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VECTOR BUNDLES OVER REAL ALGEBRAIC SURFACES AND THREEFOLDS

Wojciech Kucharz

Introduction

Algebraic vector subbundles of product vector bundles, called strongly algebraic vector bundles, over affine real algebraic varieties have remarkable properties reflecting an interplay between algebra, geometry, and topology. The theory of strongly algebraic vector bundles has attracted the attention of several mathematicians. However, only in some special cases is it understood when a given continuous vector bundle over a real algebraic variety is C^0 isomorphic to a strongly algebraic one (cf. [3], [4], [5], [10], [11], [12], [19], [23], [26]).

In this paper necessary and sufficient conditions are given for a continuous vector bundle over a compact affine nonsingular real algebraic surface or threefold to be C^0 isomorphic to a strongly algebraic vector bundle. The result is used to compare algebraic and topological K-theory.

The paper is organized as follows. The main results are formulated in Section 1 and proved in Section 4. The proofs depend on Section 3, where a construction of strongly algebraic vector bundles corresponding to algebraic subvarieties of codimension 2 is given. In Section 2, which is based on Section 1, relationships between algebraic and topological K-theory are studied.

Convention. Algebraic varieties and regular maps between them are understood in the sense of Serre [24]. Varieties are not assumed to be irreducible; subvarieties are assumed to be closed. Real algebraic varieties are equipped with the strong topology induced by the Euclidean topology on the reals.

1. Continuous and strongly algebraic vector bundles

Proofs contained in this section are based on Theorems 1.1, 1.8, and Propositions 1.10 and 1.16, which are proved in Section 4.

Throughout the section it is assumed that all vector bundles are real and have the same rank over various connected components of a base space.

Let X be an affine real algebraic variety. An algebraic real vector bundle over X is said to be *strongly algebraic* if it is algebraically isomorphic to an algebraic vector subbundle of a product vector bundle $X \times \mathbb{R}^m$ for some nonnegative integer m. Note that the total space of a strongly algebraic vector bundle is an affine variety. It is known [3,6] that the real Grassmann variety $G_{n,p}$ of p-dimensional vector subspaces of $\mathbb{R}^n$ is affine and nonsingular and the canonical Hopf vector bundle $\gamma_{n,p}$ over it is strongly algebraic. Since the notion of a strongly algebraic vector bundle plays an important role in our consideration, let us recall the following result (cf. [3], [6], [29] for proofs).

PROPOSITION A: *Let ξ be an algebraic vector bundle of rank p over an affine real algebraic variety X. Then the following conditions are equivalent*:
(a) *ξ is strongly algebraic.*
(b) *For each x in X there exist regular global sections $s_1, \dots, s_p$ of ξ linearly independent at x.*
(c) *There exists a regular map $f: X \to G_{n,p}$ such that ξ is algebraically isomorphic to the pullback vector bundle $f^*\gamma_{n,p}$.*

Moreover, the global section functor defines a one-to-one correspondence between the set of algebraic isomorphism classes of strongly algebraic vector bundles over X and the set of isomorphism classes of finitely generated projective $R(X)$-modules. □

Here $R(X)$ denotes the ring of regular functions on X. We recall that if X is an algebraic subvariety of $\mathbb{R}^n$, then the ring $R(X)$ consists of all functions of the form f/g, where f and g are the restrictions to X of polynomial functions on $\mathbb{R}^n$ and g is nowhere zero on X.

Our purpose is to give a characterization of continuous vector bundles over affine real algebraic varieties which are C^0 isomorphic to strongly algebraic ones. To this end we need some preparation. Let X be an affine real algebraic variety of dimension n. Denote by $H_k^{alg}(X, \mathbf{Z}_2)$ (resp. $H_k^{Alg}(X, \mathbf{Z}_2)$)the subgroup of $H_k(X, \mathbf{Z}_2)$ generated by homology classes represented by compact algebraic (resp. compact algebraic nonsingular) k-dimensional subvarieties of X (cf. [4], [6], and [8]). If, moreover, X is compact and nonsingular, then we denote by $H_{alg}^{n-k}(X, \mathbf{Z}_2)$ and $H_{Alg}^{n-k}(X, \mathbf{Z}_2)$ the subgroups of the cohomology group $H^{n-k}(X, \mathbf{Z}_2)$, which correspond to $H_k^{alg}(X, \mathbf{Z}_2)$ and $H_k^{Alg}(X, \mathbf{Z}_2)$, respectively, via Poincaré duality. The following result gives a necessary condition for a continuous vector bundle to be C^0 isomorphic to a strongly algebraic one (cf. [4] and [26] for proofs.)

PROPOSITION B: *Let ξ be a continuous vector bundle over a compact affine nonsingular real algebraic variety X. Assume that ξ is C^0 isomorphic to a strongly algebraic vector bundle. Then for each nonnegative integer k, the k-th Stiefel-Whitney characteristic class $w_k(\xi)$ of ξ is in $H_{alg}^k(X, \mathbf{Z}_2)$. If* $p = \operatorname{rank} \xi$, *then $w_p(\xi)$ is in $H_{Alg}^p(X, \mathbf{Z}_2)$. Moreover, every continuous line* *bundle over X whose first Stiefel-Whitney characteristic class is in $H^1_{alg}(X, \mathbf{Z}_2)$ is C^0 isomorphic to a strongly algebraic vector bundle.* □

Shiota [26] conjectured that given X as in Proposition B and a continuous vector bundle ξ over X such that $w_k(\xi)$ belongs to $H^k_{alg}(X, \mathbf{Z}_2)$ for each nonnegative integer k, ξ is C^0 isomorphic to a strongly algebraic vector bundle. Here his conjecture is shown to be true if dim $X = 2$ and "almost" true if X is orientable (as a smooth manifold) of dimension 3.

THEOREM 1.1: *Let X be a compact affine nonsingular real algebraic surface. A continuous vector bundle over X is C^0 isomorphic to a strongly algebraic vector bundle if and only if its first Stiefel-Whitney characteristic class is in $H^1_{alg}(X, \mathbf{Z}_2)$.*

As direct consequence we have.

