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REMARK ON THE DEFINITION OF AN AUTOMORPHIC FORM

V. Averbuch

Introduction

Let k be a global field, G a reductive algebraic group over k . Let K be a maximal compact subgroup of G_A , and let K^f be its finite component.

In [1], an automorphic form on G is defined as a smooth \mathbb{C} -valued function on G_A , which satisfied the following conditions.

- (1) f is K^f -finite and Hecke-finite.
- (2) f is a function on $G_k \backslash G_A$
- (3) f transforms according to a character under the action of the centre.
- (4) f satisfies a moderate growth condition in the Siegel domain.
(Precise definition is given in the beginning of Part 1 for GL_n over \mathbb{Q} and in the beginning of Part 2 for the general case).

These conditions are not independent. If $\text{char } k > 0$, then (4) can be deduced from (1)–(3). If $\text{char } k = 0$, then Hecke-finiteness at finite places follows from other conditions and Hecke-finiteness at Archimedean places. Here we prove the following theorem:

THEOREM: *If $\text{char } k = 0$, then (4) can be deduced from (2), (3), K^f -finiteness and Hecke-finiteness at some big finite place $-p$, s.t. f is invariant under K_p .*

Big finite place is any place p at which $N_p = \#O_p/m_p \geq C$ where C depends on G and on k .

I want to thank Prof. Ilya Piatetski-Shapiro. He suggested this problem to me and gave me many advices during the work.

Part 1. Case of $G = GL_n$, $k = \mathbb{Q}$

We prove the main theorem first in this case because of two reasons. First, the proof in the general case is essentially the same, but includes some technical details. Second, the intermediate results of this case are needed in the general case.

NOTATION: B is the standard Borel subgroup, $\alpha_1, \dots, \alpha_{n-1}$ simple roots, $\chi_1, \dots, \chi_{n-1}$ the fundamental weights. For each χ_i , there is a character $\mu_i: G_A \rightarrow \mathbb{R}^*$, trivial on G_k , s.t. $|\chi_i \mu_i|$ is trivial on the centre.

Let K be the standard maximal compact subgroup. We have the Iwasawa decomposition $G_A = B_A \cdot K_A$.

If χ is an algebraic character on B , we can consider $|\chi|$ as a function on G_A , using the Iwasawa decomposition, $|\chi(bk)| = |\chi(b)|$.

Recall our definition of the Hecke ring $H(G_p)$. It consists of all compactly supported smooth functions on G_p . Multiplication is given by convolution. If K' is an open compact subgroup of G_p , we can define $H^{K'}(G_p)$ as the subring consisting of all two-sided K' -invariant functions. If $g \in G_p$, we can define an element $f^g \in H^{K'}(G_p)$, by

$$f^g(g') = \begin{cases} 1/\mu(K') & g' \in K'gK' \\ 0 & \text{otherwise} \end{cases}$$

Here μ is the standard Haar measure. (The measure of maximal compact subgroup is 1).

We have a decomposition

$$K'gK' = \cup_i g_i K' \quad (\text{disjoint union})$$

Since K' is compact and open, the union is finite. If f is any smooth K' -invariant function on G_p or G_A , the action of f^g on f is given by

$$(\pi(f^g)f)(g') = \sum f(g'g_i)$$

We recall some facts about fundamental domains. For $c > 0$, set

$$\sigma(c) = \{ b \in B_A \mid |\alpha_i(b)| \geq c \quad \forall_i \}$$

These sets are called Siegel domains.

THEOREM 1: *There is $c_0 > 0$, s.t. $G_A = G_Q \cdot \sigma(c_0)K$. Any $\sigma(c)$ which satisfies this condition is called a fundamental domain. We fix c_0 to be s.t. $\sigma(c_0)$ is a fundamental domain.*

For each $C > 0$, let

$$M^C = \left\{ b \cdot k \mid b \in \sigma(c_0), k \in K, \prod_i |\chi_i \mu_i(b)| \leq C \right\}.$$

B_Q acts on M^C from the left, and the quotient is compact, modulo the centre.

