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INDECOMPOSABLE PROJECTIVE MODULES ON AFFINE DOMAINS

V. Srinivas

Let A be an affine domain i.e. an integral domain of finite type over an algebraically closed field k. If A has Krull dimension d, then by the stability theorem of Bass [Ba], any indecomposable projective A-module has rank at most d. In this paper, we consider the conditions under which A has indecomposable projective A-modules of rank d. Our main result is:

THEOREM: *Let k be a universal domain, and assume that* $d = \dim A \leqslant 3$. *Assume that* Spec A *has at worst isolated normal singularities; further if* $\dim A = 3$, *then* char, $k \neq 2, 3, 5$. *Then there exist indecomposable projective A-modules of rank d if and only if* $F_0K_0(A) \neq 0$.

Here $F_0K_0(A)$ is the subgroup of the Grothendieck group of projective A-modules generated by the classes of residue fields of smooth points; it is isomorphic (upto 2-torsion) to the Chow group of zero cycles of Spec A (see §1 for the definition). The restriction to char, $k \neq 2, 3, 5$ if $d = 3$ is because we need to use the theorem of resolution of singularities for 3-folds, which is proved under the above restriction in [A]. In a footnote in their paper [MK], M.P. Murthy and Mohan Kumar state that Abhyankar can now prove resolution for 3-folds in all characteristics; assuming this we may drop the hypothesis on the characteristic.

The hypothesis on the singularities is needed for technical reasons in the proof, but may be unnecessary: I do not know any counter example for non-normal A or for normal A of dimension 3 with non-isolated singularities. However, it seems plausible that k must be a 'big' field in order that $F_0K_0(A)$ be non-zero; I do not know an example of an affine domain of dimension $\geqslant 2$ over $\overline{\mathbb{Q}}$ with $F_0K_0(A) \neq 0$. Many examples exist of such affine domains over the algebraic closure of $\mathbb{Q}(x, y)$ (x, y indeterminates) with $F_0K_0(A) \neq 0$. But our methods do not seem to be directly applicable to such rings *. We discuss examples of rings with $F_0K_0(A) \neq 0$ in §4 and §5.

The proof is based on the following idea. Suppose P is projective of rank $d = \dim A \geqslant 2$, and for a suitable notion of the Chow ring together

* One can combine our result with the method of [BS] to get such results.

with a theory of Chern classes, we have $C_1(P) = \ldots C_{d-1}(P) = 0$. Then if $P \cong Q \oplus L$ where $L \in \operatorname{Pic} A$, rank $Q = d-1$, we have an identity in the Chow ring

$$(1 + C_1(L)) \cdot (1 + C_1(Q) + \cdots + C_{d-1}(Q)) = 1 + C_d(P)$$

This forces $C_i(Q) = (-1)^i C_1(L)^i$, and $C_d(P) = (-1)^{d-1} C_1(L)^d$. Thus our Theorem will follow once we show that if $F_0 K_0(A) \neq 0$, then there exists a projective module P with $C_i(P) = 0$ for $i < d$, and such that $C_d(P) \neq 0$ is *not* rationally equivalent to a cycle of the form $(-1)^{d-1} C_1(L)^d$. The idea of the proof is to show that if $F_0 K_0(A) \neq 0$, then the Chow group of 0-cycles $CH^d(A)$ is 'infinite dimensional' in a suitable sense, while the classes of cycles of the form $(-1)^{d-1} C_1(L)^d$ lie in a countable union of 'algebraic sets' in $CH^d(A)$, each component of which has dimension bounded by a number depending only A (this relies on the theory of the Picard variety). This is done using ideas of Roitman [R] which have to modified somewhat to deal with affine, possibly singular varieties (Roitman basically deals with smooth projective varieties). Roitman's ideas directly apply when A is smooth of dimension 2, but our modifications seem to be needed even for smooth A if dim $A = 3$.

Finally, we do not know if the theorem ought to be true if dim $A = 4$, though we know no counter examples. If we try an analogous argument, and let P be projective of rank 4 with $C_1(P) = C_2(P) = C_3(P) = 0$, then suppose that $P = P_1 \oplus P_2$ with rank $P_i = 2$. A Chern class computation yields $C_1(P_2) = -C_1(P_1)$, $C_2(P_2) = C_1(P_1)^2 - C_2(P_1)$, and $C_4(P) = \frac{1}{2} C_1(P_1)^4 - C_2(P_1)^2$. The 0-cycles of the form $C_1(P_1)^4$ range over a countable union of algebraic varieties of bounded dimension. However, $CH^2(A)$ could be infinite dimensional, so we seem to have no control on the set of classes of the form $C_2(P_1)^2$ in $CH^4(A)$.

This problem was suggested to the author by M.P. Murthy, in the case $d = 2$. I wish to thank N. Mohan Kumar and M.V. Nori for stimulating discussions; in particular, Mohan Kumar suggested the trick of looking at projectives with $C_i = 0$ for $i < d$, which reduces the problem to line bundles even if $d = 3$.

§1. Equivalence relations on zero cycles (after Roitman)

Let X be a normal projective variety over an algebraically closed field k, and let $U \subset X$ be the set of smooth points. The group of zero cycles on X, denoted by $Z_0(X)$, is defined to be the free abelian group on the points of U. Let $R(X) \subset Z_0(X)$ be the subgroup of $Z_0(X)$ generated by cycles of the form $(f)_C$ where $C \subset X$ is a curve with $C \cap (X - U) = \emptyset$ and f is a non-zero rational function on C; $(f)_C$ denotes the divisor of f on C (note that $C \cap (X - U) = \emptyset$ just means $C \subset U$; we write it as above to emphasise that C is a *closed* subset of X). The Chow group of

0-cycles is defined to be the quotient $Z_0(X)/R(X)$; we denote it below by $A_0(X)$ (rather than $CH^d(X)$, where $d = \dim X$). Similarly if A is a normal affine domain, let $V \subset \operatorname{Spec} A$ be the set of smooth points, $Z_0(A)$ the free abelian group on points of V, and $A_0(A)$ the quotient of $Z_0(A)$ modulo the group generated by $(f)_C$ where $C \subset \operatorname{Spec} A$ is a curve, $C \cap ((\operatorname{Spec} A) - V) = \emptyset$, and f is a non-zero rational function on C.

If X, U are as above, the nth symmetric product $S^n(U)$ parametrizes effective zero cycles of degree n on X i.e. cycles $\sum n_i(P_i)$ with $n_i > 0$, $\sum n_i = n$, $P_i \in U$. There are natural maps

$$\Psi_{m,n}: S^m(U) \times S^n(U) \to Z_0(X)$$

$$(A, B) \to [A] - [B]$$

where $[A]$ denotes the zero cycle corresponding to $A \in S^m(U)$. In [S1], Lemma (1.1) we showed that if

$$\gamma_{m,n}: S^m(U) \times S^n(U) \to A_0(X)$$

is induced by $\Psi_{m,n}$, then $\gamma_{m,n}^{-1}(0)$ is a countable union of locally closed sets in $S^m(U) \times S^n(U)$, and used this to study 'infinite dimensionality' of $A_0(X)$. Here we study infinite dimensionality for cycle groups associated to more general equivalence relations.

