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MACAULAY'S THEOREM AND LOCAL TORELLI FOR WEIGHTED HYPERSURFACES

Loring Tu

The Torelli problem for a family of varieties deals with the question of whether the periods of a variety determine its isomorphism class in the family; in other words, whether the period map from the base space of the family to a period matrix space is injective. For smooth hypersurfaces of a fixed degree in a projective space, several weaker variants of the Torelli problem have been solved. With the exception of a few degrees, Griffiths [10] proved that the differential of the period map is injective (the local Torelli theorem) and Donagi [6] proved that the period map has degree one (the generic Torelli theorem). On the other hand, Catanese [3] and Todorov [15] have constructed complete intersections in a weighted projective space which turn out to be counterexamples to the Torelli problem for surfaces of general type. Since the weighted hypersurfaces are in some sense the intermediate case between the usual hypersurfaces and the weighted complete intersections, it is therefore of interest to study the Torelli question for the weighted hypersurfaces.

In [10] Griffiths gave a cohomological description of the local Torelli problem and a polynomial description of the Hodge theory of a hypersurface. In terms of Grifiths' descriptions a sufficient condition for the local Torelli theorem is as follows. Let X be the smooth hypersurface of degree d in P^{n+1} given by the homogeneous polynomial f. Then the differential of the period map at X is none other than the cup product map:

$$H^1(X, \Theta) \to \bigoplus_{p=1}^{n} \mathrm{Hom}\big(H^{p,n-p}(X), H^{p-1,n-p+1}(X)\big), \tag{0.1}$$

where Θ is the tangent bundle of X. Define the *Jacobian ring* of X to be the graded algebra

$$R = \mathbb{C}[x_0, \dots, x_{n+1}]/(\partial f/\partial x_0, \dots, \partial f/\partial x_{n+1})$$

and denote by R_a the graded piece of degree a in R. All the cohomology groups in (0.1) can be identified with appropriate graded pieces of the

Jacobian ring; moreover, under this identification the cup product corresponds to polynomial multiplication. Hence, the injectivity of the multiplication map

$$R_d \to \mathrm{Hom}\left(R_{(n-p+1)d-(n+2)}, R_{(n-p+2)d-(n+2)}\right)$$

for some $p = 0, 1, \dots, n$ implies the local Torelli theorem. In this way Griffiths deduces the local Torelli theorem for hypersurfaces in P^{n+1} as a consequence of Macaulay's theorem in algebra.

MACAULAY'S THEOREM: *Let $f_0, \dots, f_{n+1}$ be a regular sequence of homogeneous polynomials of degree $d_0, \dots, d_{n+1}$ respectively in $\mathbb{C}[x_0, \dots, x_{n+1}]$ and let $R = \mathbb{C}[x_0, \dots, x_{n+1}]/(f_0, \dots, f_{n+1})$. Then R is a finite-dimensional graded $\mathbb{C}$-algebra with top degree $\sigma = \Sigma(d_i - 1)$ and the multiplication map*

$$\mu_{a,b} : R_a \times R_b \to R_{a+b}$$

is nondegenerate for $a + b \leqslant \sigma$.

At first sight the local Torelli problem for the weighted hypersurfaces appears to be a routine extension of Griffiths' result for the usual hypersurfaces. It is made interesting by the fact that Macaulay's theorem is in general false in the weighted case. Macaulay's theorem has proven to be of interest in other context. It is for instance a crucial ingredient in Donagi's proof of generic Torelli for hypersurfaces. Thus, we have two intertwining goals in this paper. First, we would like to determine some conditions under which Macaulay's theorem holds for a weighted ring. This is the content of Theorem 2.8. Secondly, we would like to show that under appropriate hypotheses on the weights and the degree d the local Torelli theorem holds for quasismooth weighted hypersurfaces of degree d.

To state the second result, recall that a *weighted hypersurface X* is the solution set of a weighted homogeneous polynomial $f(x_0, \dots, x_{n+1}) = 0$ in a weighted projective space $P(q_0, \dots, q_{n+1})$. Such a hypersurface is said to be *quasismooth* if the partial derivatives $\partial f/\partial x_0, \dots, \partial f/\partial x_{n+1}$ do not vanish simultaneously on X. For the weighted projective space $P(q_0, \dots, q_{n+1})$ let

$$m = \text{the least common multiple of the weights}$$
$$= \mathrm{lcm}(q_0, \dots, q_{n+1})$$

and

$$s = \text{the sum of the weights} = \Sigma q_i.$$

Our second main theorem is as follows.