Now we can state the main theorem.

MAIN THEOREM: *Let f be a function on $G_Q \setminus G_A$ s.t.*

- (1) *f is smooth and K_p -invariant, and transforms by a character under action of the centre.*
- (2) *f is Hecke finite at place p .*

Let $L(c) = \max_{g \in M^c} |f(g)|$.

If $p \geq C$, where C is some constant, which depends only on G , then $L(c)$ has a moderate growth as $c \rightarrow \infty$, t.i. there are constants A_1, A_2, A_3 , s.t. $L(c) \leq A_1 \cdot c^{A_2}$ for $c \geq A_3$.

We need one simple property of the decomposition $G_A = G_Q \cdot \sigma(c_0) \cdot K$.

LEMMA 1: Let $b, b' \in B_A$, $b' = q \cdot b \cdot k$, where $q \in G_Q$, $k \in K$. Let χ be a dominant weight. Then,

$$|\chi(b')| \leq \max_{w \in W} |(w\chi)(b)|$$

Here W is the Weyl group

PROOF: We shall first prove the case of $\chi = \chi_{n-1}$. Since $|\det b'| = |\det b|$, it is sufficient to show, that $|b'_n| \geq \min |b_i|$, where b_i, b'_i are the diagonal entries of b, b' . Let $\vec{b}_1, \dots, \vec{b}_n$ be the row vectors of b . It is easy to see, that

$$|b'_n| = \prod_p |b'_n|_p = \prod_p |(\sum q_{ni} \vec{b}_i)_p|_p$$

Let j be minimal, s.t. $q_{nj} \neq 0$. Since b is upper triangular,

$$|(\sum q_n \vec{b}_i)_p|_p \geq |q_{nj} b_j|_p$$

So

$$|b'_n| \geq \prod_p |q_{nj} b_j|_p = |b_j|$$

So, the case of $\chi = \chi_{n-1}$ is proved.

Now, let us use the representation $g \rightarrow \Lambda^{n-1}g$. Since

$$\chi_{n-1}(\Lambda^{n-1}b) = \chi_1(b) \cdot (\det b)^{n-2}$$

the case $\chi = \chi_1$, be follows from the previous case. Let now χ be any dominant weight. Consider the representation $V(\chi)$ with highest weight χ . Using this representation we see, that

$$|\chi(b')| \leq |\chi_1(b)|,$$

where χ_1 is some weight in the representation $V(\chi)$. It can be easily seen, that there exists $w \in W$, s.t.

$$|\chi_1(b)| \leq |w\chi(b)|.$$

We need some properties of Hecke operators.

THEOREM 2:

(1) Let $a_i = \text{diag}(p^{-1}, \dots, p^{-1}, 1, \dots, 1)$ (i times p^{-1}). Let $Ka_iK = \cup b_jK$ be a disjoint union with $b_j \in B_p$. Then each b_j has i eigenvalues with absolute value p and $n - i$ eigenvalues with absolute value 1. Among them, there is only one whose first i eigenvalues have absolute value p . This element lies in aK .

(2) If $g, g' \in G_p$, then $g \in Kg'K \Leftrightarrow$ for all j , the maximal absolute value of $j \cdot j$ minors of g is equal to that of g' .

This theorem is well-known and we shall not prove it. We use this theorem in the following formulation. Let us denote by A^+ the set of all diagonal matrixes, absolute values of whose eigenvalues are not increasing.

THEOREM 3:

Let $a \in A^+$, let $KaK = aK \cup (\cup b_jK)$ be a disjoint decomposition with $b_j \in B$.