Let $W \subset Z_0(X)$ be a subgroup. Then W defines an equivalence relation on 0-cycles, namely $[A] \sim [B]$ iff $[A] - [B] \in W$. Let $A_0^W(X) = Z_0(X)/W$ be the corresponding quotient. We define W to be *admissible* if (i) $W \supset R(X)$ (ii) for any m, n if $\varphi_{m,n}: S^m(U) \times S^n(U) \to A_0^W(X)$ is induced by $\Psi_{m,n}$, then $\varphi_{m,n}^{-1}(0)$ is a countable union of locally closed sets.*

We recall some terminology of Roitman. For a variety X over a universal domain, a c-closed set in X is a countable union of closed subsets of X. For any c-closed set $F = \bigcup F_i$ in X, we define $\dim F = \sup (\dim F_i)$. This concept is well defined, and depends only on the set F and not on the specific decomposition into sets F_i (since an irreducible variety over a universal domain is not the union of a countable set of proper subvarieties). Similarly, a c-locally closed set in X is a countable union of locally closed subsets of X. For any c-locally closed set $G = \bigcup G_i$ with G_i locally closed, we define the c-closure of G to be $\bigcup \overline{G_i}$. This is also characterised as the smallest c-closed subset of X containing G, since if $F = \bigcup F_j \supset G$ where F_j are closed, then each G_i is contained in a *finite* union of the F_j, which is a closed set (and so contains $\overline{G_i}$). In particular the c-closure of G does not depend on the decomposition of G

* By the methods of [LW] one can prove that $\varphi_{m,n}^{-1}(0)$ is a countable union of closed sets if W is admissible, but we do not need this.

into locally closed subsets. For any c-locally closed set G, we define $\dim G$ to be the dimension of its c-closure. Finally, a c-open set in X is defined to be the complement of a c-closed set.

We have the following lemma, essentially due to Roitman.

LEMMA (1.1): *Let X be a variety over a universal domain, $\Gamma \subset X \times X$ the graph of an equivalence relation on X. Suppose that Γ is c-locally closed in $X \times X$. Then there is a c-open set $V \subset X$ and an integer $d \geqslant 0$ such that for any $x \in V$, the set*

$$\Gamma_x = \{ y \in X \mid (x, y) \in \Gamma \}$$

is a c-locally closed subset of X of dimension d, such that all irreducible components of the c-closure of Γ_x have dimension d.

(We note that every c-closed set in X has a unique irredundant decomposition into irreducible closed sets, so it makes sense to talk of the components of a c-closed set).

PROOF: Let $\bar{\Gamma}$ be the c-closure of Γ. Then $\bar{\Gamma}$ is also the graph of an equivalence relation on X. One sees easily that there exists a c-closed set $F \subset X$ such that for $x \in X - F$, $(\bar{\Gamma})_x = \{ y \in X \mid (x, y) \in \bar{\Gamma} \}$ is just the c-closure of Γ_x. Hence we reduce to the case when Γ itself is c-closed, and so Γ_x is also c-closed. Let $\Gamma = \cup \Gamma_i$ be an irredundant decomposition into irreducible closed subsets of $X \times X$.

We claim first that there exists a unique component of Γ containing the diagonal in $X \times X$. Indeed, there is a proper c-closed subset $F \subset X$ such that for all $x \in X - F$, either

(i) $(\{x\} \times \Gamma_x) \cap \Gamma_i$ is empty, or

(ii) if $\Gamma_{i,x} = \{ y \in X \mid (x, y) \in \Gamma_i \}$, then $\Gamma_{i,x}$ is a finite disjoint union of irreducible subvarieties of dimension equal to $(\dim \Gamma_i - \dim X)$; further $\Gamma_x = \bigcup_i \Gamma_{i,x}$ is an irredundant union, where i runs over indices such that $\Gamma_{i,x} \neq \emptyset$ (by irredundancy we mean that no component of $\Gamma_{i,x}$ is contained in $\Gamma_{j,x}$ for $j \neq i$).

Now suppose $y \in \Gamma_{i,x} - \bigcup_{j \neq i} \Gamma_{j,x} - F$. Then $(x, y) \in \Gamma_i - \bigcup_{j \neq i} \Gamma_j$. For any $y \in \Gamma_x$, we have $\Gamma_y = \Gamma_x$ as Γ is an equivalence relation. If $\Gamma_y = \bigcup_{i'} \Gamma_{i',y}$ is the above decomposition (which exists because $y \notin F$), then the component of $\Gamma_{i,x}$ containing y lies in a unique $\Gamma_{i',y}$, so that $(y, y) \in \Gamma_{i'} - \bigcup_{j \neq i'} \Gamma_j$. Thus $\Gamma_{i'}$ is the unique component of Γ containing the diagonal. Further, $\Gamma_{i,x}$ and $\Gamma_{i',y}$ are equidimensional schemes with a common irreducible component (namely, the component through y) and so

$$\dim \Gamma_i = \dim \Gamma_{i,x} + \dim X = \dim \Gamma_{i',y} + \dim X = \dim \Gamma_i'.$$

Since i was arbitrary, dim Γ_i is independent of i, at least for all components Γ_i which dominate X under the first projection. Thus dim $\Gamma_{i,x} =$ dim $\Gamma_{j,x}$ for all such Γ_i, Γ_j i.e. Γ_x is a union of irreducible varieties each of which has dimension $d = (\dim \Gamma_{i'} - \dim X)$. This proves the lemma.

Now let W be an admissible equivalence relation on 0-cycles on X, where X is a normal projective variety over a universal domain. Then via $\varphi_{m,n}$, W induces a c-locally closed equivalence relation on $S^m(U) \times S^n(U)$, where $U = X - X_{\text{sing}}$. Let $w_{m,n}$ be the number d given by Lemma (1.1) in this context, and let

$$d_{m,n} = \dim S^m(U) \times S^n(U) - w_{m,n} = (m+n) \dim X - w_{m,n}.$$

We define $S^0(U)$ to be a point, corresponding to the zero cycle 0, and let $d_n = d_{n,0}$, $w_n = w_{n,0}$ etc.

PROPOSITION (1.2): *There are integers d_W, j_W and $n_0 > 0$ depending only on X and W such that $0 \leqslant d_W(X) \leqslant \dim X$, and for $m + n \geqslant n_0$ we have*

$$d_{m,n} = (m+n) d_W + j_W.$$

PROOF: We first show that $w_{m,n} = w_{m+n}$. Indeed, consider

$$\Gamma^{m,n} = \{(A, B, C, D) \in S^m(U) \times S^n(U) \times S^m(U) \times S^n(U) \\ \mid \varphi_{m,n}((A, B)) = \varphi_{m,n}((C, D))\}.$$

Clearly $\varphi_{m,n}((A, B)) = \varphi_{m,n}((C, D))$ iff $\varphi_{m+n}(A + D) = \varphi_{m+n}(B + C)(\varphi_{m+n} = \varphi_{m+n,0})$. Now $\Gamma^{m,n}$ is the graph of the equivalence relation determined by $\varphi_{m,n}$, so that if $\Gamma_0^{m,n}$ is the unique component of the c-closure of $\Gamma^{m,n}$ containing the diagonal, then

$$\dim \Gamma_0^{m,n} = \dim S^m(U) \times S^n(U) + w_{m,n}.$$

Let $\Gamma^{m+n} \subset S^{m+n}(U) \times S^{m+n}(U)$ be the graph of the equivalence relation determined by φ_{m+n}, and let Γ_0^{m+n} be the unique component of the c-closure containing the diagonal; then

$$\dim \Gamma_0^{m+n} = \dim S^{m+n}(U) + w_{m+n}.$$

Let $\Psi : S^m(U) \times S^n(U) \times S^m(U) \times S^n(U) \to S^{m+n}(U) \times S^{m+n}(U)$ be given by $(A, B, C, D) \to (A + D, B + C)$. Then Ψ is finite, and $\Psi^{-1}(\Gamma^{m+n}) = \Gamma^{m,n}$. Hence $\dim \Gamma_0^{m,n} = \dim \Gamma_0^{m+n}$ (since Ψ maps the diagonal to the diagonal) i.e. $w_{m,n} = w_{m+n}$, and $d_{m,n} = d_{m+n}$.