THEOREM: *Suppose d is a positive integer of the form $s + km$, where k is an integer $\geqslant 2$. Then the local Torelli theorem holds for quasismooth hypersurfaces of degree d in $P(q_0, \ldots, q_{n+1})$.*

By making some hypotheses on m and s, we can prove the local Torelli theorem for other degrees as well. See Theorem 2.10.

In Section 1 we review the Hodge theory of weighted hypersurfaces. In Section 2 we first present a counterexample to Macaulay's theorem in the weighted case and then give a condition under which the theorem is true. Next we apply a result of Delorme [4] to show that with appropriate restrictions on the weights and degrees this condition is satisfied and therefore the local Torelli theorem follows.

It is a pleasure to thank Ron Donagi for many helpful discussions.

§1. The Hodge theory of weighted hypersurfaces

The basic facts about weighted projective spaces and weighted hypersurfaces may be found in Al-Amrani [1], Delorme [4], Dolgachev [5], Mori [13], and Steenbrink [14]. We summarize here what will be needed later.

Let $q_0, \ldots, q_{n+1}$ be positive integers and let $\mathbb{C}^* = \mathbb{C}\text{-}\{0\}$ act on $\mathbb{C}^{n+2}\text{-}\{0\}$ by

$$t \cdot (x_0, \ldots, x_{n+1}) = (t^{q_0}x_0, \ldots, t^{q_{n+1}}x_{n+1}).$$

The quotient is the weighted projective space $P(q_0, \ldots, q_{n+1})$. The weighted projective space may also be represented as the quotient of P^{n+1} by the finite group $\mathbb{Z}_{q_0} \times \ldots \times \mathbb{Z}_{q_{n+1}}$. Thus it is in general a singular variety with quotient singularities.

The *weight* of a monomial $x^K = x_0^{k_0} \ldots x_{n+1}^{k_{n+1}}$ is defined to be $\Sigma k_i q_i$. A polynomial $f(x_0, \ldots, x_{n+1}) = \Sigma a_K x^K$ is *weighted homogeneous* of degree d if it is a sum of monomials each of which has weight d. The zero locus X of f is a *weighted hypersurface*. If the partial derivatives $\partial f/\partial x_0, \ldots, \partial f/\partial x_{n+1}$ have no common zeros on X, then X is said to be *quasismooth*.

PROPOSITION 1.1: *A quasismooth hypersurface is a V-variety (a variety which is locally the quotient of a smooth variety by a finite group).*

PROOF: With the notations above, Let $P = P(q_0, \ldots, q_{n+1})$,

$$U_i = \{(x_0, \ldots, x_{n+1}) \in P \mid x_i \neq 0\},$$

$$V_i = \{(x_0, \ldots, x_{n+1}) \in \mathbb{C}^{n+2} \mid x_i = 1\},$$

and

$$C=\left\{(x_0,\dots,x_{n+1})\in\mathbb{C}^{n+2}\mid f(x_0,\dots,x_{n+1})=0\right\}.$$

We claim that $C\cap V_i$ is a smooth hypersurface in $\mathbb{C}^{n+1}$. For simplicity, let $i=0$. Then

$$C\cap V_0=\left\{(1,\ x_1,\dots,x_{n+1})\in\mathbb{C}^{n+2}\mid f(1,\ x_1,\dots,x_{n+1})=0\right\}.$$

Since f is weighted homogeneous of degree d, Euler's formula holds:

$$\Sigma q_i x_i\ \partial f/\partial x_i(x_0,\dots,x_{n+1})=d\cdot f(x_0,\dots,x_{n+1}).$$

