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ORBITS OF THE WEYL GROUP AND A THEOREM OF DECONCINI AND PROCESI

James B. Carrell *

§1. Introduction

In [DP], C. DeConcini and C. Procesi proved a remarkable connection between dual SL_n conjugacy classes of nilpotent complex matrices. The goal of this paper is to show that weaker form of this connection holds for certain pairs of nilpotent conjugacy classes for an arbitrary semi-simple algebraic group G over $\mathbb{C}$. We also obtain a short intrinsic proof of the formula of [DP]. The main techniques in this paper are analytic, namely applications of the ideas on torus actions developed in [ACL].

First, recall the SL_n case. For an $n \times n$ nilpotent complex matrix σ, let $\pi(\sigma) = \sigma_1 \geqslant \sigma_2 \geqslant \dots \geqslant \sigma_k$ be the partition of n determined by the sizes of the Jordan blocks of σ. Two nilpotent conjugacy classes O_σ and O_τ are called dual if $\pi(\sigma)$ and $\pi(\tau)$ are dual partitions of n. Let $A_\sigma = A(\overline{O}_\sigma \cap \mathscr{D}_n)$ be the coordinate ring of the scheme theoretic intersection of $\overline{O}_\sigma$ and the diagonal $n \times n$ matrices $\mathscr{D}_n$ over $\mathbb{C}$ with trace zero. Also, let X^τ be the variety of Borel subalgebras of sl_n containing τ. in [DP], it was shown that there exists an S_n-equivariant $\mathbb{C}$-algebra isomorphism

$$A_\sigma \cong H^{\cdot}(X^\tau)(\mathbb{C}\text{-coefficients}), \tag{1}$$

provided σ and τ are dual. Moreover, the S_n-module structure is the induced representation $\mathrm{Ind}_{S_{\tau_1} \times \dots \times S_{\tau_m}}^{S_n}(\mathbb{C})$ which verifies a conjecture of H. Kraft [Kr] (inspired by some questions of Kostant). In [Ta], Tanisaki simplified the proof of (1) and extended the result on induced representatons to certain symplectic conjugacy classes. Kraft explicitly pointed out, however, that $A(\overline{O} \cap \mathscr{D}_6)$ is not an induced representation for the class of type (3, 3) in $\mathfrak{sp}(6)$, [Kr].

The present treatment proceeds by showing that (1) factors into two homomorphisms, either of which may fail to be an isomorphism even when defined. One of these morphisms arises from the theory of sheets

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in a Lie algebra [BK]. The other relates the cohomology of a variety of fixed flags to a graded algebra associated to a certain Weyl group orbit.

Let $G \supset B \supset H$ be respectively a semi-simple algebraic group over $\mathbb{C}$, a Borel subgroup, and a maximal torus. By convention, Lie algebras will be denoted by the corresponding lower case script letter. Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. For $s \in \mathfrak{h}$, consider the orbit $W \cdot s \subset \mathfrak{h}$ as a finite reduced variety with ring $A(W \cdot s)$ of regular functions, i.e. $A(W \cdot s) = A(h) I(W \cdot s)$. $A(W \cdot s)$ is a W-module with a canonical W-invariant filtration $D_0 \subset D_1 \subset D_2 \subset \cdots$, where D_i consists of restrictions of polynomials on $\mathfrak{h}$ of degree $\leqslant i$. $D_i D_j \subset D_{i+j}$, so the graded ring $\operatorname{Gr} A(W \cdot s) = \oplus D_i / D_{i-1}$ is canonically defined.

THEOREM 1: *Given $s \in \mathfrak{h}$, let τ be a regular nilpotent in the Levi subalgebra $\ell = z_{\mathfrak{g}}(s)$, and let X^τ denote the variety of Borel subalgebras of X containing τ. Let i_τ: $X^\tau \to X$ be the inclusion into the flag variety of all Borel subalgebras of $\mathfrak{g}$. Then there exists a W-equivariant $\mathbb{C}$-algebra homomorphism ψ_s: $\operatorname{Gr} A(W \cdot s) \to H^\cdot(X^\tau)$. ψ_s doubles degree. The image of ψ_s is $i_\tau^* H^\cdot(X)$. Moreover, ψ_s is an isomorphism if and only if i_τ^* is surjective. In fact, the kernel and cokernel of ψ have the same dimension.*

Note that W acts on $H^\cdot(X^\tau)$ via the Springer representation ([Spr]), while the W-module $\operatorname{Gr} A(W \cdot s)$ is $\operatorname{Ind}_{W_L}^W(\mathbb{C})$, where W_L is the Weyl group of $(\ell, \mathfrak{h})$.