(1) Let χ be any dominant weight, and $w \in W$ any element of the Weyl group. Then,

$$|\chi(a)| \geq |w\chi(b_j)|.$$

(2) Suppose that \forall_i s.t. $|\alpha_i(a)| > 1$, we have χ/χ_i is a dominant weight. Then,

$$|\chi(a)| \geq p|\chi(b_i)|.$$

PROOF 1: (1) We can write χ as a product of fundamental weights, and since both sides of the inequality are multiplicative in χ , we can assume that $\chi = \chi_i$ for some i . Then $|\chi(a)|$ is the maximal absolute value of a $i \times i$ minor of a , and $|w\chi(b_j)|$ is the absolute value of some $i \times i$ minor of b_j . Since $b_j \in KaK$, (1) now follows from Theorem 2, (2).

(2) We can write a as a product of $a_0 \cdot c \cdot \prod a_i^{n_i}$, where $a_0 \in K \cap A$, c lies in the centre, a_i are as in Theorem 2 and $n_i \geq 0$. For $a = a_0 \cdot c$ the theorem is trivial because $KaK = aK$. So, suppose that $a = a' \cdot a_i$, $a' \in A^+$ and the theorem is true for a' , and let us prove the theorem for a . Let

$$Ka'K = a'K \cup \left(\bigcup_j b'_jK \right), \quad Ka_iK = a_iK \cup \left(\bigcup_j b'_jK \right)$$

Since $KaK \subset Ka'K \cdot Ka_iK$,

$$\begin{aligned} KaK &\subset \left(a'K \cup \left(\bigcup_j b'_jK \right) \right) \cdot Ka_iK \\ &= a'Ka_iK \cup (\cup b'_jKa_iK) \\ &= a'a_iK \cup \left(\bigcup_j b'_ja_iK \right) \cup \left(\bigcup_j a'b'_jK \right) \cup \left(\bigcup_{j_1, j_2} b'_{j_1}b'_{j_2}K \right). \end{aligned}$$

Since $b_l \notin aK$, b_l lies in one of the classes $b'_j a_i K$ or $a' b'_j k$ or $b'_j b'_l K$. It is easy to see, that if $b_l \in b \cdot K$, then $|\chi(b_l)| = |\chi(b)|$. Since $|\chi|$ is multiplicative on B , it is enough to prove, that

- (1) $|\chi(a')| > |\chi(b'_j)|$
- (2) $|\chi(a_i)| > |\chi(b'_j)|$.

since χ and a satisfy second condition of the theorem, χ and a' also satisfy this condition. So, (1) is checked. Also, from second condition we have: $\chi = \chi_i \cdot \chi'$, where χ' is a dominant weight. From part (1) of the theorem, $|\chi'(a_i)| \geq |\chi'(b'_j)|$. From part (1) of theorem 2 we have that

$$|\chi_i(a_i)| > |\chi_i(b'_j)|$$

Now, we can prove the main theorem.

We assumed, that f is Hecke-finite at p . Let $f_1 = f, \dots, f_N$ be the basis of the finite-dimensional space that f spans under the action of the Hecke ring $H^{K_p}(G_p)$. They all are K_p -invariant from the right.

Recall our sets M^c , which were defined at beginning of the part. Let us define

$$L'(c) = \max_{i, g \in M^c} |f_i(g)|.$$

$L'(c)$ exists, since $B_q \backslash M^c$ is compact, modulo the centre. We shall show, that there are constants c_1, c_2, c_3 s.t. for $c \geq c_3$,

$L'(c) \leq \max(c_1 L'(c/c_2), L'(c/2))$. Here c_1, c_2 depend on p , and this will prove the theorem. Furthermore, we can take $c_2 = p/A'$, where A' is a constant. Let us estimate $L'(c)$, with $c \gg 1$.

Let us take $g \in M^c$, $g \notin M^{c/2}$, $g = b \cdot k$. Since f_i are K_p invariant, we can assume $k_p = 1$.

