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CYCLOTOMY AND AN EXTENSION OF THE TANIYAMA GROUP

Greg W. Anderson

§0. Introduction

The theme of this paper is *factorization*: e.g. the factorization of the values of certain Hecke characters of cyclotomic fields into gaussian sums, the factorization of the special values of the corresponding Hecke L-series into values of the classical gamma function, and the factorization of the Hodge structures of Fermat hypersurfaces into "fractional" Hodge structures. The goal of the paper is to explain such factorization phenomena in terms of a factorization of the motives of Fermat hypersurfaces into *ulterior motives*.

Concerning the goal of the paper, we can be a little more specific. The category of motives for absolute Hodge cycles defined over $\mathbb{Q}$ can be identified with the category of finite dimensional representations defined over $\mathbb{Q}$ of a certain proreductive affine group scheme $\mathscr{G}$ over $\mathbb{Q}$; the quotient T of $\mathscr{G}$ the representations of which classify those motives constructible from spectra of numberfields and abelian varieties over $\mathbb{Q}$ with potential complex multiplication is, by a theorem of Deligne [7], canonically isomorphic to the Taniyama group of Langlands [10]. (See §4 and §5 for definitions and further explanation.) Moreover, one can show (see §9) that the motives

$$H^*_{\text{prim}}(X^n_m) \overset{\text{def}}{=} H^*(X^n_m)/H^*(\mathbb{P}^{n-1}),$$

where X^n_m is the Fermat hypersurface of degree m and dimension $n-2$, have the property that, viewed as representations of $\mathscr{G}$, all are inflations of representations of T. Our object in this paper is to construct an exact sequence

$$0 \to 2\pi i\hat{\mathbb{Z}} \to \tilde{T} \to T \to 1$$

of affine group schemes over $\mathbb{Q}$, where $2\pi i\hat{\mathbb{Z}}$ denotes the profinite completion of the kernel of the exponential exp: $\mathbb{C} \to \mathbb{C}^*$, and an (infinite-dimensional) representation $\mathbb{E}$ of $\tilde{T}$ defined over $\mathbb{Q}$ such that

$$\mathbb{E}^{2\pi i\hat{\mathbb{Z}}} = 0 \qquad (0.1)$$

and for all m, $n > 1$ there exists an isomorphism

$$H^*_{\mathrm{prim}}(\mathrm{X}^{\mathrm{n}}_{\mathrm{m}}) \otimes \mathrm{H}^2(\mathbb{P}^1) \tilde{\to} \left(\mathbb{E}^{\otimes \mathrm{n}}_{\mathrm{m}}\right)^{2\pi \mathrm{i}\hat{\mathbf{Z}}} \tag{0.2}$$

of representations of $\tilde{T}$, where

$$\mathbb{E}_m \stackrel{\mathrm{def}}{=} \mathbb{E}^{2\pi \mathrm{i} m \hat{\boldsymbol{Z}}}.$$

The objects $\mathbb{E}_m$ are each finite-dimensional; more precisely

$$\dim_{\mathbf{Q}} \mathbb{E}_m = m - 1.$$

We shall call the objects $\mathbb{E}_m$ *ulterior motives* because while by (0.1) they are not themselves motives, motives may be constructed from them via the operations of linear algebra as is made evident by (0.2). The structure of $\tilde{T}$ and $\mathbb{E}$ is described in detail by Theorem 8 of §6, the main theorem of the paper.

As an application of the main result we prove a conjecture of Lichtenbaum [3] (the "Γ-hypothesis" inspired by observations of Weil [22]) giving the critical values of certain Hecke L-series (those attached to the "Jacobi sum Hecke characters" studied by Weil [18,21], Deligne [5], Kubert-Lichtenbaum [9], Kubert [8], and others) up to an undetermined rational factor as a monomial in values of the classical gamma function $\Gamma(s)$ for rational values of s. (See §2 for a definition of the class of Jacobi sum Hecke characters and a formulation of the Γ-hypothesis.) In essence, the proof of the Γ-hypothesis (given in §8) is a reduction to Deligne's conjecture [6] joined with the observation that, in consequence of basic results of Siegel [15] and Blasius [1], Deligne's conjecture is actually a theorem in all cases of relevance to the Γ-hypothesis.

Now Blasius' result is a quite delicate relation between the periods of algebraic integrals on CM abelian varieties and the special values of Hecke L-series; that such relations should exist is not a new idea and it would be misleading not to indicate its background briefly. The results of Damerell [4] point in this direction. Shimura has developed this idea to a high degree, in particular proving in [13] a vast generalization of Damerell's result. Shimura's work provided the foundation that Blasius built upon and a great deal of the evidence upon which Deligne based his conjecture.

A portion of the paper is expository: In order to fix language and notation, the key points of the theory of tannakian categories and of the theory of motives for absolute Hodge cycles are summarized in §3 and §4, respectively. A general discussion of just how the theory of motives for absolute Hodge cycles might be brought to bear upon the problem of evaluating Hecke L-series is given in §5.

The key concept of the paper (introduced in §6) is that of an *arithmetic Hodge structure* (*AHS*), a notion intermediate between that of a motive for absolute Hodge cycles over $\mathbb{Q}$ and that of a Hodge structure in the sense that the functor assigning to each motive its underlying Hodge structure factors through the category of arithmetic Hodge structures. An AHS has "de Rham" and "Betti" cohomological realizations, but (in general) no "ℓ-adic" realizations. The Hodge numbers $h^{p,q}$ of an arithmetic Hodge structure are defined for all pairs p, q of *rational* numbers summing to an integer and need not vanish when p and q fail to be integral.

Equipped with the notion of an AHS, we define $\tilde{T}$ and $\mathbb{E}$ as follows. $\tilde{T}$ is defined to be the group associated to the tannakian subcategory of the category of AHS's generated by the class of AHS's consisting of every AHS underlying a motive "potentially of CM type" and every member of a certain "nested" family $\{E_m\}_{m=1}^{\infty}$ of "extrageometrical" AHS's. The family $\{E_m\}$ is "nested" in the sense that E_m is a subobject of E_n whenever m divides n, and "extrageometrical" in the sense that E_m cannot arise as the AHS underlying a motive as its "Hodge type" is $(1/m, (m-1)/m), \ldots, ((m-1)/m, 1/m)$. The representations $\mathbb{E}_m$ of $\tilde{T}$ are defined to be those associated to the E_m and $\mathbb{E}$ is defined to be the direct limit of the $\mathbb{E}_m$. Relation (0.1) is deduced from the extrageometrical property of the E_m and relation (0.2) from an analogous relation among arithmetic Hodge structures. A crucial role is played by Deligne's theory [7] of absolute Hodge cycles on abelian varieties: We use it to deduce that the morphism $\tilde{T} \to T$ dual to the functor assigning to each motive potentially of CM type its underlying arithmetic Hodge structure is faithfully flat.

The proof of the main theorem is carried out in §9, §10, and §11. The had work is concentrated in §9 and §10, where the cohomology of the Fermat hypersurfaces is subjected to close scrutiny. A geometrical insight of Shioda-Katsura [14] plays a key role. In §11 we simply tie up the loose ends.

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§1. Notation and conventions

1.0. We employ the following more or less standard notation:

$\mathbb{Z} \stackrel{\text{def}}{=}$ the rational integers

$\mathbb{Q} \stackrel{\text{def}}{=}$ the rational numbers

$\mathbb{R} \overset{\text{def}}{=}$ the real numbers

$\mathbb{C} \overset{\text{def}}{=}$ the complex numbers

$\hat{\mathbb{Z}} \overset{\text{def}}{=}$ the profinite completion of $\mathbb{Z}$

$\mathbb{Z}_p \overset{\text{def}}{=}$ the p-adic completion of $\mathbb{Z}$

$\mathbb{Q}_p \overset{\text{def}}{=}$ the p-adic completion of $\mathbb{Q}$

$\mathbb{F}_q \overset{\text{def}}{=}$ the field of q elements

1.1. Given fields K, L, M and embeddings σ: $K \to L$, τ: $L \to M$ we denote the image of $x \in K$ under σ by x^σ and the composition of σ with τ by $\tau\sigma$. In order that these conventions be consistent we must have

$$(x^\sigma)^\tau = x^{(\tau\sigma)} \tag{1.1.1}$$

for all $x \in K$.

1.2. A *numberfield* is understood to be a finite extension of $\mathbb{Q}$ embedded in $\mathbb{C}$. The union of all numberfields is denoted by $\overline{\mathbb{Q}}$. Given a numberfield k, $\mathfrak{g}(k)$ denotes the galois group of $\overline{\mathbb{Q}}$ over k; the galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$ is denoted simply by $\mathfrak{g}$.

1.3. We choose a square root of -1 in $\mathbb{C}$ and denote it by i. Complex conjugation is denoted by ρ.

1.4. For each real number x we write

$[[x]] \overset{\text{def}}{=}$ the greatest integer less than or equal to x,

$\langle x \rangle \overset{\text{def}}{=} x - [[x]]$,

$e(x) \overset{\text{def}}{=} \exp(2\pi i x)$.

1.5. For each rational prime p we fix an ultrametric absolute value $x \mapsto |x|_p$: $\mathbb{C} \to \mathbb{R}_{\geqslant 0}$ extending the p-adic absolute value of $\mathbb{Q}$ with respect to which $\mathbb{C}$ is complete. (Such an absolute value exists because the completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$ is algebraically closed of the same cardinality as $\mathbb{C}$, hence abstractly isomorphic to $\mathbb{C}$.) We identify $\mathbb{Q}_p$ with the closure of $\mathbb{Q}$ in $\mathbb{C}$ relative to the topology defined by $|?|_p$. We put

$$D(p) \overset{\text{def}}{=} \{\sigma \in \mathfrak{g} \mid \forall x \in \overline{\mathbb{Q}},\ |x^\sigma|_p = |x|_p\},$$

$$I(p) \overset{\text{def}}{=} \{\sigma \in \mathfrak{g} \mid \forall x \in \overline{\mathbb{Q}},\ |x|_p \leqslant 1 \Rightarrow |x - x^\sigma|_p < 1\}.$$

We fix $F(p) \in D(p)$ such that

$$\forall x \in \overline{\mathbb{Q}},\ |x|_p \leqslant 1 \Rightarrow |x^p - x^{F(p)^{-1}}|_p < 1, \tag{1.5.1}$$

referring to $F(p)$ as the *geometric Frobenius* at p.

1.6. Given sets X and Y

$$X \sim Y \overset{\text{def}}{=} \{x \in X \mid x \notin Y\}.$$

1.7. All rings are commutative and possess a unit element (unless otherwise noted). Every module over a ring R satisfies

$$1_R m = m$$

for all $m \in M$, where 1_R denotes the unit element of R. Every homomorphism f: $R \to S$ of rings verifies $f(1_R) = 1_S$. The category of finitely generated modules over a ring R is denoted by $\mathscr{MOD}(R)$.

1.8. Given a ring R, the multiplicative group of R is denoted either by $\mathbb{G}_m(R)$ or R^*. Given also a positive integer n, the group of $n^{\underline{\text{th}}}$ roots of unity in R is denoted by $\boldsymbol{\mu}_n(R)$.

1.9. The *cyclotomic character* χ_{cyc}: $\mathfrak{g} \to \hat{\mathbb{Z}}^*$ is defined to be the unique continuous homomorphism such that for all $\sigma \in \mathfrak{g}$ and complex roots ζ of unity

$$\zeta^{\chi_{\text{cyc}}(\sigma)} \overset{\text{def}}{=} \zeta^{\sigma}.$$

1.10. For each numberfield k, $\mathcal{O}_k$ denotes the ring of integers of k and for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$ we write

$$\mathbb{N}\mathfrak{p} \overset{\text{def}}{=} [\mathcal{O}_k : \mathfrak{p}].$$

By a *prime* $\mathfrak{p}$ of k we always understand a *prime ideal* $\mathfrak{p}$ of $\mathcal{O}_k$.

1.11. Given a field k, a k-vectorspace V of finite dimension n over k and a k-linear map f: $V \to V$, the *trace* of f (the sum of the diagonal entries of any n by n matrix representing f) is denoted by $\text{tr}_k(f \mid V)$, the *determinant* of f (the determinant of any n by n matrix representing f) by $\det_k(f \mid V)$.

1.12. Given a field k and a vectorspace V over k, $GL_k(V)$ denotes the functor assigning to each k-algebra R the group of R-linear automorphisms of the R-module $V \underset{k}{\otimes} R$.

1.13. Given a category $\mathscr{A}$ and a group G, a *G-object* of $\mathscr{A}$ is an object X of $\mathscr{A}$ equipped with a structure map $G \to \mathrm{Aut}_{\mathscr{A}}(X)$. Given a *covariant* functor F: $\mathscr{A} \to \mathscr{B}$ and a G-object X of $\mathscr{A}$, we shall consider $F(X)$ to be a G-object under the structure map $G \to \mathrm{Aut}_{\mathscr{B}}(F(X))$ obtained by composing $G \to \mathrm{Aut}_{\mathscr{A}}(X)$ with the homomorphism

$$f \mapsto F(f) \colon \mathrm{Aut}_{\mathscr{A}}(X) \to \mathrm{Aut}_{\mathscr{B}}(F(X)).$$

Given a *contravariant* functor F': $\mathscr{A} \to \mathscr{C}$ and a G-object X of $\mathscr{A}$, we shall consider $F'(X)$ to be a G-object under the structure map $G \to \mathrm{Aut}_{\mathscr{C}}(F'(X))$ obtained by composing $G \to \mathrm{Aut}_{\mathscr{A}}(X)$ with the homomorphism

$$f \mapsto F'(f^{-1}) \colon \mathrm{Aut}_{\mathscr{A}}(X) \to \mathrm{Aut}_{\mathscr{C}}(F'(X)).$$

1.14. The symbol "□" signals the end of a proof or the omission of a proof.

§2. Jacobi sum Hecke characters and the Γ-hypothesis

2.0. We shall define the notion of a *Jacobi sum Hecke character* and formulate the *Γ-hypothesis* of Lichtenbaum [4] concerning the special values of the L-series associated to Jacobi sum Hecke characters.

2.1. Let k be a numberfield. We denote the idèle group of k by $\mathscr{J}_k$. A *Hecke character* of k of *type* A_0 is understood to be a homomorphism ψ: $\mathscr{J}_k \to \overline{\mathbb{Q}}^*$ such that the kernel of ψ is open in $\mathscr{J}_k$ and such that for a suitable function Θ: $\mathfrak{g}/\mathfrak{g}(k) \to \mathbb{Z}$ and all $x \in k^*$

$$\psi(x) = \prod_{\sigma \in \mathfrak{g}/\mathfrak{g}(k)} (x^{\sigma})^{\Theta(\sigma)}.$$

(Cf. the exercise on p. II-17 of [12].) The function Θ is uniquely determined by ψ and called the *infinity type* of ψ. The group of Hecke characters of k of type A_0 is denoted by $A_0(k)$. The restriction of a Hecke character $\psi \in A_0(k)$ to the copy of $(k \otimes \mathbb{R})^*$ embedded in $\mathscr{J}_k$ is denoted by ψ_∞ and called the *infinite component* of ψ. The *conductor* $\mathfrak{f}$ of $\psi \in A_0(k)$ is the integral ideal of k dividing every integral ideal $\mathfrak{a}$ of k with the property that $\psi(x) = 1$ for all idèles $x \in \mathscr{J}_k$ satisfying $x_v > 0$ at all real places v of k and $|x_v - 1|_v \leqslant |\mathfrak{a}|_v$, $|x_v|_v = 1$ at all finite places

v of k. Given $\psi \in A_0(k)$ of conductor $\mathfrak{f}$ and a prime $\mathfrak{p}$ of k not dividing $\mathfrak{f}$ we write

$$\psi(\mathfrak{p}) \stackrel{\text{def}}{=} \psi(\pi_{\mathfrak{p}})$$

where $\pi_{\mathfrak{p}} \in \mathscr{I}_k$ is any uniformizing element at $\mathfrak{p}$. The *weight* w of $\psi \in A_0(k)$ is defined to be the unique integer w such that for all primes $\mathfrak{p}$ of k not dividing the conductor of ψ,

$$|\psi(\mathfrak{p})| = \mathbb{N}\mathfrak{p}^{w/2}.$$

The *L-series* $L_k(s, \psi)$ associated to $\psi \in A_0(k)$ is given by the infinite product

$$L_k(s, \psi) \stackrel{\text{def}}{=} \prod \left(1 - \frac{\psi(\mathfrak{p})}{\mathbb{N}\mathfrak{p}^s}\right)^{-1}$$

extended over all primes $\mathfrak{p}$ of k not dividing the conductor of ψ.

2.2. Let $\mathbb{B}$ denote the free abelian group on the symbols $[a]$ where a runs through the nonzero elements of $\mathbb{Q}/\mathbb{Z}$. Given $\boldsymbol{a} = \Sigma n_a[a] \in \mathbb{B}$ we put

$$w(\boldsymbol{a}) \stackrel{\text{def}}{=} \sum n_a,$$

$$\langle \boldsymbol{a} \rangle \stackrel{\text{def}}{=} \sum n_a \langle a \rangle.$$

For each rational prime p, let $\mathbb{B}_p$ denote the subgroup of $\mathbb{B}$ generated by elements of the form

$$\sum_{j=1}^{f} [p^j a]$$

where f is any positive integer, $0 \neq a \in \mathbb{Q}/\mathbb{Z}$ any element annihilated by $p^f - 1$. Given a positive integral power $q = p^f$ of p, let b_q: $\boldsymbol{\mu}_{q-1}(\mathbb{C}) \to \frac{1}{p}\mathbb{Z}/\mathbb{Z}$ denote the unique function with the property that for all $\theta \in \boldsymbol{\mu}_{q-1}(\mathbb{C})$

$$\left| p\langle b_q(\theta)\rangle - \sum_{j=1}^{f} \theta^{p^j} \right|_p < 1.$$

PROPOSITION / DEFINITION 2.2.1: *For each rational prime p there exists a unique homomorphism* g_p: $\mathbb{B}_p \to \mathbb{C}^*$ *such that for all positive integral* *powers* $q=p^f$ *and* $0 \neq a \in \mathbb{Q}/\mathbb{Z}$ *annihilated by* q-1

$$g_p\left(\sum_{j=1}^{f}\left[p^j a\right]\right) = -\sum_{\theta\in\mu_{q-1}(\mathbb{C})} \theta^{-\langle a\rangle(q-1)} e\big(b_q(\theta)\big).$$

PROOF. This is a reformulation of the classical Hasse-Davenport theorem. See App. 5 of Weil's book [19] for a discussion in modern language. □

PROPOSITION 2.2.2: *For all rational primes* p *and* $\boldsymbol{a} \in \mathbb{B}_p$ *the following hold*:

(I) $|g_p(\boldsymbol{a})|_p = |p|_p^{\langle \boldsymbol{a}\rangle}$

(II) $|g_p(\boldsymbol{a})| = p^{w(\boldsymbol{a})/2}$.

