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ZEROS OF HOLOMORPHIC VECTOR FIELDS ON SINGULAR SPACES AND INTERSECTION RINGS OF SCHUBERT VARIETIES

E. Akyildiz *, J.B. Carrell ** and D.I. Lieberman

Introduction

The purpose of this paper is to initiate a study of the cohomology rings of invariant subvarieties of a smooth projective variety X with a holomorphic vector field V having nontrivial zero set Z . We will first consider the case in which V is generated by a torus action on X , showing that if V is tangent to the set of smooth points of a closed subvariety Y of X such that $Y \cap Z$ is finite, then the graded ring $i^*H^*(X; C)$, $i: Y \rightarrow X$ being the inclusion, is the image under a C -algebra homomorphism ψ of the graded algebra associated to a certain filtration of $H^0(Y \cap Z; C)$. In certain cases, for example when Z is finite and i^* surjective, ψ is an isomorphism.

Applying this to the vector fields on flag varieties $X = G/B$ studied in $[C_1]$ and $[A]$ gives a surprising description of the cohomology algebra of a Schubert variety which is now explained. Suppose G is a semi-simple complex Lie group, B a Borel subgroup and $X = G/B$ the associated flag variety. Let \mathfrak{h} be a Cartan subalgebra of $\text{Lie}(G)$ in $\text{Lie}(B)$, and let W be the associated partially ordered Weyl group of G . For any regular element $h \in \mathfrak{h}$, consider the regular orbit $W \cdot h \subset \mathfrak{h}$ as a finite reduced subvariety of \mathfrak{h} with ring of regular functions $A(W \cdot h) = A(\mathfrak{h})/I(W \cdot h)$, the ring of complex polynomials on \mathfrak{h} modulo those vanishing on $W \cdot h$.

The ascending filtration on $A(\mathfrak{h})$ coming from the degree of a polynomial gives an ascending filtration F of $A(W \cdot h)$ whose associated graded ring $\text{Gr } A(W \cdot h)$ is isomorphic with $H^*(X; C)$ (see $[C_1]$). The upshot of our result on torus actions is that if $X_w = \bigcup_{v \leq w} BvB/B$ is the generalized Schubert variety in X determined by $w \in W$, then $H^*(X_w; C) \cong \text{Gr } A([e, w] \cdot h)$, where $[e, w] \cdot h = \{v \cdot h \mid v \leq w\}$ and the C -algebra on the right is the graded algebra associated to the ring of

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regular functions on the subvariety $[e, w] \cdot h$ of $W \cdot h$ with natural ascending filtration defined as above. In addition, the natural map $i^*: H^*(X; C) \rightarrow H^*(X_w; C)$ is precisely the restriction

$$j_h^*: \text{Gr } A(W \cdot h) \rightarrow \text{Gr } A([e, w] \cdot h)$$

where $j_h: [e, w] \cdot h \rightarrow W \cdot h$ is the inclusion.

The formulas for $H^*(X; C)$ and $H^*(X_w; C)$ have in a slightly modified form been extended to infinite dimensional flag varieties jointly by B. Kostant and S. Kumar. Namely, they have shown that the algebra C^W of all complex valued functions on W has a filtration whose associated graded algebra is $H^*(X; C)$ and that the induced filtration on $C^{[e, w]}$ gives rise to $H^*(X_w; C)$ in the usual way. See also the remarks at the end of §3.

Generalizing in another direction the second author has studied the graded rings $\text{Gr } A(W \cdot s)$, where s is a regular element of some subalgebra \mathfrak{o} of \mathfrak{h} . It turns out that if P is a parabolic in G containing B , $P = LU$ is its Levi decomposition, and u is any regular unipotent element of L , then the torus $Z(L)$ acts on the subvariety of flags X_u in X fixed by u with isolated fixed points, and this yields a homomorphism of graded algebras $\psi_s: \text{Gr } A(W \cdot s) \rightarrow H^*(X_u; C)$, where s is any regular element of $\mathfrak{o} = \text{Lie}(Z(L))$. As above, ψ_s is an isomorphism if and only if $i: X_u \rightarrow X$ is surjective on the level of cohomology. This result is used in $[C_2]$ to give a general version of a theorem of DeConcini and Procesi [DP].