Since $\prod |\chi_i(b)| \gg 1$, there is some simple root α_i s.t. $|\alpha_i(b)| \gg 1$.

We use the Hecke operator $T_i = Ka_iK$. If $Ka_iK = a_iK \cup (\cup b_l K)$, we have

$$f_j(ga_i) + \sum_l f_j(gb_l) = T_i f_j(g),$$

so

$$f_j(g) = T_i f_j(ga_i^{-1}) - \sum_l f_j(ga_i^{-1}b_l).$$

In order to prove our estimate, it is sufficient to check, that every argument in the right side can be written as $q \cdot g'$ with $q \in G_Q$ and $g' \in M^{c/c_2}$:

It is easy to see, that $ga_i^{-1} \in M^{c/p}$.

Let us write $ga_i^{-1}b_l = q \cdot g'$, $g' \in \sigma(c_0) \cdot K$.

From Lemma 1, $|\chi_K(g')| \leq \max |(w\chi_K)(ga_i^{-1}b_l)|$. (although g' , $ga_i^{-1}b_l$ do not lie in B , we already defined $|\chi|$ on the group G).

LEMMA 2: (i) $|w\chi_K(ga_i^{-1}b_l)| \leq A|\chi_K(g)|$, A doesn't depend on p .

(ii) $|w\chi_i(ga_i^{-1}b_l)| \leq |\chi_i(g)|/p$.

PROOF: (i) Since $g_p \in B$,

$$|w\chi_K(ga_i^{-1}b_l)| = |w\chi_K(g)| \cdot \left| \frac{w\chi_K(b_l)}{w\chi_K(a_i)} \right|.$$

We can write

$$\chi_K/w\chi_K = \prod \alpha_m^{n_m}, \quad n_m \geq 0.$$

Case 1: $n_i > 0$. In this case, $|w\chi_K(g)| \ll |\chi_K(g)|$, and $|w\chi_K(a_i^{-1}b_l)|$ is bounded, so the assertion is proved.

Case 2: $n_i = 0$. Then, $|w\chi_K(g)/\chi_K(g)| = \prod |\alpha_m(g)|^{-n_m}$. Since $n_m \geq 0$ and $|\alpha_m(g)|$ are bounded from below, since $g \in \sigma(c_0) \cdot K$, this expression is bounded from above.

Since $n_i = 0$, $|w\chi_K(a_i)| = |\chi_K(a_i)|$. By Theorem 3,

$$\left| \frac{w\chi_K(b_l)}{w\chi_K(a_i)} \right| = \left| \frac{w\chi_K(b_l)}{\chi_K(a_i)} \right| \leq 1.$$

Q.E.D.

(ii) Again, we write $\chi_i/w\chi_i = \prod \alpha_m^{n_m}$, $n_m \geq 0$.

We have two cases:

Case 1: $n_i > 0$. Then, as we saw before,

$$|w\chi_i(ga_i^{-1}b_l)| \ll |\chi_i(g)|.$$

Case 2: $n_i = 0$. Then $w\chi_i = \chi_i$. So, we need only to check that

$$|\chi_i(a_i)| \geq p|\chi_i(b_l)|.$$

Q.E.D.

This is proved in Theorem 3, (2)

So, we see that $|\chi_K(g')| \leq \max_w |w\chi_K(ga_i^{-1}b_l)| \leq A|\chi_K(g)|$, and $|\chi_i(g')| \leq P^{-1}|\chi_i(g)|$.

So, $\prod |\chi_i(g')| \leq A' \cdot p^{-1} \cdot \prod |\chi_i(g)|$.

Hence, we can take $c_2 = p/A'$. If $p \gg 1$, $c_2 > 1$.

Part 2. Notation and basic facts

If k is a number field, we always normalise it's absolute values by the condition, that the product formula should be satisfied, and at any non-archimedean place the value group must be generated by the number of elements in the residue field.

Recall the following facts ([2]).