PROOF: Prop. 2.2.2 (I) is a reformulation of Stickelberger's theorem. See Weil's article [20] for a proof of (I) in essentially the same form as we have presented it. Prop. 2.2.2 (II) is well known. □

2.3. Given $\boldsymbol{a} = \sum n_a[a] \in \mathbb{B}$ and $\sigma \in \mathfrak{g}$ put

$m(\boldsymbol{a}) \overset{\text{def}}{=}$ cardinality of the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by the set $\{a \in \mathbb{Q}/\mathbb{Z} \sim \{0\} \mid \mathrm{n_a} \neq 0\}$,

$$\sigma\boldsymbol{a} \overset{\text{def}}{=} \sum n_a\left[\chi_{\text{cyc}}(\sigma)a\right].$$

Put

$$\mathbb{B}^0 \overset{\text{def}}{=} \left\{\sum n_a[a] \in \mathbb{B} \mid \sum n_a a = 0\right\}.$$

For each numberfield k put

$$\mathbb{B}_k \overset{\text{def}}{=} \{\boldsymbol{a} \in \mathbb{B} \mid \sigma\boldsymbol{a} = \boldsymbol{a} \text{ for all } \sigma \in \mathfrak{g}(k)\},$$

$$\mathbb{B}_k^0 \overset{\text{def}}{=} \mathbb{B}^0 \cap \mathbb{B}_k.$$

Note that for each rational prime p,

$$\mathbb{B}_p = \mathbb{B}^{D(p)} \cap \{\boldsymbol{a} \in \mathbb{B} \mid p \nmid m(\boldsymbol{a})\}. \tag{2.3.1}$$

Given a numberfield k and a prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$ put

$$D(k, \mathfrak{p}) \stackrel{\text{def}}{=} \left\{\sigma \in \mathfrak{g} \mid \mathfrak{p}^\sigma = \left\{x \in \mathcal{O}_k^\sigma \mid |x|_p < 1\right\}\right\}.$$

Given a numberfield k and $\boldsymbol{a} \in \mathbb{B}_k$ we define a function $\Theta_k(\boldsymbol{a})$: $\mathfrak{g}/\mathfrak{g}(k) \to \mathbb{Q}$ by the formula

$$\Theta_k(\boldsymbol{a})(\sigma) = \langle \sigma^{-1}\boldsymbol{a} \rangle.$$

Given a numberfield k, $\boldsymbol{a} \in \mathbb{B}_k$ and a prime $\mathfrak{p}$ of k not dividing $m(\boldsymbol{a})$ we put

$$g_k(\boldsymbol{a}, \mathfrak{p}) \stackrel{\text{def}}{=} g_p\left(\sum_{\sigma \in D(k,\mathfrak{p})/\mathfrak{g}(k)} \sigma^{-1}\boldsymbol{a} \right),$$

where p denotes the rational prime lying below $\mathfrak{p}$. The following theorem defines the *Jacobi sum Hecke characters* and is the distillation of work of several authors: Weil [18,21], Deligne [15], Kubert-Lichtenbaum [9], and Kubert [8].

THEOREM 1: *For each numberfield k and $\boldsymbol{a} \in \mathbb{B}_k^0$ there exists a unique Hecke character $J_k(\boldsymbol{a})$ of k of type A_0 with the following properties*:

(I) *The conductor of $J_k(\boldsymbol{a})$ divides a power of $m(\boldsymbol{a})$.*

(II) *For all primes $\mathfrak{p}$ of k not dividing $m(\boldsymbol{a})$*,

$$J_k(\boldsymbol{a})(\mathfrak{p}) = g_k(\boldsymbol{a}, \mathfrak{p}).$$

A new proof of this result will be given later in the paper, based on the notion of an "ulterior motive".

REMARK 2.3.2: It follows from Stickelberger's theorem that the infinity type of $J_k(\boldsymbol{a})$ is $\Theta_k(\boldsymbol{a})$. Further, using the techniques of the proof of Theorem 1, one can show (although we shall not) that $J_k(\boldsymbol{a})_\infty = 1$.

REMARK 2.3.3: In order to facilitate comparison with [5], we make the following observations: For I a finite set with a $\mathfrak{g}(k)$-action and $(\chi_i)_{i \in I}$ a family of characters of $\boldsymbol{\mu}_N \subset \overline{\mathbb{Q}}^*$ such that

$$1 = \prod \chi_i, \quad \chi_{\sigma(i)} = \chi_i \circ \sigma^{-1}$$

for all $i \in I$ and $\sigma \in \mathcal{J}(k)$, one attaches in [5] a compatible system or ℓ-adic representations of $\mathcal{J}(k)$. If one twists it with the character of order 2 giving the signature of the action of $\mathcal{J}(k)$ upon I, the result is independent of the action of $\mathcal{J}(k)$ upon I, depending only upon the multiplicity with which each character χ of $\boldsymbol{\mu}_N$ appears among the χ_i;

the class of representations so obtained corresponds to the class of Jacobi sum Hecke characters of k defined here.

2.4. For each $\boldsymbol{a} \in \mathbb{B}$ we define

$$\Gamma(\boldsymbol{a}) \overset{\text{def}}{=} \prod \Gamma(\langle a \rangle)^{n_a},$$

$$\Phi(\boldsymbol{a}) \overset{\text{def}}{=} \{\sigma \in \mathfrak{p} \mid \langle \sigma \boldsymbol{a} \rangle \geqslant \langle \sigma \pi \boldsymbol{a} \rangle\},$$

where

$$\boldsymbol{a} = \sum n_a [a].$$

Now fix an abelian numberfield k and $\boldsymbol{a} \in \mathbb{B}_k^0$. Put

$L_k(s, \boldsymbol{a}) \overset{\text{def}}{=}$ The Hecke L-series associated to the Hecke character $J_k(\boldsymbol{a})$,

$$\Sigma_k(\boldsymbol{a}) \overset{\text{def}}{=} \left. \begin{array}{l} \{n \in 2\mathbb{Z} \mid n > w(\boldsymbol{a}) + \frac{1}{2}\} \\ \quad \text{if } k = k^+ \\ \{n \in \mathbb{Z} \mid \langle \sigma \rho \boldsymbol{a} \rangle < n \leqslant \langle \sigma \boldsymbol{a} \rangle \text{ for all } \sigma \in \Phi(\boldsymbol{a})\} \\ \quad \text{if } k \neq k^+ \end{array} \right\},$$

$$\Omega_k(n, \boldsymbol{a}) \overset{\text{def}}{=} \pi^{-nd} |\Delta^{(-)^n}|^{1/2} \prod_{\sigma \in \Phi(\boldsymbol{a})/\mathfrak{g}(k)} \Gamma(\sigma \boldsymbol{a}),$$

where in order to abbreviate we have written

$k^+ =$ the maximal totally real subfield of k,

$\Delta^+ =$ the discriminant of k^+,

$\Delta =$ the discriminant of k,

$\Delta^- = \Delta / \Delta^+$,

$d = [k^+ : \mathbb{Q}]$.

The following formula for the special values of the L-series associated to a Jacobi sum Hecke character is, in substance, the Γ-*hypothesis* conjectured by Lichtenbaum [3], subsequently proven in the case k totally real by Brattström [2], and in the case k imaginary quadratic of odd class number by Brattström-Lichtenbaum [3].

THEOREM 2: *For all abelian numberfield* k, $\boldsymbol{a} \in \mathbb{B}_k^0$, *and integers* $n \in \Sigma_k(\boldsymbol{a})$, $\Omega_k(n, \boldsymbol{a}) L_k(n, \boldsymbol{a})$ *is a rational number.*

We shall prove this formula later in the paper by means of our theory of "ulterior motives" combined with some basic results of Siegel [15] and recent important results of Blasius [1] making Deligne's conjecture [6] available in many cases.

§3. Tannakian categories; representations of affine group schemes

3.0. We briefly review the key points of the theory of tannakian categories in order to fix language and notation. The reader is referred to the books of Saavedra [11] and Waterhouse [16] for the in-depth development of the topics merely touched upon below. See also pp. 101–228 of [7].

3.1. Let k be a field, G a group-valued functor of k-algebras. We say G *acts k-linearly* on a k-vectorspace V or that V is a *representation* of G *defined over* k if for each k-algebra R and R-linear left action of $G(R)$ on $V \underset{k}{\otimes} R$ is given that depends functorially on R. We denote by $\mathscr{REP}(G/k)$ the category of representation of G defned over k and by $\mathscr{REP}_0(G/k)$ the full subcategory of objects for which the underlying k-vectorspace is finite-dimensional. The functor G is said to be an *affine group scheme* over k if the underlying set-valued functor is representable; if of finite type over k as well, G is called an *affine algebraic group*. We have the following basic finiteness properties of affine group schemes and representations.

PROPOSITION 3.1.1: *Every object of* $\mathscr{REP}(G/k)$ *is the direct limit of objects belonging to* $\mathscr{REP}_0(G/k)$.

PROOF: See Chapter 3 of Waterhouse's book [16]. □

PROPOSITION 3.1.2: *Every affine group scheme over k is the inverse limit of affine algebraic groups over k.*

PROOF: Chap. 3 of [16]. □

3.4. A *neutralized tannakian category* (NTC) over k is understood for the purposes of this paper to be a triple $(\mathscr{C}, \otimes, \omega)$ consisting of a category $\mathscr{C}$, a functor $\otimes$: $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called the *tensor product* of the NTC, and a functor ω: $\mathscr{C} \to \mathscr{MOD}(k)$ called the *neutralizing functor* of the NTC, satisfying conditions (3.2.1–6) below:

(3.2.1) The diagram

$$\begin{array}{ccc} \mathscr{C} \times \mathscr{C} & \xrightarrow{\ \otimes\ } & \mathscr{C} \\ {\scriptstyle \omega \times \omega} \downarrow & & \downarrow {\scriptstyle \omega} \\ \mathscr{MOD}(k) \times \mathscr{MOD}(k) & \underset{\otimes_k}{\longrightarrow} & \mathscr{MOD}(k) \end{array}$$

commutes.

(3.2.2) $\mathscr{C}$ is equivalent to a small abelian category, ω is faithful, additive and exact, the groups $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ are equipped with k-vectorspace structure functorially in objects X and Y of $\mathscr{C}$, and the maps

$$f \mapsto \omega(f)\colon \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_k(\omega(X), \omega(Y))$$

are k-linear.

(3.2.3) For all objects X and Y of $\mathscr{C}$ there exists an isomorphism ψ: $X \otimes Y \xrightarrow{\sim} Y \otimes X$ such that the diagram

$$\begin{array}{ccc} \omega(X \otimes Y) & = & \omega(X) \underset{k}{\otimes} \omega(Y) \\ {\scriptstyle \omega(\psi)} \downarrow & & \downarrow {\scriptstyle u \underset{k}{\otimes} v \mapsto v \underset{k}{\otimes} u} \\ \omega(Y \otimes X) & = & \omega(Y) \underset{k}{\otimes} \omega(X) \end{array}$$

commutes. (ψ is unique and therefore functorial in X and Y.)

(3.2.4) For all objects X, Y and Z of $\mathscr{C}$ there exists an isomorphism φ: $(X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ such that the diagram

$$\begin{array}{ccc} \omega((X \otimes Y) \otimes Z) & = & (\omega(X) \underset{k}{\otimes} \omega(Y)) \underset{k}{\otimes} \omega(Z) \\ {\scriptstyle \omega(\varphi)} \downarrow & & \downarrow {\scriptstyle (u \underset{k}{\otimes} v) \underset{k}{\otimes} \mapsto u \underset{k}{\otimes} (v \underset{k}{\otimes} w)} \\ \omega(X \otimes (Y \otimes Z)) & = & \omega(X) \underset{k}{\otimes} (\omega(Y) \underset{k}{\otimes} \omega(Z)) \end{array}$$

commutes. (φ is unique and therefore functorial in X, Y and Z.)

(3.2.5) There exists an object U of $\mathscr{C}$ such that $\dim_k(\omega(U)) = 1$, U is isomorphic to $U \otimes U$ and the functor

$$X \mapsto X \otimes U\colon \mathscr{C} \to \mathscr{C}$$

is an equivalence of categories. (Abusing language, we call such an object U of $\mathscr{C}$ a *unit object*.)

(3.2.6) For all objects X of $\mathscr{C}$ there exists an object $\hat{X}$ of $\mathscr{C}$ and a morphism q: $X \otimes \hat{X} \to U$, where U is any unit object, such that the induced map

$$\omega(q)\colon \omega(X) \underset{k}{\otimes} \omega(\hat{X}) \to \omega(U)$$

is a perfect pairing of k-vectorspaces. (Abusing language, we say that objects $\hat{X}$ of $\mathscr{C}$ as above are *dual* to X.)

Examples of neutralized tannakian categories over k abound: Given any affine group scheme G over k, the category $\mathscr{REP}_0(G/k)$ equipped with the evident tensor product and neutralized by the forgetful functor to $\mathscr{MOD}(k)$ is an NTC over k. There are no other examples in a sense to be made precise presently.

3.3. Let $(\mathscr{C}, \otimes, \omega)$ be a neutralized tannakian category over k. For each k-algebra R let the functor ω: $\mathscr{C} \to \mathscr{MOD}(R)$ be defined by the rule

$$\omega^R(X) \overset{\text{def}}{=} \omega(X) \underset{k}{\otimes} R$$

and let $\mathscr{AUT}_k(\mathscr{C}, \otimes, \omega)(R)$ denote the group of R-linear automorphisms g: $\omega^R \xrightarrow{\sim} \omega^R$ making the diagram

$$\begin{array}{ccc} \omega^R(X) \underset{R}{\otimes} \omega^R(Y) & \to & \omega^R(X \otimes Y) \\ \Big\downarrow g(X) \underset{R}{\otimes} g(Y) & & \Big\downarrow g(X \otimes Y) \\ \omega^R(X) \underset{R}{\otimes} \omega^R(Y) & \to & \omega^R(X \otimes Y) \end{array}$$

commute for all objects X and Y of $\mathscr{C}$, where the horizontal arrows are given by the rule

$$\left(u \underset{k}{\otimes} r\right) \underset{R}{\otimes} \left(v \underset{k}{\otimes} r'\right) \mapsto \left(u \underset{k}{\otimes} v\right) \underset{R}{\otimes} rr'.$$

The group-valued functor of k-algebras $\mathscr{AUT}_k(\mathscr{C}, \otimes, \omega)$ is called the *automorphism group* of $(\mathscr{C}, \otimes, \omega)$ over k.

THEOREM 3: *The automorphism group of a neutralized tannakian category $(\mathscr{C}, \otimes, \omega)$ over k is an affine group scheme over k and the evident functor $\mathscr{C} \to \mathscr{REP}_0(\mathscr{AUT}_k(\mathscr{C}, \otimes, \omega))$ an equivalence of categories.*

PROOF: This is a simplified version of Théorème 1 on p. 6 of [11]. □

3.4. A neutralized tannakian category $(\mathscr{C}, \otimes, \omega)$ over k is said to be *semisimple* if every exact sequence in $\mathscr{C}$ splits. An affine group scheme G over k is said to be *pro-reductive* if G is the inverse limit of reductive affine algebraic groups over k.

PROPOSITION 3.4.1: *The following properties of a neutralized tannakian category* $(\mathscr{C}, \otimes, \omega)$ *over* k *are equivalent provided that* k *is of characteristic zero*:

(I) $(\mathscr{C}, \otimes, \omega)$ *is semisimple*,

(II) $\mathscr{A}\mathscr{U}\mathscr{T}_k(\mathscr{C}, \otimes, \omega)$ *is pro-reductive*.

PROOF: See the "dictionnaire" on pp. 156–157 of [11].

3.5. Let $\{X_i\}_{i\in I}$ be an indexed family of objects of a neutralized tannakian category $(\mathscr{C}, \otimes, \omega)$ over k, and for brevity put $G = \mathscr{A}\mathscr{U}\mathscr{T}_k(\mathscr{C}, \otimes, \omega)$. We say that $\{X_i\}$ *generates* $(\mathscr{C}, \otimes, \omega)$ if every object of $\mathscr{C}$ is isomorphic to an object constructed from the X_i through the formation of tensor product, dual, direct sum and subquotient, and iterates of these processes.

PROPOSITION 3.5.1: *The following are equivalent*:

(I) $\{X_i\}$ *generates* $(\mathscr{C}, \otimes, \omega)$.

(II) *For all k-algebras R the evident map*

$$G(R) \to \prod_{i\in I} \mathrm{Aut}_R\Big(\omega(X_i) \bigotimes_k R\Big)$$

is injective.

PROOF: In view of Theorem 3 above, the proposition is simply a statement about the representation theory of affine group schemes which the reader can find a proof of in Chap. 3 of [16]. □

3.6. Let $(\mathscr{C}, \otimes, \omega)$ and $(\mathscr{D}, \otimes, \eta)$ be neutralized tannakian categories over k and F: $\mathscr{C} \to \mathscr{D}$ an additive functor making the diagrams

$$\begin{array}{ccc} \mathrm{C} & \xrightarrow{F} & \mathscr{D} \\ & {\scriptstyle\omega}\searrow \quad \swarrow{\scriptstyle\eta} & \\ & \mathscr{M}\mathscr{O}\mathscr{D}(k) & \end{array} \qquad \begin{array}{ccc} \mathscr{C}\times\mathscr{C} & \xrightarrow{F\times F} & \mathscr{D}\times\mathscr{D} \\ \otimes\downarrow & & \downarrow\otimes \\ \mathscr{C} & \xrightarrow[F]{} & \mathscr{D} \end{array} \qquad (3.6.1)$$

commute. To abbreviate notation put

$$G = \mathscr{A}\mathscr{U}\mathscr{T}_k(\mathscr{C}, \otimes, \omega)$$

$$H = \mathscr{A}\mathscr{U}\mathscr{T}_k(\mathscr{D}, \otimes, \eta)$$

and let $f\colon H \to G$ denote the k-morphism induced in evident fashion by $F\colon \mathscr{C} \to \mathscr{D}$. The morphism $f\colon H \to G$ is said to be *dual* to the functor $F\colon \mathscr{C} \to \mathscr{D}$.