COROLLARY 1.2: *Let X be as in Theorem* 1.1. *Then every continuous orientable vector bundle over X is C^0 isomorphic to a strongly algebraic vector bundle.*

PROOF: The first Stiefel-Whitney characteristic class of a continuous orientable vector bundle over X is equal to 0, and hence belongs to $H^1_{alg}(X, \mathbf{Z}_2)$. □

COROLLARY 1.3: *Let X be as in Theorem* 1.1. *Then the following conditions are equivalent*:

(*i*) *Every continuous vector bundle over X is C^0 isomorphic to a strongly algebraic vector bundle.*

(*ii*) *Every continuous line bundle over X is C^0 isomorphic to a strongly algebraic line bundle.*

(*iii*) $H^1_{alg}(X, \mathbf{Z}_2) = H^1(X, \mathbf{Z}_2)$

PROOF: (i) ⇒ (ii) is obvious. To prove (ii) ⇒ (iii) it suffices to use the fact that every element in $H^1(X, \mathbf{Z}_2)$ is the firrst Stiefel-Whitney characteristic class of some continuous line bundle over X. Finally (iii) ⇒ (i) follows from Theorem 1.1. □

COROLLARY 1.4: *Let X be a compact affine nonsingular real algebraic variety homeomorphic to the 2-sphere or real projective plane. Then every continuous vector bundle over X is C^0 isomorphic to a strongly algebraic one.*

PROOF: Indeed, $H^1(X, \mathbf{Z}_2) = H^1_{alg}(X, \mathbf{Z}_2) = 0$ if X is homeomorphic to the 2-sphere. For X homeomorphic to the real projective plane

$H^1(X, \mathbf{Z}_2) = H^1_{alg}(X, \mathbf{Z}_2) = \mathbf{Z}_2$, for every compact affine nonsingular nonorientable real algebraic variety contains an algebraic hypersurface which determines a nonzero homology class [4,5]. □

Let $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 = 1\}$ be the unit n-sphere. For $X = S^2$ the conclusion of Corollary 1.4 is known [11], [19]. However, for X only homeomorphic to S^2 the methods of [11] and [19] cannot be applied and the proof given in [3] (for X homeomorphic to S^n, n arbitrary) is incorrect.

EXAMPLE 1.5: Denote by U_k and V_k the algebraic surfaces obtained by blowing up k points from S^2 and $S^1 \times S^1$, respectively. Let $X = U_k$ or V_k. Clearly, $H^1_{alg}(X, \mathbf{Z}_2) = H^1(X, \mathbf{Z}_2)$ and hence every continuous vector bundle over X is C^0 isomorphic to a strongly algebraic one.

For arbitrary algebraic surfaces, the picture is more complicated.

EXAMPLE 1.6: Given a compact connected smooth surface M, not diffeomorphic to the 2-sphere, the real projective plane or the Klein bottle, there exists an affine nonsingular real algebraic surface X diffeomorphic to M with $H^1_{alg}(X, \mathbf{Z}_2) \neq H^1(X, \mathbf{Z}_2)$, [17], cf. also [4], [22], [25]. Thus there exist continuous vector bundles over X which are not C^0 isomorphic to strongly algebraic vector bundles. On the other hand M is diffeomorphic to some affine nonsingular real algebraic variety Y such that every continuous vector bundle over Y is C^0 isomorphic to a strongly algebraic one. The last statement holds true without any assumption about the dimension of M [3].

From Theorem 1.1 one can also derive a result concerning an approximation of continuous or smooth maps by regular ones.

COROLLARY 1.7. *Let X be a compact affine nonsingular real algebraic surface and let n be an integer $\geqslant 2$. Then the following conditions are equivalent*:

(*i*) *Each continuous map from X to $G_{n,p}$, $1 \leqslant p \leqslant n-1$, can be approximated in the C^0 topology by regular maps.*

(*ii*) *Each smooth map from X to $G_{n,p}$, $1 \leqslant p \leqslant n-1$, can be approximated in the C^∞ topology by regular maps.*

(*iii*) $H^1(X, \mathbf{Z}_2) = H^1_{alg}(X, \mathbf{Z}_2)$

PROOF: The implications (i) ⇒ (iii) and (ii) ⇒ (iii) are obvious. Indeed, any element v in $H^1(X, \mathbf{Z}_2)$ can be written as $v = w_1(f^*\gamma_{n,p})$ for some smooth map f from X to $G_{n,p}$. It follows from the assumptions and Proposition A that the vector bundle $f^*\gamma_{n,p}$ is C^0 isomorphic to a strongly algebraic vector bundle. By Proposition B, v is in $H^1_{alg}(X, \mathbf{Z}_2)$. (iii) ⇒ (i) and (iii) ⇒ (ii) follow from Theorem 1.1 and [7]. □

For vector bundles over algebraic threefolds, we have the following.

THEOREM 1.8: *Let X be a compact affine nonsingular real algebraic threefold. Assume that X is orientable as a smooth manifold. Then a given continuous vector bundle over X is C^0 isomorphic to a strongly algebraic vector bundle if and only if its first and second Stiefel-Whitney characteristic classes are in $H^1_{alg}(X, \mathbf{Z}_2)$ and $H^2_{Alg}(X, \mathbf{Z}_2)$, respectively.*

COROLLARY 1.9: *Let X be as in Theorem* 1.8. *Then the following conditions are equivalent*:

(*i*) *Every continuous vector bundle over X is C^0 isomorphic to a strongly algebraic vector bundle.*

(*ii*) *Every continuous vector bundle of rank* 2 *over X is C^0 isomorphic to a strongly algebraic vector bundle.*

(*iii*) $H^1_{alg}(X, \mathbf{Z}_2) = H^1(X, \mathbf{Z}_2)$ *and* $H^2_{Alg}(X, \mathbf{Z}_2) = H^2(X, \mathbf{Z}_2)$.

