COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 60, nº 1 (1986), p. 53-63

<http://www.numdam.org/item?id=CM_1986__60_1_53_0>

[1] Compositio Mathematica 60: 53–63 (1986)

BOUNDS FOR THE NUMBER OF CONNECTED COMPONENTS AND THE FIRST BETTI NUMBER MOD TWO OF A REAL ALGEBRAIC SURFACE

R. Silhol

1. Introduction and notations

The object of this paper is to answer part of Hilbert's 16th problem for surfaces, namely find explicit bounds for the number of connected components of the real part of a given algebraic surface and for the size of the $H^1(X(\mathbb{R}), \mathbb{Z}/2)$. We give such bounds in terms of the Hodge decomposition of the associated complex surface $X(\mathbb{C})$. We also give examples that show that, at least for small values of $h^{0,2}$, these bounds are the best possible.

The results proved here are consequences of the works of Kharlamov (see for example [3]) and Rokhlin [6], see also for a detailed account Risler [5] or Wilson [10]. We also use a few points exposed in [8].

Throughout X will be a *smooth and projective algebraic surface over* $\mathbb{R}$. $X(\mathbb{C})$ *the set of complex points*, i.e. the associated complex surface, and $X(\mathbb{R})$ *the set of real points*. Note that because X is projective both $X(\mathbb{C})$ and $X(\mathbb{R})$ are compact, for this reason we will use systematically cohomology instead of homology, using Poincaré duality to translate theorems originally stated for homology.

We will use the following notations:
$B_i = \dim H^i(X(\mathbb{C}), \mathbb{Q})$ *the* i^{th} *Betti number of the complex part*;

$$B_i(\mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H^i(X(\mathbb{C}), \mathbb{Z}/2);$$

$$h^*(X(\mathbb{C})) = \sum B_i(\mathbb{Z}/2);$$

$$h^i(X(\mathbb{R})) = \dim H^i(X(\mathbb{R}), \mathbb{Z}/2);$$

$$h^*(X(\mathbb{R})) = \sum h^i(X(\mathbb{R})) \quad \textit{and} \quad \chi(X(\mathbb{R})) = \sum (-1)^i h^i(X(\mathbb{R}))$$

the Euler characteristic; *by* $h^{p,q}$ *we will as usual mean the dimension of the complex vector space* $H^{p,q}(X(\mathbb{C}))$ *obtained in the Hodge decomposition of* $H^{p+q}(X(\mathbb{C}, \mathbb{C})$; $\tau = \tau(X(\mathbb{C}))$ *will denote the index or signature of the cup product form on* $H^2(X(\mathbb{C}), \mathbb{R})$.

Let $G = \mathrm{Gal}(\mathbb{C} \mid \mathbb{R}) = \{1, \sigma\}$. Via complex conjugation G operates on the groups $H^i(X(\mathbb{C}), \mathbb{Q})$. *We will write* $H^i(X(\mathbb{C}), \mathbb{Q})^G$ *for the subgroup fixed under this action of G and* $b_i = \dim H^i(X(\mathbb{C}), \mathbb{Q})^G$.

We will always assume $X(\mathbb{R})$ *to be non-empty.*

2. Review of the basic results used in the sequel

The first formula we will use is the Harnack-Thom inequality, proved using Smith's theory (see for example Wilson [10] or Risler [5]):

$$h^*(X(\mathbb{R})) = h^*(X(\mathbb{C})) - 2\sum a_i \tag{1}$$

where the a_i are zero or positive integers (in this paper we will not need to know anything more about these a_i's).

If $\Sigma a_i = 0$ in which case we will say that X is an M-surface (more generally if $\Sigma a_i = r$ we will say that X is an $(M - r)$-surface), we have:

$$\chi(X(\mathbb{R})) \equiv \tau(X(\mathbb{C})) \pmod{16}. \tag{2}$$

For a proof see for example [10] p. 59.