PROPOSITION 3.6.2: *The following are equivalent*:

(I) $f\colon H \to G$ *is a closed immersion.*

(II) *Every object of* $\mathscr{D}$ *is isomorphic to a subquotient of* $F(X)$ *for a suitable object* X *of* $\mathscr{C}$.

PROOF: See the "dictionnaire" on pp. 156–157 of [11]. □

PROPOSITION 3.6.3: *The following are equivalent*:

(I) $f\colon H \to G$ *is faithfully flat.*

(II) *For every object* X *of* $\mathscr{C}$ *and subobject* Z *of* $F(X)$ *there exists a subobject* Y *of* X *such that* $\omega(Y) = \eta(Z)$.

PROOF: Consulting the "dictionnaire" on pp. 156–157 of [11], we find that (I) is equivalent to the conjunction of (II) and the statement "F is fully faithful". We therefore have only to show that, assuming (II), F is already fully faithful. For this it in turn suffices to prove that for all objects X of $\mathscr{C}$

$$\omega(X)^G = \omega(X)^H. \tag{3.6.4}$$

Let U be a unit object of $\mathscr{C}$. Then $\omega(U)$ is one-dimensional over k and

$$\omega(U)^G = \omega(U) = \omega(U)^H.$$

If (3.6.4) is violated for some X, it follows that $\omega(U \oplus X)$ contains an H-stable line that is not G-stable. From this contradiction, (3.6.4) and the proposition follow. □

REMARK 3.6.5: In the case that H is pro-reductive, condition (II) reduces to the condition that F be fully faithful.

3.7. Let k be of characteristic zero, G a pro-reductive affine group scheme over k and $\mathscr{A}$ a full subcategory of $\mathscr{REP}_0(G/k)$ the class of objects of which generates $\mathscr{REP}_0(G/k)$ and such that for all V and W belonging to $\mathscr{A}$, $V \underset{k}{\otimes} W$ and $V \oplus W$ are isomorphic in $\mathscr{REP}_0(G/k)$ to objects of $\mathscr{A}$. Let $\tilde{G}$ denote the affine group scheme over k the group of points of which in a k-algebra R consists of all R-linear automorphisms

$\tilde{g}$ of the functor $V \to V \underset{k}{\otimes} R\colon \mathscr{A} \to \mathscr{MOD}(R)$ rendering the diagram

$$\begin{array}{ccc} (V \underset{k}{\otimes} R) \underset{R}{\otimes} (W \underset{k}{\otimes} R) & \xrightarrow{\sim} & U \underset{k}{\otimes} R \\ \Big\downarrow{\scriptstyle \tilde{g}(V) \underset{R}{\otimes} \tilde{g}(W)} & & \Big\downarrow{\scriptstyle \tilde{g}(U)} \\ (V \underset{k}{\otimes} R) \underset{R}{\otimes} (W \underset{k}{\otimes} R) & \xrightarrow{\sim} & U \underset{k}{\otimes} R \end{array} \tag{3.7.1}$$

commutative for all objects V, W and U of $\mathscr{A}$ and isomorphisms $f\colon V \underset{k}{\otimes} W \xrightarrow{\sim} U$ in $\mathscr{REP}_0(G/k)$, the horizontal arrows of (3.7.1) being given by the rule

$$\left(v \underset{k}{\otimes} r\right) \underset{R}{\otimes} \left(w \underset{k}{\otimes} r'\right) \to f\left(v \underset{k}{\otimes} w\right) \underset{k}{\otimes} rr'.$$

Proposition 3.7.2: *Under the evident map, G is isomorphic to $\tilde{G}$.*

Proof: For brevity put

$$G' \overset{\text{def}}{=} \mathscr{AUT}_k\left(\mathscr{REP}_0(G/k),\ \underset{k}{\otimes}\ , \text{forget}\right),$$

an affine group scheme over k. The evident map $G \to G'$ is an isomorphism by Theorem 3, Prop. 3.5.3 and Prop. 3.5.4. It therefore suffices to show that for each k-algebra R and point $\tilde{g} \in \tilde{G}(R)$ there exists a unique point $g' \in G'(R)$ such that for all objects V of $\mathscr{A}$, $g'(V) = \tilde{g}(V)$. This latter task is a diagram chase left to the reader. □

3.8. Brauer's theorem is valid for a somewhat larger class of algebraic groups than that of the finite groups. Namely, we have

Proposition 3.8.1: *Let G be an algebraic group over an algebraically closed field of characteristic zero whose connected component is a torus. Then the Grothendieck group of $\mathscr{REP}_0(G/k)$ is generated by the representations of G induced by one-dimensional representations of subgroups of finite index in G.*

Proof: Cf. Weil [17]. □

§4. Motives

4.0. We shall attach a precise meaning to the term *motive* and set up a supporting formalism. The theory sketched here is developed at length in

[7], albeit from a different point of view.

4.1. Let X be a smooth *quasi-projective* $\mathbf{Q}$-scheme. We put

$H_B^*(X) \overset{\text{def}}{=}$ the singular cohomology of the manifold of complex points of X with coefficients taken in $\mathbf{Q}$,

$H_{DR}^*(X) \overset{\text{def}}{=}$ the hypercohomology of the algebraic de Rham complex $\Omega^{\cdot}_{X/\mathbf{Q}}$,

$H_\ell^*(X) \overset{\text{def}}{=}$ the cohomology of the constant ℓ-adic sheaf $\mathbf{Q}_\ell$ on the étale site of $X \otimes \overline{\mathbf{Q}}$.

The canonical comparison isomorphisms $H_B^*(X) \otimes \mathbf{Q}_\ell \xrightarrow{\sim} H_\ell^*(X)$ and $H_B^*(X) \otimes \mathbf{C} \xrightarrow{\sim} H_{DR}^*(X) \otimes \mathbf{C}$ are denoted I_ℓ and I, respectively.

4.2. Each of the cohomology groups attached above to a smooth *projective* $\mathbf{Q}$-scheme X has auxiliary structure: The continuous action of complex conjugation on the manifold of complex point of X induces an involution of $H_B^*(X)$ denoted by ρ_B^*. The continuous action of $\mathfrak{g}$ on $X \otimes \overline{\mathbf{Q}}$ relative to the étale topology induces a continuous $\mathbf{Q}_\ell$-linear action of $\mathfrak{g}$ on $H_\ell^*(X)$. For each integer n, $H_{DR}^n(X)$ is equipped with a filtration

$$\ldots \subseteq F^{p+1}H_{DR}^n(X) \subseteq F^pH_{DR}^n(X) \subseteq \ldots$$

called the *Hodge filtration*.

4.3. The auxiliary structures possessed by the cohomology groups attached to a smooth projective $\mathbf{Q}$-scheme X are subject to certain compatibility conditions: Putting

$$H^{(p,q)}(X) \overset{\text{def}}{=} I^{-1}(F^pH_{DR}^{p+q}(X) \otimes \mathbf{C}) \cap (1 \otimes \rho) I^{-1}(F^qH_{DR}^{p+q}(X) \otimes \mathbf{C})$$

one obtains a direct sum decomposition

$$H_B^n(X) \otimes \mathbf{C} = \bigoplus_{p+q=n} H^{(p,q)}(X)$$

called the *Hodge decomposition* each direct summand of which verifies

$$(1 \otimes \rho) H^{(p,q)}(X) = H^{(q,p)}(X).$$

Further, the diagrams

$$\begin{array}{ccc} H_B^*(X)\otimes\mathbb{Q}_\ell & \xrightarrow{I_\ell} & H_\ell^*(X) \\ \downarrow{\scriptstyle \rho_B^*\otimes 1} & & \downarrow{\scriptstyle \rho} \\ H_B^*(X)\otimes\mathbb{Q}_\ell & \xrightarrow[I_\ell]{} & H_\ell^*(X) \end{array} \tag{4.3.1}$$

$$\begin{array}{ccc} H_B^*(X)\otimes\mathbb{C} & \xrightarrow{I} & H_{DR}^*(X)\otimes\mathbb{C} \\ \downarrow{\scriptstyle \rho_B^*\otimes\rho} & & \downarrow{\scriptstyle 1\otimes\rho} \\ H_B^*(X)\otimes\mathbb{C} & \xrightarrow[I]{} & H_{DR}^*(X)\otimes\mathbb{C} \end{array} \tag{4.3.2}$$

commute.

4.4 Let X and Y be smooth projective $\mathbb{Q}$-schemes and let π_1: $X\times Y\to X$ and π_2: $X\times Y\to Y$ denote first and second projections, respectively. The *Künneth isomorphism* $\kappa_B=\kappa_B(X,Y)$: $H_B^*(X)\otimes H_B^*(Y)\xrightarrow{\sim} H_B^*(X\times Y)$ is given by the formula

$$\kappa_B(\xi\otimes\eta)\overset{\text{def}}{=}\pi_1^*\xi\cup\pi_2^*\eta,$$

where $\cup$ denotes cup product. Analogous functorial isomorphisms exist for the cohomology theories H_ℓ^* and H_{DR}^* and are compatible with the comparison isomorphisms I_ℓ and I, respectively.

4.5. Let X and Y be smooth projective $\mathbb{Q}$-schemes. A $\mathbb{Q}$-linear map f: $H_B^*(X)\to H_B^*(Y)$ is called an *absolute Hodge correspondence* if there exist for each rational prime ℓ a $\mathbb{Q}_\ell$-linear map f_ℓ: $H_\ell^*(X)\to H_\ell^*(Y)$ and a $\mathbb{Q}$-linear map f_{DR}: $H_{DR}^*(X)\to H_{DR}^*(Y)$ with the following properties:

f_ℓ *is* $\mathfrak{g}$*-equivariant.* (4.5.1)

$$f_{DR}(F^pH_{DR}^n(X))\subseteq F^pH_{DR}^n(Y). \tag{4.5.2}$$

The diagram

$$\begin{array}{ccc} H_B^*(X)\otimes\mathbb{Q}_\ell & \xrightarrow{f\otimes 1} & H_B^*(Y)\otimes\mathbb{Q}_\ell \\ \downarrow{\scriptstyle I_\ell} & & \downarrow{\scriptstyle I_\ell} \\ H_\ell^*(X) & \xrightarrow[f_\ell]{} & H_\ell^*(Y) \end{array} \tag{4.5.3}$$

commutes.

The diagram

$$\begin{array}{ccc} H_B^*(X)\otimes\mathbb{C} & \xrightarrow{f\otimes 1} & H_B^*(Y)\otimes\mathbb{C} \\ {\scriptstyle I}\big\downarrow & & \big\downarrow{\scriptstyle I} \\ H_{DR}^*(X)\otimes\mathbb{C} & \xrightarrow[f_{DR}\otimes 1]{} & H_{DR}^*(Y)\otimes\mathbb{C} \end{array} \tag{4.5.4}$$

commutes.

The set of absolute Hodge correspondences f: $H_B^*(X)\to H_B^*(Y)$ is denoted $\mathscr{Z}^{ah}(X, Y)$. Let $\mathscr{A}$ denotes the category the objects of which are the smooth projective $\mathbb{Q}$-schemes but for which the *morphism sets* are given by the rule

$$\mathrm{Hom}_{\mathscr{A}}(X, Y)\overset{\mathrm{def}}{=}\mathscr{Z}^{ah}(X, Y).$$

4.6. The *motivic galois group* $\mathscr{G}$ is defined to be the affine group scheme over $\mathbb{Q}$ the group of points of which in each $\mathbb{Q}$-algebra R consists of all R-linear automorphisms g of the functor $X\mapsto H_B^*(X)\otimes R$: $\mathscr{A}\to \mathscr{MOD}(R)$ rendering the diagram

$$\begin{array}{ccc} (H_B^*(X)\otimes R)\underset{R}{\otimes}(H_B^*(Y)\otimes R) & \to & H_B^*(X\times Y)\otimes R \\ {\scriptstyle g(X)\underset{R}{\otimes} g(Y)}\big\downarrow & & \big\downarrow{\scriptstyle g(X\times Y)} \\ (H_B^*(X)\otimes R)\underset{R}{\otimes}(H_B^*(Y)\otimes R) & \to & H_B^*(X\times Y)\otimes R \end{array} \tag{4.6.1}$$

commutative for all smooth projective $\mathbb{Q}$-schemes X and Y, the horizontal arrows of (4.6.1) being given by the rule

$$(\xi\otimes r)\underset{R}{\otimes}(\eta\otimes r')\mapsto \kappa_B(\xi\otimes\eta)\otimes rr'.$$

THEOREM 4:

(I) *$\mathscr{G}$ is pro-reductive.*

(II) $\mathscr{Z}^{ah}(X, Y)=\mathrm{Hom}_{\mathscr{G}}(H_B^*(X), H_B^*(Y))$.

PROOF: The motivic galois group over $\mathbb{Q}$ as defined on p. 213 of [7] is proreductive, operates upon $H_B^*(X)$ for all smooth projective $\mathbb{Q}$-schemes X, verifies the evident analogue of Thm. 4(II), and has the property that the representations of form $H_B^*(X)$ generate its category of finite-dimensional representations in the sense of paragraph 3.5. We conclude via Prop. 3.8.2 that the commutative diagrams (4.6.1) provide a set of defining equations for it. In short, the motivic galois group as defined in [7] coincides with the motivic galois group as we have defined it above. Theorem 4 follows immediately. □

4.7. A *motive* is defined to be a finite-dimensional $\mathbb{Q}$-vectorspace equipped with a $\mathbb{Q}$-linear action of $\mathcal{G}$, i.e., an object of $\mathcal{REP}_0(\mathcal{G}/\mathbb{Q})$. By Theorem 4 this notion of motive is essentially the same as the notion of *motive for absolute Hodge cycles over* $\mathbb{Q}$ as defined in [7]. Generally we write $\mathcal{M}$ instead of $\mathcal{REP}_0(\mathcal{G}/\mathbb{Q})$, referring to $\mathcal{M}$ as the *category of motives*. The functor H^n: $\mathcal{A} \to \mathcal{M}$ is defined to be that which assigns to each smooth projective $\mathbb{Q}$-scheme X the $\mathbb{Q}$-vectorspace $H_B^n(X)$ equipped with the evident action of $\mathcal{G}$. The functor ω_B: $\mathcal{M} \to \mathcal{MOD}(\mathbb{Q})$ is defined to be that which assigns to each representation of $\mathcal{G}$ of finite dimension the underlying $\mathbb{Q}$-vectorspace. The functor $\otimes$: $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is defined to be the usual tensor product of representations. A motive is said to be *effective* if isomorphic to a direct summand of $H^*(X)$ for some smooth projective $\mathbb{Q}$-scheme X. The *rank* of a motive M is defined to be the dimension over $\mathbb{Q}$ of $\omega_B(M)$. On the basis of Theorem 4 the reader can easily check

LEMMA 4.7.1: *For each motive* M *there exists an effective motive* N *of rank one such that* $M \otimes N$ *is effective.* □

REMARK 4.7.2: Consideration of Poincaré duality shows that the rank one motive N above can be taken to be a tensor power of $H^2(\mathbb{P}^1)$.

4.8. A profinite group G is said to act admissibly on a set S if for all elements s of S the subgroup $\{\sigma \in G \mid \sigma s = s\}$ is open. For all 0-dimensional smooth projective $\mathbb{Q}$-schemes X there exists a unique $\mathbb{Q}$-linear admissible action of $\mathfrak{g}$ on $H_B^0(X)$ rendering the diagrams

$$\begin{array}{ccc} H_B^0(X)\otimes\mathbb{Q}_\ell & \xrightarrow{I_\ell} & H_\ell^0(X) \\ {\scriptstyle \sigma\otimes 1}\downarrow & & \downarrow{\scriptstyle \sigma} \\ H_B^0(X)\otimes\mathbb{Q}_\ell & \xrightarrow[I_\ell]{} & H_\ell^0(X) \end{array}$$

$$\begin{array}{ccc} H_B^0(X)\otimes\mathbb{C} & \xrightarrow{I} & H_{DR}^0(X)\otimes\mathbb{C} \\ {\scriptstyle \sigma\otimes s}\downarrow & & \downarrow{\scriptstyle 1\otimes s} \\ H_B^0(X)\otimes\mathbb{C} & \xrightarrow[I]{} & H_{DR}^0(X)\otimes\mathbb{C} \end{array} \qquad (4.8.1)$$

commutative for all $\sigma \in \mathfrak{g}$, $s \in \mathrm{Aut}(\mathbf{C})$ extending σ, and rational primes ℓ. We then have

$$\mathscr{Z}^{ah}(X, Y) = \mathrm{Hom}_{\mathfrak{g}}\big(H_B^0(X), H_B^0(Y)\big) \tag{4.8.2}$$

for all zero-dimensional smooth projective $\mathbf{Q}$-schemes X and Y. It follows that for all smooth projective $\mathbf{Q}$-schemes X of dimension zero the image of $\mathscr{G}$ in $GL_{\mathbf{Q}}(H_B^0(X))$ lies in the image of $\mathfrak{F}$ by an application of Prop. 3.8.2. Let φ: $\mathscr{G} \to \mathfrak{g}$ denote the unique morphism rendering the diagram

$$\begin{array}{ccc} & & \mathscr{G} \\ & \overset{\varphi}{\swarrow} & \downarrow \\ \mathfrak{g} & \longrightarrow & GL_{\mathbf{Q}}(H_B^0(X)) \end{array} \tag{4.8.3}$$

commutative for all smooth projective $\mathbf{Q}$-schemes X of dimension zero.

4.9. For each rational prime ℓ and $\sigma \in \mathfrak{g}$ let $\alpha_\ell(\sigma) \in \mathscr{G}(\mathbf{Q}_\ell)$ be the unique point such that for all smooth projective $\mathbf{Q}$-schemes X the diagram

$$\begin{array}{ccc} H_B^*(X) \otimes \mathbf{Q}_\ell & \xrightarrow{I_\ell} & H_\ell^*(X) \\ \Big\downarrow{\scriptstyle \alpha_\ell(\sigma)} & & \Big\downarrow{\scriptstyle \sigma} \\ H_B^*(X) \otimes \mathbf{Q}_\ell & \xrightarrow[I_\ell]{} & H_\ell^*(X) \end{array} \tag{4.9.1}$$

commutes, thereby defining a homomorphism α_ℓ: $\mathfrak{g} \to \mathscr{G}(\mathbf{Q}_\ell)$ which is a section of φ: $\mathscr{G} \to \mathfrak{g}$ (albeit a nonalgebraic one). For each motive M put

$$\omega_\ell(M) \overset{\text{def}}{=} \omega_B(M) \otimes \mathbf{Q}_\ell.$$

Via the homomorphism α_ℓ the $\mathbf{Q}_\ell$-vectorspace $\omega_\ell(M)$ is canonically equipped with a continuous $\mathbf{Q}_\ell$-linear action of $\mathfrak{g}$. The point $\alpha_\ell(\rho)$ of $\mathscr{G}(\mathbf{Q}_\ell)$ is by virtue of diagram (4.3.1) defined over $\mathbf{Q}$, independent of ℓ and therefore denoted by $\alpha(\rho)$.

