenever $\ell(y) \geqslant 2$. Identify

$$\{v_{i\ell}, w_{ym} \mid 1 \leqslant i \leqslant h;\ \ell = 1, 2;\ y \in \mathrm{Ed}(\Gamma_0 \backslash \Delta) \text{ satisfying } \ell(y) \geqslant 2 \text{ and } 1 \leqslant m < \ell(y)\}$$

with Ver $\tilde{G}$ by letting an element α in the former set correspond to a component $[\alpha]$ of $(\widetilde{M_{\mathscr{B}} \times \mathbb{Z}_p})_0$ in Ver $\tilde{G}$. The intersection matrix for $(\widetilde{M_{\mathscr{B}} \times \mathbb{Z}_p})_0$, $A = (A_{\alpha\beta}) = ([\alpha] \cdot [\beta])_{\alpha,\beta \in \mathrm{Ver}\, \tilde{G}}$, is readily obtained from G:

(i) $[v_{i1}] \cdot [v_{j2}] = N_{ij}^1$ for $i \neq j$.

$$[w_{y1}] \cdot [o(\hat{y})] = [w_{y2}] \cdot [t(\hat{y})] = 1 \text{ if } \ell(y) = 2.$$

$$[w_{y1}] \cdot [o(\hat{y})] = [w_{y1}] \cdot [w_{y2}] = [w_{y2}] \cdot [t(\hat{y})] = 1$$

if $\ell(y) = 3$. [1.3]

(ii) A is symmetric.
(iii) All off-diagonal entries of A not already determined by i) and ii) are 0.
(iv) The diagonal entries of A are determined so that any row (or column) sum is 0. Thus $[w_{ym}]^2 = -2$ and

$$[v_{i\ell}]^2 = -\sum_{k=1}^{3} \sum_{j=1}^{h} N_{ij}^k.$$

§2. The group of connected components

Let L be the free abelian group on the set Ver $\tilde{G}$. Let $L_0 = \left\{ \sum_{v \in \mathrm{Ver}\, \tilde{G}} n_v v \in L \mid \sum n_v = 0 \right\}$. The intersection matrix A represents a transformation $\mathscr{A} : L \to L$ relative to the standard basis. We have $\mathscr{A}L \subset L_0$ by [1.3 iv]. According to Raynaud [8], $\Phi \approx L_0/\mathscr{A}L$ canonically. Since $L \approx L_0 \oplus \mathbb{Z}$ (noncanonically), $L/\mathscr{A}L \approx \mathbb{Z} \oplus \Phi$. To describe Φ we need some linear algebra preliminaries. For $i \neq j$ let $R_i \to R_i + aR_j$ (respectively $C_i \to C_i + aC_j$) denote the operation of adding a constant multiple a of the jth row (column) of a given matrix Z to the ith row (column). Let Z^{ij} denote the matrix obtained from Z by deleting the ith row and the jth column. If Z is a square matrix we denote its characteristic polynomial by P_Z.

2.1. LEMMA: *Suppose X and Y are $n \times n$ matrices. Then*

$$\text{(i) } \det\begin{pmatrix} X & Y \\ Y & X \end{pmatrix} = \det(X - Y)\det(X + Y).$$

Suppose in addition that X is symmetric with zero row sum and that Y is diagonal. Then

$$\text{(ii) } (-1)^{n-1}\det(X^{ij}) = \frac{(-1)^{i+j}}{n}P_X'(0).$$

$$\text{(iii) } (-1)^{n-1}P_{XY}'(0) = \frac{1}{n}P_X'(0)P_Y'(0).$$

PROOF: Adding the first block row to the second transforms $\begin{pmatrix} X & Y \\ Y & X \end{pmatrix}$ to $\begin{pmatrix} X & Y \\ X+Y & X+Y \end{pmatrix}$; subtracting then the second block column from the first gives $\begin{pmatrix} X-Y & Y \\ 0 & X+Y \end{pmatrix}$, proving (i). Now suppose X is symmetric with zero row sum. For a fixed i let X_j denote the jth column of the $(n-1) \times n$ matrix obtained by omitting the ith row of X. By assumption $\sum_{j=1}^{n} \mathrm{X}_j = 0$ so that $\det(X^{ij}) = \det(X_1 \dots \hat{X}_j \dots X_n) = \det(-(X_2 + \cdots + X_n), \quad X_2 \dots \hat{X}_j \dots X_n) = \det(-X_j, \quad X_2 \dots \hat{X}_j \dots X_n) = (-1)^{j+1}\det(X_2 \dots X_n) = (-1)^{j+1}\det(X^{1j})$. Since X is symmetric $\det(X^{ij}) = (-1)^{i+j}\det(X^{11})$. However $(-1)^{n-1}P_X'(0) = \sum_{\ell=1}^{n}\det(X^{\ell\ell}) = n\det(X^{11})$, so (ii) follows. Finally suppose in addition that Y is diagonal. Note that $(XY)^{\ell\ell} = X^{\ell\ell}Y^{\ell\ell}$, so that

$$(-1)^{n-1}P_{XY}'(0) = \sum_{\ell=1}^{n}\det\left((XY)^{\ell\ell}\right) = \det(X^{11})\sum_{\ell=1}^{n}\det(Y^{\ell\ell})$$

$$= \frac{1}{n}P_X'(0)P_Y'(0),$$

proving (iii).