The second step of the factorization is the definition of a map $A(\overline{O}_\sigma \cap \mathfrak{h}) \to \operatorname{Gr} A(W \cdot s)$ for a suitable nipotent $\sigma \in \mathfrak{g}$. This step appears implicitly in [Kr, Prop. 4]. Begin by fixing a parabolic subalgebra $\mathfrak{p}$ containing ℓ as a Levi subalgebra, and let σ be a Richardson element of the nilradical $nil\,\mathfrak{p}$ (i.e. the P conjugacy class of σ is Zariski dense in $nil\,\mathfrak{p}$). One imposes two conditions on σ:

(N) the closure $\overline{O}_\sigma$ of the nilpotent G orbit is normal in $\mathfrak{g}$, and

(S) $Z_P(\sigma) = Z_G(\sigma)$, i.e. the P and G stabilizers of σ coincide.

PROPOSITION 1: *Assuming (N) and (S), there exists a W-equivariant, surjective, graded $\mathbb{C}$-algebra homomorphism ϕ_σ: $A_\sigma = A(\overline{O}_\sigma \cap \mathfrak{h}) \to \operatorname{Gr} A(W \cdot s)$.*

Combining this with Theorem 1 *we obtain our "generalization" of* (1).

COROLLARY 1: *Let P be a parabolic in G whose nilradical contains a Richardson element σ satisfying (N) and (S). Then there exists a surjective degree doubling W-equivariant $\mathbb{C}$-algebra homomorphism*

$$\mu\colon A(\overline{O}_\sigma \cap \mathfrak{h}) \to i_\tau^* H^\cdot(X)$$

where τ is a regular nilpotent in a Levi subalgebra of $\mathfrak{g}$.

The homomorphism μ admits a definition which is independent of s. Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}$, and let β: $A(\mathcal{N} \cap \mathfrak{h}) \to H^\cdot(X)$ be the

W-equivariant isomorphism obtained as the composition of isomorphisms

$$A(\mathcal{N}\cap \mathfrak{h}) \overset{\kappa}{\rightarrow} A(\mathfrak{h})/I_W \overset{\beta'}{\rightarrow} H^{\cdot}(X).$$

Here I_W is the ideal generated by $\{f\in A(\mathfrak{h})^W \mid f(0)=0\}$, κ is the natural map [Ko], and β' is the isomorphism of Borel which assigns to any element χ of the group $X(H)$ of characters on H the first Chern class $c_1(L_\chi)$ of the holomorphic line bundle L_χ on X. It will be clear from the definition of μ and the proof of Theorem 1 that we obtain a commutative diagram

$$\begin{array}{ccc} A(\mathcal{N}\cap \mathfrak{h}) & \overset{\beta}{\underset{\sim}{\rightarrow}} & H^{\cdot}(X) \\ \downarrow & & \downarrow \\ A(\bar{O}_\sigma \cap \mathfrak{h}) & \overset{\mu_\sigma}{\rightarrow} & H^{\cdot}(X^\tau) \end{array} \qquad (2)$$

As an application, we now prove (1). Let σ and τ lie on dual nilpotent orbits in $\mathfrak{sl}_n$. Then τ may be chosen as a regular nilpotent in $\ell = \mathfrak{sl}_{\tau_1} \times \cdots \times \mathfrak{sl}_{\tau_m}$ and σ as a Richardson in $\mathfrak{nil}\ \mathfrak{p}$, where $\mathfrak{p}$ is the parabolic corresponding to ℓ. In $\mathfrak{sl}_n$ condition (N) holds by [KP] and (S) is well known. Thus we obtain a surjection $A_\sigma \rightarrow i_\tau^* H^{\cdot}(X)$, where here X is the flag variety of SL_n. By a result of Spaltenstein i_τ^* is always surjective [Sp], hence it will suffice to show that $\dim A_\sigma \leqslant \dim H^{\cdot}(X^\tau)$. By [Ta], $\dim A_\sigma \leqslant \dfrac{n!}{\tau_1! \cdots \tau_m!}$ which is precisely $\dim H^{\cdot}(X^\tau)$.