4.10. Each motive M is equipped with a unique grading

$$M = \bigoplus_{n \in \mathbf{Z}} M^n$$

functorial in M, verifying

$$(M \otimes N)^n = \bigoplus_{i+j=n} M^i \otimes N^j$$

and in the case $M = H^*(X)$ for some smooth projective $\mathbb{Q}$-scheme X,

$$M^n = H^n(X).$$

If $M = M^w$ for some integer w, M is said to be *pure* of *weight* w.

4.11. For each motive M there exists a unique $\mathbb{C}$-linear direct sum decomposition

$$\omega_B(M) \otimes \mathbb{C} = \oplus\, \omega^{(p,q)}(M)$$

functorial in M, such that

$$\omega^{(p,q)}(M \otimes N) = \bigoplus_{p'+p''=p} \; \bigoplus_{q'+q''=q} \omega^{(p',q')}(M) \underset{\mathbb{C}}{\otimes} \omega^{(p'',q'')}(N).$$

and in the case $M = H^*(X)$,

$$\omega^{(p,q)}(M) = H^{(p,q)}(X).$$

Note that necessarily

$$(1 \otimes \rho)\,\omega^{(p,q)}(M) = \omega^{(q,p)}(M). \tag{4.11.1}$$

For each motive M and integer p we put

$$F^p(\omega_B(M) \otimes \mathbb{C}) \stackrel{\text{def}}{=} \bigoplus_{p'=p}^{\infty} \; \bigoplus_{q=-\infty}^{\infty} \omega^{(p',q)}(M).$$

4.12. A $\mathbb{Q}$-subspace V_0 of a $\mathbb{C}$-vectorspace V is said to be a $\mathbb{Q}$-*lattice* in V if the map $V_0 \otimes \mathbb{C} \to V$ induced by inclusion is an isomorphism. For each motive M there exists a unique $\mathbb{Q}$-lattice $\omega_{DR}(M) \subseteq \omega_B(M) \otimes \mathbb{C}$ depending functorially on M, verifying

$$\omega_{DR}(M \otimes N) = \omega_{DR}(M) \otimes \omega_{DR}(N)$$

and in the case $M = H^*(X)$ satisfying

$$I(\omega_{DR}(M)) = H^*_{DR}(X) \subseteq H^*_{DR}(X) \otimes \mathbb{C}.$$

It follows that for each motive M the diagram

$$\begin{array}{ccc} \omega_{DR}(M) & \subseteq & \omega_B(M) \otimes \mathbb{C} \\ \Big\| & & \Big\downarrow {\scriptstyle \alpha(\rho) \otimes \rho} \\ \omega_{DR}(M) & \subseteq & \omega_B(M) \otimes \mathbb{C} \end{array} \tag{4.12.1}$$

commutes. Further, putting for each integer p

$$F^p\omega_{DR}(M) \overset{\text{def}}{=} \omega_{DR}(M) \cap F^p(\omega_B(M) \otimes \mathbb{C}),$$

we have that

$$F^p\omega_{DR}(M) \text{ is a } \mathbb{Q}\text{-lattice in } F^p(\omega_B(M) \otimes \mathbb{C}). \tag{4.12.2}$$

It is convenient to define I: $\omega_B(M) \otimes \mathbb{C} \xrightarrow{\sim} \omega_{DR}(M) \otimes \mathbb{C}$ to be the inverse of the isomorphism $\omega_{DR}(M) \otimes \mathbb{C} \xrightarrow{\sim} \omega_B(M) \otimes \mathbb{C}$ induced by the inclusion of $\omega_{DR}(M)$ in $\omega_B(M) \otimes \mathbb{C}$.

4.13. The *Tate motive* $\mathbb{Z}(1)$ is defined to be the dual of $H^2(\mathbb{P}^1)$. For each integer n one defines $\mathbb{Z}(n)$ so that

$$\mathbb{Z}(n+m) \text{ is isomorphic to } \mathbb{Z}(n) \otimes \mathbb{Z}(m). \tag{4.13.1}$$

Given a motive M and an integer n one writes

$$M(n) \overset{\text{def}}{=} M \otimes \mathbb{Z}(n). \tag{4.13.2}$$

4.14. The *Hodge numbers* of a motive M are given by the rule

$$h^{(p,q)}(M) \overset{\text{def}}{=} \dim_{\mathbb{C}} \omega^{(p,q)}(M),$$

the *index* $\imath(M)$ of a motive M by the formula

$$\imath(M) \overset{\text{def}}{=} \operatorname{tr}_{\mathbb{Q}}(\alpha(\rho) \mid \omega_B(M)).$$

4.15. A motive M is said to be *critical* if pure of some weight w and, whenever $h^{(p,q)}(M) \neq 0$ for some integers p and q, one of the following conditions holds:

$$\min(p, q) < 0 \leqslant \max(p, q) \tag{4.15.1}$$

$$p = q = \tfrac{1}{2}w > -\tfrac{1}{2} \quad \text{and} \quad \imath(M) = -h^{(p,p)}(M). \tag{4.15.2}$$

$$p = q = \tfrac{1}{2}w < -\tfrac{1}{2} \quad \text{and} \quad \imath(M) = h^{(p,p)}(M). \tag{4.15.3}$$

A motive is critical in the sense just defined if and only if of pure weight and "critique" in the sense defined by Deligne on p. 322 of [6], as a calculation based on the table on p. 329 of [6] shows.

4.16. In order to attach an L-series to a motive M one ought to assume that M satisfies the:

Hypothesis of Strict Compatibility (Hypothesis SC). For each rational prime p there exists a polynomial $Q_p(M, t) \in \mathbb{Q}[t]$ *such that for all rational primes* ℓ *distinct from p, the characteristic polynomial* $\det \mathbb{Q}_\ell(1 - tF(p) \mid \omega_\ell(M)^{I(p)}) \in \mathbb{Q}_\ell[t]$ *coincides with* $Q_p(M, t)$. *Further, the degree of* $Q_p(M, t)$ *coincides with the rank of M for all but finitely many p.* (Conjecturally, every motive satisfies Hypothesis SC.)

For each motive M satisfying Hypothesis SC we put

$$L(s, M) \stackrel{\text{def}}{=} \prod_p Q_p(M, p^{-s})^{-1},$$

the L-series associated to M, which has meaning as a formal Dirichlet series and which, on the basis of the Riemann hypothesis for varieties over finite fields, can be shown to converge absolutely for $\mathrm{Re}(s) \gg 0$.

4.17. Let M be a critical motive of weight w. Put

$\omega_B^+(\mathrm{M}) \stackrel{\text{def}}{=}$ the $+1$-eigenspace in $\omega_B(M)$ under the action of $\alpha(\rho)$,

$$\omega_{DR}^+(M) \stackrel{\text{def}}{=} \left.\begin{array}{lll} \omega_{DR}(M) / \bigcup_{p \geqslant \frac{1}{2}w} F^p \omega_{DR}(M) & \text{if} & w \geqslant -1 \\ \omega_{DR}(M) / \bigcup_{p > \frac{1}{2}w} F^p \omega_{DR}(M) & \text{if} & w < -1 \end{array}\right\}.$$

The isomorphism I^+: $\omega_B^+(M) \otimes \mathbb{C} \xrightarrow{\sim} \omega_{DR}^+(M) \otimes \mathbb{C}$ is defined to be the arrow rendering the diagram

$$\begin{array}{ccc} \omega_B^+(M) \otimes \mathbb{C} & \xrightarrow{I^+} & \omega_{DR}^+(M) \otimes \mathbb{C} \\ \downarrow & & \uparrow\!\uparrow \\ \omega_B(M) \otimes \mathbb{C} & \xrightarrow[I]{} & \omega_{DR}(M) \otimes \mathbb{C} \end{array}$$

commutative. Put

$$c^+(M) \stackrel{\text{def}}{=} \det(I^+) \in \mathbb{R}^* / \mathbb{Q}^*,$$

the determinant being calculated with respect to any choice of $\mathbb{C}$-bases in $\omega_B^+(M) \otimes \mathbb{C}$ and $\omega_{DR}^+(M) \otimes \mathbb{C}$ that are also $\mathbb{Q}$-bases in $\omega_B^+(M)$ and $\omega_{DR}^+(M)$, respectively. The following conjecture is due to Deligne [6].

CONJECTURE D: *If M is a critical motive satisfying Hypothesis SC for which $L(s, M)$ admits meromorphic continuation to the whole complex s-plane and for which $L(0, M)$ is finite,*

$$c^+(M)^{-1}L(0, M) \in \mathbb{Q}.$$

REMARK: Conjecturally, the hypotheses of Conjecture D are all consequences of the hypothesis "M critical". The possibility that $L(s, M)$ *vanishes* at $s = 0$ for a critical motive M is not ruled out by Conjecture D.

§5. How to relate Hecke characters and Hecke L-series to motives

5.0. We gather some results in the theory of motives here with the goal in mind of transforming Theorems 1 and 2 into statements that can be proven by counting points on varieties over finite fields and evaluating algebraic integrals. Figuring prominently in the discussion to follow are Deligne's theory [7] of absolute Hodge cycles on abelian varieties and recent results of Blasius [1] making Conjecture D available in many cases.

5.1. To give a *Hodge structure* is to give a $\mathbb{Q}$-vectorspace V of finite dimension equipped with a grading

$$V = \bigoplus_{n \in \mathbb{Z}} V^n \tag{5.1.1}$$

and a $\mathbb{C}$-linear direct sum decomposition

$$V^n \otimes \mathbb{C} = \bigoplus_{p+q=n} V^{(p,q)} \tag{5.1.2}$$

each direct summand of which satisfies

$$(1 \otimes \rho) V^{(p,q)} = V^{(q,p)}. \tag{5.1.3}$$

The category of Hodge structures is denoted by $\mathscr{HOD}$. Equipped with the evident tensor product and neutralized by the functor ω_{hod} which assigns to each Hodge structure the underlying $\mathbb{Q}$-vectorspace, $\mathscr{HOD}$ is an NTC over $\mathbb{Q}$. By the observations made in paragraph 4.11 above, it is clear that underlying each motive is a Hodge structure.

5.2. A Hodge structure V is said to be *of complex multiplication* (CM) *type* if there exists a semisimple commutative $\mathbb{Q}$-subalgebra E of $\text{End}_{\mathscr{HOD}}(V)$ admitting a positive involution and such that V is free over E of rank one. Denoting by $\mathscr{HCM}$ the full subcategory of $\mathscr{HOD}$

consisting of the objects of CM type and noting that $(\mathscr{HCM}, \otimes, \omega_{\text{hod}})$ is a NTC over $\mathbf{Q}$, put

$$\mathscr{S} \stackrel{\text{def}}{=} \mathscr{AUT}_{\mathbf{Q}}(\mathscr{HCM}, \otimes, \omega_{\text{hod}}).$$

The affine group scheme $\mathscr{S}$ over $\mathbf{Q}$, known as the *connected Serre group*, is commutative, pro-reductive and connected [7,12].

5.3. We say that a smooth projective $\mathbf{Q}$-scheme X is *potentially* of *complex multiplication* (PCM) *type* if the following conditions hold:

The Hodge structure underlying $H^*(X)$ is of CM type. (5.3.1)

The manifold of complex points of X is a disjoint union of complex tori. (5.3.2)

Note that, in particular, every 0-dimensional smooth projective $\mathbf{Q}$-scheme is of PCM type. The full subcategory $\mathscr{PCM}$ of $\mathscr{M}$ is defined to be that which is generated (in the sense of ¶3.6) by the objects of the form $H^*(X)$ for X a smooth projective $\mathbf{Q}$-scheme of PCM type. A motive belonging to $\mathscr{PCM}$ is said to be of PCM type. Note that the Hodge structure underlying each motive of PCM type is of CM type.

5.4. Put

$$T \stackrel{\text{def}}{=} \mathscr{AUT}_{\mathbf{Q}}(\mathscr{PCM}, \otimes, \omega_B)$$

an affine group scheme over $\mathbf{Q}$ canonically a quotient of $\mathscr{G}$, hence pro-reductive. The morphism $T \to \mathfrak{g}$ through which φ: $\mathscr{G} \to \mathfrak{g}$ factors is denoted again by φ, the homomorphism $\mathfrak{g} \to T(\mathbf{Q}_\ell)$ obtained by composing α_ℓ: $\mathfrak{g} \to \mathscr{G}(\mathbf{Q}_\ell)$ with the quotient map $\mathscr{G} \to T$ denoted again by α_ℓ. We define i: $\mathscr{S} \to T$ to be the morphism dual to the functor assigning to each motive of PCM type of the underlying CM Hodge structure.

5.5. The following structure theorem for T encapsulates what we need to know of Deligne's theory [7] of absolute Hodge cycles on abelian varieties.

THEOREM 5: *The sequence*

$$1 \to \mathscr{S} \xrightarrow{i} T \xrightarrow{\varphi} \mathfrak{g} \to 1$$

is exact.

PROOF: This is in essence Prop. 6.28 on p. 219 of [7] together with the remark on the following page of that work. □

5.6. Fix a number field k and put

$$T_k \overset{\text{def}}{=} \varphi^{-1}(\mathfrak{g}(k)).$$

For any affine group scheme G over $\mathbb{Q}$ put

$$\mathrm{Hom}(G, \mathbb{C}^*) \overset{\text{def}}{=} \text{the group of homomorphisms of } G \text{ to } \mathbb{G}_m \text{ defined over } \mathbb{C}.$$

Let $\mathfrak{p}$ denote a prime of k, let p denote the rational prime lying below $\mathfrak{p}$ and let f denote the positive integer satisfying $\mathbb{N}\mathfrak{p} = p^f$. A subgroup $I(\mathfrak{p}, k) \subseteq \mathfrak{g}(k)$ is termed an *inertia group* at $\mathfrak{p}$ if there exists $\sigma \in D(k, \mathfrak{p})$ such that

$$I(\mathfrak{p}, k) = \sigma^{-1} I(p) \sigma \cap \mathfrak{g}(k).$$

An element $F(\mathfrak{p}, k) \in \mathfrak{g}(k)$ is called a *geometric Frobenius* at $\mathfrak{p}$ if there exists $\sigma \in D(k, \mathfrak{p})$ such that

$$F(\mathfrak{p}, k) \in \sigma^{-1} F(p)^f I(p) \sigma \cap \mathfrak{g}(k).$$

(The open subset $D(k, \mathfrak{p}) \subseteq \mathfrak{g}$ was defined in paragraph 2.3.)

THEOREM 6: *There exists a unique isomorphism*

$$\psi \mapsto \psi^{\#}: A_0(k) \xrightarrow{\sim} \mathrm{Hom}(T_k, \mathbb{C}^*)$$

with the following properties:

(I) *For all primes $\mathfrak{p}$ of k, all rational primes ℓ not divisible by $\mathfrak{p}$ and Hecke characters $\psi \in A_0(k)$, $\mathfrak{p}$ divides the conductor of ψ if and only if*

$$\psi^{\#}(\alpha_\ell(I(\mathfrak{p}, k))) \neq \{1\}$$

for some inertia group $I(\mathfrak{p}, k)$ at $\mathfrak{p}$.

(II) *For all primes $\mathfrak{p}$ of k, all rational primes ℓ not divisible by $\mathfrak{p}$ and Hecke characters $\psi \in A_0(k)$ of conductor not divisible by $\mathfrak{p}$*

$$\psi(\mathfrak{p}) = \psi^{\#}(\alpha_\ell(F(\mathfrak{p}, k)))$$

for any geometric Frobenius $F(\mathfrak{p}, k)$ at $\mathfrak{p}$.

Proof: Serre [12] has defined a commutative pro-reductive affine group scheme S over $\mathbb{Q}$ and a system of ℓ-adic representations $\epsilon_\ell\colon \mathfrak{g}(k) \to S(\mathbb{Q}_\ell)$ such that the statement obtained by replacing "T_k" with "S" and "α_ℓ" with "ϵ_ℓ" in Theorem 6 is true. Langlands [10] has defined an exact sequence

$$1 \to \mathcal{S} \to \mathcal{T} \to \mathfrak{g} \to 1 \tag{5.6.1}$$

of pro-reductive affine group schemes $\mathbb{Q}$ split ℓ-adically for each rational prime ℓ by a continuous homomorphism $\mathfrak{g} \to \mathcal{T}(\mathbb{Q}_\ell)$, naming $\mathcal{T}$ the *Taniyama group*. Let λ denote the arrow $\mathcal{T} \to \mathfrak{g}$ of (5.6.1), β_ℓ the ℓ-adic splitting $\mathfrak{g} \to \mathcal{T}(\mathbb{Q}_\ell)$, put

$$\mathcal{T}_k \overset{\text{def}}{=} \lambda^{-1}(\mathfrak{g}(k)),$$

and let $\gamma_\ell\colon \mathfrak{g}(k) \to \mathcal{T}_k^{ab}(\mathbb{Q}_\ell)$ denote the map obtained by composing β_ℓ with the projection of $\mathcal{T}_k$ to its maximal commutative quotient $\mathcal{T}_k^{ab}$. Langlands has related his construction to Serre's by proving the existence of an isomorphism $S \xrightarrow{\sim} \mathcal{T}_k^{ab}$ rendering the diagram

$$\begin{array}{ccc} & & S(\mathbb{Q}_\ell) \\ & \overset{\epsilon_\ell}{\nearrow} & \downarrow \\ \mathfrak{g}(k) & & \\ & \underset{\gamma_\ell}{\searrow} & \\ & & \mathcal{T}_k^{ab}(\mathbb{Q}_\ell) \end{array} \tag{5.6.2}$$

commutative for all ℓ. It follows that the statement obtained by replacing "T_k" with "$\mathcal{T}_k$" and "α_ℓ" with "β_ℓ" in Theorem 6 is true. Deligne has shown [7, p. 262] that there exists an isomorphism $T \xrightarrow{\sim} \mathcal{T}$ rendering the triangles

$$\begin{array}{ccc} T & & \\ \downarrow & \overset{\varphi}{\searrow} & \\ & & \mathfrak{g} \\ & \underset{\lambda}{\nearrow} & \\ \mathcal{T} & & \end{array} \tag{5.6.3}$$

$$\begin{array}{ccc} & & T(\mathbb{Q}_\ell) \\ & \overset{\alpha_\ell}{\nearrow} & \downarrow \\ \mathfrak{g} & & \\ & \underset{\beta_\ell}{\searrow} & \\ & & \mathcal{T}(\mathbb{Q}_\ell) \end{array} \tag{5.6.4}$$

commutative, the latter for all ℓ. □

5.7. For each numberfield k and character $\lambda \in \mathrm{Hom}(T_k, \mathbb{C}^*)$ let $\mathrm{Ind}_{k/\mathbb{Q}}(\lambda)$ denote the representation of T defined over $\mathbb{C}$ obtained by inducing λ. As $T \otimes \mathbb{C}$ is the inverse limit of algebraic groups over $\mathbb{C}$ the connected components of which are tori via Proposition 3.9.1 we deduce the

BRAUER INDUCTION LEMMA: *The Grothendieck group of $\mathcal{REP}_0(T \otimes \mathbb{C}/\mathbb{C})$ is generated by the representations of the form $\mathrm{Ind}_{k/\mathbb{Q}}(\lambda)$ for k a number field and $\lambda \in \mathrm{Hom}(T_k, \mathbb{C}^*)$.* □

The following important corollaries of Theorem 6 are deduced with the aid of the Brauer Induction Lemma.