For arbitrary holomorphic vector fields, namely those not generated by torus actions, we need the further assumption that Z is finite. It was shown in $[CL_2]$ that the ring $A(Z)$ of functions on Z (viewed as a possibly unreduced variety) has a filtration whose associated graded algebra $\text{Gr } A(Z) \cong H^*(X; C)$. If V is tangent to the set of smooth points of a closed subvariety Y , then the ring of regular functions $A(Y \cap Z)$ on the scheme-theoretic intersection of Y and Z inherits a filtration G_i from $A(Z)$. We show that the associated graded algebra $\text{Gr } A(Y \cap Z)$ maps homomorphically onto $i^*H^*(X; C) \subset H^*(Y; C)$ and interpret when this is an isomorphism.

The paper is arranged as follows: in §1 we study cohomology rings of smooth projective varieties having a torus action with nonisolated fixed points, essentially extending $[CL_2]$. In §2, we state the main theorems, and in §3 we work out the cohomology rings of Schubert varieties. In §4, and §5 we prove the main results.

§1. Holomorphic vector fields on smooth varieties

Throughout this paper, X denotes a compact Kaehler manifold and V a holomorphic vector field on X with non-trivial zero set Z . Let Ω_X^p (resp.

O_X) denote the sheaf of germs of holomorphic p -forms (resp. functions) on X , and let $i(V): \Omega_X^p \rightarrow \Omega_X^{p-1}$ be the contraction operator associated to V . The structure sheaf O_Z of Z is by definition $O_X/i(V)\Omega_X^1$. That is, Z is the possibly unreduced variety defined by the sheaf of ideals $I(Z) = i(V)\Omega_X^1$ in O_X . Since $i(V)^2 = 0$, one may consider the complex of sheaves (where $n = \dim X$).

$$0 \rightarrow \Omega_X^n \rightarrow \Omega_X^{n-1} \rightarrow \dots \rightarrow \Omega_X^1 \rightarrow O_X \rightarrow 0, \tag{1.1}$$

having differential $i(V)$, and giving rise to a spectral sequence with $E_1^{-p,q} = H^q(X; \Omega_X^p)$. The basic property of this spectral sequence is that all differentials vanish since X is compact Kaehler and $Z \neq \emptyset$ ([CL₁]; see also [G-H]).

Since $i(V)$ is a derivation of the complex (1.1), one immediately obtains results about the cohomology ring of X . To describe these, let \underline{H}_X denote the hypercohomology ring of X arising from (1.1) and recall that for every p and q , $\underline{H}_X^p \underline{H}_X^q \subset \underline{H}_X^{p+q}$. If

$$F_0 \underline{H}_X^p \subset F_{-1} \underline{H}_X^p \subset \dots \subset F_{-n} \underline{H}_X^p = \underline{H}_X^p$$

is the filtration associated to spectral sequence with $E_1^{-p,q} = H^q(X; \Omega_X^p)$, then

$$F_i \underline{H}_X^p F_j \underline{H}_X^q \subset F_{i+j} \underline{H}_X^{p+q} \tag{1.2}$$

Consider the bigraded ring $\text{Bigr } H_X = \sum_{p,q} A_{p,q}$, where

$$A_{p,q} = F_{-p} \underline{H}_X^{q-p} / F_{-p+1} \underline{H}_X^{q-p} \tag{1.3}$$

associated to (1.2). The degeneracy of $E_1^{-p,q} = H^q(X; \Omega_X^p) \Rightarrow \underline{H}_X^{q-p}$ implies a bigraded ring isomorphism

$$\sum_{p,q} H^q(X; \Omega_X^p) \cong \sum_{p,q} A_{p,q} \tag{1.4}$$

In general, the isomorphism (1.4) has no obvious geometric content. However, if either Z is finite or V is generated by a C^* action on X , then (1.4) computes the classical cohomology of X over X on the zeros of V . Set $A(Z) = H^0(X; O_Z)$ and denote the complex cohomology $H^*(X; C)$ of a space X by $H^*(X)$. The next two theorems describe these computations.