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§2. Torus actions on varieties of fixed flags

The point of this section will be to establish the existence of a torus with finitely many fixed points on X^σ, for any regular nilpotent σ in a Levi subalgebra ℓ, and to determine the combinatorial properties of the fixed point set. It is convenient, for this purpose, to consider X as the variety of a Borel subgroups of G with G acting on X by conjugation. For $M \subset G$, let X^M denote the set of all Borels containing M. Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ with group H, and let $\ell \supset \mathfrak{h}$ be a Levi subalgebra with group L. Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and W_L the Weyl group of $(\ell, \mathfrak{h})$.

THEOREM 2: *Let u be a regular unipotent in L. Then the torus $S = Z(L)$ acts on X^u with exactly $[W: W_L]$ fixed points. In fact, if we fix a Borel B*

containing u and S, then there exists a one to one correspondence ϕ_u: $X^u \cap X^S \to W_L \backslash W$ defined as follows: if $B' \in X^u \cap X^S$ and $w = w(B')$ is the unique element of W so that $wBw^{-1} = B'$, then $\phi_u(B') = \bar{w}$ (the right coset of w modulo W_L). The set $W_u = \{w \in W \mid wBw^{-1} \in X^u \cap X^S\}$ gives a complete set of representatives for $W_L \backslash W$.

PROOF: First, it is well known that X^S has $[W: W_L]$ components, each being a flag variety $L/L \cap B$. L being connected, u acts on each component of X^S and has exactly one fixed point by regularity. Hence $X^u \cap X^S$ has exactly $[W: W_L]$ points.

Next, Let H be the maximal torus corresponding to $\mathfrak{h}$. Also suppose $B \supset H$. There exists a surjection ϕ_u: $X^u \cap X^H \to W_L \backslash W$ defined exactly as above. To show surjectivity, it is enough to show that for any $w \in W$ there is a $\tilde{w} \in W_L$ so that $B' = (\tilde{w}w)B(\tilde{w}w)^{-1}$ contains u and H. It suffices to show $u \in B'$. Note that $wBw^{-1} \cap L$ is a Borel in L containing H, and hence there exists a $\tilde{w} \in W_L$ so that $\tilde{w}(wBw^{-1} \cap L)\tilde{w}^{-1} = B \cap L$. But $u \in B \cap L$ and $\tilde{w}(wBw^{-1} \cap L)\tilde{w}^{-1} = (\tilde{w}w)B(\tilde{w}w)^{-1} \cap L$, so $u \in B' = (\tilde{w}w)B(\tilde{w}w)^{-1}$. Hence ϕ_u is surjective. But clearly, $X^u \cap X^H \subset X^u \cap X^S$, so by $\# X^u \cap X^H \geqslant [W: W_L] = \# X^u \cap X^s$ we conclude $X^u \cap X^S = X^u \cap X^H$, and ϕ_u as defined above is a bijection, which completes the proof.

COROLLARY 2: *Let $P \supset B$ be a parabolic with Levi subgroup L. Then*

$$\dim H^{\cdot}(X^u) = \dim H^0(X^S) = \dim H^{\cdot}(G/P) = [W: W_L].$$

PROOF: It suffices to verify the first equality. Use the fact that for an algebraic action of a torus S on a projective variety V, the Euler characteristic $\chi(V)$ equals $\chi(V^S)$. In particular, $\chi(X^u) = [W: W_L]$. But, every X_u has vanishing odd rational homology, so $\chi(X^u) = \dim H^{\cdot}(X^u)$.

§3. Torus actions and cohomology

In this section we will explain the main result of [ACL] which will be the important tool in the proof of Theorem 1. Let X be a smooth projective variety over $\mathbb{C}$ of dimension n on which an algebraic torus S acts algebraically. The fixed point set $Z = X^S$ is automatically nontrivial, but not necessarily finite. Further, let Y denote a closed S-invariant subvariety of X so that $Y \cap Z$ is finite (it is automatically nonempty). For simplicity, assume also that $H^q(X; \Omega_X^p)$ vanishes if $p \neq q$, Ω_X^p being the sheaf on X of germs of holomorphic p-forms. As a consequence of this, $H^{\text{odd}}(X)$ vanishes and $H^{2p}(X) = H^p(X; \Omega_X^p)$ for all $p \geqslant 0$. Note that all cohomology has complex coefficients.