PROOF: Given v in $H^1(X, \mathbf{Z}_2)$, one can find a continuous line bundle ξ over X with $w_1(\xi) = v$ [14]. Analogously given u in $H^2(X, \mathbf{Z}_2)$, there exists a continuous vector bundle η of rank 2 for which $w_1(\eta) = 0$ and $w_2(\eta) = u$. Indeed since $H^3(X, \mathbf{Z}) = \mathbf{Z}$, by the universal coefficient theorem [27], p. 246, the natural reduction (modulo 2) homomorphism from $H^2(X, \mathbf{Z}) \otimes \mathbf{Z}_2$ to $H^2(X, \mathbf{Z}_2)$ is an isomorphism. On the other hand, any element in $H^2(X, \mathbf{Z})$ can be written as the Euler class of a continuous oriented vector bundle of rank 2 and the reduction modulo 2 of the Euler class is the second Stiefel-Whitney class [20]. These remarks prove (ii) ⇒ (iii), while (iii) ⇒ (i) follows from Theorem 1.8, and (i) ⇒ (ii) is trivial. □

For nonorientable threefolds, the picture seems to be more complicated. We only have the following.

PROPOSITION 1.10: *Let X be a compact affine nonsingular real algebraic threefold. Then a given continuous orientable vector bundle of rank* 2 *over X is C^0 isomorphic to a strongly algebraic vector bundle if and only if its second Stiefel-Whitney characteristic class corresponds, via Poincaré duality, to the homology class represented by a one-dimensional nonsingular algebraic subvariety of X with trivial normal bundle.*

In view of the above results, it would be interesting to know if $H^2_{Alg}(X, \mathbf{Z}_2) = H^2_{alg}(X, \mathbf{Z}_2)$ for all compact affine nonsingular real algebraic threefolds.

REMARK 1.11: If X is a compact affine nonsingular real algebraic variety, then $H^1_{alg}(X, \mathbf{Z}_2) = H^1_{Alg}(X, \mathbf{Z}_2)$, [4], [26]. Moreover, the cup product of k elements in $H^1_{alg}(X, \mathbf{Z}_2)$ belongs to $H^k_{Alg}(X, \mathbf{Z}_2)$ [26].

For some threefolds one can show that $H^2_{alg} = H^2_{Alg}$. In particular, we have the following.

COROLLARY 1.12: *Let X_1, X_2 be two compact connected affine nonsingular real algebraic varieties. Assume that* $\dim X_1 = 1$, $\dim X_2 = 2$, *and X_2 is orientable as a smooth manifold. Then a given continuous vector bundle over $X = X_1 \times X_2$ is C^0 isomorphic to a strongly algebraic vector bundle if and only if its first and second Stiefel-Whitney characteristic classes are in $H^1_{alg}(X, \mathbf{Z}_2)$ and $H^2_{alg}(X, \mathbf{Z}_2)$, respectively.*

PROOF: It suffices to show that $H_1^{alg}(X, \mathbf{Z}_2) = H_1^{Alg}(X, \mathbf{Z}_2)$. Note that $H_1(X, \mathbf{Z}_2)$ is generated by the homology classes represented by $\{x_1\} \times Y$ and $X_1 \times \{x_2\}$, where $x_1 \in X_1$, $x_2 \in X_2$, and Y is a smooth one-dimensional submanifold of X_2. Since the homology class represented by $X_1 \times \{x_2\}$ is in $H_1^{Alg}(X, \mathbf{Z}_2)$ it remains to show that the homology class represented by $\{x_1\} \times Y$ is in $H_1^{Alg}(X, \mathbf{Z}_2)$ provided it belongs to $H_1^{alg}(X, \mathbf{Z}_2)$. Let $p: X_1 \times X_2 \to X_2$ be the natural projection. Then the induced homomorphism p_* from $H_1(X, \mathbf{Z}_2)$ to $H_1(X_2, \mathbf{Z}_2)$ maps u onto the homology class v represented by Y. One sees easily, using [4], Proposition 2.3, that v is in $H_1^{alg}(X_2, \mathbf{Z}_2)$. By [4] or [26] there exists a smooth embedding h of Y into X_2, arbitrarily close to the inclusion map in the C^∞ topology, such that $V = h(Y)$ is a nonsingular algebraic subvariety of X_2. Clearly, $\{x_1\} \times V$ represents u and hence u belongs to $H_1^{Alg}(X, \mathbf{Z}_2)$. □

EXAMPLE 1.13: Let X_i, $i = 1, 2, 3$, be a compact connected affine nonsingular real algebraic curve. Then every continuous vector bundle over $X = X_1 \times X_2 \times X_3$ is C^0 isomorphic to a strongly algebraic one. Indeed, it is clear that $H^1_{alg}(X, \mathbf{Z}_2) = H^1(X, \mathbf{Z}_2)$ and $H^2_{Alg}(X, \mathbf{Z}_2) = H^2(X, \mathbf{Z}_2)$.

EXAMPLE 1.14: Let X be an affine nonsingular real algebraic variety homeomorphic to the three-dimensional real projective space. Then every continuous vector bundle over X is C^0 isomorphic to a strongly algebraic one if and only if $H^1_{alg}(X, \mathbf{Z}_2) = H^1(X, \mathbf{Z}_2)$. Indeed, the "if" part follows from Corollary 1.9 and the converse from Remark 1.11.

We have also a result concerning an approximation of continuous or smooth maps on real algebraic threefolds by regular maps. The proof will be omitted since it is analogous to the proof of Corollary 1.7.

COROLLARY 1.15: *Let X be a compact affine nonsingular real algebraic threefold. Assume that X is orientable as a smooth manifold. Then the following conditions are equivalent*:

(*i*) *For all positive integers n, p with $p \leqslant n-1$, each continuous map from X to $G_{n,p}$ can be approximated in the C^0 topology by regular maps.*

(*ii*) *For all positive integers n, p with $p \leqslant n-1$, each smooth map from X to $G_{n,p}$ can be approximated in the C^∞ topology by regular maps.*

(*iii*) $H^1(X, \mathbf{Z}_2) = H^1_{alg}(X, \mathbf{Z}_2)$ *and* $H^2(X, \mathbf{Z}_2) = H^2_{Alg}(X, \mathbf{Z}_2)$.