Let S be a maximal split torus. Let ϕ be the root system (possibly non-reduced) relative to S , and $\alpha_1, \dots, \alpha_l$ a base. We shall call α_i the simple roots. Let B be the minimal parabolic subgroup relative to the base $(\theta_1, \dots, \alpha_l)$. Let Z be the centralizer of S in G , and let K be a maximal compact subgroup. We have the Iwasawa decomposition $G_A = B_A K_A$.

Let \tilde{k} be a finite extension of k , in which G is split, and let S_1 be a maximal torus of $G(\tilde{k})$ which contains $S(\tilde{k})$.

If α is a root of $G(\tilde{k})$ relative to S_1 , then the restriction of α to S is a character from S to k . There is a base $\tilde{\alpha}_1, \dots, \tilde{\alpha}_K$ in $G(\tilde{k})$ relative to S_1 , s.t. restriction of each $\tilde{\alpha}_i$ to S is either a positive root or 1.

Let \tilde{B} be the Borel subgroup of $G(\tilde{k})$ relative to the base $\tilde{\alpha}_1, \dots, \tilde{\alpha}_K$. Let \tilde{K} be a maximal compact subgroup of $G(\tilde{k})$ which contains K . We have $G(\tilde{k}_A) = \tilde{B}_A \cdot \tilde{K}_A$.

For each simple root $\tilde{\alpha}_i$, there is a copy of $SL_2(\tilde{k})$ imbedded in $G(\tilde{k})$, whose standard unipotent subgroup coincides with the root space, corresponding to $\tilde{\alpha}_i$. Let $H_i \subset G(\tilde{k})$ be the standard maximal compact subgroup of this copy.

We can find a finite set $\tilde{\Sigma}$ of places p , s.t. $|H_i|_p \subset \tilde{K}_p$ for $p \notin \tilde{\Sigma}$.

Let Σ be the set of restrictions of $\tilde{\Sigma}$ to k . We define Siegel domains

$$\sigma(c) = \{s \cdot z \cdot u \mid |\alpha_i(s)| \geq c, z \in Z_0, u \in U\}$$

where Z_0 is a compact subset in Z_A s.t. $Z_A = Z_Q \cdot Z_0 \cdot S_A$, and U is the unipotent radical of B .

THEOREM 1*: *There is a constant $c_0 > 0$, s.t.*

$$G_A = G_k \cdot \sigma(c) \cdot K$$

for $0 < c \leq c_0$. For the proof, see [3].

Any such $\sigma(c)$ will be called the fundamental domain. Let \bar{U} be the Lie algebra of U , and $V_i \subset \bar{U}$ be the subspace generated by root spaces with respect to S , whose roots are $\geq \alpha_i$, and let χ_i denote the determinant of the adjoint action of B in V_i .

If χ is any algebraic character of B , it gives a character of \tilde{B} , and a function $|\tilde{\chi}|$ as $G(\tilde{k})_A$. Restriction of this function to G_A satisfies

$$|\tilde{\chi}|_{|G_A} = |\chi|^n, \quad \text{where } n = [\tilde{k} : k].$$

Finally, if p is non-archimedean place of k , we denote by $|p|$ the number of elements in the residue field of p .

We now state the main theorem.

Again, we introduce the sets

$$M^c = \{b \cdot k \mid b \in \sigma(c_0), k \in K : \prod |\chi_i(b)| \leq c\}$$

$S_k \cdot U_k$ acts on M^c from the left, and the quotient is compact, modulo the centre.

MAIN THEOREM (FOR GENERAL CASE):

Let f be a function on $G_k \backslash G_A$ s.t.

- (1) f is a smooth function on G_A and K_p -invariant for some place p .
- (2) f transforms by a character under the action of the centre.
- (3) f is Hecke-finite at p .

