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# ON ABSOLUTELY EXTREMAL POINTS

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## Abstract

Given three doubly asymptotic points $x$, $y$, $z$ in a minimal flow $X$, we construct an affine embedding $\varphi: X \to Q$ such that $\varphi(x) = \frac{1}{2}(\varphi(y) + \varphi(z))$. Thus $x$ is not absolutely extremal. We produce an example of a metric minimal flow $X$ with the property that for every $x \in X$ a triple $x$, $y$, $z$ as above exists, thereby showing that no point of $X$ is absolutely extremal.

## Introduction

We recall the definitions of affine embedding and absolute extremality for flows, introduced in [1]. If $(T, X)$ is a flow ($T$ is a self homeomorphism of the compact space $X$) and $(T, Q)$ an affine flow (i.e., $Q$ is a compact convex set and $T$ an affine homeomorphism) then an equivariant continuous map $\varphi: X \to Q$ is called an *affine embedding* if $\overline{\text{co}}\, \varphi(X) = Q$. A point $x \in X$ is called *absolutely extremal* if for every affine embedding $\varphi: X \to Q$ $\varphi(x)$ is in $\partial Q$, the set of extreme points of $Q$.

Suppose $(T, X)$ is metric and minimal (i.e., every orbit is dense) then for every affine embedding $\varphi: X \to Q$ the set $\{x \in X: \varphi(x) \in \partial Q\}$ is a dense $G_\delta$. It was shown in [1] that if $(T, X)$ is metric and minimal then every distal point of $X$ is absolutely extremal. Again under our assumptions on $(T, X)$ the set of distal points is either empty or a dense $G_\delta$. These facts led the first author to ask in [1] whether every minimal metric flow must have absolutely extremal points.

The easiest examples where non-absolutely extremal points exist are given by certain almost automorphic flows where the flow $X$ is presented as a set of sequences in $l^\infty(Z)$ and the identity map of $X$ into $Q = \overline{\text{co}}(X) \subset l^\infty(Z)$ gives a natural affine embedding [1]. Some doubly asymptotic points of $X$ turns out to be non-extreme in $Q$. In this note, we show that in any minimal flow a point with two doubly asymptotic points is not absolutely extremal. ($x$, $y$ are doubly asymptotic if $\lim_{|n| \to \infty} d(T^n x, T^n y) = 0$). We construct a minimal metric flow, every point of which has a continuum of doubly asymptotic points; thus providing an example of a metric minimal flow no point of which is absolutely extremal.

The principle of construction is due to Grillenberger (see e.g. [4]), who first showed how to define a minimal set with some desired property as an intersection of a family of subshifts of finite type. A continuous version of Grillenberger's construction and applications of this method are described in [3] and [2]. The present paper can be considered as a sequel to [1] and we refer the reader to [1] for further motivation.

## Section 1. An affine embedding associated with doubly asymptotic points

1.1 PROPOSITION: *Let* $(X, T)$ *be an infinite metric minimal flow,* $x_0, y_0, z_0 \in X$, *doubly asymptotic points. Then there exists an affine embedding* $\varphi: X \to Q$ *such that* $\varphi(x_0) = \frac{1}{2}(\varphi(y_0) + \varphi(z_0))$.

PROOF: In $C^*(X)$ we let $V$ be the weak * closed linear space spanned by the set

$$V_0 = \left\{ \delta_{T^n_{x_0}} - \tfrac{1}{2}\left( \delta_{T^n_{y_0}} + \delta_{T^n_{z_0}} \right) : n \in Z \right\}.$$

We let $\pi: C^*(X) \to E = C^*(X)/V$ be the quotient map and define $\varphi: X \to E$ by $\varphi(x) = \pi(\delta_x) = \delta_x + V$. Put $Q = \overline{\text{co}}(\varphi(X))$ and let

$$W = \left\{ \eta \in C^*(X) : \eta = \sum_{n \in Z} a_n \left( \delta_{T^n_{x_0}} - \tfrac{1}{2}\left( \delta_{T^n_{y_0}} + \delta_{T^n_{z_0}} \right), \right.\right.$$

$$\left. \times \sum_{n \in Z} |a_n| < \infty \right\}.$$

We claim that $V = W$; to see this let $\sum_{n \in Z} |a_n| < \infty$ be given. Put