If X is an $(M\text{-}1)$-surface (that is if $\Sigma a_i = 1$) we have (see [10] p. 60):

$$\chi(X(\mathbb{R})) \equiv \tau(X(\mathbb{C})) \pm 2 \pmod{16}. \tag{3}$$

The next formula we will be conserned with comes from a different point of view. We have:

$$\chi(X(\mathbb{R})) = \sum_{i \equiv 0 (\mathrm{mod}\ 2)} (2b_i - B_i)$$

Which for surfaces gives:

$$\chi(X(\mathbb{R})) = 2b_2 - B_2 + 2 \quad (\text{see [8] prop. (1.3)}). \tag{4}$$

We give a sketch of the proof: Let $Y = X(\mathbb{C}/G)$ be the topological quotient. Then as is well known:

$$\chi(X(\mathbb{R})) = 2\chi(Y) - \chi(X(\mathbb{C})).$$

By a classical result (see for example Floyd [1] p. 38) we have:

$$H^1(Y, \mathbb{Q}) \cong H^i(X(\mathbb{C}), \mathbb{Q})^G.$$

Hence putting the two together:

$$\chi(X(\mathbb{R})) = \sum (-1)^i (2b_i - B_i).$$

We are going to show that for i odd we have $2b_i = B_i$. This will end the proof.

For this consider:

$$H^i(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = H^i(X(\mathbb{C}), \mathbb{C}) = \sum_{p+q=i} H^{p,q}(X(\mathbb{C})),$$

and the action on this space of $F_\infty = \sigma \otimes \mathrm{id}$ (where σ is as before the action induced by complex conjugation on $H^i(X(\mathbb{C}), \mathbb{Q})$).

It is not hard to see that F_∞ transforms (p, q)-differential forms into (q, p)-forms and harmonic forms into harmonic forms (see [8] p. 474). In other words:

$$F_\infty H^{p,q} = H^{q,p}. \tag{5}$$

As an immediate consequence we have:

$$\text{if } i \text{ is odd } b_i = \sum_{\substack{p+q=i\\ p<q}} h^{p,q} = (1/2)B_i,$$

which is the desired result.

From (5) we can also deduce the following useful result:

$$\text{if } i \text{ is even } \sum_{\substack{p+q=i\\ p<q}} h^{p,q} \leqslant b_i \leqslant B_i - \sum_{\substack{p+q=i\\ p<q}} h^{p,q}.$$

In the case of surfaces this gives:

$$h^{0,2} \leqslant b_2 \leqslant B_2 - h^{0,2}. \tag{6}$$

We can give many examples of surfaces where the lower bound of (6) is reached. The upper bound however can be improved. For this look at the exact sequence of sheaves:

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O}_{X_{\mathbb{C}}} \xrightarrow{\exp 2\pi} \mathcal{O}^*_{X_{\mathbb{C}}} \to 0 \tag{7}$$

where $\mathcal{O}_{X_{\mathbb{C}}}$ (resp. $\mathcal{O}^*_{X_{\mathbb{C}}}$) is the sheaf of holomorphic functions (resp. invertible holomorphic functions) on $X(\mathbb{C})$ and the first map is multiplication by $\sqrt{-1} = i$. Because of this i factor the above sequence is not compatible with the action of complex conjugation. To render it compatible we must twist the action of σ on the constant sheaf $\mathbb{Z}$. This is obtained by composing the action of σ with the automorphism $n \mapsto -n$ in the stalks of the sheaf $\mathbb{Z}$. We write $\mathbb{Z}(1)$ for this twisted structure. The cohomology groups $H^i(X(\mathbb{C}), \mathbb{Z})$ and $H^i(X(C), \mathbb{Z}(1))$ are identical as groups but as is explained in [7] p. 444 σ-invariant classes of $H^i(X(\mathbb{C}), \mathbb{Z})$ correspond to $(-\sigma)$-invariant classes of $H^i(X(\mathbb{C}), \mathbb{Z}(1))$ and vice versa. In particular if rank $H^i(X(\mathbb{C}), \mathbb{Z})^G = b_i$ then rank $H^i(X(\mathbb{C}), \mathbb{Z}(1))^G = B_i - b_i$.