THEOREM 3: ([ACL]). *Associated to any regular element s of $\mathfrak{s}$, there exists a filtration of $H^{\cdot}(Z)$*

$$H^{\cdot}(Z)=F_n\supset F_{n-1}\supset\cdots\supset F_1\supset F_0\supset\cdots(n=\dim X)$$

so that

(*a*) $F_iF_j\subset F_{i+j}$

(*b*) $F_i/F_{i-1}\cong H^{2i}(X)$, *and*

(*c*) $H^{2\cdot}(X)\cong \mathrm{Gr}\, H^{\cdot}(Z)=\sum_{i\geqslant 0}F_i/F_{i-1}$ (*as graded rings*).

Let $j: Y\cap Z\to Z$ be the inclusion, $G_k=j^*F_k\subset H^0(Y\cap Z)$, an assume j^* is surjective. Then there exists a surjective homomorphism of graded rings

$$\psi_Y:\mathrm{Gr}\, H^0(Y\cap Z)=\sum_{i\geqslant 0}G_i/G_{i-1}\to I(X,Y)$$

where $I(X,Y)$ denotes the graded ring $\sum_{k\geqslant 0}I(X,Y)_k$ with

$$I(X,Y)_k=\mathrm{Im}\left[H^{2k}(X)\to H^{2k}(Y)\right].$$

Finally, if all odd Betti numbers of Y vanish, then ψ_Y is an isomorphism if and only if $I(X,Y)=H^{\cdot}(Y)$.

To see how to calculate $H^{\cdot}(X)$ in Gr $H^{\cdot}(Z)$, chose a Leray cover $\mathscr{U}$ of X and form the complex $K_X^{\cdot}=\sum_{q-p=\cdot}C^q(\mathscr{U},\Omega_X^p)$ with differential D: $K_X^j\to K_X^{j+1}$ given on $C^q(\mathscr{U},\Omega_X^p)$ by $\delta+(-1)^p i(V)$. By [CL_2], $\underline{\underline{H}}_X^k=H^k(K_X^{\cdot},D)$ vanishes if $k\neq 0$, and $\underline{\underline{H}}_X^0$ is a ring with filtration

$$\underline{\underline{H}}_X^0=\bar{F}_N\supset\bar{F}_{n-1}\supset\cdots\supset\bar{F}_0\supset\cdots$$

so that $\bar{F}_i\bar{F}_j\subset\bar{F}_{i+j}$, and there exists a graded ring isomorphism

$$\mathrm{Gr}\,\underline{\underline{H}}_X^0=\oplus\bar{F}_i/\bar{F}_{i-1}\cong\oplus H^i\left(X;\,\Omega_X^i\right).$$

The inclusion map i_Z: $Z\to X$ induces a quasi-isomorphism i_Z^*: $K_X^{\cdot}\to K_Z^{\cdot}$, hence $\underline{\underline{H}}_X^0\cong\underline{\underline{H}}_Z^0=\sum_{p\geqslant 0}H^p(Z;\,\Omega_Z^p)$. But by various vanishing theorems, in particular [CS], the right hand side is $H^{\cdot}(Z)$. In particular, if $F_j=i_Z^*(\bar{F}_j)$, then Gr $\underline{\underline{H}}_X^0\cong$ Gr $\underline{\underline{H}}_Z^0$ which finishes th calculation of $H^{\cdot}(X)$.

