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SOME REMARKS ON PLANE CURVES OVER FIELDS OF FINITE CHARACTERISTIC

Rita Pardini

Introduction

Let X be a plane curve over an algebraically closed field \mathbb{K} , let $\varphi: X \rightarrow \mathbb{P}^2(\mathbb{K})^*$ be the dual map that associates to every regular point of X its tangent line, and let X^* be the closure of $\varphi(X)$, i.e. the dual curve of X . A classical result states that when the ground field \mathbb{K} has characteristic zero then $\varphi: X \rightarrow X^{**}$ is birational. This is not always true when the ground field has finite characteristic: those curves for which φ turns out to be separable are called reflexive.

This situation has been studied in a wider context by A. Wallace in [4] and we have made use here of some of its results. We have considered only nonsingular plane curves and in this case we are able to prove some statements that are stronger than the corresponding ones given in [4].

In particular, in section 2 of this paper we obtain the explicit form of the equation of a nonsingular nonreflexive plane curve and a condition on its degree. This enables us to study in a more detailed way in section 3 the case of nonsingular nonreflexive plane curves of degree $p + 1$, where p is the characteristic of the field \mathbb{K} ; the main result is that all such curves are projectively isomorphic. Moreover, for such curves, X^* is projectively isomorphic to X and there are a finite number of points $x \in X$ such that the multiplicity of intersection of the tangent line $T_x(X)$ with X at the point x is equal to $p + 1$. We count the number of these points by observing that they coincide with the generalized Weierstrass points and using the results obtained by Laksov in [2], [3].

In addition, for $n = p + 1$, it is true that $X = X^{**}$, while in section 4 we show that for a nonsingular nonreflexive curve X of degree $n = dp + 1$, $d > 1$, it is true that $X^{**} \neq X$ in general.

In the last section, we work out an example that shows that for nonreflexive curves of degree greater than $p + 1$, the dual curve X^* can either be reflexive or not and, in case the dual curve is not reflexive, its dual can either coincide with X or not, so that it is not possible to predict in general the behaviour of the dual of a nonreflexive curve.

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§1

Let \mathbb{K} be an algebraically closed field and let $X \subseteq \mathbb{P}^2(\mathbb{K})$ be an irreducible curve, whose degree is n and whose homogeneous equation is F .

If F_i denotes the partial derivative of F with respect to the i -th variable and $\mathbb{P}^2(\mathbb{K})^*$ denotes the dual of $\mathbb{P}^2(\mathbb{K})$, then

$$\begin{aligned} \varphi: X &\rightarrow \mathbb{P}^2(\mathbb{K})^* \\ x &\rightarrow (F_0(x), F_1(x), F_2(x)) \end{aligned}$$

is a rational map, called the dual map. The closure X^* of $\varphi(X)$ in $\mathbb{P}^2(\mathbb{K})^*$ is called the dual curve of X . We see that if X is nonsingular, φ is a morphism.

DEFINITION 1.1: Let $\varphi: X \rightarrow X^*$, $\varphi': X^* \rightarrow X^{**}$ be dual maps. We say that X is reflexive if:

- (a) $X^{**} = X$
- (b) $\varphi' \cdot \varphi = id_X$

Later, we will give an example of curve X such that only condition (a) of the definition is satisfied. The following proposition, whose proof can be read in [4], gives us a useful criterion of reflexivity:

PROPOSITION 1.2: *X is reflexive iff φ is separable.* We have the following immediate consequence of the above proposition.

COROLLARY 1.3: *If $\text{char } \mathbb{K} = 0$, then X is reflexive.* The following example shows that nonreflexive curves do exist:

EXAMPLE 1.4: Let p be the characteristic of the ground field \mathbb{K} , let X be the plane curve defined by

$$x_0 x_1^p + x_1 x_2^p + x_2 x_0^p = 0.$$

Then $\varphi: X \rightarrow X^*$, defined by $(x_0, x_1, x_2) \rightarrow (x_1^p, x_2^p, x_0^p)$ is not separable, hence X is not reflexive.

In order to understand better the above described situation, we will now examine the local expression of φ in the neighborhood of a nonsingular point $z \in X$. (Analogous computations can be found in [5], Ch. IV §4 and Ch. V §8.) Let $z = (1, a_0, b_0)$ and $F_2(z) \neq 0$. We choose affine coordinates $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$ in $\mathbb{P}^2(\mathbb{K})$ and $x' = \frac{y_0}{y_2}$, $y' = \frac{y_1}{y_2}$ in $\mathbb{P}^2(\mathbb{K})^*$;

then, $t = x - a_0$ is a local parameter on X in z and the following is a place of X with center z :

$$x = t + a_0$$

$$y = b_0 + b_1t + b_2t^2 + \dots$$

and φ can be written locally as:

$$x' = a_0b_1 - b_0 + 2a_0b_2t + (3a_0b_3 + b_2)t^2 + (4a_0b_4 + 2b_3)t^3 + \dots$$

$$y' = -b_1 - 2b_2t - 3b_3t^2 + \dots$$

We see that if $\text{char } \mathbb{K} = 2$ then φ is always inseparable.