Now look at the map:

$$H^1(X(\mathbb{C}), \mathcal{O}^*_{X_\mathbb{C}}) = \mathrm{Pic}(X(\mathbb{C})) \to H^2(X(\mathbb{C}), \mathbb{Z}(1))$$

coming from the long exact sequence associated to (7) modified as indicated above. The image of $\mathrm{Pic}(X)$ under this map is $\mathrm{NS}(X(\mathbb{C}))$ the Néron-Severi group and because we have done what was needed to make the exact sequence compatible with the action of G the image of $\mathrm{Pic}(X(\mathbb{C}))^G$, $\mathrm{NS}(X(\mathbb{C}))^G$, lies in $H^2(X(\mathbb{C}), \mathbb{Z}(1))^G$. On the other hand $\mathrm{NS}_0(X(\mathbb{C})) = \mathrm{NS}(X(\mathbb{C})) \otimes \mathbb{Q}$ lies in $H^{1,1}(X(\mathbb{C})) \cap H^2(X(\mathbb{C}), \mathbb{Q})$ (see for example [2] p. 162–163). Recalling the above remark that rank $H^2(X(\mathbb{C}), \mathbb{Z}(1)) = B_2 - b_2$ and combining with (6) we get:

$$h^{0,2} \leqslant b_2 \leqslant B_2 - h^{0,2} - r \tag{8}$$

where $r = \mathrm{rank}\ \mathrm{NS}(X(\mathbb{C}))^G = \mathrm{rank}\ \mathrm{NS}(X)$.

For smooth projective algebraic surfaces we always have $r \geqslant 1$. On the other hand for a generic surface (or more precisely generic in a family such that $h^{0,2} > 0$) we have $\mathrm{rank}\ \mathrm{NS}(X(\mathbb{C})) = 1$ (note that if in the family under consideration all surfaces have $h^{0,2} = 0$ then this may not be the case any more. For example for smooth cubics in $\mathbb{P}^3$ we have $r \geqslant 3$ (see §5). Thus in practice we will use:

$$h^{0,2} \leqslant b_2 \leqslant B_2 - h^{0,2} - 1. \tag{9}$$

3. Maximum value of $h^1(X(\mathbb{R}))$

We have:

$$h^1(X(\mathbb{R})) = \frac{h^*(X(\mathbb{R})) - \chi(X(\mathbb{R}))}{2}.$$

Hence by (1) and (4) and since by Poincaré duality $B_1(\mathbb{Z}/2) = B_3(\mathbb{Z}/2)$:

$$h^1(X(\mathbb{R})) = \frac{2B_1(\mathbb{Z}/2) + B_2(\mathbb{Z}/2) + B_2 - 2a_i - 2b_2}{2} \tag{10}$$

or if the cohomology of $X(\mathbb{C})$ is torsion free:

$$h^1(X(\mathbb{R}) = B_1 + B_2 - \sum a_i - b_2. \tag{11}$$

The maximum value of $h^1(X(\mathbb{R}))$ corresponds to the minimum of $\sum a_i + b_2$. To compute this minimum we will need to reformulate (2) and (3).

We have

$$\tau(X(\mathbb{C})) = \sum_{\substack{p+q\equiv 0 \\ (\mathrm{mod.}\ 2)}} (-1)^p h^{p,q} \quad \text{(see [2] p. 126)},$$

or, because of the relations between the $h^{p,q}$'s, for surfaces:

$$\tau(X(\mathbb{C})) = 2 + 2h^{0,2} - h^{1,1}.$$

Since $B_2 = h^{1,1} + 2h^{0,2}$ we get from (4) that:

$$\tau(X(\mathbb{C})) \equiv \chi(X(\mathbb{R})) \pm 2n \pmod{16}$$

is equivalent to:

$$b_2 \equiv 2h^{0,2} \pm n \pmod{8}.$$

(2) and (3) then become:

$$\text{If } X \text{ is an } M\text{-surface then } b_2 \equiv 2h^{0,2} \pmod{8} \tag{12}$$

and:

$$\text{if } X \text{ is an } (M\text{-}1)\text{-surface then } b_2 \equiv 2h^{0,2} \pm 1 \pmod{8}. \tag{13}$$