Let $F(x_1,\dots,x_{n+1})=f(1,\ x_1,\dots,x_{n+1})$ be the equation of $C\cap V_0$ in $\mathbb{C}^{n+1}$. Then $\partial F/\partial x_i=\partial f/\partial x_i$ and if $\partial F/\partial x_i$ $(i=1,,\dots,n+1)$ all vanish at a point of $C\cap V_0$, by Euler's formula $\partial f/\partial x_0$ would also vanish at that point, contradicting the quasismoothness of X. Therefore, $C\cap V_0$ is smooth. The natural map π_0: $C\cap V_0\to X\cap U_0$ represents $X\cap U_0$ as the quotient of $C\cap V_0$ by the finite group $\mathbb{Z}_{g_0}$. It follows that X is a V-variety. □

The complex cohomology of a V-variety has a pure Hodge structure in each dimension (Steenbrink [14]). In the case of the quasismooth weighted hypersurface X the Hodge structure may be described as follows. Let Σ be the singular locus of X, i: $X-\Sigma\to X$ the inclusion, $\Omega^p_{X-\Sigma}$ the sheaf of germs of holomorphic p-forms on $X-\Sigma$, and $\Theta_{X-\Sigma}$ the tangent bundle of $X-\Sigma$. Define

$$\tilde{\Omega}^p_X=i_*\Omega^p_{X-\Sigma}.$$

Then $\tilde{\Omega}^p_X$ plays the role of Ω^p for a smooth variety and the Hodge decomposition assumes the form

$$H^n(X,\mathbb{C})=\bigoplus_{p+q=n}H^q(X,\tilde{\Omega}^p_X).$$

Just as for a smooth projective hypersurface in P^{n+1} the Hodge theory of a quasismooth weighted hypersurface may be described in terms of polynomials. Let $J=J(f)=(\partial f/\partial x_0,\dots,\partial f/\partial x_{n+1})$ be the weighted Jacobian ideal and $R=\mathbb{C}[x_0,\dots,x_{n+1}]/J$ the weighted Jacobian ring.

THEOREM 1.2: (Steenbrink [14]). *Let $H^{p,n-p}$ be the primitive $(p,\ n-p)$-cohomology of the quasismooth weighted hypersurface X of degree d and R_a the graded piece of the Jacobian ring in degree a. Then there is an isomorphism*

$$H^{p,n-p}\cong R_{(n-p+1)d-s},\tag{1.3}$$

where $s=\Sigma q_i$ is the sum of the weights.

Because X is a V-variety, its Kuranishi space exists and the Zariski tangent space to the Kuranishi space is isomorphic to $H^1(X, \Theta_X)$ (Fujiki [8, (3.4)]). We let $H^1(X, \Theta_X)_{\text{proj}}$ denote the subspace of $H^1(X, \Theta_X)$ corresponding to first-order projective deformations. Then as in the unweighted case, there is a canonical identification

$$H^1(X, \Theta_X)_{\text{proj}} \cong R_d. \tag{1.4}$$

Since we are concerned only with projective transformations, by the *locall Torelli theorem* for X we will mean the injectivity of the natural map

$$v\colon H^1(X, \Theta_X)_{\text{proj}} \to \bigoplus_{p=1}^{n} \operatorname{Hom}\left(H^{p,n-p}(X),\ H^{p-1,n-p+1}(X)\right).$$

By the same argument as in the unweighted case, after making the identifications (1.3) an (1.4), v is simply polynomial multiplication.

§2. Local duality and Macaulay's theorem for a weighted ring

Let $S = \mathbb{C}[x_0, \dots, x_{n+1}]$ be the polynomial ring in the weighted variables $x_0, \dots, x_{n+1}$, and let $J = (f_0, \dots, f_{n+1})$ be the ideal generated by a sequence of weighted homogeneous polynomials of degrees $d_0, \dots, d_{n+1}$. We call the quotient ring $R = \mathbb{C}[x_0, \dots, x_{n+1}]/J$ a *weighted Jacobian ring* or a *weighted ring*. Let q_i = weight of x_i.