For studying some properties of real algebraic threefolds, the following is useful.

PROPOSITION 1.16: *Let X be a compact affine nonsingular real algebraic threefold and let C be a smooth one-dimensional submanifolds of X. Assume that X is orientable as a smooth manifold. Then the following conditions are equivalent*:

(*i*) *The homology class represented by C in $H_1(X, \mathbf{Z}_2)$ is in $H_1^{Alg}(X, \mathbf{Z}_2)$.*

(*ii*) *There exists a smooth diffeotopy of X transforming C onto a nonsingular algebraic subvariety of X.*

(*iii*) *There exists an embedding $h: C \to X$, arbitrarily close to the inclusion map $C \to X$ in the C^∞ topology, such that $h(C)$ is a nonsingular algebraic subvariety of X.*

The next example shows that Corollary 1.2 is no longer true for vector bundles over algebraic threefolds.

EXAMPLE 1.17: There exists an affine nonsingular real algebraic threefold X and a continuous orientable vector bundle ξ of rank 2 over it such that X is diffeomorphic to $S^1 \times S^2$ and ξ is not C^0 isomorphic to a strongly algebraic vector bundle. Indeed, by [17] there exists an affine nonsingular real algebraic variety X and a smooth curve $C \subset X$ such that X is diffeomorphic to $S^1 \times S^2$, C is diffeomorphic to S^1, and C is not diffeotopic to any nonsingular algebraic subvariety of X. Let ξ be a smooth orientable vector bundle of rank 2 over X whose second Stiefel-Whitney characteristic class corresponds, via Poincare duality, to the homology class represented by the cycle C (cf. the proof of Corollary 1.9). It follows from Proposition 1.16 that ξ cannot be C isomorphic to a strongly algebraic vector bundle.

2. Comparison of algebraic and topological K-theory

The results of Section 1 can be used to compare algebraic and topology K-theory. Given an affine real algebraic variety X, we denote by $V_{\mathbf{R}}(X)$

(resp. $V_{alg}(X)$) the category of C^0 (resp. algebraic) isomorphism classes of continuous (resp. strongly algebraic) vector bundles over X. Denote by $K_{\mathbb{R}}(X)$ (resp. $K_{alg}(X)$) the K group of the semigroup (with the Whitney sum as addition) $V_{\mathbb{R}}(X)$ (resp. $V_{alg}(X)$). Note that $K_{\mathbb{R}}$ is just the real topological K-theory, often denoted by KO. One can consider the natural homomorphism of semigroups

$$\varphi_X : V_{\text{alg}}(X) \to V_{\mathbb{R}}(X),$$

which assigns to the algebraic isomorphism class of a strongly algebraic vector bundle the C^0 isomorphism class of the same bundle and the induced homomorphism of groups

$$\psi_X : K_{\text{alg}}(X) \to K_{\mathbb{R}}(X).$$

The maps φ_X and ψ_X are important for understanding relationships between algebraic and topological K-theory. Let us recall briefly what is known about them:

(2.1) Both maps φ_X and ψ_X are injective if X is compact [3], [10].

(2.2) In general, even for X compact and connected, the maps are not surjective [4] and Examples 1.6 and 1.17 above.

(2.3) The C^0 isomorphism class of a continuous vector bundle ξ over X is in the image of φ_X if and only if the class represented by ξ in $K_{\mathbb{R}}(X)$ is in the image of ψ_X. In other words, ξ is C^0 isomorphic to a strongly algebraic vector bundle if and only if $\xi \oplus \eta$ is, for some continuous vector bundle η which is C^0 isomorphic to a strongly algebraic one [3]. In particular, φ_X is a bijection if and only if ψ_X is.

(2.4) Both maps are bijective if X is the unit n-sphere (it follows from (2.3) and [11]).

(2.5) If M is a compact smooth manifold, then there exists an affine nonsingular real algebraic variety X diffeomorphic to M such that φ_X and ψ_X are bijective [3].

For more information [4], [12], [23] can be consulted.

It would be interesting to describe $Coker\ \psi_X = K_{\mathbb{R}}(X)/Image\ \psi_X$ and we propose here such a description for $dim\ X \leqslant 3$. First we need some preparation. Given a continuous vector bundle ξ over a compact topological space X we denote by $[\xi]$ its class in $K_{\mathbb{R}}(X)$. The product vector bundle $X \times \mathbb{R}^n$ is denoted by ϵ^n. Thus every element in $K_{\mathbb{R}}(X)$ is of the form $[\xi] - [\epsilon^n]$ for some continuous vector bundle ξ over X and a nonnegative integer n.

CONVENTION. *From now on we shall only be dealing with compact varieties in this section. Therefore we shall identify* (*which is justified by* (2.1)) $K_{alg}(X)$ *with the image of* ψ_X *and* $K_{\mathbb{R}}(X)/K_{alg}(X)$ *with the cokernel of* ψ_X.

Now let X be a compact affine real algebraic variety. We define the homomorphism of groups

$$\beta_X: K_{\mathbb{R}}(X)/K_{alg}(X) \to H^1(X, \mathbf{Z}_2)/H^1_{alg}(X, \mathbf{Z}_2)$$

induced by the map which assigns to each continuous vector bundle over X its first Stiefel-Whitney characteristic class. It follows from Proposition B that β_X is well defined. Clearly, β_X is a surjection [14].

THEOREM 2.6: *Let X be a compact connected affine nonsingular real algebraic surface. Then*

$$\beta_X: K_{\mathbb{R}}(X)/K_{alg}(X) \to H^1(X, \mathbf{Z}_2)/H^1_{alg}(X, \mathbf{Z}_2)$$

is an isomorphism.