$$\eta_N = \sum_{|n| \leqslant N} a_n \left( \delta_{T^n_{x_0}} - \tfrac{1}{2}\left( \delta_{T^n_{y_0}} + \delta_{T^n_{z_0}} \right) \right)$$

and let $\eta$ be the infinite sum. For every $f \in C(X)$ we have

$$|f(\eta_N) - f(\eta)| \leqslant \|f\| 2 \sum_{|n| > N} |a_n| \to 0.$$

Hence $\eta_N \to \eta$ and $\eta \in V$. Thus $W \subset V$; since $V_0 \subset W$ it is enough to show that $W$ is weak * closed. By Krein–Šmulyan's theorem it suffices to show that $W \cap B_r$ is weak * closed, where $B_r = \{\nu \in C^*(X): \|\nu\| \leqslant r\}$. Since $B_r$ is metrizable we can deal with sequences. So let

$$\eta^k = \sum_{n \in Z} a_n^k \left( \delta_{T^n_{x_0}} - \tfrac{1}{2}\left( \delta_{T^n_{y_0}} + \delta_{T^n_{z_0}} \right) \right)$$

be a sequence in $W \cap B_r$ with $\eta^k \to \eta \in V$. We have

$$\operatorname*{Sup}_k \| \eta^k \| = \operatorname*{Sup}_k \operatorname{Sup}\{ | f(\eta^k) | : \| f \| = 1 \} = \operatorname*{Sup}_k 2 \sum_{n \in Z} | a_n^k | < r.$$

Using a diagonal process we can choose a subsequence $\eta^{k_i}$ such that for each $n$ $a_n^{k_i} \to b_n$. For convenience we denote this subsequence also by $\eta^k$, thus we now assume $a_n^k \to b_n$ for every $n$. Using Fatou's lemma (in $l_1(Z)$) we have

$$\sum_{n \in Z} | b_n | \leqslant \underline{\lim} \sum_{n \in Z} | a_n^k | < r.$$

Put

$$\tilde{\eta} = \sum_{n \in Z} b_n \left( \delta_{T^n_{x_0}} - \tfrac{1}{2}\left( \delta_{T^n_{y_0}} + \delta_{T^n_{z_0}} \right) \right)$$

and let $f \in C(X)$ and $\epsilon > 0$ be given. Choose $\delta > 0$ such that $d(x, y) < \delta \Rightarrow | \mathrm{f(x)} - \mathrm{f(y)} | < \epsilon$ and $N$ with

$$| n | > N \Rightarrow \mathrm{d}(T^n y_0, T^n z_0), \mathrm{d}(T^n z_0, T^n x_0), \mathrm{d}(T^n y_0, T^n x_0) < \delta.$$

Then

$$\begin{aligned} | f(\eta^k - \tilde{\eta}) | &= \Bigg| \sum_{|n| \leqslant N} (a_n^k - b_n) [ f(T^n x_0) - \tfrac{1}{2}( f(T^n y_0) + f(T^n z_0))] \\ &\quad + \sum_{|n| > N} (a_n^k - b_n) [ f(T^n x_0) - \tfrac{1}{2}( f(T^n y_0) + f(T^n z_0))] \Bigg| \\ &\leqslant 2 \| f \| \sum_{|n| \leqslant N} | a_n^k - b_n | + \epsilon \sum_{|n| > N} ( | a_n^k | + | b_n | ) \\ &\leqslant 2 \| f \| \sum_{|n| \leqslant N} | a_n^k - b_n | + \epsilon ( \| \eta^k \| + \| \tilde{\eta} \| ). \end{aligned}$$

It follows that $\eta^k \to \tilde{\eta}$ so that $\tilde{\eta} = \eta$ is in $W$ and $V = W$.

Clearly $\varphi$ is continuous and equivariant from $X$ into the affine flow $Q$. If $\varphi(x) = \varphi(y)$ then $\delta_x - \delta_y \in V$. But as $V = W$, every non-zero measure in $V$ is supported by at least three points. Thus $x=y$ and $\varphi$ is one to one. Finally

$$\varphi(x_0)=\delta_{x_0}+V=\delta_{x_0}-\left(\delta_{x_0}-\tfrac{1}{2}\left(\delta_{y_0}+\delta_{z_0}\right)\right)+V$$
$$=\tfrac{1}{2}\left(\delta_{y_0}+\delta_{z_0}\right)+V=\tfrac{1}{2}\left(\varphi(y_0)+\varphi(z_0)\right).$$