PROPOSITION 2.1: *If $f_0, \dots, f_{n+1}$ is a regular sequence in S, then R is a finite-dimensional graded algebra over $\mathbb{C}$ with top degree $\sigma = \Sigma(d_i - q_i)$ and the Poincaré polynomial of R is*

$$P_t(R) = \prod \frac{1 - t^{d_i}}{1 - t^{q_i}}.$$

PROOF: Since $f_0, \dots, f_{n+1}$ is a regular sequence, the zero locus of $J = (f_0, \dots, f_{n+1})$ in C^{n+2} is the origin. Let m be the ideal of the origin. By Hilbert's Nullstellensatz, $m^r \subset J$ for some positive integer r. Hence, there is a surjection $S/m^r \to S/J$. Since S/m^r is finite-dimensional, so is $R = S/J$. Furthermore, the Poincaré series of R is actually a polynomial. The computation of the Poincaré polynomial may be found in Bott and Tu [2, p. 294]. The degree of this polynomial is $\Sigma(d_i - q_i)$, which is therefore the top degree of R. □

REMARK: In Proposition 2.1 since the coefficient of t^σ in the Poincaré polynomial $P_t(R)$ is 1, the top degree piece R_σ is isomorphic to $\mathbb{C}$.

THEOREM 2.2: Local duality for a weighted ring. *Let $f_0, \ldots, f_{n+1}$ be a regular sequence of weighted homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_{n+1}]$ and let $R = C[x_0, \ldots, x_{n+1}]/(f_0, \ldots, f_{n+1})$. Suppose q_i = weight of x_i and d_i =* deg f_i. *Then for any a such that $0 \leqslant a \leqslant \sigma$, the pairing*

$$R_a \times R_{\sigma-a} \to R_\sigma$$

given by multiplication is nondegenerate, where $\sigma = \Sigma(d_i - q_i)$ is the top degree.

The simplest way to prove this theorem is probably to use the teory of socles. Recall that the *socle* of a graded algebra A over a field k is defined to be

$$\operatorname{Soc} A = \left\{ h \in A \mid hg = 0 \quad \text{for all } g \in \bigoplus_{i=1}^{\infty} A_i \right\}.$$

In general the socle of A may be empty or it may contain elements in various degrees, but if A is the weighted Jacobian ring $R = \mathbb{C}[x_0, \ldots, x_{n+1}]/(f_0, \ldots, f_{n+1})$ with $f_0, \ldots, f_{n+1}$ a regular sequence as in Proposition 2.1, then Soc R turns out to be a 1-dimensional vector space over k, generated by the top degree elements; more precisely, Soc $R = R_\sigma$, where σ is the top degree of R. A proof of this fact may be found in the Appendix.

PROOF OF THEOREM 2.2: The theorem is clearly true for $a = \sigma$. So we may assume $0 \leqslant a < \sigma$.

LEMMA: *Given $f \neq 0 \in R_a$, where $a < \sigma$, there is an x_i such that $fx_i \neq 0$ in R.*

PROOF: If $fx_i = 0$ for all i, then f would be in Soc R, but by Corollary A3 of the Appendix Soc R exists only in degree σ. □

Thus, given any $f \neq 0$ in R_a, by repeated application of the lemma we can multiply it successively by the variables x_i's until we land in R_σ, at which point we have a monomial x^K such that $fx^K \neq 0$ is in R_σ. This proves the nondegeneracy of the multiplication map $R_a \times R_{\sigma-a} \to R_\sigma$ on the first factor. The local duality theorem follows by symmetry. □

A counterexample to Macaulay's theorem for a weighted ring

Let x_0 and x_1 have weights 1 and 2 respectively and let $J = (x_0^2, x_1^3)$ in $\mathbb{C}[x_0, x_1]$. Then $R = \mathbb{C}[x_0, x_1]/J$ has the following set of monomials as a basis over $\mathbb{C}$:

weights	0	1	2	3	4	5
basis	1	x_0	x_1	x_0x_1	x_1^2	$x_0x_1^2$.

Consider the multiplication map

$$\mu\colon R_1 \times R_3 \to R_4.$$

Since $x_0 \cdot R_3 = 0$ but $x_0 \neq 0$, μ is not nondegenerate and Macaulay's theorem is false. Note however that local duality holds.

The truth of Macaulay's theorem is closely related to the surjectivity of the multiplication map.