PROPOSITION 1.5: *Let $\text{char } \mathbb{K} = p$, $p \neq 2$, and let $\varphi : X \rightarrow X^*$ be the dual map. Then*

- (a) φ is inseparable $\Leftrightarrow X$ has infinitely many flexes. When either of these conditions are satisfied, then $\forall x \in X$ such that x is a regular point of X $m_x(T_x(X) \cdot X) \geq p$.
- (b) φ is separable $\Rightarrow X$ has finitely many bitangents, that is there exist finitely many lines tangent to X at more than one point.

PROOF: The local expression of φ that we have worked out above shows that x is a ramification point of X iff $b_2 = 0$, i.e. iff x is a flex of X . Since a separable map has at most a finite number of ramification points, φ is separable iff X has finitely many flexes.

Moreover, we see that if φ is not separable, p must divide the ramification order of φ at every point and the condition $b_2 = \dots = b_{p-1} = 0$ is therefore a necessary one. This finishes the proof of (a).

It is easy to verify, as in [5], pages 153–4, that if φ is separable, then to every bitangent of X there corresponds a singularity of X^* . This remark proves (b). Q.E.D.

We finish this section with a proposition which illustrates the relationship between the flexes of X and the Hessian matrix of X :

PROPOSITION 1.6: *Suppose $\text{char } \mathbb{K} \neq 2$. Let $H(x)$ denote the determinant of the Hessian matrix of X , let x be a regular point of X , and let $Q(x)$ be the conic of equation*

$$\sum_{i,j=0}^2 F_{i,j}(x)W_iW_j = 0.$$

Then: x is a flex \Leftrightarrow the tangent line to X in x is a component of $Q(x)$.

Moreover:

(a) if $\text{char } \mathbb{K} = 0$ or $\text{char } \mathbb{K} = p$ and p does not divide $n - 1$ then:

$$x \text{ is a flex} \Leftrightarrow H(x) = 0;$$

(b) if $\text{char } \mathbb{K} = p$ and p divides $n - 1$, then $Q(x)$ is singular in x .

PROOF: Since $\text{char } \mathbb{K} = 2$, it is possible to use Taylor's formula up to the second order. Let $b = (b_0, b_1, b_2) \in \mathbb{P}^2(\mathbb{K})$. Then we have

$$\begin{aligned} F(x + tb) &= F(x) + t \sum_0^2 F_i(x) b_i + \frac{t^2}{2} \sum_{i,j=0}^2 F_{ij}(x) b_i b_j \\ &\quad + t^3 g(b, t, x). \end{aligned}$$

If $b \in T_x(X)$ then

$$F(x + tb) = \frac{t^2}{2} \sum_{i,j=0}^2 F_{ij}(x) b_i b_j + t^3 g(t, b, x).$$

Thus x is a flex iff $\sum_{i,j=0}^2 F_{ij}(x) b_i b_j = 0 \quad \forall b \in T_x(X)$, i.e. iff $T_x(X)$ is a component of $Q(x)$. This proves part " \Rightarrow " of (a) and the last assertion. The rest of the proof follows from an application of Euler's Theorem:

$$\begin{aligned} (*) \quad \left. \frac{\partial Q(x)}{\partial W_i} \right|_{W=x} &= \sum_{k=0}^2 F_{ki}(x) W_k \Big|_{W=x} = \sum_{k=0}^2 F_{ki}(x) x_k \\ &= (n-1) F_i(x) \quad i = 0, 1, 2. \end{aligned}$$

If x is a regular point and $\text{char } \mathbb{K} = 0$, or if $\text{char } \mathbb{K} = p$ and p does not divide $n - 1$, then $Q(x)$ is nonsingular at x and $T_x(Q(x)) = T_x(X)$.

When $H(x) = 0$, it then follows that $T_x(X)$ is a component of $Q(x)$ and x is a flex. Part a) is completely proved by this.

However, if $\text{char } \mathbb{K} = p$ and $p \mid n - 1$, then from (*) it follows that:

$$\left. \frac{\partial Q(x)}{\partial W_i} \right|_{W=x} = 0, \quad i = 0, 1, 2$$

and x is a singular point of $Q(x)$. Q.E.D.

§2

We will now state the main result of this article and some of its corollaries:

THEOREM 2.1: *Let $\text{char } \mathbb{K} = p$, $p \geq 3$. Then if X is a nonsingular plane curve with infinitely many flexes,*

$$F_{ij} = 0, \quad \text{for } i, j = 0, 1, 2.$$

PROOF: First of all, we will show $F_{ii} = 0$, for $i = 0, 1, 2$. Suppose $F_{ii} \neq 0$. By assumption, the dual morphism φ can be written as the composition of a purely inseparable morphism of degree p^k and a separable morphism, since any field extension can be factored as a separable extension followed by a purely inseparable one.