We can now calculate the order of Φ. By the theory of elementary divisors $|\Phi| = \gcd_{\alpha,\beta}(\det(A^{\alpha\beta}))$. By Lemma 2.1, $|\Phi| = |\det(A^{\alpha\beta})|$ for any α and β, which we will choose equal and among the $v_{i\ell}$. Row and column operations $R_\gamma \to R_\gamma + aR_\delta$, $C_\gamma \to C_\gamma + aC_\delta$ $(\gamma \neq \delta)$ will not change $\det(A^{\alpha\alpha})$ so long as $\delta \neq \alpha$. We will use these to simplify A.

Step 1: Suppose $\ell(y) = 2$ for $y \in \mathrm{Ed}(\Gamma_0 \backslash \Delta)$. Set $\alpha_1 = o(\hat{y})$, $\alpha_2 = t(\hat{y})$,

$\alpha_3 = w_{y1}$. Then $A_{\alpha\alpha_3} \neq 0$ only when $\alpha \in \{\alpha_i\}_{i=1}^3$. The 3×3 minor $M = (A_{\alpha_i\alpha_j})_{1 \leqslant i,j \leqslant 3}$ has the form

$$M = \begin{pmatrix} a & b & 1 \\ b & c & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Applying to A the transformations $R_{\alpha_1} \to R_{\alpha_1} + \frac{1}{2}R_{\alpha_3}$, $R_{\alpha_2} \to R_{\alpha_2} + \frac{1}{2}R_{\alpha_3}$, and then the symmetric operations on columns transforms the minor M to

$$M' = \begin{pmatrix} a+\frac{1}{2} & b+\frac{1}{2} & 0 \\ b+\frac{1}{2} & c+\frac{1}{2} & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

leaves A symmetric, and doesn't change the other elements of A.

Performing these operations for all $y \in \mathrm{Ed}(\Gamma_0 \backslash \Delta)$ with $\ell(y) = 2$ will transform the subminor

$$\left(A_{\alpha_k\alpha_\ell}\right)_{1 \leqslant k,\ell \leqslant 2} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad \text{where } \alpha_k = v_{i1}, \quad \alpha_\ell = v_{j2}$$

(or $\alpha_k = v_{i2}$ and $\alpha_\ell = v_{j1}$), $1 \leqslant i,\ j \leqslant h$, to

$$\begin{pmatrix} a + \frac{1}{2}\sum_{k=1}^{h} N_{ik}^2 & b + \frac{1}{2}N_{ij}^2 \\ b + \frac{1}{2}N_{ij}^2 & c + \frac{1}{2}\sum_{k=1}^{h} N_{kj}^2 \end{pmatrix}.$$

Step 2: Now suppose $\ell(y) = 3$ for $y \in \mathrm{Ed}(\Gamma_0 \backslash \Delta)$. Set $\alpha_1 = o(\hat{y})$, $\alpha_2 = t(\hat{y})$, $\alpha_3 = w_{y1}$, $\alpha_4 = w_{y2}$. The corresponding 4×4 minor has the form

$$M = \begin{pmatrix} a & b & 1 & 0 \\ b & c & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix}$$

and $A_{\alpha_3\alpha} = A_{\alpha_4\alpha} = 0$ for $\alpha \notin \{\alpha_i\}_{i=1}^4$. Applying $R_{\alpha_2} \to R_{\alpha_2} + \frac{1}{2}R_{\alpha_4}$, $R_{\alpha_3} \to R_{\alpha_3} + \frac{1}{2}R_{\alpha_4}$ and then $C_{\alpha_2} \to C_{\alpha_2} + \frac{1}{2}C_{\alpha_4}$, $C_{\alpha_3} \to C_{\alpha_3} + \frac{1}{2}C_{\alpha_4}$ transforms M to

$$M' = \begin{pmatrix} a & b & 1 & 0 \\ b & c+\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Applying next $R_{\alpha_1} \to R_{\alpha_1} + \frac{2}{3} R_{\alpha_3}$, $R_{\alpha_2} \to R_{\alpha_2} + \frac{1}{3} R_{\alpha_3}$, $C_{\alpha_1} \to C_{\alpha_1} + \frac{2}{3} C_{\alpha_3}$, and $C_{\alpha_2} \to C_{\alpha_2} + \frac{1}{3} C_{\alpha_3}$ gives