If $|p| \geq C_1$, where C_1 depends only on G and k , then

$$\max_{g \in M^c} |f(g)| \leq C_2 c^{C_3}$$

Of course, $\max_{g \in M^c} |f(g)|$ exists, since $S_k U_k \backslash M^c$ is compact modulo the centre.

We shall now generalize the proof of case $GL_n(Q)$.

First we prove the generalizations of Lemma 1 and Theorem 3.

LEMMA 1*: If $b, b' \in B_A$, $b' = q \cdot b \cdot k$, $q \in G_k$, $k \in K_A$, χ is a dominant weight over \tilde{k} , W the Weyl group over \tilde{k} , then

$$|\chi(b')| \leq C \max_{w \in W} |w\chi(b)|$$

Here b', b are considered as elements of $G(\tilde{k})$, and $|\chi|, |w\chi|$ are defined on $G(\tilde{k})$ as was explained before.

PROOF: In fact, the assertion is about $G(\tilde{k})$, so we can assume, that G is split and $k = \tilde{k}$. Let $V(\chi)$ be the irreducible representation with highest weight χ . If $g \in G$, we shall denote by \bar{g} its image in $GL(V(\chi))$. Also, for each weight χ_1 of representation $V(\chi)$, there is defined the standard weight $\bar{\chi}_1$ of $GL(V(\chi))$

$$\chi_1(g) = \bar{\chi}_1(\bar{g})$$

Let K_1 be a maximal compact subgroup in $GL(V(\chi))_A$ which contains the image of K . Let $K_0 \subset GL(V(\chi))_A$ be a standard maximal compact subgroup relative to a basis in which S acts diagonally. Let B_0 be the standard Borel subgroup containing B .

Let $g = q \cdot b$. Then, there is $k' \in K_0$, s.t. $\bar{g}k' \in B_0$. Also, $\bar{g}k \in B_0$. We know from Lemma 1, that

$$|\bar{\chi}(\bar{g} \cdot k')| \leq \max_{\chi' \in M} |\bar{\chi}'(\bar{b})|$$

where M is the set of all weights of $V(\chi)$. It is easy to see, that

$$\max_{\chi' \in M} |\bar{\chi}'(\bar{b})| = \max_w |w\bar{\chi}(\bar{b})| = \max_w |w(b)|$$

Now, $|\bar{\chi}(\bar{g}^k)/\bar{\chi}(\bar{g}k')| = |\bar{\chi}(k'^{-1} \cdot \bar{k})| \leq C$, where $C = \max_{K_0 \cdot K_1} |\bar{\chi}|$. So,

$$|\chi(b')| = |\bar{\chi}(\bar{g}k)| \leq C |\bar{\chi}(\bar{g} \cdot \bar{k}')| \leq C \max |w\chi(b)|.$$

Q.E.D.

THEOREM 3*: *Let $p \notin \Sigma$.*

Let $s \in S^+ = \{s \in S_p \mid |\alpha_i(s)| \geq 1\}$.

Let $K_p s K_p = s K_p \cup (\cup b_i K_p)$, $b_i \in B$ be a disjoint union.

Let w be an element of the Weyl group over \bar{k} , and χ a dominant weight over \bar{k} .

- (1) $|\chi(s)| \geq |w\chi(b_i)|$.
- (2) *Let now, χ_0 be the determinant of the action of B on the span of those root spaces \bar{V}_α or which $|\alpha(s)| > 1$. Suppose that χ/χ_0 is a dominant weight. Then*

$$|\chi(s)| \geq |p| |\chi(b_i)|.$$

PROOF: Again, the theorem is about $G(\bar{k})$, so we can assume that G is split.

- (1) Use the representation with highest weight χ .
- (2) We need a lemma.

LEMMA: *Let $b \in B$, and suppose that $|\alpha_i(b)| < 1$.*

Then, there is $b' \in KbK \cap B$ s.t.