THEOREM 1: *Suppose Z is finite. Then the ring $A(Z)$ of regular functions on Z has a decreasing filtration F_i with $F_i F_j \subset F_{i+j}$ so that $\sum_{p \geq 0} H^p(X; \Omega_X^p)$ and $\text{Gr } A(Z) = \sum_{p \geq 0} F_{-p} / F_{-p+1}$ are isomorphic graded algebras. More-*

over, $H^q(X; \Omega_X^p)$ vanishes if $p \neq q$, so the cohomology ring $H^*(X)$ is isomorphic to $\text{Gr } A(Z)$.

PROOF: See [CL₂].

Next, let V (or some nonzero multiple kV) be the infinitesimal generator of a C^* action on X with not necessarily isolated fixed points Z . Observe that Z is smooth and that O_Z is simply the sheaf of germs of holomorphic functions on Z . Let \underline{H}_Z denote the ring $\sum_k \underline{H}_Z^k$ where $\underline{H}_Z^k = \sum_{q-p=k} H^q(Z; \Omega_Z^p)$. Clearly $\underline{H}_Z^r \underline{H}_Z^s \subset \underline{H}_Z^{r+s}$.

THEOREM 2: There exists a filtration F_i of \underline{H}_Z so that

$$\text{Bigr } \underline{H}_Z \cong \sum_{p,q} H^q(X; \Omega_X^p)$$

With the left hand side defined as in (1.3).

PROOF: Let $j: Z \rightarrow X$ denote the inclusion and consider the following diagram of complexes of sheaves:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \Omega_X^n & \xrightarrow{i(V)} & \Omega_X^{n-1} & \rightarrow & \dots & \rightarrow & \Omega_X^1 & \rightarrow & O_X & \rightarrow & 0 \\ & & \downarrow j^* & & \downarrow j^* & & & & \downarrow j^* & & \downarrow j^* & & \\ 0 & \rightarrow & \Omega_Z^n & \rightarrow & \Omega_Z^{n-1} & \rightarrow & \dots & \rightarrow & \Omega_Z^1 & \rightarrow & O_Z & \rightarrow & 0 \end{array}$$

On the second row, the differential is $i(V|Z)$ which, obviously, vanishes.

LEMMA 1: The inclusion $i: Z \rightarrow X$ induces a quasi-isomorphism of complexes. Consequently, for all q ,

$$\mathcal{H}^{-q}(\Omega_X) = \ker i(V) | \Omega_X^{-q} / i(V) \Omega_X^{-q+1} \cong \Omega_Z^{-q}$$

PROOF: See Fujiki [F] and [CS].

Note that \underline{H}_Z as defined above is the hypercohomology of $\{\Omega_Z, 0\}$. Hence Lemma 1 implies there exists an isomorphism of rings

$$\underline{H}_X \cong \underline{H}_Z \tag{1.6}$$

To prove Theorem 2, use the isomorphism (1.6) to produce a filtration F_i of \underline{H}_Z and apply (1.4).

COROLLARY 1: *Suppose $H^q(Z; \Omega_Z^p)$ vanishes if $p \neq q$. Then the same is true on X . Moreover, $\underline{H}_Z = \underline{H}_Z^0 \cong \Sigma H^{2j}(Z) = H^*(Z)$. Consequently, $H^*(Z)$ has a filtration so that*

$$\text{Gr } H^*(Z) \cong H^*(X)$$

PROOF: The first statement follows immediately from Theorem 2. The second follows from the fact that the hypotheses on Z imply $H^{2j}(Z) = H^j(Z; \Omega_Z^j)$, so $\underline{H}^0(Z) = \Sigma H^j(Z; \Omega_Z^j)$.

§2. Results on singular varieties

Suppose, to begin with, that Y is a closed C^* -invariant subvariety of the projective smooth variety X with algebraic C^* -action having fixed point set Z . We do not necessarily assume Y is irreducible or Z is finite. Let $\phi: \underline{H}_Z^0 \rightarrow H^0(Y \cap Z)$ be the natural homomorphism defined in (4.1) below, and let $G_i = \phi(F_i)$, where F_i is the filtration of \underline{H}_Z^0 (obtained by restricting the filtration of \underline{H}_Z of Theorem 2). $G_i G_j \subset G_{i+j}$, so there is an associated graded algebra $\text{Gr } H^0(Y \cap Z) = \Sigma_{p \geq 0} G_{-p} / G_{-p+1}$. Finally let $i: Y \rightarrow X$ and $j: Y \cap Z \rightarrow Z$ denote the inclusions.