COROLLARY 5.7.1: *Every object of $\mathcal{PCM}$ satisfies Hypothesis SC.*

COROLLARY 5.7.2: *Given an object M of $\mathcal{PCM}$ there exists numberfields $k_1, \ldots, k_n$, Hecke characters $\psi_i \in A_0(k_i)$ and integers m_i for $i = 1, \ldots, n$ such that*

$$L(s, M) = \prod_{i=1}^{n} L_{k_i}(s, \psi_i)^{m_i}. \quad \square$$

REMARK 5.7.3: Taking into account the Riemann hypothesis for abelian varieties over finite fields, we see that Corollary 5.8.2 admits a refinement: *For M pure of weight w, the Hecke characters ψ_i may all be taken to be of weight w..*

COROLLARY 5.7.4: *For each object M of $\mathcal{PCM}$ pure of weight zero, the order of the pole of $L(s, M)$ at $s = 1$ equals the dimension over $\mathbb{Q}$ of the subspace of T-invariants in $\omega_B(M)$.* □

From Corollary 5.7.4 one can then deduce in purely formal fashion the seemingly much stronger

PROPOSITION 5.7.5: *If objects m and N of $\mathcal{PCM}$ satisfy $L(s, M) = L(s, N)$, then M and N are isomorphic.*

PROOF: We may assume that M and N are pure of the same weight; necessarily the ranks of M and N coincide. By induction on the common rank of M and N and the semisimplicity of $\mathcal{M}$ it suffices to prove $0 \neq \mathrm{Hom}_{\mathcal{M}}(M, N)$. In turn it suffices to show by Corollary 5.7.4 that $L(s, M \otimes \hat{N})$ has a pole at $s = 1$. Now at most finitely many Euler factors of $L(s, M \otimes \hat{N})$ disagree with those of $L(s, M \otimes \hat{M})$; none of the Euler factors of either $L(s, M \otimes \hat{N})$ or $L(s, M \otimes \hat{N})$ vanishes or has a pole at $s = 1$. Thus $L(s, M \otimes \hat{N})$ has a pole at $s = 1$ if and only if $L(s, M \otimes \hat{M})$ has a pole at $s = 1$; now $L(s, M \otimes \hat{M})$ has a pole at $s = 1$ by a second application of Corollary 5.7.4. □

5.8. It can sometimes happen that the L-series of an object M of $\mathscr{PCM}$ can be written in the form

$$L(s, M) = L_k(s, \psi) \tag{5.8.1}$$

for a suitable numberfield k and Hecke character $\psi \in A_0(k)$, e.g., $M = H^1(E)$ where $E/\mathbb{Q}$ is an elliptic curve complex analytically isomorphic to $\mathbb{C}/\Lambda$, Λ a lattice in an imaginary quadratic numberfield. If in (5.8.1) the numberfield k may be taken to be either a totally real numberfield or a totally imaginary quadratic extension of a totally real numberfield, we say that M is *tractible*.

THEOREM 7: *For all tractible, critical objects M of $\mathscr{PCM}$*

$$c^+(M)^{-1} L(0, M) \in \mathbb{Q},$$

i.e., Conjecture D holds for M.

PROOF: Writing $L(s, M) = L_k(s, \psi)$ as above, we distinguish the cases $k \subseteq \mathbb{R}$ and $k \not\subseteq \mathbb{R}$. The former case is a consequence of Siegel's theorem [15] asserting the rationality of the values of the partial zeta functions of totally real numberfields for nonpositive integral values of s, together with the comptability of Deligne's conjecture with the functional equation proven in §5 of [6]. The latter case is a consequence of Blasius' interpretation [1] of the values of Eisenstein series for $GL_2(F)$, F totally real, at "CM points" as periods of motives. □

REMARK 5.8.2: The theorem above is a somewhat elliptic but quite convenient formulation of the results of Blasius that we need to use. It is elliptic because one needs to have at one's disposal for each pair k, E of numberfields a theory of motives over k with coefficients in E in order to formulate and prove in full Blasius' results which concern not merely Conjecture D above, but a more general conjecture of Deligne concerning motives with coefficients. It is convenient because for our purposes, namely the construction of $\tilde{T}$ and the evaluation of a certain class of Hecke L-series up to factors in $\mathbb{Q}$, only a formalism of motives over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$ is required.

§6. Formulation of the main theorem

6.0. In this § we introduce the notion of an *arithmetic Hodge structure* by means of which we then define the group $\tilde{T}$ and the representation $\mathbb{E}$

discussed in the introduction. The main result of the paper (Theorem 8 below) gives a detailed description of the structure of $\tilde{T}$ and $\mathbb{E}$ and relates these objects to the motives of Fermat hypersurfaces.

6.1. An *Arithmetic Hodge structure* W is a finite-dimensional $\mathbb{Q}$-vectorspace W_B equipped with a grading

$$W_B = \bigoplus_{n \in \mathbb{Z}} W_B^n \tag{6.11}$$

and a $\mathbb{C}$-linear direct sum decomposition

$$W_B^n \otimes \mathbb{C} = \bigoplus_{\substack{a,b \in \mathbb{Q} \\ a+b=n}} W^{(a,b)}, \tag{6.1.2}$$

together with the specification of a $\mathbb{Q}$-lattice

$$W_{DR} \subseteq W_B \otimes \mathbb{C} \tag{6.1.3}$$

such that these three conditions hold:

$$(1 \otimes \rho) W^{(a,b)} = W^{(b,a)}. \tag{6.1.4}$$

There exists a $\mathbb{Q}$-linear involution $\rho^*\colon W_B \to W_B$ rendering the diagram

$$\begin{array}{ccc} W_B \otimes \mathbb{C} & \xrightarrow{\rho^* \otimes \rho} & W_B \otimes \mathbb{C} \\ \cup\!| & & \cup\!| \\ W_{DR} & = & W_{DR} \end{array} \tag{6.1.5}$$

commutative. (Note that ρ^* is unique.)

For all $a \in \mathbb{Q}$ and $n \in \mathbb{Z}$, $F^a W_{DR}^n$ is a $\mathbb{Q}$-lattice in $F^a(W^n \otimes \mathbb{C})$, where we have written

$$F^a(W_B^n \otimes \mathbb{C}) \stackrel{\text{def}}{=} \bigoplus_{\substack{a'+b=n \\ a'>a}} W^{(a',b)},$$

$$F^a W_{DR}^n \stackrel{\text{def}}{=} W_{DR} \cap F^a(W_B^n \otimes \mathbb{C}). \tag{6.1.6}$$

A *morphism* $f\colon V \to W$ of arithmetic Hodge structures is a $\mathbb{Q}$-linear map for which

$$(f \otimes 1) V^{(a,b)} \subseteq W^{(a,b)}, \tag{6.1.7}$$

$$(f \otimes 1) V_{DR} \subseteq W_{DR}. \tag{6.1.8}$$

The category of arithmetic Hodge structures is denoted by $\mathscr{AHOD}$. Equipped with the evident tensor product and neutralized by the functor ω_{ahod}: $\mathscr{AHOD} \to \mathscr{MOD}(\mathbb{Q})$ assigning to each arithmetic Hodge structure W the underlying $\mathbb{Q}$-vectorspace W_B, $\mathscr{AHOD}$ is an NTC over $\mathbb{Q}$. Clearly an arithmetic Hodge structure underlies each motive; the functor ω_∞: $\mathscr{M} \to \mathscr{AHOD}$ is defined to be that which assigns to each motive the underlying arithmetic Hodge structure. An arithmetic Hodge structure W for which

$$(a, b) \notin \mathbb{Z}^2 \Rightarrow W^{(a,b)} = 0$$

is said to be integral; a Hodge structure underlies each integral arithmetic Hodge structure. An arithmetic Hodge structure W satisfying the condition

$$n \neq w \Rightarrow W_B^n = 0$$

for an integer w is said to be *pure* of *weight* w; every arithmetic Hodge structure is the direct sum of pure such.

6.2. The arithmetic Hodge structure underlying the Tate motive $\mathbb{Z}(1)$, denoted by $\mathbb{Q}(1)$, has the following description up to isomorphism in $\mathscr{AHOD}$:

$$\mathbb{Q}(1)_B = \mathbb{Q} \tag{6.2.1}$$

$$\mathbb{Q}(1)_B \otimes \mathbb{C} = \mathbb{Q}(1)^{(1,1)} \tag{6.2.2}$$

$$\mathbb{Q}(1)_{DR} = (2\pi \mathrm{i})^{-1}\mathbb{Q} \subseteq \mathbb{C}. \tag{6.2.3}$$

Given an integer n and an arithmetic Hodge structure V, we put

$$V(n) \stackrel{\mathrm{def}}{=} V \otimes \mathbb{Q}(1)^{\otimes n}$$

For ready reference later on we note that the rule for "twisting" an arithmetic Hodge structure V is the following.

$$V(n)_B \stackrel{\mathrm{def}}{=} V_B. \tag{6.2.4}$$

$$V(n)^{(p,q)} \stackrel{\mathrm{def}}{=} V^{(p+n,q+n)}. \tag{6.2.5}$$

$$V(n)_{DR} \stackrel{\mathrm{def}}{=} (2\pi \mathrm{i})^{-n} V_{DR} \subseteq V_B \otimes \mathbb{C}. \tag{6.2.6}$$

6.3. Put

$2\pi i\hat{\mathbb{Z}} \overset{\text{def}}{=}$ the profinite completion of the kernel of the exponential exp: $\mathbb{C} \to \mathbb{C}^*$.

For all $a \in \mathbb{Q}/\mathbb{Z}$ and $2\pi i n \in 2\pi i\hat{\mathbb{Z}}$ we write

$$[a, 2\pi i n] \overset{\text{def}}{=} \exp(2\pi i \langle na \rangle),$$

thereby defining a perfect pairing [?, ?]: $\mathbb{Q}/\mathbb{Z} \times 2\pi i\hat{\mathbb{Z}} \to \mathbb{C}^*$ relative to which we identify $\mathbb{Q}/\mathbb{Z}$ with the Pontryagin dual of $2\pi i\hat{\mathbb{Z}}$. Let V be a $\mathbb{Q}$-vectorspace. Each $\mathbb{Q}$-linear admissible action of $2\pi i\hat{\mathbb{Z}}$ gives rise to a $\mathbb{C}$-linear $\mathbb{Q}/\mathbb{Z}$-grading

$$V \otimes \mathbb{C} = \bigoplus_{a \in \mathbb{Q}/\mathbb{Z}} V(a) \tag{6.3.1}$$

into eigenspaces under the action of $2\pi i\hat{\mathbb{Z}}$ which for all $\sigma \in \mathfrak{g}$, $s \in \mathrm{Aut}(\mathbb{C})$ extending σ, and $a \in \mathbb{Q}/\mathbb{Z}$ has the property

$$(1 \otimes s)V(a) = V(\chi_{\text{cyc}}(\sigma)a). \tag{6.3.2}$$

Conversely, each $\mathbb{C}$-linear $\mathbb{Q}/\mathbb{Z}$-grading of V of the form (6.3.1) satisfying (6.3.2) is obtained from a unique $\mathbb{Q}$-linear admissible action of $2\pi i\hat{\mathbb{Z}}$.

6.4. For each $0 \neq a \in \mathbb{Q}/\mathbb{Z}$ let $\gamma(a)$: $2\pi i\hat{\mathbb{Z}} \to \mathbb{C}^*$ denote the function

$$\gamma(a)(2\pi i n) \overset{\text{def}}{=} [-a, 2\pi i n]\Gamma(\langle -a \rangle).$$

For each positive integer m we define an arithmetic Hodge structure E_m as follows:

$(\mathrm{E_m})_{\mathrm{B}}$ is the $\mathbb{Q}$-vectorspace of functions e: $2\pi i\hat{\mathbb{Z}} \to \mathbb{Q}$ constant on cosets of $2\pi i m\hat{\mathbb{Z}}$ and averaging to zero over $2\pi i\hat{\mathbb{Z}}$. (6.4.1)

$$E_m^{(\langle a \rangle, \langle -a \rangle)} \overset{\text{def}}{=} \mathbb{C}\gamma(a) \quad \text{for } 0 \neq a \in \frac{1}{m}\mathbb{Z}/\mathbb{Z}. \tag{6.4.2}$$

$$(E_m)_{DR} \overset{\text{def}}{=} \sum_{0 \neq a \in \frac{1}{m}\mathbb{Z}/\mathbb{Z}} \mathbb{Q}\gamma(a) \tag{6.4.3}$$

The axioms (6.1.4–6) are easy to check. Note that E_m is pure of weight 1.

Note also that for all positive integers m and n with m dividing n, the inclusion $(E_m)_B \subseteq (E_n)_B$ underlies a morphism of arithmetic Hodge structures.

6.5. Let $\mathscr{PCM}^{\sim}$ denote the full subcategory of $\mathscr{AHOD}$ the class of objects of which is the smallest containing every object isomorphic to an object of the class, closed under the formation of tensor product, direct sum, dual and subquotient which contains E_m for every positive integer m and $\omega_\infty(M)$ for every object M of $\mathscr{PCM}$. We put

$$\tilde{T} \stackrel{\text{def}}{=} \mathscr{AUT}_{\mathbf{Q}}(\mathscr{PCM}^{\sim}, \otimes, \omega_{\text{ahod}})$$

Let $\mathbb{E}_m$ denote the $\mathbf{Q}$-vectorspace $(E_m)_B$ underlying E_m equipped with the evident action of $\tilde{T}$ and put

$$\mathbb{E} \stackrel{\text{def}}{=} \varinjlim \mathbb{E}_m.$$

The functor ω_∞ restricted to $\mathscr{PCM}$ takes values in $\mathscr{PCM}^{\sim}$ by definition and thus induces a morphism ψ: $\tilde{T} \to T$. For each object W of $\mathscr{PCM}^{\sim}$ and $a \in \mathbf{Q}/\mathbf{Z}$, if we write

$$W(a) \stackrel{\text{def}}{=} \bigoplus_{p,q \in \mathbf{Z}} W^{(p+\langle a \rangle, q-\langle a \rangle)}$$

we obtain a decomposition of $W \otimes \mathbf{C}$ of the form (6.2.1) verifying (6.2.2) depending functorially and tensor-compatibly on W, a decomposition recovered under the correspondence of paragraph 6.2 from a unique morphism j: $2\pi i\hat{\mathbf{Z}} \to \tilde{T}$. Let

$$\mathbb{E} \otimes \mathbf{C} = \bigoplus_{a \in \mathbf{Q}/\mathbf{Z}} \mathbb{E}(a)$$

be the direct sum decomposition of $\mathbb{E} \otimes \mathbf{C}$ into eigenspaces under the action of $2\pi i\hat{\mathbf{Z}}$ via j. Note that

$$\mathbb{E}(0) = 0, \tag{6.5.1}$$

while

$$\dim_{\mathbf{C}} \mathbb{E}(a) = 1 \tag{6.5.2}$$

for each $0 \neq a \in \mathbf{Q}/\mathbf{Z}$. For each positive integer m and integer $n > 1$ put

$$X_m^n \stackrel{\text{def}}{=} \operatorname{Proj}(\mathbf{Q}[x_1, \ldots, x_n]/(x_1^m + \cdots + x_n^m = 0)),$$

the Fermat hypersurface of degree m and dimension $n-2$ over $\mathbb{Q}$, let $\imath$: $X_m^n \to \mathbb{P}^{n-1}$ denote the inclusion of X_m^n in the ambient projective space and put

$$H^*_{\text{prim}}(X_m^n) \stackrel{\text{def}}{=} H^*(X_m^n)/\imath^* H^*(\mathbb{P}^{n-1}),$$

an object of $\mathscr{M}$, the motive of "primitive" cohomology of the degree m dimension $n-2$ Fermat hypersurface over $\mathbb{Q}$.

6.6. We are now ready to state the principal result of the paper.

THEOREM 8:

(I) *The sequence*

$$1 \to 2\pi \mathrm{i}\hat{\mathbb{Z}} \xrightarrow{j} \tilde{T} \xrightarrow{\psi} T \to 1$$

is exact.

(II) *The motive $H^*_{\text{prim}}(X_m^n)(-1)$ belongs to $\mathscr{PCM}$, hence is a representation of T and under inflation via ψ a representation of $\tilde{T}$. There exists an isomorphism*

$$H^*{}_{\text{prim}}(X_m^n)(-1) \xrightarrow{\sim} \left(\mathbb{E}_m^{\otimes n}\right)^{2\pi \mathrm{i}\hat{\mathbb{Z}}}$$

of representations of $\tilde{\mathrm{T}}$.