It follows that the divisor D_i of F_i on X has the form:

$$D_i = p^k \sum_{j=1}^s Q_{ij}, \quad \text{for } i = 0, 1, 2.$$

Note that $\{F_i = 0\}$ cannot contain X , because its degree is smaller than the degree of X and X is irreducible.

We can choose homogeneous coordinates in $\mathbb{P}^2(\mathbb{K})$ in such a way that the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are not on X .

Let $\mathcal{P}_0 \in X$, $\mathcal{P}_0 = (1, a_0, b_0)$, be a point such that $F_1(\mathcal{P}_0) = 0$. Since X is nonsingular, F_2 does not vanish at \mathcal{P}_0 and if we use affine coordinates $x = \frac{x_1}{x_0}$, $y = \frac{y_2}{y_0}$, we can choose $t = x - a_0$ as a local parameter on X at \mathcal{P}_0 .

The following equality holds in the local ring $\mathcal{O}_{X, \mathcal{P}_0}$: $F_1(1, x, y) = at^{rp}$, where $a \in \mathcal{O}_{X, \mathcal{P}_0}$. This is the same as saying that there is an open set $V \subseteq \mathbb{P}^2(\mathbb{K})$, such that $\mathcal{P}_0 \in V$, and an element $b \in \mathbb{K}[V]$, such that:

$$F_1 = at^{rp} + bF \quad \text{in } \mathbb{K}[V].$$

We see that the derivative of a function on V with respect to the variable t is the same as the derivative with respect to x and hence:

$$\begin{aligned} F_{11} &= rpat^{p-1} + \frac{\partial a}{\partial t} t^{rp} + \frac{\partial b}{\partial t} F + bF_1 \\ &= \left(\frac{\partial a}{\partial t} + ba \right) t^{rp} + \left(b^2 + \frac{\partial b}{\partial t} \right) F, \end{aligned}$$

i.e.

$$F_{11} = \left(\frac{\partial a}{\partial t} + ba \right) t^{r_p} \quad \text{in } \mathcal{O}_{X, \mathcal{P}_0}.$$

Therefore, if F_{11} does not vanish identically on the curve, it has a zero at least of the same order as F_1 at every point $\mathcal{P}_0 \in X$ such that $F_1(\mathcal{P}_0) = 0$, and so the divisor D_{11} corresponding to F_{11} is greater than or equal to D_1 . This is not possible, because $\deg D_{11} = n(n-2)$, while $\deg D_1 = n(n-1)$. Thus $F_{11} = 0$. $F_{00} = F_{22} = 0$ can be proved in the same way.

We have seen (Proposition 1.6) that a point $\mathcal{P}_0 \in X$ is a flex iff the conic $Q(\mathcal{P}_0)$ defined by $\sum_{i,j=0}^2 F_{ij}(\mathcal{P}_0)W_iW_j = 0$ is reducible and the tangent line to X at \mathcal{P}_0 is one of its components.

This is the same as saying that there are $b_0(\mathcal{P}_0), b_1(\mathcal{P}_0), b_2(\mathcal{P}_0) \in \mathbb{K}$ such that

$$\sum_{i,j=0}^2 F_{ij}(\mathcal{P}_0)W_iW_j = \left(\sum_{i=0}^2 b_iW_i \right) \left(\sum_{j=0}^2 F_j(\mathcal{P}_0)W_j \right) = 0.$$

It follows that: $F_{ii}(\mathcal{P}_0) = b_jF_i(\mathcal{P}_0)$, $i = 0, 1, 2$.

Since X is nonsingular by assumption, we have $F_i \neq 0$ on X except for finitely many points.

We have: $0 = F_{ii}(\mathcal{P}_0) = b_jF_i(\mathcal{P}_0)$ hence $b_i(\mathcal{P}_0) = 0$ on X except for finitely many point of X . It follows that $2F_{ij} = b_iF_j + b_jF_i = 0$, $i, j = 0, 1, 2$ on X with the possible exception of finitely many points, hence everywhere on X . Q.E.D.

COROLLARY 2.2: *Let char $\mathbb{K} = p$, $p \geq 3$. If X is a nonsingular curve with infinitely many flexes then p divides $n-1$.*

PROOF: From Euler's Theorem we have:

$$\sum_{j=0}^2 F_{ij}(x)x_j = (n-1)F_i, \quad i, j = 0, 1, 2.$$

On the other hand, from the above theorem it follows that $F_{ij} = 0$, $i, j = 0, 1, 2$ and so it is necessary that $n-1 \equiv 0 (p)$. Q.E.D.