Remark: (12) and (13) can in fact be proved directly. For this first note that, because of (5), the index of the cup product form restricted to $H^2(X(\mathbb{C}), \mathbb{R})^G$ is equal to $h^{0,2} - (b_2 - h^{0,2}) = 2h^{0,2} - b_2$. Then one uses the existence of the Wu-classe to prove that this restricted form is of type II (see [10] p. 59) and one can conclued by a standard argument on the determinant (see [10] p. 60).

In other cases X is at most an (M-2)-surface. In fact we are going to show that:

$$\text{if } b_2 \equiv 2h^{0,2} \pm 3 \pmod{8} \text{ then } X \text{ is at most an } (M\text{-}3)\text{-surface.} \tag{14}$$

This will follow from:

Lemma 1: $\Sigma a_i \equiv b_2 \pmod{2}$.

From $h^*(X(\mathbb{R})) + \chi(X(\mathbb{R})) \equiv 0 \pmod{4}$, (1) and (4) we get:

$$2\left(\sum a_i - b_2\right) \equiv 2B_1(\mathbb{Z}/2) + B_2(\mathbb{Z}/2) - B_2 \pmod{4}.$$

On the other hand $2(B_1(\mathbb{Z}/2) - B_1) = B_2(\mathbb{Z}/2) - B_2$ (invariance of the Euler characteristic) and $B_1 \equiv 0 \pmod{2}$ (Hodge relations). Hence:

$$2\left(\sum a_i - b_2\right) \equiv 4B_1(\mathbb{Z}/2) \equiv 0 \pmod{4} \quad \text{and the lemma.}$$

We are now in mesure to give bounds for $h^1(X(\mathbb{R}))$. To simplify notations we will only formulate the results in the case when the cohomology of $X(\mathbb{C})$ is torsion free, but it is clear that in the case when there is torsion one can deduce an analogus statement using (10) in place of (11).

THEOREM 1: *If X is a smooth projective algebraic surface over $\mathbb{R}$ and if $H^*(X(\mathbb{C}), \mathbb{Z})$ is torsion free then we have for the real part*:

$$h^1(X(\mathbb{R})) \leqslant B_1 + B_2 - h^{0,2} \quad \textit{if } h^{0,2} \equiv 0 \pmod{8} \tag{15}$$

$$h^1(X(\mathbb{R})) \leqslant B_1 + B_2 - h^{0,2} - 1 \quad \textit{if } h^{0,2} \equiv \pm 1 \pmod{8} \tag{16}$$

$$h^1(X(\mathbb{R})) \leqslant B_1 + B_2 - h^{0,2} - 2 \quad \textit{if } h^{0,2} \equiv \pm 2 \textit{ or } 4 \pmod{8} \tag{17}$$

$$h^1(X(\mathbb{R})) \leqslant B_1 + B_2 - h^{0,2} - 3 \quad \textit{if } h^{0,2} \equiv \pm 3 \pmod{8}. \tag{18}$$

We are going to prove (18), the proof of the other cases being exactly the same.

From (6) we get $b_2 \geqslant h^{0,2}$. If $b_2 = h^{0,2}$ then the hypothesis is $b_2 \equiv 2h^{0,2} \pm 3$ and (18) follows from (11) and (14). If $b_2 = h^{0,2} + 1$ then $b_2 \equiv 2h^{0,2} - 2$ or $b_2 \equiv 2h^{0,2} + 4$. The surface is then by (12) and (13) at most an $(M$-2)-surface, and (18) is again statisfied. If $b_2 = h^{0,2} + 2$ the surface is at most an $(M$-1)-surface and this again implies (18). If $b_2 \geqslant h^{0,2} + 3$ the inequality (18) is automatically satisfied.