Let $V^n \subset U \times S^n(U)$ be the set

$$V^n = \{(x, A) \mid \exists B \in S^{n-1}(U) \text{ with } \varphi_n(A) = \varphi_n(B + x)\}.$$

Similarly for $A \in S^n(U)$ let V_A^n be (the image in U of) the fiber over A of the projection $V^n \to S^n(U)$. For any n, V^n is a c-locally closed set in $U \times S^n(U)$, and for any A, V_A^n is a c-locally closed set in U. Thus for some non-empty c-open set $U_1 \subset S^n(U)$ and for all $A \in U_1$, we have

$$\dim V_A^n = \min_{B \in S^n(U)} \dim V_B^n.$$

Let $\delta_n = \dim V_A^n$ for any $A \in U_1$. For any $A \in S^n(U)$ and any $x \in U$ we have $V_A^n \subset V_{A+x}^{n+1}$, so that $\delta_n \leqslant \delta_{n+1} \leqslant \dim X$. Thus $\delta_n = \delta_{n+1}$ for all $n \geqslant n_0$. Let δ be this common limiting value, and let $d_W = \dim X - \delta$. Consider the map (addition)

$$\pi : U \times S^n(U) \to S^{n+1}(U).$$

If $\Gamma_A^{n+1} = \{B \in S^{n+1}(U) \mid \varphi_n(A) = \varphi_n(B)\}$ for any $A \in S^{n+1}(U)$, then $p_1(\pi^{-1}(\Gamma_A^{n+1})) = V_A^{n+1}$, where p_1 is the projection $U \times S^n(U) \to U$. For $x \in V_A^{n+1}$, the fiber of $(p_1)|_{\pi^{-1}(\Gamma_A^{n+1})}$ is just Γ_B^n, where B is the effective 0-cycle A-$[x]$. Since $w_n = \dim \Gamma_B^n$ for B in some c-open subset of U, and similarly $w_{n+1} = \dim \Gamma_A^{n+1}$ for general A, we have $w_{n+1} = w_n + \delta$ i.e. $d_{n+1} = d_n + d_W$ for $n \geqslant n_0$. This proves the proposition.

REMARK: This proof only used the property (ii) in the definition of an admissible equivalence relation.

LEMMA (1.3): *Either $d_W = 0$ or $d_W \geqslant 2$. If $d_W = 0$, then for some n image$(\varphi_{n,n})$ is a subgroup of $A_0^W(X)$ with cyclic quotient.*

PROOF: Suppose $d_W \leqslant 1$. Then for $n \geqslant n_0$, we have $\dim V_A^{n+1} \geqslant \dim X - 1$ for any $A \in S^{n+1}(U)$, by definition of $d_X = \dim X - \delta$. Thus there exists a smooth curve $C \subset X$ with $C \cap (X - U) = \emptyset$ and $C \cap V_A^{n+1} \neq \emptyset$ for all $A \in S^{n+1}(U)$ (in fact any general smooth complete intersection curve C will do). Then for any $A \in S^m(U)$ with $m > n_0$ we can find $A_1 \in S^{n_0}(U)$ and $A_2 \in S^{m-n_0}(C)$ such that

$$\varphi_m(A) = \varphi_{n_0}(A_1) + \varphi_{m-n_0}(A_2).$$

Since W is an admissible equivalence relation, we have diagrams for all n

$$\begin{array}{ccc} S^n(C) & \to & S^n(U) \\ \gamma_n \downarrow & & \downarrow \varphi_n \\ \operatorname{Pic} C & \to & A_0^W(X) \end{array}$$

If we choose a base point $P_0 \in C$, and $g = $ genus (C), then by the

Riemann-Roch theorem, for $n > g$ and any $B \in S^n(C)$, there exists $B_1 \in S^g(C)$ such that

$$\gamma_n(B) = \gamma_g(B_1) + \gamma_{n-g}((n-g)P_0) \text{ in Pic } C.$$

Thus for any $m > n_0 + g$, and any $A \in S^m(U)$, there exists $B \in S^{n_0+g}(U)$ such that

$$\varphi_m(A) = \varphi_{n_0+g}(B) + (m - n_0 - g)\varphi_1(P_0).$$

Thus image $\varphi_{m,m}$ = image φ_{n_0+g,n_0+g} for $m > n_0 + g$ i.e. $d_X = 0$ and image $\varphi_{m,m}$ is a subgroup of $A_0^W(X)$ whose quotient is the cyclic group generated by the image of $\varphi_1(P_0)$.

We recall another definition. For any variety Z, a set theoretic map $Z \to A_0^W(X)$ is defined to be a *regular map* if there exists a diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & S^m(U) \times S^n(U) \\ f\downarrow & & \downarrow \varphi_{m,n} \\ Z & \to & A_0^W(X) \end{array}$$

where f, g are morphisms and f is surjective. Since W is admissible, if $C \subset X$ is a smooth curve with $C \cap (X - U) = \emptyset$, the natural homomorphism Pic $C \to A_0^W(X)$ is a regular map. Given any finite set of points in U, there is a smooth curve C as above through these points. Thus any given zero cycle is in the image of such a regular homomorphism. Since the jacobian $J(C)$ is divisible, if $A_0^W(X)' \subset A_0^W(X)$ is the subgroup generated by cycles of degree 0, then $A_0^W(X)'$ is divisible. Clearly $\varphi_{n,n}$ factors through $A_0^W(X)'$.

PROPOSITION (1.4): *Let $d_W = 0$. Then there is a surjective regular homomorphism $f: A \to A_0^W(X)'$, where A is an abelian variety which is a quotient of $Alb(X)$, and f has countable kernel.*

PROOF: We first remark that there is a surjective regular homomorphism $f: A \to A_0^W(X)$ from an abelian variety A, such that ker f is countable. This is proved for $W = R(X)$ (i.e. rational equivalence) in [S1], Lemma (1.3). But the only property of $R(X)$ used in the proof is that it is admissible in our sense.