THEOREM 3: *Assume $Y \cap Z$ is finite and that $j^*: H^0(Z) \rightarrow H^0(Y \cap Z)$ is surjective. Then there exists an algebra homomorphism $\psi: \text{Gr } H^0(Y \cap Z) \rightarrow H^*(Y)$, so that $\psi(G_{-p} / G_{-p+1}) \subset H^{2p}(Y)$, making the following diagram commute:*

$$\begin{array}{ccc} \text{Gr } \underline{H}_Z^0 & \rightarrow & \sum_{p \geq 0} H^p(X; \Omega_X^p) \\ \downarrow \phi' & & \downarrow i^* \\ \text{Gr } H^0(Y \cap Z) & \rightarrow & H^*(Y) \end{array}$$

where ϕ' is the homomorphism of graded algebras determined by ϕ . The mapping ϕ' is a surjection, so $\text{Im } \psi = \Sigma i^* H^p(X; \Omega_X^p)$. If all the odd Betti numbers of Y vanish, then ψ is an isomorphism if and only if either ψ is injective or surjective. In particular, ψ is an isomorphism if and only if $\Sigma i^* H^p(X; \Omega_X^p) = H^*(Y)$.

Note that the assumption that j^* is surjective in degree zero is equivalent to supposing that no component of Z meets Y more than once. Without this assumption, it is not the case that $\cup G_i = H^0(Y \cap Z)$. In case of surjectivity, $H^0(Y \cap Z)$ and $\text{Gr } H^0(Y \cap Z)$ have the same dimension.

A case of particular interest is when $H^p(X; \Omega_X^q)$ is trivial provided $p \neq q$. In this case $H^{2p}(X) = H^p(X; \Omega_X^p)$ for all $p \geq 0$ and $H^{\text{odd}}(X) = \{0\}$, but more importantly, the same is true on Z , by [CS]. Hence $\underline{H}_Z^0 = H^*(Z)$, the filtration F is a filtration of $H^*(Z)$ and the map $\underline{H}_Z^0 \rightarrow H^0(Y \cap Z)$ is j^* . We obtain therefore a commutative diagram of homomorphisms

$$\begin{array}{ccc}
 \text{Gr } H^*(Z) & \longrightarrow & H^*(X) \\
 \phi' \downarrow & & \downarrow i^* \\
 \text{Gr } H^0(Y \cap Z) & \longrightarrow & H^*(Y)
 \end{array} \tag{2.2}$$

where ϕ' is the graded homomorphism associated to j^* . This diagram computes the algebra $i^*H^*(X)$ on $Y \cap Z$ as $\text{Gr } H^0(Y \cap Z) / \ker \psi$. More can be said, however.

PROPOSITION: *Assume all odd Betti numbers of Y vanish, and $j^*: H^0(Z) \rightarrow H^0(Y \cap Z)$ is surjective. Then*

- (a) $\dim H^*(Y) = \dim H^0(Y \cap Z) = \dim \text{Gr } H^0(Y \cap Z)$,
- (b) $\dim \ker \psi = \text{coker } \psi = \dim H^*(Y) - \sum_{p \geq 0} \dim i^*H^p(X; \Omega^p)$, and
- (c) $\dim \ker i^* \geq \sum_{p \neq q} \dim H^p(X; \Omega^q)$.

We now take up the case of holomorphic vector fields. Unfortunately, there does not in general appear to be a simple analogue of Theorem 3 so we will stay in the case where V is a holomorphic vector field on X with isolated zeroes Z . Recall that $A(Z) = H^0(X; \mathcal{O}_Z)$ is the ring of functions on Z . For V -invariant subvariety Y of X with ideal sheaf $I(Y)$ (i.e. $VI(Y) \subset I(Y)$), define $\mathcal{O}_{Y \cap Z}$ to be $\mathcal{O}_Z / \text{Im}(I(Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Z \rightarrow \mathcal{O}_Z)$ and $A(Y \cap Z)$ to be $H^0(X; \mathcal{O}_{Y \cap Z})$. We say that Y satisfies the algebraic Hopf index theorem if $\dim_{\mathbb{C}} A(Y \cap Z) = \chi(Y)$, the Euler characteristic of Y .