The next step is to use this calculation to compute $c_1(L)$ in Gr $H^{\cdot}(Z)$, where L is any holomorphic line bundle on X. Let V be the holomorphic vector field on X determined by a regular element s of $\mathfrak{s}$. The fundamental fact is that since V has zeros, there exists a lifting of the derivation V: $O_X\to O_X$ on the sheaf of germs of holomorphic functions

O_X on X to a $\tilde{V}$: $O_X(L) \to O_X(L)$ such that if $f \in O_{X,x}$ and $\sigma \in O_X(L)_x$, then

$$\tilde{V}(f\sigma) = V(f)\sigma + f\tilde{V}(\sigma) \quad (\text{see } [\mathrm{CL}_2]).$$

To calculate the class $\tilde{c}(L) \in \bar{F}_1 \subset \underline{\underline{H}}^0_X$ which defines $c_1(L)$ in the associated graded, one chooses a local holomorphic connection D_α for $L \mid U_\alpha$ for any $U_\alpha \in \mathscr{U}$. Thus D_α: $O_{U_\alpha}(L) \to O_{U_\alpha}(L) \otimes \Omega^1_{U_\alpha}$ satisfies $D_\alpha(f\sigma) = \sigma \otimes df + fD_\alpha(\sigma)$. Then a Cech cocycle representing $\tilde{c}(L)$ in K^0_X is $\{\theta_{\alpha,\beta}\} + \{L_\alpha\}$, where $\theta_{\alpha,\beta} = D_\alpha - D_\beta$ and $L_\alpha = \tilde{V} - i(V)D_\alpha$. in K^0_Z, $\tilde{c}(L)$ is represented by $\{i^*_Z\theta_{\alpha\beta}\} + \tilde{V} \mid Z$, where i_Z: $Z \to X$ is the inclusion. In the associated gradeds, both classes give $c_1(L) \in H^2(X)$.

Suppose now that Y is an S-invariant subvariety of X so that $Y \cap Z$ is finite, and let $j: Y \cap Z \to Z$ be the inclusion. Note first that $\tilde{V} \mid Z$ is a holomorphic function on Z. For $\tilde{V} \mid Z \in H^0(Z, \mathrm{Hom}(O_X(L), O_X(L))) = H^0(Z, O_Z)$. Consequently, since Z is compact, $\tilde{V} \mid Z \in H^0(Z)$. Thus $j^*(\{i^*_Z\theta_{\alpha\beta}\} + \tilde{V} \mid Z) = j^*(\tilde{V} \mid Z) \in G_1 \subset H^0(Y \cap \tilde{Z})$, and $j^*(V \mid Z)$ determines $i^*_Y c_1(L) \in H^2(Y)$ (modulo the kernel of the mapping $G_1/G_0 \to \mathrm{Im}\ i^*_Y \subset H^2(Y)$).

We now return to the case where X is the flag variety of an arbitrary semio-simple group G. Let $s \in \mathscr{h}$, and suppose σ is a regular nilpotent in $\ell = z_{\mathscr{g}}(s)$. Denote by S the torus $Z(L)$. We will apply the above construction to the bundle L_χ on X associated to a character $\chi \in X(H)$. Let V be the vector field associated to s, and denote the lift of V to $O_X(L_\chi)$ by V_χ. Recall that in Theorem 2 we exhibited a bijection $\phi^{*\,-1}_\sigma$: $H^0(X^\sigma \cap X^S) \cong H^0(X^S) \to \mathbb{C}^{W_L \setminus W}$, the ring of complex functions on $W_L \setminus W$.

Lemma: *Let $\mathscr{b}'$ be a Borel subalgebra of $\mathscr{g}$ in $X^\sigma \cap X^S$ and suppose $\psi_\sigma(\mathscr{b}') = \bar{w}$. Then*

$$V_\chi(\mathscr{b}') = -d\chi(w^{-1} \cdot s)$$

Consequently, $j^*_\sigma V_\chi(\mathscr{b}') = -d\chi(w^{-1} \cdot s)$ where j_σ: $X^\sigma \cap X^S \to X^S$ is the inclusion.

The proof is identical with lemma 1 of [C], so we will omit it.