REMARK 2.3: The assumption that X be nonsingular is necessary in Theorem 2.1, as it is shown by the example of the curve defined by the equation $y = x^p$.

In fact, X^* is a line and so X is not reflexive. From propositions 1.2 and 1.5, we see that X must have infinitely many flexes, although it does not satisfy Corollary 2.2.

Here is another immediate consequence of Theorem 2.1.

COROLLARY 2.4: *The nonsingular plane curves on \mathbb{K} of degree $kp + 1$ which have infinitely many flexes are parameterized by an open subset of a projective space of dimension $N = \frac{3(k+2)(k+1)}{2} - 1$.*

PROOF: It is easy to see that, in order to meet the conclusion of Theorem 2.1, the equation of a nonsingular curve of degree $kp + 1$ with infinitely many flexes must be of the form:

$$F(x_0, x_1, x_2) = x_0 a(x_0^p, x_1^p, x_2^p) + x_1 b(x_0^p, x_1^p, x_2^p) + x_2 c(x_0^p, x_1^p, x_2^p),$$

where a, b, c are homogeneous polynomials of degree k in 3 variables.

We finish the proof by noticing that the homogeneous polynomials of degree k in 3 variables have $\frac{(k+1)(k+2)}{2}$ coefficients and that there are nonsingular plane curves of degree $kp + 1$ with infinitely many flexes for every $k \in \mathbb{N}$. For instance:

$$x_0 x_1^{kp} + x_1 x_2^{kp} + x_2 x_0^{kp} = 0. \quad \text{Q.E.D.}$$

§3

We will now discuss in a more detailed way the case of nonsingular plane curves of degree $p + 1$ which have infinitely many flexes; in particular, we will show that all such curves are projectively equivalent.

PROPOSITION 3.1: *Let $\text{char } \mathbb{K} = p$, $p \geq 3$, and $X \subseteq \mathbb{P}^2(\mathbb{K})$ a nonsingular plane curve of degree $p + 1$. Then: X has infinitely many flexes $\Rightarrow X^*$ is a nonsingular plane curve of degree $p + 1$. Conversely, X has finitely many flexes $\Rightarrow X^*$ has degree $p(p + 1)$.*

PROOF: First of all we notice that the degree of the linear system on X corresponding to the morphism $\varphi: X \rightarrow X^*$ is $p(p + 1)$; then, since the degree of a morphism equals the cardinality of the generic fiber, it follows:

$$(*) \quad p(p + 1) = \deg X^* \deg \varphi.$$

If X has infinitely many flexes, φ is inseparable by Proposition 1.2. In this case, φ is also one-to-one and onto, since, by proposition 1.5(a), at every point $x \in X$ the multiplicity of intersection of the tangent line $T_x(X)$ with X is greater than or equal to p and so X cannot have any multiple tangent.

Then φ can be written as the composition of a purely inseparable morphism of degree p and a birational morphism and the genus of X^* is the same as the genus of X . (see [0], page 302). From the equality (*) it follows that $\deg X^* = p + 1$ and, by the genus formula, X^* is nonsingular. Conversely, if X has finitely many flexes then φ is separable and almost everywhere injective, hence birational, by proposition 1.5 and X^* has degree $p(p + 1)$ by (*). Q.E.D.

PROPOSITION 3.2: *Let $\text{char } \mathbb{K} = p$, $p \geq 3$, and X be a nonsingular curve of degree $p + 1$ with infinitely many flexes. Let $\varphi: X \rightarrow X^*$, $\varphi': X^* \rightarrow X^{**}$ by dual morphisms and define $h: X \rightarrow X$ by $x \rightarrow T_x(X) \cap (X - \{x\})$. Then*

- (a) X^* has infinitely many flexes and $X = X^{**}$;
- (b) by (a) we can consider the following diagram which is commutative

$$\begin{array}{ccc}
 & & h \\
 & & \longrightarrow \\
 X & \xrightarrow{\quad} & X \\
 \searrow \varphi & & \nearrow \varphi' \\
 & & X^*
 \end{array}$$

- (c) there are at most finitely many points $x \in X$ such that $m_x(T_x(X) \cdot X) = p + 1$.

PROOF: First notice that if we define $h(x) = x$ for those points $x \in X$ such that $m_x(T_x(X) \cdot X) = p + 1$, h is a morphism. In fact, consider affine coordinates $z = \frac{x_1}{x_0}$, $w = \frac{x_2}{x_0}$ and assume that the equation F of X is normalized so that the coefficient of w^{p+1} is equal to 1. Assume that $x = (a, b) \in X$ is a point such that $F_2(a, b) \neq 0$. The equation of the tangent line to X at x is $w - b = -\frac{F_1(x)}{F_2(x)}(z - a)$.