THEOREM 4: *Keeping the above notation and assumptions, suppose $G_{-p} = j^*F_{-p}$ where F is the filtration of $A(Z)$ and $j^*: A(Z) \rightarrow A(Y \cap Z)$ is the natural surjective map. Then there exists a homomorphism $\psi: \text{Gr } A(Y \cap Z) \rightarrow H^*(Y)$ so that the induced homomorphism $\phi: \text{Gr } A(Z) \rightarrow \text{Gr } A(Y \cap Z)$ corresponds to $i^*: H^*(X) \rightarrow H^*(Y)$. ψ is an isomorphism provided i^* is surjective and $A(Y \cap Z)$ satisfies the algebraic Hopf index theorem.*

Sometimes $A(Z)$ is already graded and isomorphic with $H^(X)$. This is, in particular, the case when V is generated by a unipotent subgroup of an SL_2 action on X (see [ACLS_{1,2}]). There is a one to one correspondence between ideals in $A(Z)$ and V -equivariant subvarieties Y of X . Thus we get information about the topology of invariant subvarieties from the ideals in $A(Z)$.*

§3. Computation of intersection rings of Schubert varieties

Let $G \supseteq B \supseteq H$ denote a semi-simple complex Lie group, a Borel subgroup and a maximal torus respectively, and let $X = G/B$ be the associated flag variety. We will write points of X as gB , $g \in G$. Also let W be the Weyl group of (G, H) . H acts on X with exactly $|W|$ fixed points, namely $X^H = \{wB \mid w \in W\}$, where by definition wB is $n_w B$ for any representative n_w in $N_G(H)$ of w .

Recall that the Schubert varieties in X are the closures of the B -orbits BwB of the fixed points wB of H . Let $X_w = \overline{BwB}$ be the Schubert variety determined by w . A more useful definition is that $X_w = \bigcup_{v \leq w} BvB$ where \leq denotes the partial order on W determined by fixing B . Clearly, H leaves every Bruhat cell BwB invariant and $X^H \cap BwB = \{wB\}$. Consequently each X_w is H invariant and $X^H \cap X_w = \{vB \mid v \leq w\}$.

Any regular element h of $\mathfrak{h} = \text{Lie}(H)$ determines a holomorphic vector field V_h on X with $Z_h = \text{zero}(V_h) = X^H$, by the definition of regularity. Moreover, V_h is tangent to every X_w . In order to analyse the filtration of $H^0(X, \mathcal{O}_{Z_h}) = H^0(X^H)$, consider the ring $A(W \cdot h)$ of regular functions on the orbit $W \cdot h$ of h , i.e. $A(W \cdot h) = A(\mathfrak{h})/I(W \cdot h)$, where $I(W \cdot h)$ is the ideal of all $f \in A(\mathfrak{h})$, the ring of polynomials on \mathfrak{h} , such that $f|_{W \cdot h} = 0$. It was shown in [C₁] that the homomorphism $\psi_h: A(W \cdot h) \rightarrow H^0(X^H)$ defined for homogeneous f by $\psi_h(f)(wB) = (-1)^{\deg f(w \cdot h)}$ maps the natural ascending filtration F_i of $A(W \cdot h)$ coming from degree in $A(\mathfrak{h})$ onto the filtration F_{-i} of $H^0(X^H)$, i.e. $\psi_h F_i = F_{-i}$.

From this follows the first statement of the next theorem.