This completes the proof. □

## Section 2. A metric minimal flow every point of which has a continuum of asymptotic points

Let $\Omega=[0,1]^Z$ denote the compact metric space of two sided $[0,1]$ valued sequences with the metric $d(x,y)=\sup_{n\in Z} 2^{-|n|}|x_n-y_n|$. For a closed $W\subset[0,1]^n$ and $i\in Z$ we let

$$C_i(W)=\{x\in\Omega:\forall j\in Z,\ x[i+jn,\ i+(j+1)n-1]\in W\}$$

and $C(W)=\bigcup_{i=1}^{n} C_i(W)$. We define inductively a sequence $n_k$ and closed sets $W_k\subset W_{k-1}^{n_k}$ as follows. Let $W_0=[0,1]$. Given $W_{k-1}$ we choose an arbitrary but fixed $2^{-k}$-net $\{u_1,u_2,\dots,u_{l_k}\}$ of $W_{k-1}^2$, where the metric on a finite dimensional cube $[0,1]^n$ is $d(w,v)=\operatorname{Sup}_{1\leqslant i\leqslant n}|w_i-v_i|$. Let $n_k=100l_k$ and define

$$W_k=\{w\in W_{k-1}^{n_k}:\ \text{there exist odd indices } 1\leqslant i_1,i_2,\dots,i_{l_k}\leqslant n_k \text{ such that } w_{i_j}w_{i_j+1}=u_j \text{ for } j=1,2,\dots,l_k \text{ where } w=w_1w_2\dots w_{n_k},\ w_i\in W_{k-1}\}.$$

We call the set $\{i_1,i_2,\dots,i_{l_k}\}$ a *u-set for w*. Put $X=\bigcap_{k=1}^{\infty} C(W_k)$.

2.1 PROPOSITION: *Let $T$ be the shift on $X$, then $(X,T)$ is a minimal flow.*

PROOF: Follows directly from the way $X$ was defined.

2.2 PROPOSITION: *For every $k$, $W_k$ is pathwise connected.*

PROOF: Assume $W_{k-1}$ is connected. Let $w,w'\in W_k$, $w=w_1w_2\dots w_{n_k}$, $w'=w_1'w_2'\dots w_{n_k}'$, $w_i,w_i'\in W_{k-1}$. Assume first that there exists a $u$-set $A=\{i_1,i_2,\dots,i_{l_k}\}$ common to $w$ and $w'$. We let $w(t)=w_1(t)w_2(t)\dots w_{n_k}(t)$ be defined by $w_{i_j}(t)w_{i_j+1}(t)\equiv u_j$ if $i_j\in A$ and where for all other $i$'s $w_i(t)$ is a path in $W_{k-1}$ connecting $w_i$ and $w_i'$. Clearly $w(t)\in W_k$ for every $t\in[0,1]$. For the general case let $A=\{i_1,i_2,\dots,i_{l_k}\}$

and $A' = \{i'_1, i'_2, \dots, i'_{l_k}\}$ be $u$-sets for $w$ and $w'$ respectively. Choose $A'' = \{i''_1, i''_2, \dots, i''_{l_k}\} \subset \{1, 2, \dots, n_k\}$ a set of odd indices disjoint from $A \cup A'$ and define $v = v_1 v_2 \dots v_{n_k}$, $v' = v'_1 v'_2 \dots v'_{n_k}$ as follows

$$v_i v_{i+1} = \begin{cases} u_j & \text{if } i = i_j \text{ or } i = i''_j \\ w_i w_{i+1} & \text{otherwise} \end{cases}$$

$$v'_i v'_{i+1} = \begin{cases} u_j & \text{if } i = i'_j \text{ or } i = i''_j \\ w'_i w'_{i+1} & \text{otherwise.} \end{cases}$$

Clearly $v$, $v' \in W_k$. Now as $A$ is a common $u$-set for $w$ and $v$, $A''$ a common $u$-set for $v$ and $v'$ and $A'$ a common $u$-set for $v'$ and $w''$, we conclude by the first part of the proof, that there exists a path in $W_k$ connecting $w$ and $w'$. □

DEFINITION: Let $K$ be a natural number $1 \leqslant r$, $s \leqslant l_k$, $r \neq s$. A *chain* from $u_r$ to $u_s$ is a set $\{j_0, j_1, \dots, j_l\}$ of indices such that $j_0 = r$, $j_l = s$ and $d(u_{j_n}, u_{j_{n+1}}) < 2^{-k}$, $0 \leqslant n < l$.