Given this homomorphism, choose a base point $P \in U$, and let $\Gamma \subset U \times A$ be the set

$$\Gamma = \{(x, a) \mid \varphi_{1,1}((x, P)) = f(a)\}$$

Then Γ is c-locally closed, and since f is surjective, Γ surjects onto U

under the projection. Hence some component Γ_0 of the c-closure of Γ dominates U. Let $\Gamma_1 = \Gamma_0 \cap \Gamma$; then $\Gamma_1 \to U$ has countable, and hence finite, fibers (since f has countable fibers). There is an open set $U_0 \subset U$ such that if $\Gamma_2 = \Gamma_1 \times_U U_0$, then $\pi : \Gamma_2 \to U_0$ is a finite flat morphism of varieties. Let n be the separable degree and p^m the inseparable degree of this morphism. If $x \in U_0$ is general, then $\pi^{-1}(x) = \{(x, a_1), \dots, (x, a_n)\}$ for n distinct points $a_1, \dots, a_n \in A$. Then $x \to p^m(a_1 + \cdots + a_n)$ gives a well defined morphism $\Psi : U_0 \to A$ such that $f \circ \Psi : U_0 \to A_0^W(X)'$ is the natural map $\varphi_{1,1}$ multiplied by $n.p^m$. Hence $f \circ \Psi(U_0)$ generates $A_0^W(X)'$ (since $A_0^W(X)'$ is divisible, and $\varphi_{1,1}(U_0 \times \{P\})$ generates $A_0^W(X)'$). Since f has countable fibers, $\Psi(U_0)$ generates A. Regarding Ψ as a rational map $X \to A$, Ψ factors through $\mathrm{Alb}(X)$ by the universal property of the Albanese variety. Since the image of $\mathrm{Alb}(X) \to A$ contains $\Psi(U)$, $\mathrm{Alb}(X) \to A$ is surjective.

We apply these results to the following situation. Let X be a normal projective variety over a universal domain, and let $V \subset X$ be an open subset containing X_{sing}. Then we have an exact sequence

$$Z_0(X - V) \to A_0(X) \to A_0(V) \to 0$$

where $A_0(V)$ is the free abelian group on points of $V - V_{\mathrm{sing}}$ modulo the subgroup generated by cycles $(f)_C$ where $C \subset V$ is a (closed) curve, $C \cap V_{\mathrm{sing}} = \emptyset$, and f is a non-zero rational function on C. The above presentation for $A_0(V)$ follows because $V_{\mathrm{sing}} = X_{\mathrm{sing}}$. From the presentation, if we set $A_0^W(X) = A_0(V)$, then W is an admissible equivalence relation.

LEMMA (1.5): *Suppose* $\dim X \geqslant 2$, $d_W = 0$ *and* $X - V$ *generates Alb* X. *Then* $A_0^W(X) = 0$.

PROOF; Clearly $A_0^W(X) = A_0^W(X)'$, so it suffices to show that if $f : A \to A_0^W(X)'$ is a surjective regular homomorphism with countable kernel, then $A = 0$. As in the proof of Prop. (1.4), we claim there is a diagram

$$\begin{array}{ccc} \mathrm{Alb}\, X & \xrightarrow{h} & A \\ {\scriptstyle g}\uparrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[n.p^m\ \varphi_{1,1}]{} & A_0^W(X)' \end{array}$$

A priori we only have such a diagram with U_0 instead of U, but the rational map $X \to \mathrm{Alb}\ X$ with $P \to 0$ is a morphism restricted to U. Thus we would have two regular maps $U \to A_0^W(X)'$, namely $n.p^m\ \varphi_{1,1}$ and $f \circ h \circ g$, which agree on U_0. We claim that they must agree everywhere on U. Indeed, if $C \subset U$ is a complete smooth curve with $C \cap U_0 \neq \emptyset$, then we have 2 regular maps $C \to A_0^W(X)'$ namely $np^m\varphi_{1,1}|_C$ and $f \circ h \circ g|_C$. Both induce regular homomorphisms Pic $C \to A_0^W(X)'$, the former because W is admissible, and the latter because $\mathrm{Pic}^0 C = \mathrm{Alb}\ C$, so that $g|_C: C \to A$ induces a homomorphism Pic $C \to A$. These two maps Pic $C \to A_0^W(X)'$ are equal, because Pic C is generated by the classes of points in $C \cap U_0$. Thus the two original maps on U agree on C. But such a curve C can be found through any point of U, since X is normal. *

Now $X - V \subset U$, and $f \circ h \circ g(X-V) = 0$ in $A_0^W(X)$ (by definition of W, and the equality of the 2 maps). Thus $h \circ g(X-V) \subset (\ker f)$ which is a countable subgroup of A i.e. $h \circ g(X-V)$ is finite. But $g(X-V) \subset \mathrm{Alb}\ X$ generates Alb X, by choice of V, and Alb $X \twoheadrightarrow A$. Hence $A = 0$ i.e. $A_0^W(X)^- = 0$. This proves the lemma.

§2. Families of 0-cycles associated to Picard families

Let X be a normal projective variety of dimension d over an algebraically closed field k, and let T be a normal variety. Assume given $\mathscr{L} \in \mathrm{Pic}(X \times T)$, and for each $t \in T$, consider the rational equivalence class of the zero cycle $(-1)^{d-1}C_t(\mathscr{L}_t)^d \in A_0(X)$, where $\mathscr{L}_t = \mathscr{L}|_{X \times \{t\}}$. Here if X is smooth, the intersection theory is the usual one (see Fulton [F]) while if X is singular, we use the theory of Levine [L1] (we discuss this theory briefly in §3). In this section, the only property of the intersection theory needed is that if $D_1, \dots, D_d$ are effective Cartier divisors on X meeting properly, and $\cap\,\mathrm{supp}\ D_i \subset U$, then $D_1 \cap \dots \cap D_d$ (as schemes) represents the rational equivalence class of $[D_1][D_2]\dots[D_d]$ in $A_0(X)$. Returning to our situation, we have a map of sets $\varphi: T \to A_0(X)$, given by $t \to (-1)^{d-1}C_1(\mathscr{L}_t)^d$.

Lemma (2.1): *Under the above conditions, $\varphi: T \to A_0(X)$ is a regular map in the following sense: there exist a finite open cover $\{U_i\}$ of T, positive integers m_i, n_i and morphisms $f_i: U_i \to S^{m_i}(U) \times S^{n_i}(U)$ giving commutative diagrams*

$$\begin{tikzcd} & S^{m_i}(U) \times S^{n_i}(U) \arrow[dr, "\gamma_{m_i,n_i}"] & \\ U_i \arrow[ur, "f_i"] \arrow[rr, "\varphi|_{U_i}"'] & & A_0(X) \end{tikzcd}$$

(where γ_{m_i,n_i} are the natural maps).

* We could have used the c-closedness of W-equivalence to directly prove the equality of the 2 maps; but the proof of c-closedness is more subtle.