PROPOSITION 2.3: *Let R be a weighted ring for which local duality holds and let σ be the top degree of R. Given nonnegative integers a and b satisfying $a + b \leqslant \sigma$, if*

$$S_b \times S_{\sigma-(a+b)} \to S_{\sigma-a}$$

is surjective, then $R_a \to \mathrm{Hom}(R_b,\ R_{a+b})$ is injective.

PROOF: Suppose $S_b \times S_{\sigma-(\alpha+b)} \to S_{\sigma-a}$ is surjective. From the commutative diagram

$$\begin{array}{ccc} S_b \times S_{\sigma-(a+b)} & \to & S_{\sigma-b} \\ \downarrow & & \downarrow \\ R_b \times R_{\sigma-(a+b)} & \to & R_{\sigma-a} \end{array}$$

we see that

$$R_b \times R_{\sigma-(a+b)} \to R_{\sigma-a} \tag{2.4}$$

is also surjective. Suppose $u \in R_a$ and $u.R_b = 0$. By the surjectivity of (2.4), $u.R_{\sigma-a} = u.R_b.R_{\sigma-(a+b)} = 0$. By local duality $u = 0$ in R_a, so $R_a \to \mathrm{Hom}(R_b,\ R_{a+b})$ is injective. □

For the weighted projective space $P(q_0, \dots, q_{n+1})$ let

$$m = \mathrm{lcm}(q_0, \dots, q_{n+1})$$

and

$$s = \Sigma q_i.$$

If $J = (j_1, \dots, j_\nu)$ is a subset of $\{0, 1, \dots, n+1\}$ we set as in Delorme [4]

$$m(q \mid J) = \mathrm{lcm}(q_{j_1}, \dots, q_{j_\nu})$$

and

$$G = -s + \frac{1}{n+1} \sum_{2 \leqslant \nu \leqslant n+2} \binom{n}{\nu-2}^{-1} \sum_{|J| = \nu} m(q \mid J). \tag{2.5}$$

For the unweighted projective space P^{n+1}, $s = n+2$, $m = 1$, and $G = -1$. A routine computation, making use of the inequality $m(q \mid J) \leqslant m$, simplifies (2.5) to the more palatable estimate:

$$G \leqslant -s + m(n+1). \tag{2.6}$$

THEOREM 2.7: (Delorme [4, prop. 2.2, p. 207]). *Let l be a nonnegative integer $\geqslant G+1$. For any nonnegative integer k every weighted monomial of degree $l + km$ is divisible by a weighted monomial of degree km.*

This, in conjunction with Proposition 2.3, turns out to be precisely what we need for a formulation of the weighted Macaulay's theorem.

THEOREM 2.8: (Weighted Macaulay's theorem). *Let $R = \mathbb{C}[x_0, \ldots, x_{n+1}]/J$ be the weighted ring defined by the ideal J of a regular sequence $f_0, \ldots, f_{n+1}$. Set $d_i = \deg f_i$, $q_i = \text{weight } x_i$, and $\sigma = \Sigma(d_i - q_i)$. The natural map*

$$R_a \to \operatorname{Hom}(R_b, R_{a+b})$$

is injective

(*i*) *if b is a multiple of m and $\sigma - (a+b) \geqslant \max(G+1, 0)$, or*
(*ii*) *if $\sigma - (a+b)$ is a multiple of m and $b \geqslant G+1$.*

PROOF: (i) By Delorme's theorem if $l \geqslant G+1$, then $S_{km} \times S_l \to S_{l+km}$ is surjective. Set $l = \sigma - (a+b)$ and $km = b$. Since $\sigma - (a+b) \geqslant \max(G+1, 0)$, the hypothesis of Delorme's theorem is satisfied. Hence, $S_b \times S_{\sigma-(a+b)} \to S_{\sigma-a}$ is surjective. By Proposition 2.3, the natural map $R_a \to \operatorname{Hom}(R_b, R_{a+b})$ is injective. The proof of (ii) is similar with the role of b and $\sigma - (a+b)$ interchanged. □

COROLLARY 2.9: *Let R be as in Theorem 2.8. Suppose $d_i = d - q_i$ for some d. Let p be an integer between 1 and n inclusive for which $\gcd(m, p)$ divides s. Then there are infinitely many nonnegative integers $k \geqslant ((n+1)p/(n+1-p)) - (s/m)$ such that $d = (km+s)/p$ is a positive integer. For any such d the natural map*

$$R_d \to \operatorname{Hom}\left(R_{(n-p+1)d-s}, R_{(n-p+2)d-s}\right)$$

is injective.