$$\begin{pmatrix} a+\frac{2}{3} & b+\frac{1}{3} & 0 & 0 \\ b+\frac{1}{3} & c+\frac{2}{3} & 0 & 0 \\ 0 & 0 & -3/2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Performing these operations for all $y \in \mathrm{Ed}(\Gamma_0 \backslash \Delta)$ with $\ell(y) = 3$ will transform the subminor

$$\left(A_{\alpha_k \alpha_\ell}\right)_{1 \leqslant k, \ell \leqslant 2} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad \text{where } \alpha_k = v_{im}, \quad \alpha_\ell = v_{j,3-m}$$

for $m = 1, 2$; $1 \leqslant i,\ j \leqslant h$, to

$$\begin{pmatrix} a + \frac{2}{3} \sum_{k=1}^{n} N_{ik}^3 & b + \frac{1}{3} N_{ij}^3 \\ b + \frac{1}{3} N_{ij}^3 & c + \frac{2}{3} \sum_{k=1}^{h} N_{jk}^3 \end{pmatrix}.$$

Step 3: Suppose that Ver $\tilde{G}$ is ordered so that the first h rows (and columns) of A correspond to $\{v_{i1}\}_{i=1}^{h}$ (in order) and the next h rows and columns similarly correspond to $\{v_{i2}\}_{i=1}^{h}$. After Steps 1 and 2 A is transformed to a matrix with block form $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$, where U is a $2h \times 2h$ matrix. For $1 \leqslant \ell \leqslant 3$ let n_ℓ be the number of oriented edges of length ℓ in $\mathrm{Ed}(\Gamma_0 \backslash \Delta)$. The matrix V is diagonal with $n_2 + n_3$ entries equal to -2 and n_3 entries equal to $-\frac{3}{2}$. U has the block form $U = \begin{pmatrix} J & N \\ N & J \end{pmatrix}$, where $N = N^1 + \frac{1}{2} N^2 + \frac{1}{3} N^3$ (see Section 1). By our calculation J is the diagonal matrix given by

$$J_{ii} = A_{ii} + \tfrac{1}{2} \sum_{j=1}^{h} N_{ij}^2 + \tfrac{2}{3} \sum_{j=1}^{h} N_{ij}^3 \quad \text{for } 1 \leqslant i \leqslant h.$$

Hence by [1.3, iv]

$$J_{ii} = -\sum_{j=1}^{h} \left(N_{ij}^1 + \tfrac{1}{2} N_{ij}^2 + \tfrac{1}{3} N_{ij}^3 \right).$$

It follows that U is a symmetric zero row sum matrix. By [1.1] $N = BF^{-1}$ and by [1.2] $-J = (p+1)F^{-1}$. Hence $U = \hat{U}\hat{F}^{-1}$, where $\hat{U} = \begin{pmatrix} -(p+1)I & B \\ B & -(p+1)I \end{pmatrix}$ and $\hat{F} = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$. Using Lemma 2.1, (iii) we now obtain

$$|\Phi| = |\det(A^{11})| = |\det(U^{11})\det(V)| = 2^{n_2}3^{n_3}\frac{1}{2h}|P'_U(0)|$$

$$= 2^{n_2}3^{n_2}|P'_{\hat{U}}(0)/P'_{\hat{F}}(0)|.$$

Firstly, $|P'_{\hat{F}}(0)| = 2|P_F(0)P'_F(0)| = 2(\text{tr } F^{-1})(\det F)^2$. Next, using Lemma 2.1 (i), $P_{\hat{U}}(x) = \det\begin{pmatrix} (x+p+1)I & -B \\ -B & (x+p+1)I \end{pmatrix} = (\det((x+p+1)I + B)\det((x+p+1)I - B)) = (-1)^h P_B(-x-p-1) P_B(x+p+1)$. Differentiating at $x = 0$ gives $P'_{\hat{U}}(0) = (-1)^h P_B(-p-1) P'_B(p+1)$, since $p+1$ is an eigenvalue for B, so that $P_B(p+1) = 0$. Hence we have proven:

2.2. Theorem:

$$|\Phi| = \frac{2^{n_2}3^{n_3}}{2(\text{tr } F^{-1})\cdot(\det F)^2}|P_B(-p-1)P'_B(p+1)|.$$