PROOF: Since T is quasi-compact, we need only prove the following: for each $t \in T$ there exists a neighbourhood U_t of t, positive integers m, n and a morphism $f_t : U_t \to S^m(U) \times S^n(U)$ such that $\varphi|_{U_t} = \gamma_{m,n} \circ f_t$. Let $\mathscr{H} \in \operatorname{Pic} X$ be ample. Then $p_1^*\mathscr{H} \in \operatorname{Pic}(X \times T)$ is relatively ample for $p_2 : X \times T \to T$. Hence for large n, we may assume that $\mathscr{H}^n$ and $\mathscr{L}_t \otimes \mathscr{H}^n$ are very ample, $H^i(X, \mathscr{H}^n) = 0$ for $i > 0$, and $(R^i p_2)_*(\mathscr{L} \otimes p_1^*\mathscr{H}^n) = 0$ for $i > 0$. By the base change theorem (see [M1], pg. 46) if $k(t)$ is the residue field of $\mathscr{O}_{t,T}$ then

$$(p_2)_*(\mathscr{L} \otimes p_1^*\mathscr{H}^n)_t \otimes k(t) \cong H^0(X, \mathscr{L}_t \otimes \mathscr{H}^n)$$

where

$$(p_2)_*(\mathscr{L} \otimes p_1^*\mathscr{H}^n)_t = H^0(X \times \operatorname{Spec} \mathscr{O}_{t,T}, j^*(\mathscr{L} \otimes p_1^*\mathscr{H}^n))$$

and $j : X \times \operatorname{Spec} \mathscr{O}_{t,T} \to X \times T$ is the natural map. Fix d effective divisors $H_1, \dots, H_d$ in the linear system $|\mathscr{H}^n|$ on X such that $H_1 \cap \dots \cap H_d$ is a finite set of points contained in U; in particular all the partial intersections are proper i.e. $\dim H_{i_1} \cap \dots \cap H_{i_k} = d - k$ for any k distinct indices $i_1, \dots, i_k$. Next, choose d divisors $D_1, \dots, D_d$ in $|\mathscr{L}_t \otimes \mathscr{H}^n|$ on X such that all the intersections $D_{i_1} \cap \dots \cap D_{i_k} \cap H_{j_1} \cap \dots H_{j_l}$ are proper, and all zero dimensional intersection cycles are supported on U (we can choose the D_i inductively, using the remark that if $X \subset \mathbb{P}^N$, $Y_1, \dots, Y_m \subset X$ subvarieties: then the general hyperplane section of X intersects all the Y_i properly). Now $\mathscr{O}_X(D_i - H_i) \cong \mathscr{L}_t$, so the intersection class $(-1)^{d-1} C_1(\mathscr{L}_t)^d$ is just the class associated to the zero cycle

$$(-1)^{d-1}(D_1 - H_1) \cdot (D_2 - H_2) \cdots\cdots (D_d - H_d) \in Z_0(X);$$

the cycle is well defined and supported on U by choice. Let $s_1, \dots, s_d \in H^0(X, \mathscr{L}_t \otimes \mathscr{H}^n)$ be sections corresponding to the divisors D_i. From the consequence of the base change theorem noted above, we can find a neighbourhood V of $t \in T$ and sections $\tilde{s}_i \in H^0(X \times V, \mathscr{L} \otimes p_1^*\mathscr{H}^n|_{X \times V})$ such that $\tilde{s}_i$ restricts to s_i on $X \times \{t\}$. If $\tilde{D}_i$ is the divisor of $\tilde{s}_i$, and $\tilde{H}_i = H_i \times V$, then all the intersections $\tilde{D}_{i_1} \cap \dots \cap \tilde{D}_{i_k} \cap \tilde{H}_{j_1} \cap \dots \cap \tilde{H}_{j_l}$ are proper, at least on some smaller open set of the form $X \times W$, $t \in W \subset V$ (since the dimension of the fibers of a surjective proper morphism of varieties is an upper semi-continuous function on the base [H1] II Ex. 3.22). By shrinking W further we can ensure that for $k + l = d$, all intersections

$$(\tilde{D}_{i_1} \cap \dots \cap \tilde{D}_{i_k} \cap \tilde{H}_{j_1} \cap \dots \cap \tilde{H}_{j_l}) \cap (X \times W) \subset U \times W.$$

If $\Gamma = \Gamma(i_1, \dots, i_k;\ j_1, \dots, j_l) = (\tilde{D}_{i_1} \cap \dots \cap D_{i_k} \cap \tilde{H}_{j_1} \cap \dots \cap \tilde{H}_{j_l}) \cap (X \times W)$, where $k + l = d$, then $\Gamma \to W$ is flat and proper of relative dimension 0 (flatness is immediate from the Koszul resolution for the ideal sheaf of $\Gamma \subset X \times W$; note that $\Gamma \subset U \times W$ where U is smooth). Since W is normal, if $\Psi : \Gamma \to W$ has degree q, then we have an induced morphism $W \to S^q(U)$ given by $w \to [\Psi^{-1}(w)]$ as an effective 0-cycle of degree q on U. Give two morphisms $g : W \to S^q(U)$, $h : W \to S^r(U)$, we have a morphism $g + h : W \to S^{q+r}(U)$ obtained by addition of 0-cycles, $(g+h)(w) = g(w) + h(w)$. Hence for suitable positive integers m, n we have a morphism $f_W : W \to S^m(U) \times S^n(U)$ such that

$$f_W(w) = (-1)^{d-1}(\tilde{D}_1 - \tilde{H}_1), \ldots, (\tilde{D}_d - \tilde{H}_d) \cdot (X \times \{w\})$$

as 0-cycles, where the zero cycle $A - B$ with A, B effective of degrees m, n corresponds to $(A, B) \in S^m(U) \times S^n(U)$. Thus $U_t = W$, $f_t = f_W$ have the required properties.

Corollary (2.2): *Let X be a normal projective variety over an algebraically closed field k. Then there exists a countable family of varieties W_i, pairs of positive integers m_i, n_i and morphisms $g_i : W_i \to S^{m_i}(U) \times S^{n_i}(U)$ such that if $f_i = \gamma_{m_i, n_i} \circ g_i$, $f_i : W_i \to A_0(X)$, then*

(*i*) $\bigcup_{i=1}^{\infty} f_i(W_i) \subset A_0(X)$ *contains the rational equivalence classes of all 0-cycles of the form $(-1)^{d-1}C_1(\mathscr{L})^d$, $\mathscr{L} \in \operatorname{Pic} X$, $d = \dim X$.*

(*ii*) $\dim W_i = q$ *is independent of i.*

Proof: By a result of Chevalley [C], there is an abelian variety A and $\mathscr{L} \in \operatorname{Pic}(X \times A)$ such that $(A, \mathscr{L})$ is a Picard family i.e. if $\operatorname{Pic}^0 X \subset \operatorname{Pic} X$ is the subgroup of bundles algebraically equivalent to 0, then the map $A \to \operatorname{Pic} X$ given by $a \to \mathscr{L}_a = \mathscr{L}|_{X \times \{a\}}$ gives an isomorphism of groups $A \xrightarrow{\cong} \operatorname{Pic}^0 X$; further, any family of line bundles on X parametrized by a normal variety T is induced by a morphism $T \to A$. The Neron-Severi group $\mathrm{NS}(X) = \operatorname{Pic} X / \operatorname{Pic}^0 X$ is finitely generated (see Kleiman [K], for example). Thus if $\mathscr{L}_j \in \operatorname{Pic} X$ give a (countable) set of coset representatives for $\operatorname{Pic} X / \operatorname{Pic}^0 X$, then $p_1^* \mathscr{L}_j \otimes \mathscr{L} \in \operatorname{Pic}(X \times A)$ give a countable set of families of line bundles on X, each with parameter space A, such that every line bundle on X occurs exactly once in a unique family. Corresponding to each such family, we can associate a finite cover $\{W_{ij}\}$ of A and morphisms $g_{ij} : W_{ij} \to S^{m_{ij}}(U) \times S^{n_{ij}}(U)$ such that $\gamma_{m_{ij}, n_{ij}} \circ g_{ij}(w)$ is a zero cycle representing $(-1)^{d-1}C_1(\mathscr{L}_w)^d$, where $\mathscr{L}_w = p_1^* \mathscr{L}_i \otimes \mathscr{L}|_{X \times \{w\}}$. We can reindex the $\{W_{ij}\}_{i,j}$ by the positive integers, giving a set $\{W_i\}$ of varieties with all the claimed properties, with $\dim W_i = q = \dim \operatorname{Pic}^0 X$.