(III) *For all $\sigma \in \mathfrak{g}$, $\tilde{t} \in \tilde{T}(\mathbb{C})$ such that $\varphi(\psi(\tilde{t})) = \sigma$, and $0 \neq a \in \mathbb{Q}/\mathbb{Z}$,*

$$\tilde{t}\mathbb{E}(a) = \mathbb{E}\left(\chi_{\text{cyc}}(\sigma)^{-1}a\right).$$

(IV) *For all pairs p and ℓ of distinct rational primes there exists $\tilde{F}(p, \ell) \in \tilde{T}(\mathbb{C})$ such that*

$$\alpha_\ell(F(p)) = \psi(\tilde{F}(p, \ell))$$

and such that for all positive integers f and $0 \neq a \in \mathbb{Q}/\mathbb{Z}$ annihilated by $p^f - 1$,

$$\operatorname{tr}_{\mathbb{C}}\left(\tilde{F}(p, \ell)^f \,|\, \mathbb{E}(a)\right) = g_p\left(\sum_{\nu=1}^{f} [p^\nu a]\right).$$

The proof is deferred. We shall first explain how to deduce Theorems 1 and 2 with the aid of Theorem 8.

§7. The Proof of Theorem 1

7.1. We begin with a lemma of a purely representation-theoretic nature. Let n be a positive integer, V a complex vectorspace equipped with a

$\mathbb{Z}/n\mathbb{Z}$-grading

$$V = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} V(j)$$

where each $V(j)$ is one-dimensional over $\mathbb{C}$, and $\gamma \in \mathrm{Aut}_{\mathbb{C}}(V)$ is an automorphism cyclically permuting the direct summands $V(j)$ according to the rule

$$\gamma V(j) = V(j+1).$$

Let $\mathbb{C}[V]$ denote the symmetric tensor algebra on V over $\mathbb{C}$ and put

$$W \stackrel{\text{def}}{=} \bigotimes_{j \in \mathbb{Z}/n\mathbb{Z}} V(j) \subseteq \mathbb{C}[V].$$

LEMMA 7.1.2:

$$\mathrm{tr}_{\mathbb{C}}(\gamma^n \mid V(0)) = \mathrm{tr}_{\mathbb{C}}(\gamma \mid W).$$

PROOF: Select $0 \neq v \in V(0)$ arbitrarily.

$$\begin{aligned} \mathrm{tr}(\gamma \mid W)\left(\bigotimes_{j=0}^{n-1} \gamma^j v\right) &= \gamma\left(\bigotimes_{j=0}^{n-1} \gamma^j v\right) \\ &= \bigotimes_{j=1}^{n} \gamma^j v \\ &= (\gamma^n v)\left(\bigotimes_{j=1}^{n-1} \gamma^j v\right) \\ &= \mathrm{tr}(\gamma^n \mid V(0))\left(\bigotimes_{j=0}^{n-1} \gamma^j v\right). \quad \square \end{aligned}$$

7.2. Put

$$\mathbb{B}^+ \stackrel{\text{def}}{=} \left\{ \sum n_a[a] \in \mathbb{B} \mid n_a \geqslant 0 \right\},$$

$$\mathbb{Q}[\mathbb{E}] \stackrel{\text{def}}{=} \text{the symmetric tensor algebra on } \mathbb{E} \text{ over } \mathbb{Q},$$

$$\mathbb{C}[\mathbb{E}] \stackrel{\text{def}}{=} \mathbb{Q}[\mathbb{E}] \otimes \mathbb{C}.$$

For each $\boldsymbol{a} = \sum \mathrm{n}_{\mathrm{a}}[\mathrm{a}] \in \mathbb{B}^+$ put

$$\mathbb{E}(\boldsymbol{a}) \stackrel{\text{def}}{=} \bigotimes_{a \neq 0} \mathbb{E}(a)^{\otimes n_a} \subseteq \mathbb{C}[\mathbb{E}],$$

obtaining a $\mathbb{C}$-linear direct sum decomposition

$$\mathbb{C}[\mathbb{E}] = \bigoplus_{\boldsymbol{a} \in \mathbb{B}^+} \mathbb{E}(\boldsymbol{a}). \tag{7.2.1}$$

PROPOSITION 7.2.2: *For all pairs p and ℓ of distinct rational primes there exists $\tilde{F}(p, \ell) \in \tilde{T}(\mathbb{C})$ such that*

$$\alpha_\ell(F(p)) = \psi(\tilde{F}(p, \ell))$$

and for all positive integers f and $\boldsymbol{a} \in \mathbb{B}^+$, provided that

$$p \nmid m(\boldsymbol{a}),\ F(p)^f \boldsymbol{a} = \boldsymbol{a},$$

we have

$$\mathrm{tr}_{\mathbb{C}}\left(\tilde{F}(p, \ell)^f \,|\, \mathbb{E}(\boldsymbol{a})\right) = g_p\left(\sum_{j=1}^{f} F(p)^j \boldsymbol{a}\right).$$

PROOF: Take $\tilde{F}(p, \ell)$ as in Theorem 8(IV). Put $q = p^f$. We may assume that for a suitable positive integer n

$$\boldsymbol{a} = \sum_{j \in \mathbb{Z}/n\mathbb{Z}} [a_j],$$

where $a_0, \ldots, a_{n-1}$ are n distinct elements of $\mathbb{Q}/\mathbb{Z}$ and $qa_j = a_{j+1}$. Then by Lemma 7.1.2

$$\mathrm{tr}_{\mathbb{C}}\left(\tilde{F}(p, \ell)^f \,|\, \mathbb{E}(\boldsymbol{a})\right) = \mathrm{tr}_{\mathbb{C}}\left(\tilde{F}(p, \ell)^{fn} \,|\, \mathbb{E}(a_0)\right).$$

By Theorem 8(IV),

$$\mathrm{tr}_{\mathbb{C}}\left(\tilde{F}(p, \ell)^{fn} \,|\, \mathbb{E}(a_0)\right) = g_p\left(\sum_{j=0}^{fn-1} [p^j a_0]\right)$$

$$= g_p\left(\sum_{j=0}^{f-1} F(p)^j \boldsymbol{a}\right). \quad \square$$

PROPOSITION 7.2.3: *For all* $\sigma \in \mathfrak{g}$. $\tilde{\imath} \in \tilde{T}(\mathbb{C})$ *such that* $\varphi(\psi(\tilde{\imath})) = \sigma$, $s \in Aut(\mathbb{C})$ *extending* σ *and* $\boldsymbol{a} \in \mathbb{B}^{+}$,

$$\tilde{\imath}\mathbb{E}(\boldsymbol{a}) = \mathbb{E}(\sigma^{-1}\boldsymbol{a}), \quad (1 \otimes s)\mathbb{E}(\boldsymbol{a}) = \mathbb{E}(\sigma\boldsymbol{a}).$$

PROOF: Immediate consequence of Theorem 8(III). □

LEMMA 7.2.4: *For all* $\boldsymbol{a} \in \mathbb{B}^{+} \cap \mathbb{B}^{0}$, *rational primes* p *not dividing* $m(\boldsymbol{a})$, $\mathfrak{i} \in I_p$, *rational primes* ℓ *distinct from* p *and* $\tilde{\mathfrak{i}} \in \tilde{T}(\mathbb{C})$ *such that* $\psi(\tilde{\mathfrak{i}}) = \alpha_{\ell}(\mathfrak{i})$, *one has* $\mathfrak{i}\boldsymbol{a} = \boldsymbol{a}$ *and*

$$\mathrm{tr}_{\mathbb{C}}(\tilde{\mathfrak{i}} \,|\, \mathbb{E}(\boldsymbol{a})) = 1.$$

PROOF: For all $a \in \mathbb{Q}/\mathbb{Z}$ of order prime to p, $\chi(\mathfrak{i})a = a$. Hence $\mathfrak{i}\boldsymbol{a} = \boldsymbol{a}$. Put

$$W \stackrel{\text{def}}{=} \sum_{\sigma \in \mathfrak{g}} \mathbb{E}(\sigma\boldsymbol{a}) \subseteq \mathbb{C}[\mathbb{E}].$$

Every vector of W is fixed under the action of $2\pi i\hat{\mathbb{Z}}$ via j. Further, by Theorem 8(III), W is a finite-dimensional $\tilde{T}(\mathbb{C})$-stable $\mathbb{C}$-subspace of $\mathbb{C}[\mathbb{E}]$. By Theorem 8(II), W is isomorphic as a representation of $\tilde{T}(\mathbb{C})$ to a subquotient of $H^{*}_{\mathrm{prim}}(X^{n}_{m})(-1) \otimes \mathbb{C}$ for $m = m(\boldsymbol{a})$ and $n = w(\boldsymbol{a})$. The lemma now follows by observing that since X^{n}_{m} has a model smooth over $\mathbb{Z}[1/m]$, the ℓ-adic representation $\mathfrak{g} \to \mathrm{Aut}_{\mathbb{Q}_{\ell}}(H^{*}_{\ell}(X^{n}_{m}))$ is unramified at p. □

LEMMA 7.2.5: *For any number field k and* $\boldsymbol{a} \in \mathbb{B}^{0}_{k}$ *there exists* $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{B}^{0}_{k} \cap \mathbb{B}^{+}$ *such that* $\boldsymbol{a} = \boldsymbol{b} - \boldsymbol{c}$. □

7.3. By means of Theorem 6 and Lemma 7.2.5 the proof of Theorem 1 is reduced to the demonstration of

TRACE FORMULA 7.3.1: *For any numberfield* k, $\boldsymbol{a} \in \mathbb{B}^{0}_{k} \cap \mathbb{B}^{+}$, *prime* $\mathfrak{p}$ *of* k *not dividing* $m(\boldsymbol{a})$, *rational prime* ℓ *not divisible by* $\mathfrak{p}$, *geometric Frobenius* $F(\mathfrak{p}, k) \in \mathfrak{g}(k)$ *at* $\mathfrak{p}$ *and* $\tilde{F}(\mathfrak{p}, k) \in \tilde{T}(\mathbb{C})$ *such that* $\psi(\tilde{F}(\mathfrak{p}, k)) = \alpha_{\ell}(F(\mathfrak{p}, k))$, *one has*

$$\mathrm{tr}_{\mathbb{C}}(\tilde{F}(\mathfrak{p}, k) \,|\, \mathbb{E}(\boldsymbol{a})) = g_k(\boldsymbol{a}, \mathfrak{p}).$$

PROOF: Let p denote the rational prime below $\mathfrak{p}$ and f the positive integer such that $\mathbb{N}\mathfrak{p} = p^f$. We may write

$$F(\mathfrak{p}, k) = \sigma^{-1} F(p)^f \mathfrak{i} \sigma$$

$$\tilde{F}(\mathfrak{p}, k) = \tilde{\sigma}^{-1} \tilde{F}(p, \ell)^f \tilde{\mathfrak{i}} \tilde{\sigma}$$

where $\sigma \in D(k, \mathfrak{p})$, $\mathfrak{i} \in I(p)$, $\tilde{F}(p, \ell)$ is as in Prop. 7.2.2, $\tilde{\sigma} \in \tilde{T}(\mathbb{C})$ satisfies $\psi(\tilde{\sigma}) = \alpha_\ell(\sigma)$, and $\tilde{\mathfrak{i}} \in \tilde{\mathrm{T}}(\mathbb{C})$ satisfies $\psi(\tilde{\mathfrak{i}}) = \alpha_\ell(\mathfrak{i})$. We have now simply to calculate thus:

$$\begin{aligned}
\mathrm{tr}_{\mathbb{C}}(\tilde{F}(\mathfrak{p}, k) \mid \mathbb{E}(\boldsymbol{a})) &= \mathrm{tr}_{\mathbb{C}}\left(\tilde{\sigma}^{-1} \tilde{F}(p, \ell)^f \tilde{\mathfrak{i}} \tilde{\sigma} \mid \mathbb{E}(\boldsymbol{a})\right) \\
&= \mathrm{tr}_{\mathbb{C}}\left(\tilde{F}(p, \ell)^f \tilde{\mathfrak{i}} \mid \mathbb{E}(\sigma^{-1}\boldsymbol{a})\right) \\
&= \mathrm{tr}_{\mathbb{C}}\left(\tilde{F}(p, \ell)^f \mid \mathbb{E}(\sigma^{-1}\boldsymbol{a})\right) \\
&= g_p\left(\sum_{j=1}^{f} F(p)^j \sigma^{-1}\boldsymbol{a}\right) \\
&= g_p\left(\sum_{\sigma \in D(k,\mathfrak{p})/\mathfrak{g}(k)} \sigma^{-1}\boldsymbol{a}\right) \\
&= g_k(\boldsymbol{a}, \mathfrak{p}). \quad \square
\end{aligned}$$

This completes the proof of Theorem 1 (modulo Theorem 8).

§8. The Proof of Theorem 2

8.1. In order to prove Theorem 2 we lose no generality by assuming that

$$\boldsymbol{a} \in \mathbb{B}^+ \tag{8.1.1}$$

$$\mathfrak{g}(k) = \{\sigma \in \mathfrak{g} \mid \sigma\boldsymbol{a} = \boldsymbol{a}\}. \tag{8.1.2}$$

this reduction of the proof is justified by the following five lemmas.

LEMMA 8.1.3: *For all abelian numberfields k and $\boldsymbol{a} \in \mathbb{B}_k^0$ there exists $\boldsymbol{b} \in 2(1+\rho)\mathbb{B} \cap \mathbb{B}_k^0$ such that $\boldsymbol{a} + \boldsymbol{b} \in \mathbb{B}^+$ and* $\mathfrak{g}(k) = \{\sigma \in \mathfrak{g} \mid \sigma(\boldsymbol{a}+\boldsymbol{b}) = \boldsymbol{a}+\boldsymbol{b}\}$. □

LEMMA 8.1.4: *For all $\boldsymbol{b} \in 2(1+\rho)\mathbb{B}$, $\langle \boldsymbol{b} \rangle$ is an even integer.* □

LEMMA 8.1.5: *For all $\boldsymbol{b} \in 2(1+\rho)\mathbb{B}_p$, $g_p(\boldsymbol{b}) = p^{\langle \boldsymbol{b} \rangle}$.*

PROOF: For all positive integers f and nonzero $a \in \mathbb{Q}/\mathbb{Z}$ annihilated by $p^f - 1$,

$$g_p\left(\sum_{j=1}^{f}[pa]\right)^{\rho} = \pm g_p\left(\sum_{j=1}^{f}[-pa]\right). \quad \square$$

LEMMA 8.1.6: *For all* $\boldsymbol{b} \in 2(1+\rho)\mathbb{B}$ *and* $\sigma \in \mathfrak{g}$

$$\left(\Gamma(\boldsymbol{b})/\pi^{\langle \boldsymbol{b}\rangle}\right)^{\sigma} = \Gamma(\sigma\boldsymbol{b})/\pi^{\langle \sigma\boldsymbol{b}\rangle}.$$

PROOF: For all $0 \neq \mathrm{a} \in \mathbb{Q}/\mathbb{Z}$,

$$\frac{\Gamma(\langle a\rangle)^2\Gamma(\langle -a\rangle)^2}{\pi^2} = -\frac{4\boldsymbol{e}(\langle a\rangle)}{(\boldsymbol{e}(\langle a\rangle)-1)^2}. \quad \square$$

LEMMA 8.1.7: *For all abelian numberfields* k, $\boldsymbol{a} \in \mathbb{B}_k^0$ *and* $\boldsymbol{b} \in 2(1+\rho)\mathbb{B} \cap \mathbb{B}_k^0$, *and* $n \in \Sigma_k(\boldsymbol{a})$ *the following hold*:

(I) $\Sigma_k(\boldsymbol{a}+\boldsymbol{b}) = \langle \boldsymbol{b}\rangle + \Sigma_k(\boldsymbol{a})$

(II) $L_k(s+\langle \boldsymbol{b}\rangle, \boldsymbol{a}+\boldsymbol{b}) = L_k(s, \boldsymbol{a})$

(III) $\Omega_k(n+\langle \boldsymbol{b}\rangle, \boldsymbol{a}+\boldsymbol{b})/\Omega_k(n, \boldsymbol{a}) \in \mathbb{Q}$

PROOF: Assertion (I) is clear. For all primes $\mathfrak{p}$ of k dividing neither $m(\boldsymbol{a}+\boldsymbol{b})$ nor $m(\boldsymbol{a})$ one has

$$g_k(\boldsymbol{a}+\boldsymbol{b}, \mathfrak{p}) = \mathbb{N}\mathfrak{p}^{\langle \boldsymbol{b}\rangle} g_k(\boldsymbol{a}, \mathfrak{p})$$

by Lemma 8.1.3 and the definitions: assertion (II) now follows. To prove (III) put

$$\boldsymbol{c} \overset{\text{def}}{=} \sum_{\sigma \in \Phi(\boldsymbol{a})/\mathfrak{g}(k)} \sigma\boldsymbol{b}.$$

Assuming as we may that $\Sigma_k(\boldsymbol{a}) \neq \emptyset$, via Lemma 8.1.4 we conclude that

$$\boldsymbol{c} = \sum_{\sigma \in \mathfrak{g}/\mathfrak{g}(k^+)} \sigma\boldsymbol{b} \in \mathbb{B}_{\mathbb{Q}} \cap 2(1+\rho)\mathbb{B},$$

$$\Omega_k(n+\langle \boldsymbol{b}\rangle, \boldsymbol{a}+\boldsymbol{b})/\Omega_k(n, \boldsymbol{a}) = \Gamma(\boldsymbol{c})/\pi^{\langle \boldsymbol{c}\rangle} \in \mathbb{Q}. \quad \square$$

8.2. By Lemma 7.2.3 there must exist a $\tilde{T}$-stable direct summand M of $\mathbb{Q}[\mathbb{E}]$ such that

$$M \otimes \mathbb{C} = \bigoplus_{\sigma \in \mathfrak{g}/\mathfrak{g}(k)} \mathbb{E}(\sigma \boldsymbol{a}). \tag{8.2.1}$$

Since the restriction of the representation M to $2\pi i\hat{\mathbb{Z}}$ via j is a trivial representation by Thm. 8(I) the representation M arises by inflation via ψ of an object of $\mathscr{PCM}$, an object again denoted by M. The calculations undertaken in the course of the proof of Theorem 1 suffice to establish that

$$L(s, M) = L_k(s, \boldsymbol{a}) \tag{8.2.2}$$

It follows, in particular, that M is tractible in the sense of paragraph 5.8.