REMARK: Theorem 1 and its proof is still true, practically word for word, if we take for $X(\mathbb{C})$ a Kähler surface with an anti-holomorphic involution and for $X(\mathbb{R})$ the fixed part under the action of this involution.

4. Examples of surfaces with $h^1(X(\mathbb{R}))$ maximum

It is easy to find examples where the bound is reached in case (15), for example $\mathbb{P}^2$, certain cubics in $\mathbb{P}^3$, and more generally all rational surfaces obtained by blowing up real points of $\mathbb{P}^2$. Other examples inclued ruled surfaces over curves of arbitrary genus, but in all these examples we have $h^{0,2} = 0$. It would be interesting to find examples where the bound is reached and $h^{0,2} = 8, 16, \ldots$.

For (16) we have an example of surface of degree 4 in $\mathbb{P}^3$ (that is a $K3$ surface and $h^{0,2} = 1$) given by Kharlamov and such that the real part is a 10 hole torus T_{10}. Since in this case the cohomology is torsion free and $B_1 = 0$ the bound is reached. In [8] §5 we also gave an example of a $K3$ surface where the real part is connected and the Euler characteristic $\chi(X(\mathbb{R})) = -18$. Another example is given by an abelian surface whose real part is formed of 4 tori. For such a surface we have $B_1 = 4$, $B_2 = 6$ and $h^{0,2} = 1$.

For (17) the simplest example would be a surface of degree 5 in $\mathbb{P}^3$. For such a surface we have: $B_1 = 0$, $B_2 = 53$ and $h^{0,2} = 4$. The maximum for $h^1(X(\mathbb{R}))$ is then 47. But the methods of construction at our disposal today do not allow to conclued on the existence of such a surface of degree 5 with $h^1(X(\mathbb{R})) = 47$.

We are going to build an example for (17) using the method of [8]. For this consider in $\mathbb{P}^2 \times \mathbb{P}^1$ (with semi-homogeneous coordinates $(x, y, z; t, u)$) the surface defined birationally by:

$$y^2 z = x^3 + \left(\frac{t^3}{u(t-u)(t+u)} \right) xz^2.$$

This surface is not smooth but we take for X the minimal desingularisation, or otherwise said the minimal regular model. This is always possible over $\mathbb{R}$ (see [8] §4) and the resulting surface is an elliptic surface fibered over $\mathbb{P}^1$. The discriminant of this elliptic fibering is: $\Delta = 4\dfrac{t^9}{u^3(t-u)^3(t+u)^3}$ and the j invariant of the fibers is constant. There are 4 singular fibers above the points: (0, 1), (1, 0), (1, 1) and $(-1, 1)$. At each of these points Δ, considered as a rational function on $\mathbb{P}^1$, has valuation $v(\Delta) \equiv 9$ (mod. 12). We also have $v(j) = 0$. From theorem (4.1) of [8] then follows that for each singular fiber L_i, $\chi(L_i(\mathbb{R})) = -7$. By classical theory we also have $\chi(L_i(\mathbb{C})) = 9$. So finally, since for any smooth fiber L we have $\chi(L(\mathbb{C})) = \chi(L(\mathbb{R})) = 0$ ($L(\mathbb{C})$ is a torus and $L(\mathbb{R})$ is a circle or the disjoint union of two circles), $\chi(X(\mathbb{C})) = 36$ and $\chi(X(\mathbb{R})) = -28$. From the construction we also get (since for each singular fiber L_i, $L_i(\mathbb{R})$ is connected (see Th. (4.1) of [8]), that $X(\mathbb{R})$ is connected and hence $h^1(X(\mathbb{R})) = 30$.