§3. Proof of the Theorem

In fact we can prove a slightly stronger result.

THEOREM: *Let X/k be a projective variety of dimension $d \geqslant 2$ over a universal domain with isolated normal singularities. Let $V = \operatorname{Spec} A$ be an affine open subset of X containing the singular locus. Suppose $A_0(V) \neq 0$ (equivalently $F_0K_0(V) \neq 0$). Then there exist (uncountably many) projective A-modules P of rank d such that P cannot be written as $Q \oplus L$ with $L \in \operatorname{Pic} A$.*

We note that this implies the theorem stated in the introduction, since the hypothesis of that theorem imply that $V = \operatorname{Spec} A$ can be compactified to a projective variety X such that $X_{\text{sing}} \subset V$ (this uses resolution of singularities). Thus $F_0K_0(V) \neq 0$ implies the existence of indecomposable projectives. Conversely, if $F_0K_0(V) = 0$ and $d \leqslant 3$, then every projective module of rank d has a trivial direct summand i.e. a unimodular element. This is obvious for $d = 1$ and follows from the Murthy-Swan cancellation theorem if $d = 2$ [MS]. If $d = 3$, this is the main result of [MK], if A is smooth over k; it has been generalized to arbitrary normal A in [L2]. We note that the conditions $F_0K_0(V) \neq 0$ and $A_0(V) \neq 0$ are equivalent – indeed we have surjections $\Psi : A_0(V) \to F_0K_0(V)$ and $C_d : F_0K_0(V) \to A_0(V)$ such that $\Psi \circ C_d$ and $C_d \circ \Psi$ both equal multiplication by $(-1)^{d-1}(d-1)!$; but both groups are divisible.

PROOF OF THE THEOREM: In the notation of §1, set $A_0^W(X) = A_0(V)$. Since X is projective and $V \subset X$ is an affine open subset, $X - V$ generates Alb X (if $Y \subset X$ is a general 2-dimensional linear space section, then Alb $Y \twoheadrightarrow$ Alb X [La], and $Y - (V \cap Y)$ is the complement of an affine open set in Y which thus supports an ample divisor on Y [H2]). Since $A_0^W(X) \neq 0$, by Lemma (1.5) we have $d_W \geqslant 2$ i.e. for $m + n \geqslant n_0$,

$$d_{m,n} = (m+n)d_W + j_W \to \infty \text{ as } (m+n) \to \infty$$

In particular, for large n, we have d_n $(= d_{n,0}) > q = \dim \operatorname{Pic}^0 X$, the number given by Corollary (2.2). We claim that there exist elements of $\varphi_n(S^n(U)) \subset A_0^W(X) = A_0(V)$ which are not of the form $(-1)^{d-1}C_1(\mathscr{L})^d$ for any $\mathscr{L} \in \operatorname{Pic} X$. Indeed, we have only to verify that $\bigcup_{i=1}^{\infty} h \circ f_i(W_i)$ does not contain $\varphi_n(S^n(U))$, where $h = A_0(X) \twoheadrightarrow A_0^W(X)$ is the natural map. The set

$$\Gamma_{i,n} = \{(A, B) \in W_i \times S^n(U) \mid h \circ f_i(A) = \varphi_n(B)\}$$

is a c-locally closed set. We claim that

$$\bigcup_i p_2(\Gamma_{i,n}) \subsetneqq S^n(U),$$

which will prove our earlier claim. Assume instead that $\bigcup_i p_2(\Gamma_{i,n}) = S^n(U)$. Then some irreducible component W of the c-closure of some $\Gamma_{i,n}$ dominates $S^n(U)$ under the projection $\Gamma_{i,n} \to S^n(U)$. By definition of d_n, there exists a c-closed subset $F \subsetneqq S^n(U)$ such that if $x \in S^n(U) - F$, then every irreducible component of the c-closure of $\varphi_n^{-1}(\varphi_n(x))$ has dimension $nd - d_n$. Thus if $y \in W$ such that image $(y) \in S^n(U) - F$, then the fiber of $W \to W_i$ through y has dimension $\leqslant nd - d_n$. Hence $\dim W_i \geqslant \dim W - (nd - d_n) \geqslant \dim S^n(U) - nd + d_n = d_n > q = \dim W_i$, a contradiction.

Thus, the theorem will be proved once we show that there exists a projective A-module of rank $d = \dim A$ with a prescribed top Chern class in $A_0(V)$, $V = \operatorname{Spec} A$, and vanishing lower Chern classes, with respect to a suitable theory of the Chow ring and Chern classes. For non-singular V, there is a standard theory (see [F]). We briefly review the properties of Levine's theory [L1] which are needed. First, $CH^1(V) \cong \operatorname{Pic} V$ and $CH^d(V) = A_0(V)$ (the second equality uses the normality of V). The first Chern class map C_1 satisfies $C_1(\mathcal{O}_X(D)) = [D] \in CH^1(X)$ as usual. If P, Q are projective modules, $C(P)$, $C(Q) \in CH'(V)$ the total Chern classes, then $C(P \oplus Q) = C(P)$, $C(Q)$. The 'Riemann-Roch theorem without denominators' is valid in the following sense: there is a filtration $F^iK_0(V)$ and maps $\Psi_i: CH^i(V) \to F^iK_0(V)/F^{i+1}K_0(V)$ such that if $\bar{C}_i: F^iK_0(V)/F^{i+1}K_0(V) \to CH^i(V)$ is induced by the ith Chern class C_i, then $\Psi_i \circ \bar{C}_i$ and $\bar{C}_i \circ \Psi_i$ are both multiplication by $(-1)^{i-1}(i-1)!$. For a subvariety $Z \subset V$ with $Z \cap V_{\text{sing}} = \emptyset$, codim $Z = i$, there is a class $[Z] \in CH^i(V)$ such that $C_i(\mathcal{O}_Z) = (-1)^{i-1}(i-1)![Z]$ and $C_j(\mathcal{O}_Z) = 0$ for $j < i$. Lastly, if $D_1, \ldots, D_d \subset V$ are effective Cartier divisors meeting properly with $(\bigcap D_i) \cap V_{\text{sing}} = \emptyset$, then $[D_1] \cdot [D_2] \cdot \ldots \cdot [D_d] = [\bigcap D_i] \in CH^d(V) = A_0(V)$. These properties suffice for our purpose.