THEOREM 5: *The cohomology ring of X is the graded ring associated to the filtration of $A(W \cdot h)$ induced by degree in $A(\mathfrak{h})$; i.e. $H^*(X) \cong \text{Gr } A(W \cdot h)$. Under this isomorphism, the element $[\chi]$ of $A(W \cdot h)$ defined by $d\chi|_{W \cdot h}$, where χ is any character on H , corresponds via ψ_h modulo F_0 (i.e. constants) to the first Chern class $c_1(L_\chi)$ of the line bundle L_χ on X associated to χ . For any $w \in W$, $H^*(X_w)$ is the graded ring associated to the degree filtration of $A([e, w] \cdot h)$, where $[e, w] = \{v \in W \mid v \leq w\}$. Moreover, this isomorphism commutes with the natural restrictions, i.e. there is a commutative diagram of C -algebra homomorphisms*

$$\begin{array}{ccc}
 \text{Gr } A(W \cdot h) & \xrightarrow{\sim} & H^*(X) \\
 \text{res} \downarrow & & \downarrow \text{res} \\
 \text{Gr } A([e, w] \cdot h) & \xrightarrow{\sim} & H^*(X_w)
 \end{array}$$

PROOF: We have checked that ψ_h is an isomorphism of filtered rings. From the commutative diagram

$$\begin{array}{ccc}
 A(W \cdot h) & \xrightarrow{\psi_h} & H^0(X^H) \\
 \text{res} \downarrow & & \downarrow \text{res} \\
 A([e, w] \cdot h) & \xrightarrow{\text{res } \psi_h} & H^0([e, w]B)
 \end{array}$$

and the fact that the filtration \tilde{F}_{-i} of $H^0([e, w]B)$ is F_{-i} , where F_{-i} is the filtration of $H^0(X^H)$, it follows that $\text{res } \psi_h$ is also an isomorphism of filtered rings. In particular, $\text{Gr } A([e, w] \cdot h) \cong \text{Gr } H^0([e, w]B) \cong H^*(X_w)$ by Theorem 3.

REMARKS: (a) In the proof, we used the surjectivity of $H^*(X) \rightarrow H^*(X_w)$ for any w , which follows from the fact that every X_w is a union of Bruhat cells. We could as easily have stated the theorem for any Zariski closed Y in X which is a union of Bruhat cells.

(b) Classically, $H^*(X)$ is described as the coinvariant algebra $A(\mathfrak{h})/I_W$ where I_W is the ideal generated by the homogeneous W -invariants of positive degree. For regular h , $I(W \cdot h)$ is generated by the W -invariants vanishing at h , and from this it is not hard to see directly that $\text{Gr } A(W \cdot h) \cong A(\mathfrak{h})/I_W$. Thus the first statement of Theorem 5 can be derived independently of torus actions. The statements about Schubert varieties are not as immediate. In order to find a homogeneous ideal in $A(\mathfrak{h})$ containing I_W whose quotient algebra is $H^*(X_w)$, consider the ideal $\text{gr } I([e, w] \cdot h)$ generated by leading terms of elements of $I([e, w] \cdot h)$. We have

$$A(\mathfrak{h})/\text{gr } I([e, w] \cdot h) \cong \text{Gr } A([e, w] \cdot h)$$

and hence, by Theorem 5, $H^*(X_w) \cong A(\mathfrak{h})/\text{gr } I([e, w] \cdot h)$.

(c) The cohomology rings of flag varieties in the case of affine Weyl groups no longer have a coinvariant algebra description. Recently, however, Kostant and Kumar have showed that the rings of all complex valued functions C^W (resp $C^{[e, w]}$) on W (resp. $[e, w]$) have filtrations whose associated graded rings are $H^*(X)$ (resp. $H^*(X_w)$). The geometric methods used here are replaced by the infinite dimensional version of Lie algebra cohomology due to Kumar, [Ku₁]. For details, see the announcement [Ku₂].

There is a similar computation for Schubert varieties in $G/P = X_p$ for any parabolic $P \supset B$. Let W_p denote the Weyl group of (P, H) and recall

that elements \bar{w} of W/W_p parameterize the B -orbits in X_p ([BGG]). Let $X_{\bar{w}} = \bar{B}wP \subset X_p$ be the Schubert variety associated to \bar{w} . Note that $A(W \cdot h)^{W_p}$ has an increasing filtration. It was shown in [A] that $\text{Gr } A(W \cdot h)^{W_p} \cong H^*(X_p)$, the isomorphism being compatible with the inclusions $A(W \cdot h)^{W_p} \subset A(W \cdot h)$ and $H^*(X_p) \rightarrow H^*(X)$. Since $A(W \cdot h)^{W_p}$ is by definition $A(W_p \setminus W \cdot h)$ this means there is a commutative diagram

$$\begin{array}{ccc} \text{Gr } A(W_p \setminus W \cdot h) & \rightarrow & H^*(X_p) \\ \downarrow & & \downarrow \\ \text{Gr } A(W \cdot h) & \rightarrow & H^*(X) \end{array}$$