Using the results of Eichler [3] and Kurihara [5] we can rewrite Theorem 2.2 in a more convenient form. Let

$$e_2 = \prod_{q|\text{Disc } \mathscr{B}}\left(1-\left(\frac{-4}{q}\right)\right), \qquad e_3 = \prod_{q|\text{Disc } \mathscr{B}}\left(1-\left(\frac{-3}{q}\right)\right).$$

2.3. Theorem:

$$|\Phi| = \frac{1}{2m(\hat{\mathscr{B}})c(\hat{\mathscr{B}})2^{e_2}3^{e_3}}|P_B(-p-1)P'_B(p+1)|$$

where $c(\hat{\mathscr{B}}) = 8$ *if* Disc $\hat{\mathscr{B}} = 2$, $c(\hat{\mathscr{B}}) = 3$ *if* Disc $\hat{\mathscr{B}} = 3$, *and* $c(\hat{\mathscr{B}}) = 1$ *otherwise.*

Proof: By Eichler's mass formula tr $F^{-1} = m(\hat{\mathscr{B}})$. Suppose Disc $\hat{\mathscr{B}} \geqslant 5$. Then $f(v) \in \{1, 2, 3\}$ for all $v \in \text{Ver}(\Gamma_0\backslash\Delta)$; set $h_\ell = \#\{v \in \text{Ver}(\Gamma_0\backslash\Delta) \mid f(v) = \ell\}$. By Kurihara [5], Section 4 we have

$$h_2 = \tfrac{1}{2}\prod_{q|\text{Disc } \hat{\mathscr{B}}}\left(1-\left(\frac{-4}{q}\right)\right) \text{ and } h_3 = \tfrac{1}{2}\prod_{q|\text{Disc } \hat{\mathscr{B}}}\left(1-\left(\frac{-3}{q}\right)\right).$$

From Kurihara's table ([5], Proposition 4-2) we obtain

$$\frac{(\det F)^2}{2^{n_2}3^{n_3}} = \frac{2^{2h_2}3^{2h_3}}{2^{h_2(1+(-4/p))}3^{h_3(1+(-3/p))}} = 2^{e_2}3^{e_3}.$$

Suppose next Disc $\hat{\mathscr{B}} = 3$. Then F is the 1×1 matrix (6) and Kurihara's table gives

$$\frac{(\det F)^2}{2^{n_2}3^{n_3}} = \frac{36}{2^{(1+(-4/p))}3^{(1/2)(1+(-3/p))}} = 3 \cdot 2^{e_2} \cdot 3^{e_3}.$$

Finally if Disc $\hat{\mathscr{B}} = 2$, $F = (12)$ and

$$\frac{(\det F)^2}{2^{n_2}3^{n_3}} = \frac{144}{2^{(1/2)(1+(-4/p))}3^{(1+(-3/p))}} = 8 \cdot 2^{e_2} \cdot 3^{e_3}.$$

The theorem follows.

2.4. Remark: In the course of the proof of Theorem 2.2 we inverted only 2 and 3. Likewise the proof of Lemma 2.1, i) shows that one can transform $\begin{pmatrix} X & Y \\ Y & X \end{pmatrix}$ to $\begin{pmatrix} X-Y & 0 \\ 0 & X+Y \end{pmatrix}$ by elementary row and column transformations $R_i \to R_i + aR_j$, $C_i \to C_i + aC_j$ with $a \in \mathbb{Z}[\frac{1}{2}]$. Hence setting

$$M = \mathbb{Z}\left[\tfrac{1}{6}\right]^h, \qquad M_0 = \left\{(a_1, \dots, a_h) \in M \mid \sum \frac{a_i}{f(v_i)} = 0\right\},$$

we have

$$\Phi \otimes \mathbb{Z}\left[\tfrac{1}{6}\right] \approx M_0/(B-(p+1)I)M \oplus M/(B+(p+1)I)M.$$

§3. The connected component

Since all components of the special fiber $(\widetilde{M_{\mathscr{B}} \times \mathbb{Z}_p})_0$ are rational the connected component $\mathscr{J}_p^0$ is a torus.

3.1. Theorem: $\mathscr{J}_p^0 \approx H^1((\Gamma_+ \backslash \Delta), \mathbb{Z}) \otimes \mathbb{G}_m$. *The action of Frobenius is* $w_p \otimes \mathrm{Frob}_{\mathbb{G}_m}$.

Proof: We need only remark that $\Gamma_+ \backslash \Delta$, $\tilde{G}$, and the graph of the special fiber as defined in Deligne and Rapoport [1], p. 164, are all naturally homotopic, so that [1], 3.7b applies.

3.2. COROLLARY: *Let $\ell \neq p$ be a prime. Then the Tate module*

$$Ta_\ell(\mathscr{J}_p^0) \approx H^1((\Gamma_+ \backslash \Delta), \mathbb{Z}_\ell)$$

with Frobenius acting as pw_p.

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