We now show that given any $\delta \in A_0(V)$, there exists a projective module P with rank $P = d$, $C_1(P) = \cdots = C_{d-1}(P) = 0$ and $C_d(P) = \delta$. If $I \subset A$ is the ideal of an effective (reduced) 0-cycle supported on $V - V_{\text{sing}}$, then there is a resolution

$$0 \to P' \to F_{d-1} \to \cdots \to F_1 \to A \to A/I \to 0$$

where F_i are free and P' is projective. Then $C_i(P') = 0$ for $i < d$, and $C_d(P') = (-1)^{d-1}C_d(A/I) = (d-1)![A/I]$. But any 0-cycle is rationally equivalent to a cycle of the form $(d-1)![A/I]$ for an effective reduced cycle $[A/I]$, since any zero cycle on V is supported on a smooth affine curve C with $C \cap V_{\text{sing}} = \emptyset$, and a similar claim is true for Pic C. Thus we can find a projective module P with $C_i(P') = 0$ for $i < d$, $C_d(P') = \delta$. Now if rank $P' = n \geqslant d$, by the stability theorem of Bass [Ba] we can write

$$P' \cong P \oplus A^{n-d}$$

where P has rank d. Then $C_i(P) = 0$ for $i < d$, $C_d(P) = \delta$. Applying this construction to a class $\delta \in \varphi_n(S^n(U)) - \cup h \circ f_i(W_i)$ as above, we see that P is indecomposable. This proves the theorem.

§4. Examples in dimension 2

The examples of affine varieties of dimension $\geqslant 2$ with nonzero Chow group of 0-cycles are given by infinite dimensionality theorems for Chow groups of projective varieties. The best result is in dimension 2. Let X be a normal projective surface over a universal domain k. We say that $A_0(X)$ is *infinite dimensional* if none of the maps $\gamma_{m,n}$ of §1 are surjective. Equivalently, the number $d(X) = d_{R(X)}$ given by Prop. (1.2) is positive. Then in fact $d(X) = 2$, by Lemma (1.3). One can improve lemma (1.4) as follows: $d(X) = 0 \Leftrightarrow A_0(X)' \cong \text{Alb } X$ (where $A_0(X)' \subset A_0(X)$ is the subgroup generated by cycles of degree 0). Here Alb X is universal for rational maps from X to abelian varieties; in particular it is a birational invariant. Thus Alb $X \cong$ Alb Y for any resolution of singularities $Y \to X$. In fact, one can show that $d(X) = 0 \Leftrightarrow$ the composite $A_0(X)' \twoheadrightarrow A_0(Y)' \twoheadrightarrow \text{Alb } Y = \text{Alb } X$ is an isomorphism (see [S1], Theorem 1). But one easily sees that if $U \subset X$ is an affine open set containing X_{sing}, then $A_0(X)' \cong \text{Alb } Y \Leftrightarrow A_0(U) = 0$ (see loc. cit.: the point is that $X - U$ supports an ample divisor). In particular, $A_0(U) \neq 0$ if either $A_0(Y)' \not\cong \text{Alb } Y$ (i.e. $A_0(Y)$ is infinite dimensional) or $A_0(X) \to A_0(Y)$ is not injective.

For smooth surfaces in arbitrary characteristic (generalising Mumford [M2] for $k = \mathbb{C}$) Bloch [Bl] has shown that $A_0(Y)$ is infinite dimensional if

$$NS(Y) \otimes_{\mathbf{Z}} \mathbf{Q}_l \not\cong H^2_{et}(Y, \mathbf{Q}_l(1)),$$

and he also conjectures the converse. If $k = \mathbb{C}$, this is equivalent to the condition $\Gamma(Y, \Omega^2_Y) \neq 0$. If X is a normal projective surface, it is shown in [S1] that $A_0(X) \to A_0(Y)$ is not injective in the following cases. Let $E \subset Y$ be the reduced exceptional divisor, which we may assume (blow up Y further) has smooth components and normal crossings. Then

(i) If $k = \mathbb{C}$, and $\text{Pic}^0 Y \to \text{Pic}^0(nE)$ is nor surjective for some $n > 0$, then $A_0(X) \to A_0(Y)$ is not injective.

(ii) if k is a universal domain of arbitrary characteristic, and $\text{Pic}^0 Y \to \text{Pic}^0 E$ is not surjective, then $A_0(X) \to A_0(Y)$ is not injective.

The condition (i) is equivalent to the condition that $H^2(X, \mathcal{O}_X) \to H^2(Y, \mathcal{O}_Y)$ is not injective, by the Leray spectral sequence and the formal function theorem. Thus if $k = \mathbb{C}$, $A_0(X)$ is infinite dimensional provided $H^2(X, \mathcal{O}_X) \neq 0$ (we give another proof of this without K-theory in [S2]). We conjecture that the converse holds, and that the converse to (ii) is also valid. Assuming the above conjectures we have a fairly clear picture as to when an affine normal surface $V = \operatorname{Spec} A$ has $A_0(V) \neq 0$.

We conclude our discussion of 2 dimensional examples by giving 2 examples of surfaces with infinite dimensional Chow group of 0-cycles, and having non-trivial Picard varieties such that the intersection product

$$\operatorname{Pic} X \otimes \operatorname{Pic} X \to A_0(X) \text{ has "large" image.}$$

EXAMPLE (a): $X = E \times E$ where E/k is an elliptic curve. If $k = \mathbb{C}$, then $\Gamma(X, \Omega_X^2) = \mathbb{C}$, so that $A_0(X)$ is infinite dimensional. If char $k = p > 0$ and E is not supersingular, then $\mathrm{NS}(X) \otimes \mathbb{Z}_l \not\cong H^2(X, \mathbb{Q}_l(1))$, so a similar result holds. If E is supersingular, in fact $A_0(X)$ is finite dimensional, as observed by Shioda (see [Bl]). In any case, $\operatorname{Pic} X \otimes \operatorname{Pic} X \twoheadrightarrow A_0(X)$ under the intersection product.

EXAMPLE (b): Let E/k be an elliptic curve with a given group law, where char. $k \neq 2$, and let $x \in E$ be a point of order 2. Let $Z = (E \times E)/(\mathbb{Z}/2\mathbb{Z})$ be the associated hyperelliptic surface, where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts on $E \times E$ by the formula