Therefore the expression of h can be computed from the formulas:

$$F\left(a, b - \frac{F_1(x)}{F_2(x)}(z - a)\right) = (z - h_1(a, b))(z - a)^p$$

$$b - h_2(a, b) = -\frac{F_1(x)}{F_2(x)}(a - h_1(a, b))$$

The Hurwitz formula $2g - 2 = \deg h \cdot (2g - 2) + \deg R$, where R is the ramification divisor, forces $\deg h = 1$ and $R = 0$, since R is an effective divisor.

Therefore h is one-to-one. Since X is irreducible, h is also onto.

This is the same as saying that every $x \in X$ lies on two tangent lines to X if $m_x(T_x(X) \cdot X) = p$, and on only one if $m_x(T_x(x) \cdot X) = p + 1$. Then $\forall x \in X$ the line $l \subseteq \mathbb{P}^2(\mathbb{K})^*$ corresponding by duality to x intersects X^* in at most two distinct points. Since, by Proposition 3.1, X^* is nonsingular of degree $p + 1$, l is tangent to X^* and the locus of tangent lines of X^* is X .

By Proposition 3.1, X^* has infinitely many flexes, too. If $l \in X^*$, $\varphi'(l)$ must be a point of X which lies on l . Then it is either the point at which l is tangent to X or the residual intersection of l and X . Since two different morphisms of curves can coincide at most in a finite number of points, it follows that $\varphi' \cdot \varphi = 1$ or $\varphi' \cdot \varphi = h$.

Because φ and φ' are inseparable, the first possibility has to be excluded and we have: $\varphi' \cdot \varphi = h$. Q.E.D.

PROPOSITION 3.4: *Let char $\mathbb{K} = p$, $p \neq 2$, and let $X \subseteq \mathbb{P}^2(\mathbb{K})$ be a nonsingular plane curve of degree $p + 1$ with infinitely many flexes.*

Then there exists $x \in X$ such that $m_x(T_x(X) \cdot X) = p + 1$.

PROOF: Let $I = \{(P, l) \mid P \in l\} \subseteq \mathbb{P}^2(\mathbb{K}) \times \mathbb{P}^2(\mathbb{K})^*$, I is a nonsingular variety of dimension 3.

By Corollary 2.4, the nonsingular plane curves of degree $p + 1$ with infinitely many flexes are parametrized by an open subset of $\mathbb{P}^8(\mathbb{K})$.

Let $W = \{(X, P, l) \mid m_P(l \cdot X) \geq p + 1\}$. An easy computation of dimensions shows that W is an irreducible 8-dimensional variety.

In fact, the group $\mathbb{PGL}(2, \mathbb{K})$ acts naturally on W and this action preserves the fibers of the projection map $\pi: W \rightarrow I$. So all the fibers of π are isomorphic and it is enough to study the fiber over one point of I .

Take $Q = ((1, 0, 0); (0, 1, 0)) \in I$; using the expression for an element of W that we have shown in the proof of Corollary 2.3, we see that $(X, Q) \in W$ iff the coefficients of the monomials x_0^{p+1} , $x_2 x_0^p$ and $x_0 x_2^p$ in the equation of X are equal to zero. We will exhibit later in the proof a nonsingular curve X such that $(X, Q) \in W$. Therefore, we can conclude that the fibers of π are isomorphic to open subsets of a 5-dimensional projective space and that W is an irreducible variety of dimension 8.

Moreover, it is easy to verify that the nonsingular plane curve X of equation $x_1 x_0^p + x_2^p x_1 + x_0 x_1^p + x_2^{p+1} = 0$ has the following properties:

- (1) the multiplicity of intersection at $(1, 0, 0)$ of the tangent line to X and X is $p + 1$;
- (2) the multiplicity of intersection at $(0, 1, 0)$ of the tangent line and X is p .

Therefore the projection $q: W \rightarrow P^8(K)$ is such that the dimension of $q^{-1}(X)$ is zero, hence it is onto. Q.E.D.

Combining the two preceding propositions, we see that every nonsingular plane curve X of degree $p + 1$ with infinitely many flexes has a finite and nonzero number of points x such that $m_x(T_x(X) \cdot X) = p + 1$. We will now show that such points are the Weierstrass points of the linear system $H^0(X, \mathcal{O}(1))$ according to the definition given by Laksov in [2] and we will determine their number.

In the rest of this section, we will use the notation introduced by Laksov in [2], [3].

PROPOSITION 3.5: *Let $\text{char } \mathbb{K} = p$, $p \neq 2$. Let X be a nonsingular plane curve of degree $p + 1$ with infinitely many flexes. Then there are exactly $p^3 + 1$ distinct points of X such that $m_x(T_x(X) \cdot X) = p + 1$.*

PROOF: Given $x \in X$, it is possible to choose a basis l_0, l_1, l_2 of $H^0(X, \mathcal{O}(1))$ such that l_0 does not vanish at x , l_1 meets X transversally at x and l_2 is tangent to X at x .