- (1) $|\alpha_i(b')| > 1$, $|\chi_i(b')| \geq |\chi_i(b)| \cdot |p|$.
- (2) $|\chi_j(b')| = |\chi_j(b)|$, $j \neq i$.

PROOF: We shall find such b' in $H_i b H_i \cap B$. First we seek for b' which satisfies (1). Then this problem reduces to the case $G = SL_2$, and this is easy. Now, H_i is contained in the parabolic subgroup on which χ_j is defined as a character, if $j \neq i$. Since $|\chi_j|_{H_i} = 1$ (since H_i is compact)

$$|\chi_j(b')| = |\chi_j(b)|, \quad j \neq i.$$

Q.E.D.

Now, take b_i and suppose that $|\chi(b_i)| \geq |\chi(s)|$. Using the Lemma, we get a sequence $b^1 = b_i, b^2, \dots$, which can stop at some b^n if $|\alpha_j(b^n)| \geq 1 \forall j$.

Since one of the characters $|\chi_j|$ always grows at $|p|$ from b^n to b^{n+1} , and on the other side $|\chi_j(b^k)| \leq |\chi_j(s)|$ by part 1 of the theorem, this sequence must stop. So, there is some b^n s.t. $|\alpha_j(b^n)| \geq 1 \forall j$.

If $|\chi(b_i)| \geq |\chi(s)|$ then $|\chi(b_i)| = |\chi(s)|$ by part 1. If $|\alpha_j(s)| > 1$, then $|\chi/\chi_j|$ is dominant, by assumption that $|\chi/\chi_0|$ is dominant. Since $|\chi_j(b^n)| \leq |\chi_j(s)|$ for all j , $|\chi_j(b^n)| = |\chi_j(s)|$ if $|\alpha_j(s)| > 1$. Weight χ_j^2/α_j is always dominant or 1. So,

$$|\chi_j^2/\alpha_j(b^n)| \leq |\chi_j^2/\alpha_j(s)|.$$

So, if $|\alpha_j(s)| > 1$, we have $|\chi_j(b^n)| = |\chi_j(s)|$, so $|\alpha_j(b^n)| \geq |\alpha_j(s)|$. If $|\alpha_j(s)| = 1$, then obviously $|\alpha_j(b^n)| \geq |\alpha_j(s)|$.

Let χ' be a dominant weight s.t. every simple root enters χ' with positive coefficients. Since

$$|\chi'(b^n)| \leq |\chi'(s)|, \quad |\alpha_j(b^n)| \geq |\alpha_j(s)|$$

we have

$$|\alpha_j(b^n)| = |\alpha_j(s)|.$$

We use now any faithful representation of G in some GL_N an Theorem 3 to show that $b^n \in Ks$.

Suppose that $b_i \neq b^n$. Then we used some simple root α_j s.t. $|\alpha_j(b^{n-1})| < 1$ and the lemma to get b^n . There are two possibilities.

- 1) $|\alpha_j(s)| = 1$. Then, by the first part of the lemma, $|\alpha_j(b^n)| > 1 \Rightarrow |\alpha_j(b^n)| > |\alpha_j(s)|$ contradiction.
- 2) $|\alpha_j(s)| > 1$. Then,

$$|\chi_j(b^n)| \geq |p| |\chi_j(b^{n-1})| \geq |p| |\chi_j(b_i)| = |p| |\chi_j(s)|$$

again contradiction.

Q.E.D.

Now, we can prove the main theorem. The results which we used in case of $GL_n(Q)$, that is Lemma 1 and Theorem 3 were already generalized.

We assume that $p \notin \Sigma$, and we choose a basis $f_1 = f, f_2, \dots, f_n$ of the space which f spans under the action of the Hecke ring. We estimate

$$L(c) = \max_{g \in \mathcal{M}^c, i} |f_i(g)|$$

We prove that $L(c) \leq \max(CL(c/c_2), L(c/2))$, where c_2 can be chosen greater than 1 for $p \gg 1$.

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