PROOF: It suffices to show that β_X is a monomorphism. Let ξ be a continuous vector bundle over X and let n be a nonnegative integer. Suppose that $([\xi]-[\epsilon^n])$ mod $K_{alg}(X)$ belongs to Ker β_X. This implies that $w_1(\xi)$ is in $H^1_{alg}(X, \mathbf{Z}_2)$ and hence, by Theorem 1.1, ξ is C^0 isomorphic to a strongly algebraic vector bundle. Thus β_X is a monomorphism. □

Again let X be a compact affine nonsingular real algebraic variety. Denote by G the subgroup of $K_{\mathbb{R}}(X)$ generated by classes (in $K_{\mathbb{R}}(X)$) of continuous orientable vector bundles over X. Let

$$\gamma_X: G/G \cap K_{alg}(X) \to H^2(X, \mathbf{Z}_2)/H^2_{Alg}(X, \mathbf{Z}_2)$$

be the homomorphism of groups induced by the map which assigns to each vector bundle over X its second Stiefel-Whitney characteristic class. It follows from elementary properties of characteristic classes and Proposition B that γ_X is a well defined homomorphism. Denote by

$$j_X: G/G \cap K_{alg}(X) \to K_{\mathbb{R}}(X)/K_{alg}(X)$$

the canonical monomorphism of groups.

THEOREM 2.7: *Let X be a compact connected affine nonsingular real algebraic threefold. Assume that X is orientable as a smooth manifold. Then γ_X is an isomorphism and*

$$0 \to H^2(X, \mathbf{Z}_2)/H^2_{Alg}(X, \mathbf{Z}_2) \xrightarrow{\alpha_X} K_{\mathbb{R}}(X)/K_{alg}(X)$$

$$\xrightarrow{\beta_X} H^1(X, \mathbf{Z}_2)/H^1_{alg}(X, \mathbf{Z}_2) \to 0$$

is an exact sequence, where $\alpha_X = j_X \circ \gamma_X^{-1}$. *The sequence splits if and only if for each element* v *in* $H^1(X, \mathbf{Z}_2)$ *the cup product* $v \cup v$ *is in* $H^2_{Alg}(X, \mathbf{Z}_2)$.

PROOF: First we have to show that γ_X is an isomorphism. Clearly, γ_X is surjective (cf. the proof of Corollary 1.9). Any element in G is of the form $[\xi] - [\epsilon^n]$, where ξ is a continuous orientable vector bundle over X and n is a nonnegative integer. Suppose that $([\xi] - [\epsilon^n]) \bmod G \cap K_{alg}(X)$ is in Ker γ_X. This means that $w_2(\xi)$ belongs to $H_2^{Alg}(X, \mathbf{Z}_2)$. Since $w_1(\xi) = 0$, ξ is, by Theorem 1.8, C^0 isomorphic to a strongly algebraic vector bundle. Thus γ_X is a monomorphism.

Since β_X is an epimorphism and $\beta_X \circ \alpha_X$ is the zero homomorphism, it remains to prove that Ker β_X is contained in Im α. Let ζ be a continuous vector bundle over X and let m be a nonnegative integer. Suppose that $([\zeta] - [\epsilon^m]) \bmod K_{alg}(X)$ belongs to Ker β_X. Let δ be a continuous line bundle with $w_1(\delta) = w_1(\zeta)$. By Proposition B one may assume that δ is strongly algebraic. Since $w_1(\zeta \oplus \delta) = 0$, $\zeta \oplus \delta$ is orientable and $[\zeta \oplus \gamma]$ is in G. Note that $[\zeta \oplus \delta] - [\delta \oplus \epsilon^m] = [\zeta] - [\epsilon^m]$. The last part of Theorem 2.7 is obvious. □

The corollaries and examples of Section 1 can be used easily to illustrate the usefulness of Theorems 2.6 and 2.7. Here we confine ourselves to only one observation.

COROLLARY 2.8: *Let* X *be as in Theorem* 2.7. *Then the group* $K_{\mathbb{R}}(X)/K_{alg}(X)$ *is finite and each of its elements have either order* 2 *or* 4.

PROOF: Given a continuous vector bundle ξ over X one checks easily that the first and second Stiefel-Whitney characteristic classes of the Whitney sum of four copies of ξ vanish. Therefore each element in $K_{\mathbb{R}}(X)/K_{alg}(X)$ has order $\leqslant 4$. Since $w_1(\xi \oplus \xi \oplus \xi) = w_1(\xi)$ and $w_2(\xi \oplus \xi \oplus \xi) = w_2(\xi) + w_1(\xi) \cup w_1(\xi)$ there are no elements of order 3. Indeed, $\xi \oplus \xi \oplus \xi$ is C^0 isomorphic to a strongly algebraic vector bundle if and only if ξ is. The last observation follows from Theorem 1.8 and Remark 1.11. Finally, since the group $K_{\mathbb{R}}(X)$ is finitely generated [16] p. 175, the group $K_{\mathbb{R}}(X)/K_{alg}(X)$ is finite. □

REMARK 2.9: The author does not know whether the exact sequence of Theorem 2.7 always splits and hence whether there exist elements of order 4 in $K_{\mathbb{R}}(X)/K_{alg}(X)$.

Note that Theorems 2.6 and 2.7 can be restated in more algebraic language. Indeed, $K_{\mathbb{R}}(X)$ and $K_{alg}(X)$ can be computed as follows. The global section functor defines the isomorphisms

$$K_{\mathbb{R}}(X) \to K_0(C(X)), \quad K_{alg}(X) \to K_0(R(X)),$$

where $C(X)$ and $R(X)$ are the rings of continuous (real-valued) and regular functions on X, respectively. Hence in Theorems 2.6 and 2.7, the cokernel of the homomorphism of groups

$$\lambda_X : K_0(R(X)) \to K_0(C(X))$$

induced by the inclusion $R(X) \subset C(X)$ is described. If X is an algebraic subvariety of $\mathbb{R}^n$, then $\lambda_X \circ \rho_X = \mu_X$, where

$$\rho_X : K_0(P(X)) \to K_0(R(X)), \quad \mu_X : K_0(P(X)) \to K_0(C(X))$$

are homomorphisms induced by inclusions of the ring $P(X)$ of polynomial functions on X in $R(X)$ and $C(X)$, respectively. Since $R(X)$ is canonically isomorphic to the localization of $P(X)$ with respect to

$$S = \{ f \in P(X) \mid f \text{ is nowhere zero} \},$$

ρ_X is an epimorphism provided that $P(X)$ is a regular ring [2], p. 499. Thus in this case *Coker* $\mu_X =$ *Coker* λ_X. In general, μ_X is not injective [10]. It is known, however, that μ_X is an isomorphism in some special cases (cf. [7], [11], [15], [23]).