8.3. For brevity put $w = w(\boldsymbol{a})$ and $m = m(\boldsymbol{a})$. The arithmetic Hodge structure underlying M is by construction the direct summand of the w-fold tensor power of E_m the underlying $\mathbb{Q}$-vectorspace of which is the space of locally constant functions f: $(2\pi i\hat{\mathbb{Z}})^w \to \mathbb{Q}$ with the following two properties:

For all $z_1, \ldots, z_w \in 2\pi i\hat{\mathbb{Z}}$ and
permutations σ: $\{1, \ldots, w\} \xrightarrow{\sim} \{1, \ldots, w\}$,

$$f(z_1, \ldots, z_w) = f(z_{\sigma(1)}, \ldots, z_{\sigma(w)}). \tag{8.3.1}$$

For all $b_1, \ldots, b_w \in \mathbb{Q}/\mathbb{Z}$

$$\int_{(2\pi i\hat{\mathbb{Z}})^w} \left(\prod_{j=1}^{w} [-b_j, z_j] \right) f(z_1, \ldots, z_w) \mathrm{d}z_1 \ldots \mathrm{d}z_w = 0$$

unless

$$\sum_{j=1}^{w} [b_j] \in \mathfrak{g}\boldsymbol{a} \subseteq \mathbb{B}^+,$$

where $\mathrm{d}z$ denotes the Haar measure on $2\pi i\hat{\mathbb{Z}}$ of unit total mass.
(8.3.2)

As this model for the arithmetic Hodge structure underlying M is cumbersome we prefer to replace it with a more simply described isomorphic copy, at the same time describing the arithmetic Hodge structures underlying all the Tate twists of M. For each $\sigma \in \mathfrak{g}$ and $n \in \mathbb{Z}$

let $\gamma_n(\sigma)$: $k \to \mathbb{C}$ denote the $\mathbb{Q}$-linear function given by the rule

$$\gamma_n(\sigma)(x) \overset{\text{def}}{=} x^\sigma \Gamma(\sigma\rho\boldsymbol{a})/(2\pi i)^n.$$

An isomorphic copy $W(n)$ of the arithmetic Hodge structure underlying $M(n)$ is obtained thus:

$$\begin{array}{l} \mathrm{W(n)_B} \text{ is the } \mathbb{Q}\text{-vectorspace of} \\ \mathbb{Q}\text{-linear functionals f: k} \to \mathbb{Q}. \end{array} \tag{8.3.3}$$

$$W(n)^{(p,q)} \overset{\text{def}}{=} \sum_{\substack{\sigma\in\mathfrak{g}/\mathfrak{g}(k)\\ \langle\sigma\boldsymbol{a}\rangle=p+n\\ \langle\sigma\rho\boldsymbol{a}\rangle=q+n}} \mathbb{C}\gamma_n(\sigma). \tag{8.3.4}$$

$$W(n)_{DR} \overset{\text{def}}{=} \sum_{\sigma\in\mathfrak{g}/\mathfrak{g}(k)} \mathbb{Q}\gamma_n(\sigma). \tag{8.3.5}$$

8.4. The Hodge numbers $h^{(p,q)}(M(n))$ and index $\mathfrak{i}(M(n))$ depend only the isomorphism class of the arithmetic Hodge structure underlying $M(n)$ and thus can be determined from $W(n)$. We have

$$h^{(p,q)}(M(n)) = \text{cardinality of } \{\sigma\in\mathfrak{g} \mid \langle\sigma\boldsymbol{a}\rangle = p+n,\ \langle\sigma\rho\boldsymbol{a}\rangle = q+n\}/\mathfrak{g}(k). \tag{8.4.1}$$

For each integer n, $\mathbb{Q}$-linear functional f: $k \to \mathbb{Q}$ and $x \in k$ put

$$(\rho_n^*(f))(x) \overset{\text{def}}{=} (-1)^n f(x^\rho),$$

noting that the diagram

$$\begin{array}{ccc} W(n)_{DR} \subseteq & W(n)_B \otimes \mathbb{C} \\ \Vert & \downarrow \rho_n^* \otimes \rho \\ W(n)_{DR} \subseteq & W(n)_B \otimes \mathbb{C} \end{array} \tag{8.4.2}$$

commutes. It follows that

$$\mathfrak{i}(M(n)) = \mathrm{tr}_{\mathbb{Q}}(\rho_n^* \mid \{f: k \to \mathbb{Q} \mid f \text{ is } \mathbb{Q}\text{-linear}\})$$

$$= \left.\begin{array}{lll} (-1)^n[k^+:\mathbb{Q}] & \text{if} & k = k^+ \\ 0 & \text{if} & k \neq k^+ \end{array}\right\}. \tag{8.4.3}$$

It follows in turn from the definition of paragraph 4.15 that

$$\{n \in \mathbb{Z} \mid M(n) \text{ is critical}\}$$
$$= \left.\begin{matrix} \Sigma_k(\boldsymbol{a}) \cup (w(\boldsymbol{a})+1-\Sigma_k(\boldsymbol{a})) & \text{if} \quad k = k^+ \\ \Sigma_k(\boldsymbol{a}) & \text{if} \quad k \neq k^+ \end{matrix}\right\}. \tag{8.4.4}$$

8.5. Now fix an integer $n \in \Sigma_k(\boldsymbol{a})$. The invariant $c^+(M(n))$ depends only on the arithmetic Hodge structure underlying $M(n)$ and thus can be determined from $W(n)$. We compute as follows: Select maximal $\mathbb{Q}$-linearly independent subsets $Y^\pm$ of $\{x \in k \mid x^\rho = \pm x\}$, respectively, and set

$$Y \overset{\text{def}}{=} Y^+ \cap Y^-.$$

For each $y \in Y$ let $\hat{y}\colon k \to \mathbb{Q}$ denote the linear functional

$$\hat{y}(x) \overset{\text{def}}{=} \operatorname{tr}_{k/\mathbb{Q}}(xy).$$

For each $y \in Y$

$$\hat{y} = \sum_{\sigma \in \mathfrak{g}/\mathfrak{g}(k)} y^\sigma \Gamma(\sigma\rho\boldsymbol{a})^{-1} (2\pi \mathrm{i})^n \gamma_n(\sigma\boldsymbol{a}), \tag{8.5.1}$$

an identity of $\mathbb{Q}$-linear maps $k \to \mathbb{C}$. It follows that the matrix with columns in one-to-one correspondence with Y and rows in one-to-one correspondence with $\mathfrak{g}/\mathfrak{g}(k)$ given by the rule

$$(\sigma\mathfrak{g}(k),\, y) \to y^\sigma \Gamma(\sigma\rho\boldsymbol{a})^{-1}(2\pi\mathrm{i})^n \tag{8.5.2}$$

represents

$$I\colon \omega_B(M) \otimes \mathbb{C} \xrightarrow{\sim} \omega_{DR}(M) \otimes \mathbb{C}$$

relative to a suitable choice of $\mathbb{Q}$-bases in $\omega_B(M(n))$ and $\omega_{\mathrm{DR}}(M(n))$, respectively, and that the $Y^{(-1)^n}$ by $\rho\Phi(\boldsymbol{a})/\mathfrak{g}(k)$ minor of this matrix represents

$$I^+\colon \omega_B^+(M) \otimes \mathbb{C} \xrightarrow{\sim} \omega_{DR}^+(M) \otimes \mathbb{C}.$$

Let $\delta^\pm$ denote the determinant of the $Y^\pm$ by $\rho\Phi(\boldsymbol{a})/\mathfrak{g}(k)$ matrix

$$(y,\, \sigma\mathfrak{g}(k)) \to y^\sigma,$$

a complex number well-defined up to a factor of ± 1.

LEMMA 8.5.3:

$$i^{nd}|\Delta_k^{(-)^n}|^{1/2}\delta^{(-)^n} \in \mathbb{Q}^*. \quad \square$$

By the lemma above and an easy calculation

$$\det(I^+) = c^+(M(n)) \equiv \Omega_k(n, \boldsymbol{a})^{-1} \bmod \mathbb{Q}^*. \tag{8.5.4}$$

Combining (8.2.2), (8.4.4) and (8.5.4) with Theorem 7, we conclude that

$$\Omega_k(n, \boldsymbol{a}) L_k(n, \boldsymbol{a}) \in \mathbb{Q},$$

as desired. This completes the proof of Theorem 2 modulo Theorem 8.

8.6. Some useful additional information can be extracted from the proof of Theorem 2. It can happen that for two different elements $\boldsymbol{a}$ and $\boldsymbol{b}$ of $\mathbb{B}_k^0$, k an abelian number field,

$$L_k(s, \boldsymbol{a}) = L_k(s, \boldsymbol{b}) \tag{8.6.1}$$

The proof of Theorem 2 yields a construction of motives M and N belonging to $\mathscr{PCM}$ for which

$$L(s, M) = L_k(s, \boldsymbol{a}), \quad L(s, N) = L_k(s, \boldsymbol{b}). \tag{8.6.2}$$

by Prop. 5.7.5 the motives M and N are isomorphic. By (8.4.4) it follows that

$$\Sigma_k(\boldsymbol{a}) = \Sigma_k(\boldsymbol{b}). \tag{8.6.3}$$

By (8.5.4) it follows that

$$\Omega_k(n, \boldsymbol{a}) \equiv \Omega_k(n, \boldsymbol{b}) \bmod \mathbb{Q}^*. \tag{8.6.4}$$

In short $\Sigma_k(\boldsymbol{a})$ depends only on $L_k(s, \boldsymbol{a})$ and for each $n \in \Sigma_k(\boldsymbol{a})$, $\Omega_k(n, \boldsymbol{a})$ modulo $\mathbb{Q}^*$ depends only on $L_k(s, \boldsymbol{a})$. In particular, Theorem 2 does not ever *force* $L_k(n, \boldsymbol{a})$ to vanish for $n \in \Sigma_k(\boldsymbol{a})$.

§9. Cohomological relations among Fermat hypersurfaces

9.0. We shall explain how the cohomology of Fermat *hypersurfaces* can be "expressed" in terms of that of Fermat *curves*. A result of this nature is stated on p. 210 of [7], but since few details of the proof are given, we shall give a complete proof here. The key to the proof is a geometrical insight of Shioda [14].

9.1. We first make precise the notion of *cohomological expressibility*. A pair (G, h^*) consisting of a pro-reductive affine group scheme G over $\mathbb{Q}$ and a contravariant $\mathcal{REP}_0(G/\mathbb{Q})$-valued functor h^* of smooth projective $\mathbb{Q}$-schemes is termed a *cohomology theory* if the following hold:

The composition of h^* with the forgetful functor $\mathcal{REP}_0(G/\mathbb{Q}) \to \mathcal{MOD}(\mathbb{Q})$ coincides with H_B^*. (9.1.1)

For each smooth projective $\mathbb{Q}$-scheme X the action of G on the $\mathbb{Q}$-vectorspace $H_B^*(X)$ underlying $h^*(X)$ preserves the cup product. (9.1.2)

A smooth projective $\mathbb{Q}$-scheme X is said to be *cohomologically expressible* in terms of a family $\{Y_i\}_{i\in I}$ of smooth projective $\mathbb{Q}$-schemes if for all cohomology theories (G, h^*) the representation $h^*(X)$ of G is isomorphic to a representation belonging to the smallest set of representations of G containing the family $\{h^*(Y_i)\}_{i\in I}$ and closed under formation of tensor product, dual, subquotient and direct sum. We shall prove

THEOREM 9: *The Fermat hypersurface X_m^n is cohomologically expressible in terms of X_m^2, X_m^3 and the projective line.*

Fix a cohomology theory (G, h^*). Put

$$f_n \overset{\text{def}}{=} \left((x_0, x_n) \to \left(x_0^2, \ldots, x_n^2\right)\right): \mathbb{P}^n \to \mathbb{P}^n,$$

$$i_n \overset{\text{def}}{=} \left((x_0, \ldots, x_{n-1}) \to (x_0, \ldots, x_{n-1}, 0)\right): \mathbb{P}^{n-1} \to \mathbb{P}^n. \tag{9.2}$$

Then f_n^*: $H_B^*(\mathbb{P}^n) \to H_B^*(\mathbb{P}^n)$ operates on $H_B^{2m}(\mathbb{P}^n)$ as the scalar 2^m. Accordingly there exists a unique subobject $h^{2m}(\mathbb{P}^n)$ of $h^*(\mathbb{P}^n)$ the underlying $\mathbb{Q}$-vectorspace of which is $H_B^{2m}(\mathbb{P}^n)$. Since for all $0 \leqslant m < n$ the map i_n^*: $H_B^{2m}(\mathbb{P}^n) \to H_B^{2m}(\mathbb{P}^{n-1})$ is an isomorphism, it follows that the isomorphism class of $h^{2m}(\mathbb{P}^n)$ is independent of n for $n \geqslant m \geqslant 0$. Since $H_B^*(\mathbb{P}^n)$ is generated by any nonzero element of $H_B^2(\mathbb{P}^n)$ as a $\mathbb{Q}$-algebra under cup product, it follows that $h^{2m}(\mathbb{P}^n)$ is isomorphic to $h^2(\mathbb{P}^1)^{\otimes m}$ for $0 \leqslant m \leqslant n$. Let f: $G \to \mathbb{G}_m$ denote the character under which each nonzero element of $H_B^2(\mathbb{P}^1)$ transforms.

LEMMA 9.2.1: *Let X be a smooth, projective $\mathbb{Q}$-scheme, $\mathcal{L}$ an invertible sheaf on X. Then the Chern class $c_1(\mathcal{L}) \in H_B^2(X)$ transforms under the action of G by the character t.*

PROOF: We may assume without loss of generality that $\mathcal{L}$ is very ample. Then under a suitable map of X to a projective space $\mathbb{P}^N$, $\mathcal{L}$ is the pull-back of an invertible sheaf on $\mathbb{P}^N$, $c_1(\mathscr{L})$ the pull-back of a cohomology class belonging to $H_B^2(\mathbb{P}^N)$. □

LEMMA 9.2.2: *Let X be a smooth, projective* $\mathbb{Q}$*-scheme,* $\mathscr{E}$ *a rank r vector bundle on X, and let* $\mathbb{P}(\mathscr{E})$ *denote the projective space bundle over X associated to* $\mathscr{E}$*. Then* $h^*(\mathbb{P}(\mathscr{E}))$ *is isomorphic to* $h^*(X)\otimes h^*(\mathbb{P}^{r-1})$.

PROOF: Let $c_1 \in H_B^2(\mathbb{P}(\mathscr{E}))$ denote the Chern class of the "twisting sheaf" over $\mathbb{P}(\mathscr{E})$, for each integer $j \geqslant 0$ let c_j denote the j-fold cup product of c_1 with itself, and let π: $\mathbb{P}(\mathscr{E}) \to X$ denote the bundle projection. It is well known that each cohomology class $\eta \in H_B^*(\mathbb{P}(\mathscr{E}))$ can be written in the form

$$\eta = \sum_{j=0}^{r-1} \pi^*\omega_j \cup c_j$$

for suitable and unique cohomology classes $\omega_j \in H_B^*(X)$. □

LEMMA 9.2.3: *Let* π: $Y \to X$ *be a morphism of smooth projective n-dimensional* $\mathbb{Q}$*-schemes such that for an open dense subscheme U of X, the morphism* $\pi^{-1}(U) \to U$ *induced by* π *is finite étale of positive degree m. Then* π^*: $H_B^*(X) \to H_B^*(Y)$ *is injective. Consequently,* $h^*(X)$ *is a subobject of* $h^*(Y)$.

PROOF: Under the hypotheses, for all C^∞ differential $2n$-forms α on the manifold $X(\mathbb{C})$ of complex points of X,

$$m\int_{X(\mathbb{C})} \alpha = \int_{Y(\mathbb{C})} \pi^*\alpha.$$

Now fix nonnegative integers p and q summing to $2n$ and a closed C^∞ differential p-form ξ on $X(\mathbb{C})$ such that $\pi^*\xi$ is exact. It will be enough to show that ξ is exact. Let ω be any closed C^∞ differential q-form on $X(\mathbb{C})$. We have

$$\int_{X(\mathbb{C})} \omega \wedge \xi = \frac{1}{m}\int_{Y(\mathbb{C})} \pi^*\omega \wedge \pi^*\xi = 0. \tag{9.2.4}$$

By Poincaré duality ξ must be exact. □

LEMMA 9.2.5: *Let X and Z be smooth, projective* $\mathbb{Q}$*-schemes. Suppose Z to be embedded as a closed subscheme of X. Let* π: $\tilde{X} \to X$ *denote the blow-up of X along Z, let* $\tilde{Z} = \pi^{-1}(Z)$*, and let* $i = \tilde{Z} \to \tilde{X}$ *denote the evident inclusion. Then the sequence*

$$H_B^*(X) \xrightarrow{\pi^*} H_B^*(\tilde{X}) \xrightarrow{i^*} H_B^*(\tilde{Z})$$

is exact. Consequently $h^*(\tilde{X})$ *is isomorphic to a direct summand of* $h^*(X) \oplus h^*(\tilde{Z})$.

PROOF: For each smooth quasi-projective $\mathbb{Q}$-scheme S and nonnegative integer n let $H_c^n(S)$ denote the space of real-valued, closed, compactly supported C^∞ differential forms of degree n on the manifold $S(\mathbb{C})$ of complex points of S, modulo the exact such. It will be enough to show that the sequence

$$H_c^*(X) \xrightarrow{\pi^*} H_c^*(\tilde{X}) \xrightarrow{i^*} H_c^*(\tilde{Z}) \tag{9.2.6}$$

is exact. Put $U \overset{\text{def}}{=} X \sim Z$, $\tilde{U} \overset{\text{def}}{=} \tilde{X} \sim \tilde{Z}$. The sequence (9.2.6) embeds in a commutative diagram

$$\begin{array}{ccccc} H_c^*(\tilde{U}) & \rightarrow & H_c^*(\tilde{X}) & \underset{i^*}{\rightarrow} & H_c^*(\tilde{Z}) \\ \uparrow & & \uparrow \pi^* & & \uparrow \\ H_c^*(U) & \rightarrow & H_c^*(X) & \rightarrow & H_c^*(Z) \end{array}$$

with exact rows and in which the left vertical arrow is an isomorphism since the restriction of π to $\tilde{U}$ is an isomorphism of $\tilde{U}$ with U. □

9.3. We shall now consider what Shioda [14] calls the *inductive structure of Fermat hypersurfaces*. For each pair of positive integers r and s let $X_m^{r,s}$ be the subscheme cut out of $\mathbb{P}^{r+s-1} = \operatorname{Proj}(\mathbb{Q}[x_1, \dots, x_r, y_1, \dots, y_s])$ by the equation

$$x_1^m + \cdots + x_r^m = y_1^m + \cdots + y_s^m.$$

LEMMA 9.3.1: $X_m^{r,s}$ *is cohomologically expressible in terms of* X_m^{r+s} *and* X_m^2.