§4. Proof of Theorem 1

Using Theorem 2, we have an identification ϕ^*_σ: $\mathbb{C}^{W_L \setminus W} \to H^0(X^\sigma \cap X^S)$, where the notation is as in §3. Recall (§1) that $A(W \cdot s)$ has a canonical filtration $D_0 \subset D_1 \subset \cdots$. Let $G_0 \subset G_1 \subset \cdots$ be the filtration of $H^0(X^\sigma \cap X^S)$ defined in Theorem 3. We will now define an isomorphism

$$\tilde{\psi}_s\colon A(W \cdot s) \to \mathbb{C}^{W_L \setminus W}$$

and show that $\tilde{\psi}_s(D_i) = \phi_\sigma^{*-1}(G_i)$ for all i. The hypotheses of Theorem 3 hold, so we obtain the desired map ψ_s as the composition

$$\mathrm{Gr}\, A(W\cdot s) \to \mathrm{Gr}\, H^0(X^\sigma \cap X^S) \to \mathrm{Im}\, i_\sigma^* = I(X,\, X^\sigma)$$

To define ψ_s, it suffices to define $\psi_s(\omega)$ for $\omega \in \mathfrak{h}^*$. We set $\psi_s(\omega)(\bar{w}) = -\omega(w^{-1}\cdot s)$.

Now W acts on $\mathbb{C}^{W_L\backslash W}$ on the right: $(w\cdot f)(\bar{v}) = f(\overline{vw})$ if $v,\ w\in W$ and $f\in \mathbb{C}^{W_L\backslash W}$.

LEMMA: *$\tilde{\psi}_s$ is a W-equivariant isomorphism.*

PROOF: A straight forward calculation.

Since $A(W\cdot s)$ is generated by D_1 and $H^\cdot(X)$ is generated by $H^2(X)$, the surjectivity assertion of Theorem 1 will be proved if we show $\tilde{\psi}_s(D_1) = \phi_\sigma^{*-1}(G_1)$. But the Lemma of §3 says that if $\chi\in X(H)$, then $\tilde{\psi}_s(d\chi) = j_\sigma^*(\tilde{c}(L_\chi))$, where $d\chi \in \mathfrak{h}^*$ is the differential of χ, so ψ_s is surjective. To finish the proof, we must show W-equivariance and that ψ_s is an isomorphism if and only if i_σ^* is surjective. The proof of equivariance will be left to the reader. The second statement follows immediately from Theorem 3 due to the fact that $H^{\mathrm{odd}}(X^\sigma)$ vanishes for all σ [BS].

§5. Proof of Proposition 1

Let $A(\mathfrak{g})$ denote the ring of polynomials on $\mathfrak{g}$ and Let $I = I(O_s) \subset A(\mathfrak{g})$ be the ideal of O_s. Recall that gr I is the ideal generated by leading terms in I. By [BK, Satz. 1.8 and pp. 80–82], the hypotheses imply that $I(\overline{O}_\sigma) = \mathrm{gr}\, I$. Thus, $I(\overline{O}_\sigma \cap \mathfrak{h}) = I(\overline{O}_\sigma) + I(\mathfrak{h}) = \mathrm{gr}\, I + I(\mathfrak{h})$. Set theoretically, $O_s \cap \mathfrak{h} = W\cdot s$. However, $O_s \cap \mathfrak{h}$ is also smooth (see e.g. [Hu, pp. 116–117]), so $I(O_s\cap \mathfrak{h}) = I(W\cdot s)$. In particular,

$$I(W\cdot s) = I + I(\mathfrak{h}).$$

For an arbitrary pair of ideals I_1 and I_2 in $A(\mathfrak{g})$, $\mathrm{gr}\, I_1 + \mathrm{gr}\, I_2 \subset \mathrm{gr}(I_1 + I_2)$. Thus, since $I(\mathfrak{h})$ is homogeneous and consequently $\mathrm{gr}(I(\mathfrak{h})) = I(\mathfrak{h})$,

$$\mathrm{gr}(I(W\cdot s)) \subset \mathrm{gr}\, I + I(\mathfrak{h}).$$

We immediately obtain a W-equivariant surjective algebra homomorphism

$$A(\overline{O}_\sigma \cap \mathfrak{h}) = A(\mathfrak{h})/\mathrm{gr}\, I + I(\mathfrak{h}) \to A(\mathfrak{g})/\mathrm{gr}(I(W\cdot s)).$$

Furthermore, since $A(\mathscr{g})$ is graded,

$$A(\mathscr{g})/\mathrm{gr}(I(W\cdot s)) \cong \mathrm{Gr}\, A(W\cdot s). \qquad (3)$$

Moreover, $W\cdot s \subset \mathscr{h}$ and W acts homogeneously on $A(\mathscr{h})$, so (3) is W-equivariant. Combining these maps yields the desired morphism.

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