Instead of real vector bundles one can also consider continuous and strongly algebraic complex vector bundles over real algebraic varieties and try to compare corresponding K-theories. For more information the reader may consult [7], [11], [12], [15], [23].

3. Subvarieties of codimension 2 and vector bundles of rank 2

The main result of this section is the following.

THEOREM 3.1: *Let X be an affine nonsingular real algebraic variety and let Y be a nonsingular algebraic subvariety of codimension 2. Assume that Y has an algebraically trivial normal vector bundle in X. Then there exist a strongly algebraic orientable vector bundle ξ of rank 2 over X and a regular section s of ξ such that s is transversal to the zero section and $Y = \{x \in X \mid s(x) = 0\}$.*

REMARK 3.2: The normal vector bundle of Y in X is strongly algebraic. Therefore if it is C^0 trivial it is also algebraically trivial, provided X compact [3], [10].

To prove Theorem 3.1 we need some preparation. Given an algebraic subvariety V of $\mathbb{R}^n$ we denote by V_C its complexification, i.e. the smallest complex algebraic subset of C^n that contains V, where $\mathbb{R}^n$ is identified with a subset of C^n in the usual way. Similarly, for a polynomial function $f: V \to \mathbb{R}$, $f_C : V_C \to C$ denotes its complexification.

The following lemma will be useful.

LEMMA 3.3: *Let V be an algebraic subvariety of $\mathbb{R}^n$ and let φ be a polynomial function on V. Given a complex regular function*

$$h: V_C - \{z \in V_C \mid \varphi_C(z) = 0\} \to C$$

there exist polynomial functions f, $g: V \to \mathbb{R}$ and a nonnegative integer k such that

$$h(z) = f_C(z)/\varphi_C^k(z) + ig_C(z)/\varphi_C^k(z)$$

for all z in $V_C - \{z \in V_C \mid \varphi_C(z) = 0\}$, where $i^2 = -1$.

PROOF: Let

$$W = \{(z, t) \in C^n \times C \mid t\varphi_C(z) = 1\}.$$

The map $j: V_C - \{z \in V_C \mid \varphi_C(z) = 0\} \to W$ defined by $j(z) = (z, 1/\varphi_C(z))$ is a regular isomorphism. The function $h \circ j^{-1}: W \to C$, being regular on a complex affine algebraic set W, is polynomial. Let u, $v: W \to C$ be the polynomial functions with real coefficients such that $h \circ j^{-1} = u + iv$. Clearly, $u \circ j$, $v \circ j$ can be written as $u \circ j = f_C/\varphi_C^k$, $v \circ j = g_C/\varphi_C^k$ for some polynomial functions f, $g: V \to \mathbb{R}$ and some nonnegative integer k. □

NOTATION: Using the same notation as in Lemma 3.3, we set $r(h) = f_C/\varphi_C^k$. One sees easily that $r(h)$ is well defined, $r(h_1 + h_2) = r(h_1) + r(h_2)$ and $r(kh) = kr(h)$ if $k = \psi_C \mid V_C - \{z \in V_C \mid \varphi_C(z) = 0\}$ for some polynomial function $\psi: V \to \mathbb{R}$.

PROOF OF THEOREM 3.1: We may assume that X is a subvariety of $\mathbb{R}^n$ for some positive integer n. Since Y has an algebraically trivial normal vector bundle in X, there exist two polynomial functions f_1, $f_2: X \to \mathbb{R}$ such that $0 \in \mathbb{R}^2$ is a regular value of the map (f_1, f_2) and

$$\{x \in X \mid f_1(x) = f_2(x) = 0\} = Y_1 \cup Y_2,$$

where $Y_1 = Y$ and Y_2 is disjoint from Y. We note that $Y_1 \cup Y_2$ and Y_1 are nonsingular algebraic subvarieties of X of the same dimension. Hence Y_2 is also an algebraic subvariety of X, [1]. Let $\psi_i: X \to \mathbb{R}$ be a polynomial function with $Y_i = \{x \in X \mid \psi_i(x) = 0\}$, $i = 1, 2$.

Now since the ideal of all regular functions on X vanishing on $Y_1 \cup Y_2$ is generated by f_1 and f_2, there exists a polynomial function $\psi: X \to \mathbb{R}$ such that ψ is nowhere zero on X, $U \subset X_C - \{z \in X \mid \psi_C(z) = \psi_{2C}(z) = 0\}$ and

$$U \cap \{z \in X_C \mid f_{1C}(z) = f_{2C}(z) = 0\} = U \cap (Y_{1C} \cup Y_{2C}),$$

where $U = X_C - \{z \in X_C \mid \psi_C(z) = 0\}$. Observe that $\{U_1, U_2\}$ is a Zariski open (complex) covering of U, where $U_1 = U - \{z \in X_C \mid \psi_{2C}(z) = 0\}$, $U_2 = U - \{z \in X_C \mid \psi_{1C}(z) = 0)\}$, and f_{1C}, f_{2C} have no common zeros on $U_1 \cap U_2$. Set $\varphi = \psi\psi_1\psi_2$. Since $U_1 \cap U_2 = X_C - \{z \in X_C \mid \varphi_C(z) = 0\}$ is a complex affine variety, there exist complex regular functions $h_1, h_2 : U_1 \cap U_2 \to \boldsymbol{C}$ such that $h_1 f_{1C} + h_2 f_{2C} = 1$ on $U_1 \cap U_2$. Let $g_1 = r(-h_2)$, $g_2 = r(h_1)$ (see the notation following the proof of Lemma 3.3). The functions $g_1, g_2 : U_1 \cap U_2 \to \boldsymbol{C}$ are regular and their restrictions to $X \cap U_1 \cap U_2$ are real valued. Moreover, det $g_{12}(z) = 1$ for all z in $U_1 \cap U_2$, where

$$g_{12}(z) = \begin{pmatrix} f_{1C}(z) & g_1(z) \\ f_{2C}(z) & g_2(z) \end{pmatrix}.$$