PROOF: Since $d_i = d - q_i$, $\sigma = (n+2)d - 2s$. By Proposition 2.3 the injectivity of the natural map

$$R_d \to \operatorname{Hom}\left(R_{(n-p+1)d-s}, R_{(n-p+2)d-s}\right)$$

follows from the surjectivity of the multiplication map

$$S_{(n-p+1)d-s} \times S_{pd-s} \to S_{(n+1)d-2s}. \tag{2.9.1}$$

Since $\gcd(m, p) \mid s$, the congruence $mk \equiv -s (\bmod\ p)$ has infinitely many solutions for k, which moreover form an arithmetic progression. For such a k, $d = (s + km)/p$ will be an integer. Set $l = (n - p + 1)d - s$ and $km = pd - s$ in Delorme's theorem. By taking k large enough we can ensure that $l \geqslant G + 1$ and hence that (2.9.1) is surjective. In fact, $k \geqslant ((n+1)p/(n+1-p)) - (s/m)$ will do. □

THEOREM 2.10: *Let p be an integer between* 1 *and n inclusive for which $\gcd(m, p)$ divides s. Then there are infinitely many nonegative integers $k \geqslant ((n+1)p/(n+1-p)) - (s/m)$ for which $d = (s + km)/p$ is a positive integer. The local Torelli theorem holds for quasismooth hypersurfaces of degree d in $P(q_0, \dots, q_{n+1})$.*

PROOF: In terms of the polynomial identifications (1.3) and (1.4) the local Torelli theorem is equivalent to the injectivity of $R_d \to \operatorname{Hom}(R_{(n-p+1)d-s}, R_{(n-p+2)d-s})$ for some p in $\{1, \dots, n\}$. So the theorem follows from Corollary 2.9.

Taking $p = 1$ in Theorem 2.10 we obtain the local Torelli theorem for quasismooth hypersurfaces of degree $s + km$ for any integer $k \geqslant 2$.

Appendix: The socle of a weighted algebra

I would like to thank Craig Huneke for many helpful discussions on the socle. According to him, among commutative algebraists the theory of the socle is considered part of the folklore. My reasons for including this appendix are twofolds. First, the socle does not appear to be widely known among algebraic geometers. Second and more importantly, there is no specific reference in the literature to the result on the socle of a weighted algebra which we need. If any lesson is to be drawn from the failure of Macaulay's theorem for a weighted ring, it is that one cannot blithely extrapolate from the unweighted to the weighted case. For these reasons it is desirable to have a more or less self-contained exposition of the socle of a weighted algebra, assuming only familiarity with the Koszul complex and the Tor functor as in Lang [12, pp. 593–604] and Hilton and Stammbach [11, Ch. IV]. The proofs below follow essentially Fröberg and Laksov [7, pp. 130–131], but are recast in the context of a weighted algebra.

Let k be a field and $S = k[x_1, \dots, x_r]$ a weighted polynomial ring over k, with q_i = weight of x_i. Given a sequence $f_1, \dots, f_r$ of polynomials in S, the Koszul complex $K.(f)$ is the sequence of S-modules

$$0 \to K_r(f) \to \cdots \to K_1(f) \to S \to S/(f_1, \ldots, f_r) \to 0,$$

where

$$K_p(f) = \text{free module with basis } \{e_{i_1} \ldots e_{i_p}\},\ i_1 < \ldots < i_p$$

and

$$d(e_{i_1} \ldots e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} e_{i_1} \ldots \hat{e}_{i_j} \ldots e_{i_p}.$$

If $f_1, \ldots, f_r$ is a regular sequence, then the Koszul complex is exact. By assigning a degree to each symbol e_i, namely deg $e_i = d_i =$ deg f_i, we can make each $K_p(f)$ into a graded S-module and each differential d into a degree 0 homomorphism. For a graded module A, define a new graded module $A[n]$ by $A[n]_m = A_{n+m}$. In this notation there is then a degree-preserving isomorphism of graded S-modules

$$K_p(f) \simeq \oplus S\left[-d_{i_1} - \ldots - d_{i_p}\right],$$

where the sum ranges over all $i_1 < \ldots < i_p$.