For every $r$ and $s$ as above, the existence of a chain from $u_r$ to $u_s$ follows from the fact that $W_k$ is pathwise connected.

DEFINITION: For $x \in X$ there exists by definition a sequence of integers $\{t_k\}$ such that $t_k \leqslant 0 < t_k + m_k$ (where $m_k$ is the length of sequences in $W_k$) and such that for every $k$ $x \in C_{t_k}(W_k)$. It is easy to see that one can choose $\{t_k\}$ so that $\forall k$ $t_{k-1} \equiv t_k (\text{mod } m_{k-1})$. Such a sequence $\{t_k\}$ will be called a *block partition for x*.

2.3 PROPOSITION: *Let $x \in X$, $\{t_k\}$ a block partition for $x$ and $w_0 \in W_{k_0}$ for some $k_0$. Then there exists $y \in X$ such that*
(1) $y[t_{k_0}, t_{k_0} + m_{k_0} - 1] = w_0$
(2) *$y$ is doubly asymptotic to $x$.*

PROOF: We define $y[t_k, t_k + m_k - 1]$ by induction on $k$. Put $y[t_{k_0}, t_{k_0} + m_{k_0} - 1] = w_0$. Let $x[t_k, t_k + m_k - 1] = w_1 w_2 \dots w_{n_k} = w$, $w_i \in W_{k-1}$, $i = 1, \dots, n_k$, and suppose $x[t_{k-1}, t_{k-1} + m_{k-1} - 1]$ is $w_n$. Let $A$ be a $u$-set for $w$. If $n, n-1 \notin A$ define $y[i] = x[i]$ for $t_k \leqslant i \leqslant t_k + m_k - 1$, $i \notin [t_{k-1}, t_{k-1} + m_{k-1} - 1]$ and then clearly $y[t_k, t_k + m_k - 1] \in W_k$.

If $n = i_r \in A$, let $m$, $1 \leqslant m \leqslant n_k - 1$ be an odd integer such that $m \notin A$. There exists an $s$, $1 \leqslant s \leqslant l_k$ such that $d(u_s, w_m w_{m+1}) < 2^{-k}$. Let $j_0, j_1, \dots, j_l$ be a chain from $u_r$ to $u_s$, $j_0 = r$, $j_l = s$; thus for $i_{j_t} \in A$ $w_{i_{j_t}} w_{i_{j_t}+1} = u_{j_t}$, $t = 1, \dots, l$. Put $w' = w'_1 w'_2 \dots w'_{n_k}$ where for an odd $i$

$$w'_i w'_{i+1} = \begin{cases} u_{j_{t-1}} & \text{if } i = i_{j_t}\ 0 < t \leqslant l \\ u_s & \text{if } i = m \\ w_i w_{i+1} & \text{otherwise} \end{cases}$$

Since for every $t$, $d(u_{J_{t-1}}, u_{J_t}) < 2^{-k}$ and also $d(u_s, w_m w_{m+1}) < 2^{-k}$ we have $d(w, w') < 2^{-k}$. Let $y[t_k, t_k + m_k - 1] = w_1' w_2' \dots w_{n-1}' y[t_{k-1}, t_{k-1} + m_{k-1} - 1] w_{n+1}' \dots w_{n_k}'$, then clearly $y[t_k, t_k + m_k - 1] \in W_k$. If $n - 1 = i_r \in A$ the construction is similar.

There are now three possibilities:

(1) $t_k \to \infty$, $t_k + m_k \to \infty$, in which case $y$ is now fully defined.

(2) There exists $k_0$ such that for $k \geqslant k_0$, $t_k = t_{k_0}$. In this case define $y[-\infty, t_{k_0} - 1] = x[-\infty, t_{k_0} - 1]$.

(3) There exists $k_0$ such that for $k \geqslant k_0$, $t_k + m_k = t_{k_0} + m_{k_0}$.

In this case define $y[t_{k_0} + m_{k_0} + 1, \infty] = x[t_{k_0} + m_{k_0} + 1, \infty]$. By definition of $y$ we have for $i < t_k$ and $i > t_k + m_k$, $|y[i] - x[i]| < 2^{-k}$. Thus $y$ and $x$ are asymptotic. □

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