$$(y, z) \to (y + x, -z)$$

Let $P \in \mathbb{Z}$ be the image of (x, x), and let E' be the quotient of E modulo translation by x. Then there is a morphism $f: Z \to E'$ all of whose fibers are isomorphic to E and there is a section $C \subset Z$ of f through P (C is the image of $E \times \{x\}$; note that $x = -x$ since x has order 2). Let $Y \to Z$ be the blow up of P, $\tilde{C}$ the strict transform of C. Consider the divisor $D = \tilde{C} + F_1 + F_2$, where F_1, F_2 are total transforms of 2 fibers of f which do not pass through P. Let F be the strict transform of the fiber through P, and E_1 the exceptional curve. The picture is as follows: Since $E \times \{x\}$ has self intersection 0 on $E \times E$, $C^2 = 0$ on Z. Thus $\tilde{C}^2 = -1$, and $D^2 = 1$. If $G \neq F$ is any irreducible curve on Y, one easily sees that $(D, G) > 0$; lastly $D \cap F = \emptyset$ so that $\mathcal{O}_X(D) \otimes \mathcal{L}\mathcal{O}_F \cong \mathcal{O}_F$. These conditions imply that for large n, the linear system $|nD|$ is basepoint free and gives a morphism $\pi: Y \to X$ to a normal surface X. Further $\pi(F)$ is a single point Q, $\pi: Y - F \to X - \{Q\}$ is an isomorphism, and $Q \in X$ is a normal elliptic singularity. Since F is contained in a fiber of the composite $Y \to Z \to E'$, we have a morphism $X \to E'$. In particular $\operatorname{Pic}^0 X \cong \operatorname{Pic}^0 Y \cong E'$. The resolution of singularities $Y \to X$ has exceptional divisor F, so that $\operatorname{Pic}^0 Y \to \operatorname{Pic}^0 F$ is zero. Thus $\operatorname{Pic}^0 Y \to \operatorname{Pic}^0 F$ is not surjective, so that by the criterion (ii) above, $A_0(X) \to A_0(Y)$ is not injective. Thus $A_0(X)$ is infinite dimensional. In particular, any affine open set $V \subset X$ containing Q supports indecomposable vector bundles of rank 2. We note that this is a "genuine" singular example, in that $A_0(Y)$ is finite dimensional [BKL], so that we cannot prove the result by just resolving singularities. We point out that though $\mathrm{Pic}^0 X \otimes \mathrm{Pic}^0 X \to A_0(X)$ is 0, $\mathrm{Pic}\ X \otimes \mathrm{Pic}\ X \to A_0(X)$ has a large image. However, by the methods used to prove the theorem, one can show that the intersection map is not surjective (the image lies in a c-closed set of dimension 1).

§5. The three dimensional case

If H is a smooth projective 3-fold over a universal domain, then $A_0(X)$ is infinite dimensional if $d(X) \neq 0$ i.e. if none of the $\gamma_{m,n}$ are surjective. But unlike in the case of surfaces, this does not guarantee that $A_0(V) \neq 0$ for any affine open set V. Thus in [BS], it is shown that if A is an abelian 3-fold and X is its Kummer 3-fold (i.e. X is obtained by resolving the singularities of $A/(\mathbb{Z}/2\mathbb{Z})$, where the involution is $x \to -x$), then there is an affine open set $V \subset X$ with $A_0(V) = 0$; however $A_0(X)$ is infinite dimensional. However, it is not clear (to me) that if $A_0(V) = 0$ for some affine open subset V of a smooth projective 3-fold X, then $A_0(V') = 0$ for any other affine open subset, like the case of surfaces. The condition $d(X) = 2$ will imply $A_0(V) = 0$ for every affine open subset. Hence a related question is:

Question: Suppose X is a smooth projective 3-fold over a universal domain, and $V \subset X$ is an affine open subset with $A_0(V) = 0$. Is $d(X) \leqslant 2$?

Roitman [R] has shown that if $k = \mathbb{C}$ and $\Gamma(X, \Omega_X^3) \neq 0$, then $d(X) = 3$; more generally if X is a smooth projective variety over $\mathbb{C}$ with $\Gamma(X, \Omega^q) \neq 0$ for some $q \geqslant 2$, then $d(X) \geqslant q$. In fact his proof proceeds by showing the stronger result that if $Y \subset X$ is a finite union of subvarieties of dimension $\leqslant q-1$, then $A_0(X - Y) \neq 0$ (see [R], pg. 584). Thus if $\dim X = n$ and $\Gamma(X, \Omega_X^n) \neq 0$, then $A_0(V) \neq 0$ for any affine open set V. Roitman's arguments can be adapted to the singular case to prove the following [S2]: let X be a normal projective variety over $\mathbb{C}$, $U = X - X_{\mathrm{sing}}$. Suppose $\Gamma(U, \Omega_U^q) \neq 0$. Then $d(X) \geqslant q$. Further if $Y \subset X$ is a finite union of subvarieties of dimension $\leqslant q-1$ with $Y \cap X_{\mathrm{sing}} = \emptyset$, then $A_0(X - Y) \neq 0$. In particular, if $\dim X = n$, $\Gamma(U, \Omega_U^n) \neq 0$ (equivalently $H^n(X, \mathcal{O}_X) \neq 0$, by [H1] III Prop. (7.5)) and X has isolated singularities, then any affine open set $V \subset X$ which contains X_{sing} has $A_0(V) \neq 0$.

For smooth surfaces over $\mathbb{C}$, Bloch's conjecture (see §4) is that if $\Gamma(X, \Omega_X^2) = 0$, then $d(X) < 2$ (since $d(X) \neq 1$, by Lemma (1.3)). Similarly one can ask:

Question: Suppose $X/\mathbb{C}$ is a smooth projective variety and $\Gamma(X, \Omega_X^r) = 0$ for $r \geqslant q$, where $q \geqslant 2$. Is $d(X) \leqslant q-1$? (One could ask a similar question for normal X).

In characteristic $p > 0$, I do not know any reasonable condition even for smooth X, which will imply $A_0(V) \neq 0$ for any affine open set $V \subset X$

(here dim $X \geqslant 3$). Assuming a suitable formalism of cycle classes in crystalline cohomology which gives an action of correspondences on crystalline cohomology groups, one might try to show that, say, if the de Rham Witt group $\Gamma(X, W\Omega_X^q) \neq 0$ (see [I]) where $q \geqslant 2$, then $d(X) \geqslant q$, and $A_0(X-Y) \neq 0$ for any closed set Y of dimension $\leqslant q-1$. However, as far as we know, cycle classes in crystalline cohomology have been constructed only for smooth subvarieties of a smooth variety [Be].

In dimension 3 we can use a result of Bloch and the author [BS] to at least give examples in characteristic $p > 0$ of smooth 3-folds X such that $A_0(V) \neq 0$ for every affine open subset $V \subset X$. In loc. cit. it is shown that if X is a smooth complete variety over a universal domain, and $V \subset X$ is an algebraic set of dimension $\leqslant 2$ such that $A_0(X-V)=0$, then algebraic and (l-adic) homological equivalence coincide for cycles of codimension 2 on X, upto torsion (by taking the product of all the l-adic cohomology groups for $l \neq p = \text{char. } k$, we can prove the result upto p^n-torsion for some n). Deligne and Katz (SGA 7, Exp. XX) have extended the methods of Griffiths [G] to characteristic $p > 0$, showing for example that if X is either a generic quintic hypersurface in $\mathbb{P}^4$, or the complete intersection of a smooth quadric and generic quintic in P^5, then there exists a 1-cycle on X whose cohomology class in $H^4(X, \mathbb{Q}_l(2))$ vanishes, such that no multiple of the cycle is algebraically equivalent to 0. Hence $A_0(V) \neq 0$ for any affine set $V \subset X$.

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Added in proof

We can now prove the following stronger result (details will appear elsewhere):

let A be an affine normal domain over a universal domain with $F_0K_0(A) \neq 0$; then there exist projective modules P of rank d = dim A such that P cannot be written as $Q \oplus L$ with L $\in$ Pic A, provided $d \geqslant 2$.