Then the sequence of the Hermite invariants is the following: $h_0 = 0$, $h_1 = 1$, $h_2 = p$ or $p + 1$ according as the multiplicity of intersection of the tangent line $T_x(X)$ and X at x is p or $p + 1$. Notice that by proposition 1.5 these are the only possibilities.

So, by Proposition 3.6, $h_2 = p + 1$ at a finite number of points, which therefore must be the Weierstrass points, and $b_0 = 0$, $b_1 = 1$, $b_2 = p$.

The number of Weierstrass points, counted according to their multiplicity, is $(b_0 + b_1 + b_2)(2g - 2) + n(r + 1)$, where g represents the genus of X , n the degree and r the projective dimension of the linear system we are considering.

In this case, $g = \frac{p(p-1)}{2}$, $r = 2$, $n = p + 1$ and the formula takes the value $p^3 + 1$.

Then the statement will follow if we can prove that every Weierstrass point has multiplicity equal to 1. Given a Weierstrass point $x \in X$, we can choose a local parameter t at x such that $l_0 = a_0 + a_{0,1}t + \dots$, $l_1 = t$, $l_2 = a_2t^{p+1} + a_{2,1}t^{p+2} + \dots$, where $a_0, a_2 \neq 0$.

The multiplicity of x as a Weierstrass point is given by the lowest of the vanishing orders at x of the non identically vanishing minors of order 3 of the following $3X(p + 1)$ matrix:

$$\begin{bmatrix} a_0 & + \dots & t & a_2t^{p+1} & + \dots \\ 0 & & 1 & a_2t^p & + \dots \\ \cdot & & \cdot & 0 & \\ \cdot & & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \\ 0 & & 0 & a_2t & + \dots \end{bmatrix}$$

and it is therefore equal to 1. Q.E.D.

We can now at last prove the result stated at the beginning of the section. We start with the following:

LEMMA 3.6: *Let char $\mathbb{K} = p$, $p \geq 3$. Let X be a nonsingular plane curve of degree $p + 1$ with infinitely many flexes. Then the subgroup $G \subseteq \mathbb{PGL}(2, \mathbb{K})$ of projectivities that induce an automorphism of X has order $\leq (p^3 + 1)^4$.*

PROOF: Since x is irreducible, on every line l there are at most $p + 1$ distinct points of X . If $p \neq 2$, we have $p^3 + 1 > 3p + 3$ and so there are at least four Weierstrass points of X such that no 3 of them are on the same line.

Since every projectivity of G induces a permutation of the Weierstrass points, the statement follows from the fact that an element of $\mathbb{PGL}(2, \mathbb{K})$ is completely determined by the images of 4 points no 3 of which are collinear. **Q.E.D.**

PROPOSITION 3.7: *Let char $\mathbb{K} = p$, $p \neq 2$. Let X be a nonsingular plane curve of degree $p + 1$ with infinitely many flexes.*

Then X is projectively equivalent to the plane curve of equation

$$x_0 x_1^p + x_1 x_2^p + x_2 x_0^p = 0.$$

PROOF: Consider the morphism $h_X: \mathbb{PGL}(2, \mathbb{K}) \rightarrow \mathbb{P}^8(\mathbb{K})$, $g \rightarrow gX$. The fibres of h_X are the right cosets of $\mathbb{PGL}(2, \mathbb{K})$ modulo the subgroup G of the projectivities which map X into itself and so, by Lemma 3.6, the fibers have dimension zero.

Since $\mathbb{PGL}(2, \mathbb{K})$ has dimension 8, $\text{im } h_X$ must include an open subset of $\mathbb{P}^8(\mathbb{K})$.

Then we finish the proof by noticing that two open subsets of $\mathbb{P}^8(\mathbb{K})$ always intersect. **Q.E.D.**

§4

We have seen in section 3 that all nonsingular plane curves of degree $p + 1$ with infinitely many flexes are such that $X^{**} = X$.

This is not true when we consider curves of higher degree; we will now show that a nonsingular plane curve X of degree $dp + 1$, $d > 1$, with infinitely many flexes, *in general* does not coincide with the bidual curve X^{**} .

REMARK 4.1: Let $x \in \mathbb{P}^2(\mathbb{K})$ be such that there are $\text{deg } X^*$ distinct points of X such that x lies on the corresponding tangent lines. Then $x \notin X^{**}$.

PROOF: Trivial. **Q.E.D.**

It is necessary to state some preliminary facts:

PROPOSITION 4.2: *Let X be a nonsingular plane curve of degree $dp + 1$ with infinitely many flexes. Then in general, X has finitely many multiple tangent lines and finitely many tangent lines such that the multiplicity of intersection in the tangency point is greater than p . (A line is a multiple tangent if it is tangent to X at more than one point).*

PROOF: It is an easy computation of dimensions, similar to that carried out in the proof of 3.4. Q.E.D.