Let $[\overline{e, w}] \cdot h$ denote the image of $[e, w] \cdot h$ under the natural map $W \cdot h$ onto $W_p \setminus W \cdot h$. Let $\text{Gr } A([\overline{e, w}] \cdot h)$ be the graded ring associated to the filtration induced from $A(W_p \setminus W \cdot h)$.

THEOREM 6: *For any $w \in W$, $H^*(X_{\bar{w}})$ is isomorphic with $\text{Gr } A([\overline{e, w}] \cdot h)$. The restriction map $H^*(X_p) \rightarrow H^*(X_{\bar{w}})$ corresponds to the natural map $\text{Gr } A(W_p \setminus W \cdot h) \rightarrow \text{Gr } A([\overline{e, w}] \cdot h)$*

The proof is similar to the proof of Theorem 5.

§4. Proof of Theorem 3

Recall that Y is a C^* -invariant subvariety of a projective manifold X with C^* -action having nontrivial fixed point set Z . Recall $H^0(Y \cap Z)$ is finite. Let $f: \tilde{Y} \rightarrow Y$ denote a C^* -equivariant resolution of singularities of Y (see [H]), and let $\tilde{f} = i \circ f$ where $i: Y \rightarrow X$ is the inclusion. Note that since \tilde{f} is an equivariant map into X , one has, for every p , induced morphisms $f^*: \underline{H}_X^0 \rightarrow \underline{H}_{\tilde{Y}}^0$ taking the filtration F_i of \underline{H}_X^0 into the filtration \tilde{F}_i of $\underline{H}_{\tilde{Y}}^0$, i.e. so that $\tilde{f}^*F_i \subset \tilde{F}_i$.

Since $\underline{H}_X^0 \cong \underline{H}_Z^0$, there is a homomorphism $\underline{H}_X^0 \xrightarrow{\phi} H^0(Y \cap Z)$ obtained by the composition

$$\underline{H}_Z^0 = \sum_{p \geq 0} H^p(Z; \Omega_Z^p) \rightarrow H^0(Z) \rightarrow H^0(Y \cap Z) \tag{4.1}$$

On the other hand, if \tilde{Z} denotes \tilde{Y}^{C^*} , then by Lemma 1, $\underline{H}_{\tilde{Y}}^0 \cong \underline{H}_{\tilde{Z}}^0$, so there exists a map $f^*: H^0(Y \cap Z) \rightarrow \underline{H}_{\tilde{Y}}^0$, due to the fact that $f(\tilde{Z}) = Y \cap Z$. In more detail, f^* is the pull back $H^0(Y \cap Z) \rightarrow H^0(\tilde{Z}) \subset \underline{H}_{\tilde{Z}}^0 \cong \underline{H}_{\tilde{Y}}^0$. Clearly, $f^*G_{-p} \subset \tilde{F}_{-p}$ where \tilde{F}_{-p} is the filtration of $\underline{H}_{\tilde{Z}}^0$. Hence there is

a commutative diagram

$$\begin{array}{ccc}
 & F_{-p} & \xrightarrow{e_p(X)} & H^p(X, \Omega_X^p) \\
 & \searrow \phi & & \swarrow \\
 G_{-p} & \xrightarrow{\tilde{f}^*} & i^*H^p(X; \Omega_X^p) & \xrightarrow{\tilde{f}^*} \\
 & \searrow f^* & \swarrow f^* & \\
 & \tilde{F}_{-p} & \xrightarrow{e_p(\tilde{Y})} & H^p(\tilde{Y}, \Omega_{\tilde{Y}}^p)
 \end{array}
 \tag{4.2}$$

where $e_p(X)$ and $e_p(\tilde{Y})$ are the appropriate edge morphisms. We wish to define the indicated map $e_p(Y)$. Thus it must be shown that if $\alpha \in F_{-p}$ and $\phi(\alpha) = 0$, then $i^*e_p(X)\alpha = 0$ in $H^{2p}(Y)$. But $\phi(\alpha) = 0$ implies $f^*(\alpha)$ vanishes in \tilde{F}_{-p} , so $e_p(\tilde{Y})f^*(\alpha) = 0$ as well. By commutativity, $\tilde{f}^*e_p(X)\alpha = 0$. This implies, by a result of Deligne [D], that $i^*e_p(X)\alpha = 0$. Therefore we may define $e_p(Y) = i^*e_p(X)\phi^{-1}$.