The *socle* of a graded k-algebra A is defined to be

$$\operatorname{Soc} A = \left\{ h \in A \mid hg = 0 \quad \text{for all } g \in \bigoplus_{i=1}^{\infty} A_i \right\}.$$

PROPOSITION A1: *Let I be an ideal in $S = k[x_1, \ldots, x_r]$ and let $R = S/I$. Then* $\operatorname{Tor}_r^S(R, k) \simeq (\operatorname{Soc} R)[-\Sigma q_i]$.

PROOF: Since $k \simeq S/(x_1, \ldots, x_r)$, the Koszul complex $K.(x)$:

$$0 \to K_r(x) \to \ldots \to K_1(x) \to S \to S/(x_1, \ldots, x_r) \to 0$$

if a free resolution of k and can be used to compute $\operatorname{Tor}_r^S(R, k)$. By definition,

$$\begin{aligned}
\operatorname{Tor}_r^S(R, k) &= Z_r(K.(x) \otimes R) \\
&= \Big\{ he_1 \ldots e_r \mid h \in R,\ d(he_1 \ldots e_r) \\
&\qquad = \Sigma(-1)^{j-1} hx_j e_1 \ldots \hat{e}_j \ldots e_r = 0 \Big\} \\
&\simeq \{ h \in R \mid hx_j = 0 \quad \text{for all } j \} \\
&= \operatorname{Soc} R.
\end{aligned}$$

Since the isomorphism above decreases the degree by Σq_i, there is a degree-preserving isomorphism $\mathrm{Tor}_r^S(R, k) \simeq (\mathrm{Soc}\, R)[-\Sigma q_i]$ of graded S-modules. □

Let I be an ideal in S. By the Hilbert syzygy theorem the graded S-module S/I has a resolution of the form

$$0 \to \bigoplus_{j=1}^{b_r} S[-n_{r,j}] \to \dots \to \bigoplus_{j=1}^{b_1} S[-n_{1,j}] \to S \to S/I \to 0, \qquad (*)$$

where each differential has degree 0 and is given by multiplication by a polynomial of positive degree on each nonzero component.

Proposition A2: *Let I be an ideal in $S = k[x_1, \dots, x_r]$, $R = S/I$, and $s = \Sigma q_i$. Then there is a degree-preserving isomorphism of graded S-modules*

$$\mathrm{Soc}\, R \simeq \bigoplus_{j=1}^{b_r} k[s - n_{r,j}].$$

(The field k can be viewed as an S-module via the isomorphism $k \simeq S/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal $(x_1, \dots, x_r)$.)

Proof: If we tensor the resolution $(*)$ by $k \simeq S/\mathfrak{m}$, all the differentials become zero. So

$$\mathrm{Tor}_r^S(R, k) = \bigoplus_{j=1}^{b_r} k[-n_{r,j}].$$

By Proposition A1,

$$\mathrm{Soc}\, R = \bigoplus_{j=1}^{b_r} k[s - r_{r,j}]. \quad \square$$

Corollary A3: *Suppose $f_1, \dots, f_r$ is a regular sequence of weighted homogenous polynomials of degrees $d_1, \dots, d_r$ in S and $R = S/(f_1, \dots, f_r)$. Then* Soc $R \simeq R_\sigma$, *where $\sigma = \Sigma(d_i - q_i)$.*

Proof: In the Koszul resolution $K.(f)$ of R, $K_r(f) \simeq S[-\Sigma d_i]$. By Proposition A2, there is a degree-preserving isomorphism $\mathrm{Soc}(R) \simeq k[-\Sigma(d_i - q_i)]$. This should be interpreted as saying that there is an isomorphism $\mathrm{Soc}\, R \xrightarrow{\sim} k$ which lowers the degree by $\sigma = \Sigma(d_i - q_i)$. Hence, Soc R is 1-dimensional and is generated by elements of degree σ.
□

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