Let η be the algebraic complex vector bundle over U determined by the covering $\{U_1, U_2\}$ and the transition function g_{12}. By Serre's theorem [24], the fibre of η over any point z in U is linearly generated by global regular sections of η evaluated at z. Clearly, the same holds true for the algebraic real vector bundles ξ over X determined by the Zariski open covering $\{X \cap U_1, X \cap U_2\}$ and the transition function $g_{12} \mid X \cap U_1 \cap U_2$. Hence, by Proposition A, ξ is a strongly algebraic vector bundle. The two maps

$$X \cap U_1 \to \mathbb{R}^2, \quad x \to (f_1(x), f_2(x)),$$

$$X \cap U_2 \to \mathbb{R}^2, \quad x \to (1, 0).$$

determine the regular global section s of ξ. One easily checks that s is transversal to the zero section and $Y = \{x \in X \mid s(x) = 0\}$. By construction, ξ is orientable. □

REMARK 3.4: Let M be a smooth manifold and let N be a smooth closed submanifold of codimension 2. Assume that N has a trivial normal bundle in M. Then there exist a smooth orientable vector bundle ξ of rank 2 over M and a smooth section s of ξ such that s is transversal to the zero section and $N = \{x \in M \mid s(x) = 0\}$.

The proof is analogous to the proof of Theorem 3.1. In fact it is much simpler, since in the proof of Theorem 3.1 one had to be careful to obtain a strongly algebraic vector bundle.

Theorem 3.1 has the following important consequence.

COROLLARY 3.5: *Let X and Y be as in Theorem* 3.1 *and let* $n = \dim X$. *If X is compact, then there exists a strongly algebraic orientable vector bundle*

ξ of rank 2 over X whose second Stiefel-Whitney characteristic class corresponds, via Poincaré duality, to the homology class in $H_{n-2}(X, \boldsymbol{Z}_2)$ represented by the cycle Y.

PROOF: By Theorem 3.1 there exist a strongly algebraic orientable vector bundle ξ of rank 2 over X and a regular section s of ξ such that s is transversal to the zero section and $Y = \{x \in X \mid s(x) = 0\}$. Hence the second Stiefel-Whitney characteristic class of ξ corresponds, via Poincaré duality, to the homology class in $H_{n-2}(X, \boldsymbol{Z}_2)$ represented by the cycle Y [9], p. 134. □

4. Proofs of the main results

The following lemma will play an important role.

LEMMA 4.1: *Let X be a compact affine nonsingular real algebraic variety of dimension $\leqslant 3$. Let ξ be a continuous vector bundle of rank 2 over X whose first and second Stiefel-Whitney characteristic classes vanish. Then ξ is C^0 isomorphic to a strongly algebraic vector bundle over X.*

PROOF: Since $w_1(\xi) = 0$, the bundle ξ is orientable [14]. Pick an orientation of ξ and denote by $e(\xi)$ the Euler characteristic class of the oriented vector bundle ξ. By the assumption $w_2(\xi) = 0$, $e(\xi) = 2u$ for some cohomology class u in $H^2(X, \boldsymbol{Z})$. Since dim $X \leqslant 3$, there exists a continuous map $f: X \to S^2$ such that $u = f^*(v)$ where v is a generator of the cohomology group $H^2(S^2, \boldsymbol{Z}) = \boldsymbol{Z}$. Indeed, u is the first Chern characteristic class of a continuous complex line bundle ζ over X [14]. By a transversality argument, it is easy to find two continuous sections of ζ whose sets of zeros are disjoint. Therefore ζ is C^0 isomorphic to the pullback line bundle of the canonical complex line bundle over the projective one-dimensional space $\boldsymbol{C}P^1$. Since $\boldsymbol{C}P^1$ is homeomorphic to S^2, $f^*(v) = u$ for a generator v of $H^2(S^2, \boldsymbol{Z})$. Hence if the tangent bundle τS^2 to S^2 is suitably oriented, $e(\tau S^2) = 2v$ and $e(\xi) = f^*(e(\tau S^2)) = e(f^*\tau S^2)$. Thus the vector bundles ξ and $f^*\tau S^2$ are C^0 isomorphic [14]. Since τS^2 is C^0 stably trivial so is ξ. By (2.3) ξ is C^0 isomorphic to a strongly algebraic vector bundle. □

PROOF OF THEOREM 1.1: As we have noted in Section 1, it suffices to show only the "if" part of the theorem. Let ξ be a continuous vector bundle over X. Assume that its first Stiefel-Whitney characteristic class is in $H^1_{alg}(X, \boldsymbol{Z}_2)$. Since clearly $H^2(X, \boldsymbol{Z}_2) = H^2_{Alg}(X, \boldsymbol{Z}_2)$, by Proposition B and Corollary 3.5, there exist strongly algebraic vector bundles η_1, η_2 over X such that rank $\eta_1 = 1$, $w_1(\eta_1) = w_1(\xi)$, rank $\eta_2 = 2$, $w_1(\eta_2) = 0$ and $w_2(\eta) = w_2(\xi) + w_1(\xi) \cup w_1\xi)$. Then $\xi \oplus \eta_1 \oplus \eta_2 = \zeta \oplus \gamma$, where ζ is a continuous vector bundle of rank 2 and γ is a C^0 trivial vector bundle.

One checks easily that $w_1(\zeta)=0$ and $w_2(\zeta)=0$. Hence, by Lemma 4.1, ζ is C^0 isomorphic to a strongly algebraic vector bundle and the conclusion follows from (2.3). □

To handle a three-dimensional case, we need the following.