Next it will be shown that $i^*H^p(X; \Omega_X^p) = e_p(Y)G_{-p}$. For that, consider the commuting diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & F_{-p+1} & \rightarrow & F_{-p} & \rightarrow & H^p(X; \Omega_X^p) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow i^* \\
 0 & \rightarrow & G_{-p+1} & \rightarrow & G_{-p} & \rightarrow & i^*H^p(X; \Omega_X^p) \rightarrow 0
 \end{array}$$

The top sequence is exact and the vertical maps are surjective, so $e_p(Y)$ is surjective also, and $e_p(Y)(G_{-p+1}) = 0$. This yields the map ψ of Theorem 3, and gives (2.1). We leave it to the reader to show ψ is an algebra homomorphism.

Next, suppose $\dim H^{\text{odd}}(Y) = 0$. The assumption $j^*H^0(Z) = H^0(Y \cap Z)$ implies that $\dim \text{Gr } H^0(Y \cap Z) = \dim H^0(Y \cap Z)$. Thus the equality $\dim \text{Gr } H^0(Y \cap Z) = \dim H^0(Y)$ will follow from

$$\dim H^0(Y \cap Z) = \# \{Y \cap Z\} = \chi(Y) = \dim H^0(Y).
 \tag{4.3}$$

The last equality follows from $\dim H^{\text{odd}}(Y) = 0$. To prove (4.3) we establish the next lemma.

LEMMA 2: *The Euler characteristic $\chi(Y)$ of an invariant subvariety Y of X is $\# \{Y \cap Z\}$, as long as $Y \cap Z$ is finite.*

PROOF: Consider the plus decomposition $Y = \bigcup_{x \in Y \cap Z} Y_x^+$, where $Y_x^+ =$

$\{y \in Y \mid \lim_{\lambda \rightarrow 0} \lambda \cdot y = x\}$. This is a locally closed decomposition of Y and each Y_x^+ is contractible. Hence $\chi(Y) = \sum_{x \in Y \cap Z} \chi(Y_x) = \# \{Y \cap Z\}$.

We leave the proof of the Proposition to the reader.

§5. Proof of Theorem 4

Let V denote a holomorphic vector field on X with isolated zeros which is tangent to Y . Consider the exact sequence

$$0 \rightarrow I(Y)_Z \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{Y \cap Z} \rightarrow 0$$

Since $I(Y)_Z$ is a coherent sheaf on Z and Z is finite, one has an exact sequence

$$0 \rightarrow H^0(X; I(Y)_Z) \rightarrow A(Z) \rightarrow A(Y \cap Z) \rightarrow 0$$

(since $H^1(X; I(Y)_Z)$ vanishes. Thus the following situation holds:

$$\begin{array}{c} A(Z) \rightarrow A(Y \cap Z) \rightarrow 0 \\ \downarrow \parallel \\ \underline{H}_X^0 \end{array}$$

Let $f: \tilde{Y} \rightarrow Y$ be a V -equivariant resolution of singularities and let f^* denote the unique map making the following diagram commute:

$$(if)^* \begin{array}{ccc} \underline{H}_X^0 \cong A(Z) & \xrightarrow{i^*} & A(Y \cap Z) \rightarrow 0 \\ \downarrow & \swarrow f^* & \\ \underline{H}_Y^0 & & \end{array}$$

(Note that f^* need not apriori exist since V lifted to \tilde{Y} may not have isolated zeros.) Now one can use this diagram to replace diagram (4.1) in (4.2). The rest of the argument will follow that of the last section, replacing the Lemma with the algebraic Hopf theorem and using the fact that $H^p(X; \Omega_X^q)$ is trivial if $p \neq q$ since V has isolated zeros on X .

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