For the topology of the complex part we must go into geometric considerations. First we have, by Noether's formula:

$$\chi(\mathcal{O}_{X_{\mathbb{C}}}) = 36/12 = 3 \quad \text{(see [2] p. 438)}.$$

Further the canonical divisor on $X(\mathbb{C})$ (a divisor associated to the canonical bundle) is the pullback of a divisor on $\mathbb{P}^1$ of degree $d = 2g(\mathbb{P}^1) + \chi(\mathcal{O}_{X_{\mathbb{C}}}) - 2 = 1$ (see [2] p. 572). Hence we have $h^{2,0} = \dim H^0(X(\mathbb{C}), \Omega^2_{X_{\mathbb{C}}}) = d - g(\mathbb{P}^1) + 1 = 2$ by Riemann-Roch. Hence $h^{0,2} = 2$, $B_1 = 2(\chi(\mathcal{O}_{X_{\mathbb{C}}}) - 2 - 1) = 0$ and $B_2 = 36 - 2 = 34$.

We are thus in case (17) and (17) gives us for maximum value of $h^1(X(\mathbb{R}))$, $34 - 2 - 2 = 30$ which is precisely the value we have found.

As an example for (18) we have the minimal regular surface associated to the surface defined birationally in $\mathbb{P}^2 \times \mathbb{P}^1$ by:

$$y^2 z = x^3 + \left(\frac{u^4}{(u-t)(u+t)(u-2t)(u+2t)} \right) z^3.$$

This is an elliptic surface with 5 singular fibers. For 4 of these we have $\chi(L_i(\mathbb{C})) = 10$ and $\chi(L_i(\mathbb{R})) = -8$ (Th. (4.1) of [8]). For the fifth we have $\chi(L_\infty(\mathbb{C})) = 8$ and by Th. (4.1) of [8] and the construction of Néron [4] p. 113 $\chi(L_\infty(\mathbb{R})) = -6$ (this because the coefficient of z^3 devided by u^4 is positive in a neighborhood of $(1, 0) \in \mathbb{P}^1$). By a computation similar to the one made for the preceding example we then get: $B_1 = 0$, $B_2 = 46$, $h^{0,2} = 3$, $X(\mathbb{R})$ connected and $h^1(X(\mathbb{R})) = 40$ which is equal to $B_1 + B_2 - h^{0,2} - 3 = 46 - 3 - 3$.

If $h^{0,2} \geqslant 4$ the surfaces we are looking for are necessarily disconnected, and the method used for the last two examples, although theoretically still appliable, leads to some very involved computations. One can, however, fairly easily build examples where $b_2 = h^{0,2}$ and this for any value of $h^{0,2}$.

5. Maximum number of connected components

The number of components of $X(\mathbb{R})$ is equal to:

$$^{\#}X(\mathbb{R}) = \frac{h^*(X(\mathbb{R})) + \chi(X(\mathbb{R}))}{4}$$

This gives:

$$^{\#}X(\mathbb{R}) = \frac{4 + 2B_1(\mathbb{Z}/2) + B_2(\mathbb{Z}/2) - B_2 + 2b_2 - 2\sum a_i}{4} \tag{19}$$

or if the cohomology of $X(\mathbb{C})$ is torsion free:

$$^{\#}X(\mathbb{R}) = \frac{B_1 + b_2 - \sum a_i}{2} + 1. \tag{20}$$

To find an upper bound for $^{\#}X(\mathbb{R})$ we need only to find an upper bound for $(b_2 - \Sigma a_i)$. Using (9), (12), (13) and (14) and the same arguments as in the proof of theorem 1 we get:

THEOREM 2: *If X is a smooth projective algebraic surface over $\mathbb{R}$ and if $H^*(X(\mathbb{C}), \mathbb{Z})$ is torsion free we have for the number of connected components $^{\#}X(\mathbb{R})$ of the real part $X(\mathbb{R})$:*

$$^{\#}X(\mathbb{R}) \leqslant \frac{B_1 + B_2 - h^{0,2} + 1}{2}$$

$$\textit{if } B_2 - 3h^{0,2} - 1 = h^{1,1} - h^{0,2} - 1 \equiv 0 \pmod{8} \tag{21}$$

$$^{\#}X(\mathbb{R}) \leqslant \frac{B_1 + B_2 - h^{0,2}}{2} \quad \textit{if } h^{1,1} - h^{0,2} - 1 \equiv \pm 1 \pmod{8} \tag{22}$$

$$^{\#}X(\mathbb{R}) \leqslant \frac{B_1 + B_2 - h^{0,2} - 1}{2}$$

$$\textit{if } h^{1,1} - h^{0,2} - 1 \equiv \pm 2 \text{ or } 4 \pmod{8} \tag{23}$$

$$^{\#}X(\mathbb{R}) \leqslant \frac{B_1 + B_2 - h^{0,2} - 2}{2} \quad \textit{if } h^{1,1} - h^{0,2} - 1 \equiv \pm 3 \pmod{8}. \tag{24}$$

REMARKS: (i) Just as in the case of Theorem 1 it is easy to formulate a version of theorem 2 valid when $H^*(X(\mathbb{C}), \mathbb{Z})$ has torsion. Just use (19) instead of (20).
(ii) Theorem 2 is still true in the case of Kähler surfaces with an antiholomorphic involution but the proof we have given is not. The reason for this is that for non-algebraic surfaces the second inequality of (8) does not make sense. To prove the theorem in this case one should use the index theorem for Kähler surfaces (see [2] p. 126).

EXAMPLES: Examples of surfaces where the bounds of theorem 2 are reached are somewhat harder to give than for Theorem 1. The reason for this lies in formula (8) -see considerations made at the end of this paragraph. Nevertheless we can still give examples, in fact families of them.

(i) Let C be a curve of genus g such that $C(\mathbb{R})$ has g + 1 components (an M-curve) and let X be the surface $\mathbb{P}^1 \times C$. We then have $^{\#}X(\mathbb{R}) = g + 1$, $B_1 = 2g$, $h^{0,2} = 0$ and $B_2 = h^{1,1} = 2$. We are thus in case (22) and $1/2(B_1 + B_2 - h^{0,2}) = g + 1$. So the maximum is reached.
(ii) Let X be an abelian surface whose real part has 4 components. We have $B_1 = 4$, $B_2 = 6$, $h^{0,2} = 1$, $h^{1,1} = 4$. We are in case (23) and it is easy to see that the maximum is reached.
(iii) There is an example of Kharlamov ([3]) of a $K3$ surface such that $X(\mathbb{R})$ has 10 components. In this case we have: $B_1 = 0$, $B_2 = 22$, $h^{0,2} = 1$ and $h^{1,1} = 20$ and it is easy to see that 10 is the maximum given by the theorem.
(iv) Let E be a curve of genus 1 and C a curve of genus 2 such that $E(\mathbb{R})$ has 2 components and $C(\mathbb{R})$, 3. Let $X = E \times C$. By Künneth's formula we then have: $^{\#}X(\mathbb{R}) = 6$, $B_1 = 6$, $B_2 = 10$ and by a computation similar to the one made for examples in §4, $h^{0,2} = 2$ and $h^{1,1} = 6$. Hence $h^{1,1} = h^{0,2} - 1 = 5 \equiv -3 \pmod{8}$ and $1/2(B_1 + B_2 - h^{0,2} - 2) = 6$.