COROLLARY 4.3: *Let X be a nonsingular plane curve of degree $dp + 1$ with infinitely many flexes. Then in general $\deg X^* = d(dp + 1)$.*

PROOF: Since X has at most finitely many multiple tangents from 4.2 it follows that the map $\varphi: X \rightarrow X^*$ is one-to-one on an open set of X . Since the generic tangent line has multiplicity of intersection p with X in the tangency point, it follows that the degree of inseparability of φ is exactly p . (See [0], §IV.2 and the formulas in Example 1.5 of this paper.)

Therefore, φ can be written as the composition of a purely inseparable morphism of degree p and of a birational morphism.

Since the degree of the linear system on X corresponding to φ is $dp(dp + 1)$, the corollary follows from the equality $dp(dp + 1) = \deg X^* \cdot \deg \varphi$. Q.E.D.

LEMMA 4.4: *Let X be a generic curve with the following properties:*

- (i) X is nonsingular
- (ii) $\deg X = dp + 1$, $d > 1$
- (iii) X has infinitely many flexes
- (iv) $(0, 0, 1) \in X$.

*Then $X \neq X^{**}$.*

PROOF: We have seen (Corollary 2.4) that the equation of X has the form $x_0 a(x_0^p, x_1^p, x_2^p) + x_1 b(x_0^p, x_1^p, x_2^p) + x_2 c(x_0^p, x_1^p, x_2^p) = 0$, where a, b, c are homogeneous polynomials of degree d .

As the ground field \mathbb{K} is algebraically closed, the equation of X can also be written in the following way: $x_0 \alpha^p(x) + x_1 \beta^p(x) + x_2 \gamma^p(x)$, where the coefficients of α, β, γ are the p -th roots of the corresponding coefficients of a, b, c .

For a curve X satisfying the hypotheses of the lemma we can assume that the following conditions hold:

- (1) γ defines a nonsingular curve
- (2) α, β, γ do not vanish at the same time
- (3) $\frac{\partial \gamma}{\partial x_2}, \gamma, x_0 \alpha^p + x_1 \beta^p$ do not vanish at the same time.

We notice that the points of X such that the tangent to X at them passes through $(0, 0, 1)$ are the solutions of:

$$\begin{cases} \gamma = 0 \\ x_0\alpha^p + x_1\beta^p + x_2\gamma^p = 0. \end{cases}$$

This system is trivially equivalent to:

$$(*) \quad \begin{cases} \gamma = 0 \\ x_0\alpha^p + x_1\beta^p = 0. \end{cases}$$

By Bézout's theorem, there are $d(dp + 1)$ solutions of this system, counted with the proper multiplicity. If we prove that every solution has multiplicity 1, we can finish the proof by means of Corollary 4.3 and Remark 4.1. The following conditions hold:

- (a) the point $(0, 0, 1)$ lies on every tangent line of the curve defined by $x_0\alpha^p + x_1\beta^p = 0$
- (b) by condition (2) α, β, γ do not vanish at the same time and so the solutions of $(*)$ are regular points both of $\{\gamma = 0\}$ and of $\{x_0\alpha^p + x_1\beta^p = 0\}$
- (c) by condition (3) the tangent lines to $\{\gamma = 0\}$ at the points that satisfy $(*)$ do not pass through $(0, 0, 1)$ and so can not be tangent to $\{x_0\alpha^p + x_1\beta^p = 0\}$, too.

So every solution of $(*)$ has multiplicity 1, and the lemma is proved.
Q.E.D.

PROPOSITION 4.5: *Let X be a nonsingular plane curve of degree $dp + 1$, $d > 1$, with infinitely many flexes. Then in general, $X \neq X^{**}$.*

PROOF: Let U be the space of nonsingular plane curves of degree $dp + 1$ with infinitely many flexes and let W be the subvariety of U consisting of the curves passing through $(0, 0, 1)$. W is irreducible of codimension 1.

Consider the morphism $\psi: Wx\mathbb{PGL}(2, \mathbb{K}) \rightarrow U$, $(X, g) \rightarrow gX$. Of course, $\psi(\overline{Wx\mathbb{PGL}(2, \mathbb{K})}) = U$ and so there is an open subset of U of curves X such that $X^{**} \neq X$. Q.E.D.