LEMMA 4.2: *Let X be a compact affine nonsingular real algebraic threefold. Assume that X is orientable as a smooth manifold. Then for each v in $H^2_{Alg}(X, \boldsymbol{Z}_2)$ there exists a strongly algebraic orientable vector bundle ξ of rank 2 over X with $w_2(\xi)=v$.*

PROOF: Let u be the homology class in $H_1^{Alg}(X, \boldsymbol{Z}_2)$ which corresponds to v via Poincaré duality. Then there exists a family $\{C_i\}_{i=1,\ldots,k}$ of nonsingular one-dimensional algebraic subvarieties of X such that $u=[C_1]+\ldots+[C_k]$, where $[C_i]$ denotes the homology class represented by the cycle C_i. By Corollary 3.5, for each $i=1,\ldots,k$ there exists a strongly algebraic orientable vector bundle ξ_i of rank 2 over X such that $w_2(\xi_i)$ is Poincaré dual to $[C_i]$. One can find a smooth section σ_i of ξ_i transversal to the zero section and such that $\{x\in X \mid \sigma_i(x)=0\}$ is disjoint from $\{x\in X \mid \sigma_j(x)=0\}$ for $j\neq i$. By [3], for each $i=1,\ldots,k$, σ_i can be approximated by regular sections. If s_i is a regular section close to σ_i and $Y_i=\{x\in X \mid s_i(x)=0\}$, then $Y_1\cup\ldots\cup Y_k$ is an algebraic nonsingular subvariety of X which represents the homology class u. Thus the conclusion follows from Corollary 3.5. □

As we know, it suffices to prove the "if" part of Theorem 1.8.

PROOF OF THEOREM 1.8: Let ξ be a continuous vector bundle over X with $w_1(\xi)$ in $H^1_{alg}(X, \boldsymbol{Z}_2)$ and $w_2(\xi)$ in $H^2_{Alg}(X, \boldsymbol{Z}_2)$. Take strongly algebraic vector bundles η_1, η_2 over X such that rank $\eta_1=1$, $w_1(\eta_1)=w_1(\xi)$, rank $\eta_2=2$, $w_1(\eta_2)=0$, and $w_2(\eta_2)=w_2(\xi)+w_1(\xi)\cup w_1(\xi)$. Then $\xi\oplus\eta_1\oplus\eta_2=\zeta_1\oplus\gamma_1$, where ζ_1 is a continuous vector bundle of rank 3 and γ_1 is a C^0 trivial vector bundle. Since $w_1(\zeta_1)=0$, ζ_1 is orientable. Fix an orientation of ζ_1. By [20], p. 98, $2e(\zeta_1)=0$, where $e(\zeta_1)$ is the Euler characteristic class of the oriented vector bundle ζ_1. But $e(\zeta_1)$ is in $H^3(X, \boldsymbol{Z})=\boldsymbol{Z}$ and therefore $e(\zeta_1)=0$. The last property implies that ζ_1 has a nowhere zero continuous global section [20], p.139 and p. 147. Hence $\xi\oplus\eta_1\oplus\eta_2=\zeta_2\oplus\gamma_2$, where ζ_2 is a continuous vector bundle of rank 2 and γ_2 is a C^0 trivial vector bundle. One sees easily that $w_1(\zeta_2)=0$ and $w_2(\zeta_2=0$ and hence ζ_2 is C^0 isomorphic to a strongly algebraic vector bundle. By (2.3) the conclusion follows. □

PROOF OF PROPOSITION 1.10: Let ξ be a continuous orientable vector bundle of rank 2 over X. Take a strongly algebraic orientable vector bundle η of rank 2 over X such that $w_2(\eta)=w_2(\xi)$. Then $\xi\oplus\eta=\zeta_1\oplus\gamma_1$, where ζ_1 is a continuous vector bundle of rank 3 and γ_1 is a C^0 trivial line bundle. One checks easily that all Stiefel-Whitney characteristic classes of ζ_1 vanish. Let X_1 be the union of all orientable connected components of X and $X_2 = X - X_1$. For the same reason as in the proof of Theorem 1.8, the bundle ζ_1 has a nowhere zero continuous section over X_1. Fix an orientation of ζ_1. Since each element in $H^3(X_2, \boldsymbol{Z})$ has order 2 and $w_3(\zeta_1) = 0$, the Euler characteristic class $e(\zeta_1 \mid X_2)$, of the restriction of the vector bundle ζ_1 to X_2, is equal to 0. Hence ζ_1 has a nowhere zero continuous section over X_2 [20]. Thus $\zeta_1 \oplus \eta = \zeta_2 \oplus \gamma_2$ for some continuous orientable vector bundle ζ_2 of rank 2 and a C^0 trivial vector bundle γ_2. Since $w_1(\zeta_2) = 0$ and $w_2(\zeta_2) = 0$, ζ_2 is C^0 isomorphic to a strongly algebraic vector bundle. By (2.3) the same holds true for the vector bundle ξ.

Now let ξ be a continuous orientable vector bundle of rank 2 over X which is C^0 isomorphic to a strongly algebraic vector bundle. Since the Stiefel-Whitney classes of a vector bundle depend only on the isomorphism class of the bundle, one may assume that ξ is strongly algebraic. It is known that there exists a regular section u of ξ transversal to the zero section [3]. The homology class in $H_1(X, \boldsymbol{Z}_2)$ represented by $Y = \{x \in X \mid u(x) = 0\}$ corresponds, via Poincare duality, to $w_2(\xi)$. Since the restriction of ξ to a neighborhood of Y is a C^∞ trivial vector bundle, Y has a trivial normal vector bundle in X. □

Proof of proposition 1.16: The implications (iii) ⇒ (ii) ⇒ (i) are obvious. To prove (i) ⇒ (iii), we note that C has a trivial normal bundle in X. By remark 3.4 there exist a smooth orientable vector bundle ξ of rank 2 over X and a smooth section s of ξ such that s is transversal to the zero section and $C = \{x \in X \mid s(x) = 0\}$. Then $w_2(\xi)$ is in $H^2_{Alg}(X, \boldsymbol{Z}_2)$ and hence, by Theorem 1.8, ξ is C^0 isomorphic to a strongly algebraic vector bundle, say η, over X. In fact one may assume that ξ is C^∞ isomorphic to η. Clearly, there exists a smooth section u of η with $C = \{x \in X \mid u(x) = 0\}$. Now (iii) follows since u can be approximated in the C^∞ topology by regular sections of η [3]. □

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