For surfaces of degree n in $\mathbb{P}^3$ theorem 2 can be fairly simply reformulated. For these surfaces we have: $B_1 = 0$, $B_2 - 3h^{0,2} - 1 =$

$(n(n-1)^2)/2$ and $B_2 - h^{0,2} = (5n^3 - 18n^2 + 25n - 6)/6$ (these formulas can easily be deduced from [2] p. 601–602). Further we have:

$$\left(n(n-1)^2\right)/2 \equiv \begin{cases} n/2 & (\text{mod. } 8) \quad \text{if } n \text{ is even} \\ 0 & (\text{mod. } 8) \quad \text{if n} \equiv 1 \ (\text{mod. } 4) \\ -2 & (\text{mod. } 8) \quad \text{if n} \equiv 3 \ (\text{mod. } 4) \end{cases}$$

Thus we have:

COROLLARY: *If X is a smooth surface of degree n in* $\mathbb{P}^3$ *then the number of connected components* $^{\#}X(\mathbb{R})$ *of the real part* $X(\mathbb{R})$ *verifies*:

$$^{\#}X(\mathbb{R}) \leqslant P(n) \quad \textit{if } n \equiv 0 \quad (\text{mod. } 16) \quad \textit{or } n \equiv 1 \quad (\text{mod. } 4) \tag{25}$$

$$^{\#}X(\mathbb{R}) \leqslant P(n) - 1/2 \quad \textit{if } n \equiv \pm 2 \quad (\text{mod. } 16) \tag{26}$$

$$^{\#}X(\mathbb{R}) \leqslant P(n) - 1$$

$$\textit{if } n \equiv \pm 4 \quad \textit{or } 8 \quad (\text{mod. } 16) \quad \textit{or } n \equiv 3 \quad (\text{mod. } 4) \tag{27}$$

$$^{\#}X(\mathbb{R}) \leqslant P(n) - 3/2 \quad \textit{if } n \equiv \pm 6 \quad (\text{mod. } 16) \tag{28}$$

where $P(n) = (5n^3 - 18n^2 + 25n)/12$.

For $n = 4$ the bound is reached, we have an example of Kharlamov of a quartic in $\mathbb{P}^3$ whose real part has 10 components. For $n > 4$ the question remains open. The best examples we have at our disposal are due to Viro [9] who proves that for n even there exists a surface in $\mathbb{P}^3$ of degree n whose real part has $(n^3 - 2n^2 + 4)/4$ components. He shows also that if $n \equiv 2$ (mod. 4) then one can build surfaces with $(7n^3 - 24n^2 + 32n)/24$ components. One should however note that, if for n big these examples are the best we know how to build, already for $n = 4$ the method of Viro can not be applied to, obtain a surface with 10 components (see remark at the end of [9]).

For $n = 3$ the bound is not reached since the theorem gives 3 while it is well known that a cubic surface in $\mathbb{P}^3$ can have at most 2 components. It is, however, interesting and rather easy to see from where the difference comes. It comes from the fact that we have used in the proof of Theorem 2 (9) instead of (8) and that for cubic surfaces $r \geqslant 3$. To see this last point it is enough to note that for such surfaces we have $\text{NS}(X(\mathbb{C})) = H^2(X(\mathbb{C}), \mathbb{Z}(1))$ (see the proof of (8)) and hence $r = B_2 - b_2$. $r \geqslant 3$ then follows from (4) and the explicit construction of such surfaces. for the construction we have choice between the classical and well known construction or start with the complex construction ($X(\mathbb{C})$ being realised over $\mathbb{C}$ as $\mathbb{P}^2$ blown up in 6 points) and use, for example, the method of [8] §3 to build real models. With this last method we can in fact directly show that $r \geqslant 3$.

Note that for surfaces of degree $\geqslant 4$ this problem does not exist, since for a generic surface of degree $\geqslant 4$ we already have $\mathrm{NS}(X(\mathbb{C})) = 1$.

Note also that if in the computations we have made we replace $B_2 - h^{0,2} - 1$ by $B_2 - h^{0,2} - r$ we get for cubics $B_2 - 3h^{0,2} - r = 7 - 3 = 4$. This means that we are in a case corresponding to (23). We then find $(B_2 - h^{0,2} - r)/2 = 2$ which is precisely the maximum value found for cubics.

A similar computation made for surfaces arising as a product of curves would have brought to the same conclusion. This tends to prove that the only obstruction lies in formula (8).

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(Oblatum 12-IV-1985)

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