§5

If X is not reflexive and $X^{**} = X$, then X^* is not reflexive, either. In effect, if X^* were reflexive, the composition of the dual morphisms $\varphi: X^* \rightarrow X^{**} = X$, and $\varphi^*: X^{**} = X \rightarrow X^{***} = X^*$ would be the identity, forcing $\varphi^*: X \rightarrow X^*$ to be separable and contradicting 1.2. If the

curve X is not reflexive and $X \neq X^{**}$, then X^* can be reflexive or not, as it is shown by the following example. Let $\text{char } \mathbb{K} = p$ and let X be the curve defined by the equation

$$x_0 x_1^{dp} + x_1 x_2^{dp} + x_2 x_0^{dp} = 0, \quad d > 1, \quad d \not\equiv 0 \pmod{p}.$$

Then X is non singular and it is not reflexive. The following conditions hold:

- (1) the dual morphism $\varphi: X \rightarrow X^*$ is one-to-one on an open subset of X ;
- (2) $\text{deg } X^* = d(dp + 1)$;
- (3) if $d \not\equiv 1 \pmod{p}$, X^* is reflexive and so $X \neq X^{**}$. Moreover, X and X^{**} have the same genus;
- (4) if $d \equiv 1 \pmod{p}$, X^* is not reflexive and also in this case we have $X \neq X^{**}$.

PROOF: Now consider the morphism:

$$\psi: X \rightarrow \tilde{X} \quad (X = \tilde{X}) \quad \text{and} \quad \tilde{h}: \tilde{X} \rightarrow X^*$$

$$(x_0, x_1, x_2) \rightarrow (x_0^p, x_1^p, x_2^p) \quad (y_0, y_1, y_2) \rightarrow (y_1^d, y_2^d, y_0^d).$$

If $\varphi: X \rightarrow X^*$ is the dual morphism, we have $\varphi = \tilde{h} \cdot \psi$. Moreover, ψ is one-to-one and inseparable of degree p , whereas \tilde{h} is separable, since p does not divide d by assumption. To prove (1), it is enough to verify that \tilde{h} is one to one on an open subset of X .

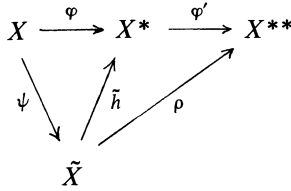
Let $y = (1, y_1, y_2)$ and $\bar{y} = (1, \bar{y}_1, \bar{y}_2) \in X$ be such that $\tilde{h}(y) = \tilde{h}(\bar{y})$. Then we have $\bar{y}_1 = \zeta_1 y_1$, $\bar{y}_2 = \zeta_2 y_2$, where $\zeta_1^d = \zeta_2^d = 1$. From the equation of \tilde{X} by elimination between the equations for y and \bar{y} we obtain:

$$(*) \quad y_2 (y_1 (1 - \zeta_1) y_2^{dp-1} + (1 - \zeta_2)) = 0.$$

If $y \neq \bar{y}$, then $1 - \zeta_1 \neq 0$ or $1 - \zeta_2 \neq 0$ and y belongs to the intersection of X with the curve defined by equation (*). This intersection is a finite set, since X is irreducible and cannot be contained in the curve defined by (*), having higher degree. Furthermore, from $\zeta_1^d = \zeta_2^d = 1$, it follows that there are at most finitely many curves defined by an equation of the form (*) and so (1) is proved. Condition (2) follows at once from (1) considering the equality

$$\text{deg } X^* \text{ deg } \varphi = dp(dp + 1)$$

and since $\deg \varphi = \deg \tilde{h} \cdot \deg \psi = p$. Let us now consider the following commutative diagram:



Since \tilde{h} is separable, φ' is separable iff ρ is.

We shall work out explicitly the expression of ρ . Let $y = (y_0, y_1, y_2) \in X$. The corresponding tangent line to X has coefficients $z = (y_1^{dp}, y_2^{dp}, y_0^{dp})$ and so the tangent line to X^* at $h(y)$ corresponds to the dual point ${}^t(d\tilde{h})^{-1}(z)$, where ${}^t(d\tilde{h})^{-1}$ indicates the transpose of the inverse of the differential of h . Then we have $\rho(y) = (y_2^{dp+d-1}y_0^{d-1}, y_0^{dp+d-1}y_1^{d-1}, y_1^{dp+d-1}y_2^{d-1})$. If p does not divide $d-1$, ρ and thus φ' are separable, and by Proposition 1.2, X^* is reflexive and birational to X^{**} .

If p divides $d-1$, ρ is inseparable and so X^* is not reflexive. The fixed part of the linear system on X corresponding to the base $(y_2^{dp+d-1}y_0^{d-1}, y_0^{dp+d-1}y_1^{d-1}, y_1^{dp+d-1}y_2^{d-1})$ is $D = (d-1)(1, 0, 0) + (d-1)(0, 1, 0) + (d-1)(0, 1, 0)$.

So the degree of the linear system on X corresponding to the morphism ρ is $(dp+1)(dp+2d-2) - 3(d-1)$.

In the equality $(dp+1)(dp+2d-2) - 3(d-1) = \deg \rho \cdot \deg X^{**}$, we notice that $\deg X = dp+1$ does not divide the left side. It then follows that $\deg X \neq \deg X^{**}$ and therefore $X \neq X^{**}$. Q.E.D.

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