PROOF: The morphism

$$\begin{aligned} &(((x_1, \dots, x_r, y_1, \dots, y_s), (\xi_0, \xi_1)) \\ &\quad \mapsto (\xi_0 x_1, \dots, \xi_0 x_r, \xi_1 y_1, \dots, \xi_1 y_s)) : X_m^{r,s} \times X_m^2 \to X_m^{r+s} \end{aligned}$$

is étale of degree m. The desired result now follows by Lemma 9.2.3. □

LEMMA 9.3.2: $X_m^{r,s}$ *is cohomologically expressible in terms of* X_m^{r+1}, X_m^{s+1}, X_m^r, X_m^s *and the projective line, whenever* r, $s > 1$.

PROOF: X_m^{r+1} contains a copy of X_m^r as a coordinate hyperplane section; let $Z_m^{r,s}$ denote the blow-up of $X_m^{r+1}\times X_m^{s+1}$ along one of the evident copies of $X_m^r\times X_m^s$. Shioda and Katsura [14] have given a generically m to 1 map $Z_m^{r,s}\to X_m^{r,s}$. The desired conclusion follows now from lemmas 9.2.3 and 9.2.5. □

Thus for any integers $r, s>1$, X_m^{r+s} is cohomologically expressible in terms of X_m^{r+1}, X_m^r, X_m^{s+1}, X_m^s, X_m^2 and the projective line. By induction the proof of Theorem 9 is completed.

§10. The structure of the cohomology of the Fermat hypersurfaces

10.0. We review some known results (the determination of the Hodge structures and period matrices of the Fermat hypersurfaces together with the traces of certain operators in the ℓ-adic cohomology of same) from a point of view facilitating their use in the proof of Theorem 8. Some general references for this section are [7] and [14].

10.1. We fix integers m, $n>1$, a rational prime p not dividing m and a rational prime ℓ distinct from p. We write simply X instead of X_m^n to denote the $(n-2)$-dimensional Fermat hypersurface over $\mathbb{Q}$ of degree m. We denote the geometric Frobenius $F(p)\in \mathfrak{g}$ simply by F.

10.2. Put

$$\mathcal{Y}\stackrel{\text{def}}{=}\operatorname{Proj}\left(\mathbb{Z}\left[\frac{1}{m}\right](y_0, y_1,\dots, y_n)/(y_0^m=y_1^m+\cdots+y_n^m)\right),$$

$$\mathcal{X}\stackrel{\text{def}}{=}\operatorname{Proj}\left(\mathbb{Z}\left[\frac{1}{m}\right](x_1,\dots, x_n)/(x_1^m+\cdots+x_n^m=0)\right),$$

$$\mathcal{U}\stackrel{\text{def}}{=}\operatorname{Spec}\left(\mathbb{Z}\left[\frac{1}{m}\right](u_1,\dots, u_n)/(u_1^m+\cdots+u_n^m=1)\right).$$

Under the map

$$i\stackrel{\text{def}}{=}(x_1,\dots,x_n)\to(0, x_1,\dots,x_n): \mathcal{X}\to\mathcal{Y}$$

we identify $\mathcal{X}$ with a smooth divisor on $\mathcal{Y}$ relative to Spec($\mathbb{Z}(1/m]$) and under the map

$$j\stackrel{\text{def}}{=}(u_1,\dots,u_n)\to(1, u_1,\dots,u_n): \mathcal{U}\to\mathcal{Y}$$

identify $\mathscr{U}$ with the open subscheme of $\mathscr{Y}$ complementary to $\mathscr{X}$. Put

$$S_n \overset{\text{def}}{=} \text{ the group of permutations of the set } \{1,\ldots,n\},$$

$$\Lambda \overset{\text{def}}{=} \bigoplus_{j=1}^{n} \mu_m.$$

The group S_n operates on $\mathscr{Y}$ by the rule

$$\sigma^{-1}(y_0, y_1,\ldots, y_n) = (y_0, y_{\sigma(1)},\ldots, y_{\sigma(n)}), \tag{10.2.1}$$

stabilizing both $\mathscr{X}$ and $\mathscr{U}$. For each $\mathbb{Z}[1/\mathrm{m}]$-algebra R the group $\Lambda(R)$ is made to operate on $\mathscr{Y}\times \operatorname{Spec}(R)$ by the rule

$$\lambda(y_0, y_1,\ldots, y_n) = (y_0, \lambda_1 y_1,\ldots, \lambda_n y_n). \tag{10.2.2}$$

This action of $\Lambda(R)$ stabilizes both $\mathscr{X}\times \operatorname{Spec}(R)$ and $\mathscr{U}\times \operatorname{Spec}(R)$. The generic fibers of $\mathscr{Y}$, $\mathscr{X}$, $\mathscr{U}$ are denoted by Y, X, U, respectively; the fibers at p of $\mathscr{Y}$, $\mathscr{X}$, $\mathscr{U}$ are denoted by Y_p, X_p, U_p, respectively.

10.3. For each $a \in (\frac{1}{m}\mathbb{Z}/\mathbb{Z})^n$ and $\lambda \in \Lambda(\mathbb{C})$ put

$$(\lambda, a) \overset{\text{def}}{=} \prod_{j=1}^{n} \lambda_j^{(m\langle a_j\rangle)},$$

$H_a^{n-2}(X) \overset{\text{def}}{=}$ the subspace of $H_B^{n-2}(X)\otimes\mathbb{C}$ transforming under the action of $\Lambda(\mathbb{C})$ according to the character $\lambda \mapsto (\lambda, a)$.

Let $\mathrm{i}\colon X \to \mathbb{P}^{n-1}$ denote the evident inclusion and put

$$\Psi \overset{\text{def}}{=} \left\{(a_1,\ldots, a_n) \in \left(\frac{1}{m}\mathbb{Z}/\mathbb{Z} \sim \{0\}\right)^n \mid \sum_{j=1}^{n} a_j \equiv 0\right\}.$$

Then the following hold:

$$H_B^*(X)^{\Lambda(\mathbb{C})} = \mathrm{i}^* H_B^*(\mathbb{P}^{n-1}). \tag{10.3.1}$$

$$H_B^*(X)\otimes\mathbb{C} = \left(\mathrm{i}^* H_B^*(\mathbb{P}^{n-1})\otimes\mathbb{C}\right)\oplus\left(\bigoplus_{a\in\Psi} H_a^{n-1}(X)\right). \tag{10.3.2}$$

For each $a \in \Psi$, $\dim_{\mathbb{C}} H_a^{n-2}(X) = 1$. (10.3.3)

For all $a \in \Psi$ and $r, s \in \mathbb{Z}$, if

$$r+1=\sum_{j=1}^{n}\langle a_j\rangle,\quad s+1=\sum_{j=1}^{n}\langle -a_j\rangle$$

then

$$H_a^{n-2}(X)\subseteq H^{(r,s)}(X). \tag{10.3.4}$$

For the proofs pp. 77–97 of [7].

10.4. The Leray spectral sequence for the inclusion j: $U(\mathbb{C})\to Y(\mathbb{C})$ of complex manifolds gives rise to a natural long exact sequence

$$\cdots\to H^k(Y(\mathbb{C}),\mathbb{Q})\to H^k(U(\mathbb{C}),\mathbb{Q})$$
$$\to H^{k-1}(X(\mathbb{C}),R^1j_*\mathbb{Q})\to\cdots$$

which when combined with a choice of isomorphism $R^1j_*\mathbb{Q}\xrightarrow{\sim} i_*\mathbb{Q}$ (such a choice boils down to choosing an isomorphism $\mathrm{Res}_B\colon H_B^1(\mathbb{G}_m)\xrightarrow{\sim} H_B^0$ (origin of $\mathbb{A}^1$)) yields a natural long exact sequence

$$\cdots\to H_B^k(Y)\to H_B^k(U)\to H_B^{k-1}(X)\to\cdots \tag{10.4.1}$$

which, in particular, is S_n- and $\Lambda(\mathbb{C})$-equivariant. The idempotent element

$$\theta_{\mathrm{prim}}\overset{\mathrm{def}}{=}\frac{1}{m^n}\sum_{\lambda\in\Lambda(\mathbb{C})}\sum_{a\in\Psi}(a,\lambda)\lambda$$

of the group ring $\mathbb{Q}[\Lambda(\mathbb{C})]$ annihilates all but two terms of the sequence (10.4.1) leaving us with the isomorphism

$$\theta_{\mathrm{prim}}H_B^{n-1}(U)\xrightarrow{\sim}\theta_{\mathrm{prim}}H_B^{n-2}(X). \tag{10.4.2}$$

The exact sequence of complexes of algebraic sheaves

$$0\to\Omega^{\cdot}_{Y/\mathbb{Q}}\to\Omega^{\cdot}_{Y/\mathbb{Q}}(\log X)\to\Omega^{\cdot}_{X/\mathbb{Q}}(-1)\to 0 \tag{10.4.3}$$

gives rise to a natural long exact sequence

$$\cdots\to H_{DR}^k(Y)\to H_{DR}^k(U)\to H_{DR}^{k-1}(X)\to\cdots \tag{10.4.4}$$

which, in particular, is Λ- and S_n-equivariant. The formation of sequences (10.4.1) and (10.4.4) *almost* respects the deRham isomorphism. Letting c be the nonzero rational multiple of $(2\pi i)^{-1}$ rendering the diagram

$$\begin{array}{ccc} H_B^1(\mathbb{G}_m)\otimes\mathbb{C} & \overset{\mathrm{Res}_B\otimes 1}{\to} & \mathrm{H}_{\mathrm{B}}^0\ (\text{origin of } \mathbb{A}^1)\otimes\mathbb{C} \\ \downarrow I & & \downarrow cI \\ H_{DR}^1(\mathbb{G}_m)\otimes\mathbb{C} & \underset{\frac{\mathrm{d}z}{z}\otimes 1\to 1\otimes 1}{\to} & H_B^0\ (\text{origin of } \mathbb{A}^1)\otimes\mathbb{C} \end{array} \tag{10.4.5}$$

commutative, one finds that the diagram

$$\begin{array}{ccccccccc} \cdots\to & H_B^k(Y)\otimes\mathbf{C} & \to & H_B^k(U)\otimes\mathbf{C} & \to & H_B^{k-1}(X)\otimes\mathbf{C} & \to & H_B^{k+1}(Y)\otimes\mathbf{C} & \to\cdots \\ & \downarrow I & & \downarrow I & & \downarrow cI & & \downarrow I & \\ \cdots\to & H_{DR}^k(Y)\otimes\mathbf{C} & \to & H_{DR}^k(U)\otimes\mathbf{C} & \to & H_{DR}^{k-1}(X)\otimes\mathbf{C} & \to & H_{DR}^{k+1}(Y)\otimes\mathbf{C} & \to\cdots \end{array} \tag{10.4.6}$$

commutes.

10.5. Put

$E_B \overset{\mathrm{def}}{=}$ the set of functions e: $(2\pi i\hat{\mathbb{Z}})^n \to \mathbb{Q}$ constant on cosets of $(2\pi i m\hat{\mathbb{Z}})^n$ with the property that for $j=1,\ldots,n$

$$\int_{2\pi i\hat{\mathbb{Z}}} e(z_1,\ldots,z_n)\mathrm{d}z_j = 0. \tag{10.5.1}$$

We give E_B the structure of S_n-module by the rule

$$(\sigma e)(z_1,\ldots,z_n) \overset{\mathrm{def}}{=} e(z_{\sigma(1)},\ldots,z_{\sigma(n)}),$$

the structure of $\Lambda(\mathbb{C})$-module by the rule

$$(\lambda e)(z_1,\ldots,z_n) \overset{\mathrm{def}}{=} e(z_1 - m\log\lambda_1,\ldots,z_n - m\log\lambda_n).$$

For each $a\in(\frac{1}{m}\mathbb{Z}/\mathbb{Z})^n$ put

$E_a \overset{\mathrm{def}}{=}$ the set of elements of $E\otimes\mathbb{C}$ transforming under the action of $\Lambda(\mathbb{C})$ according to the character $\lambda\mapsto(a,\lambda)$.

For each $r, s \in \mathbb{Q}$ such that $r + s \in \mathbb{Z}$ put

$$E^{(r,s)} \overset{\text{def}}{=} \bigoplus_{\substack{(a_1,\ldots,a_n) \in \left(\frac{1}{m}\mathbb{Z} \sim \{0\}\right)^n \\ r+1 = \sum_{j=1}^{n} \langle a_j \rangle \\ s+1 = \sum_{j=1}^{n} \langle -a_j \rangle}} E_a, \tag{10.5.2}$$

$$E_{DR} \overset{\text{def}}{=} \text{the } \mathbb{Q}\text{-span of the functions } (z_1,\ldots,z_n) \mapsto \left(\prod_{j=1}^{n} [-a_j, z_j]\Gamma(\langle -a_j \rangle)\right)/2\pi i \text{ for all } a \in \left(\frac{1}{m}\mathbb{Z}/\mathbb{Z} \sim \{0\}\right)^n. \tag{10.5.3}$$

The data (10.5.1–3) define an arithmetic Hodge structure hereafter denoted by E. It is not difficult to see that E is $E_m^{\otimes n}(1)$, where E_m is the arithmetic Hodge structure defined in paragraph 6.3.

10.6. The arithmetic Hodge structure underlying $H^*(X)$ is described by

THEOREM 10: *There exists a $\mathbb{Q}$-linear map β: $H_B^{n-2}(X) \to E_B$ with the following properties*:

(I) *The sequence*

$$0 \to H_B^{n-2}(\mathbb{P}^{n-1}) \overset{i^*}{\to} H_B^{n-2}(X) \overset{\beta}{\to} E_B$$

is exact.

(II) *The map β is $\Lambda(\mathbb{C})$-equivalent.*

(III) *For all $\sigma \in S_n$ the diagram*

$$\begin{array}{ccc} H_B^{n-2}(X) & \overset{\beta}{\to} & E_B \\ \downarrow{\scriptstyle (-1)^{\sigma}\sigma} & & \downarrow{\scriptstyle \sigma} \\ H_B^{n-2}(X) & \underset{\beta}{\to} & E_B \end{array}$$

commutes.

(IV) *The map β underlies a morphism $\omega_\infty(H^*(X)) \to E$ of arithmetic Hodge structures.*

PROOF: Put

$$\begin{aligned}\Delta &\overset{\text{def}}{=} \left\{ t \in \mathbb{R}^n \mid \inf_{j=1}^{n} t_j \geqslant 0,\ \sum_{j=1}^{n} t_j = 1 \right\}\\ &= \text{the standard } (n-1)\text{-simplex},\end{aligned}$$

$$\begin{aligned}V &\overset{\text{def}}{=} \operatorname{Spec}(\mathbb{Q}[v_1,\ldots,v_n]/(v_1+\cdots+v_n=1))\\ &= \text{a hyperplane in } \mathbb{A}^n.\end{aligned}$$

For each $z \in 2\pi i\hat{\mathbb{Z}}^n$ let $\tau(z)$ be the unique continuous map rendering the diagram

$$\begin{array}{ccc} & & U(\mathbb{C}) \\ & \overset{\tau(z)}{\nearrow} & \uparrow \; (u_1,\ldots,u_n) \mapsto (u_1^m,\ldots,u_n^m) \\ \Delta & \subseteq & V(\mathbb{C}) \end{array}$$

commutative, the value of which at the barycenter of Δ is

$$\left(\left[\frac{1}{m}, z_1\right] n^{-(1/m)}, \ldots, \left[\frac{1}{m}, z_n\right] n^{-(1/m)}\right) \in U(\mathbb{C}).$$

Let C denote the subspace of E_B consisting of those functions c: $2\pi i\hat{\mathbb{Z}}^n \to \mathbb{Q}$ with the additional property that for all $z_1,\ldots,z_n \in 2\pi i\hat{\mathbb{Z}}$,

$$\int_{2\pi i\hat{\mathbb{Z}}} c(z_1+z,\ldots,z_n+z)\,\mathrm{d}z = 0, \tag{10.6.1}$$

where as before $\mathrm{d}z$ denotes the unique Haar measure on $2\pi i\hat{\mathbb{Z}}$ of unit total mass. Note that C is both S_n-stable and $\Lambda(\mathbb{C})$-stable. For each $c \in C$ put

$$\tilde{c} \overset{\text{def}}{=} \int_{2\pi i\hat{\mathbb{Z}}^n} c(z)\tau(z)\,\mathrm{d}z_1 \cdots \mathrm{d}z_n,$$

a singular $(n-1)$-chain on $U(\mathbb{C})$ with $\mathbb{Q}$-coefficients which in view of the condition (10.5.1) placed upon the elements of E_B is in fact, an $(n-1)$-

cycle. For all $\sigma \in S_n$ and $\lambda \in \Lambda(\mathbf{C})$ the diagrams

$$\begin{array}{ccc} C & \to & H_{n-1}(U(\mathbf{C}); \mathbf{Q}) \\ \sigma \big\downarrow & & \big\downarrow (-1)^{\sigma}\sigma \\ C & \to & H_{n-1}(U(\mathbf{C}); \mathbf{Q}) \end{array} \tag{10.6.2}$$

$$\begin{array}{ccc} C & \to & H_{n-1}(U(\mathbf{C}); \mathbf{Q}) \\ \lambda \big\downarrow & & \big\downarrow \lambda \\ C & \to & H_{n-1}(U(\mathbf{C}); \mathbf{Q}) \end{array} \tag{10.6.3}$$

commute, where the horizontal arrows are given by the rule $c \mapsto \tilde{c}$ and $H_{n-1}(U(\mathbf{C}); \mathbf{Q})$ denotes the singular homology group with $\mathbf{Q}$-coefficients of $U(\mathbf{C})$ in dimension $n-1$. For each $a \in \Psi$ put

$$\omega_a \stackrel{\text{def}}{=} m^{n-1}\Gamma(\langle -a_1\rangle + \cdots + \langle -a_n\rangle)u_1^{(m\langle -a_1\rangle)} \cdots u_n^{(m\langle -a_n\rangle)}\frac{\mathrm{d}u_1}{u_1} \wedge \cdots \wedge \frac{\mathrm{d}u_{n-1}}{u_{n-1}},$$

defining a section ω_a of $\Omega^{n-1}_{U/\mathbf{Q}}$ over U. As a consequence of the identity (cf. [7])

$$\frac{\Gamma(\langle -a_1\rangle)\cdots\Gamma(\langle -a_n\rangle)}{\Gamma(\langle -a_1\rangle + \cdots + \langle -a_n\rangle)} = \int_0^1\int_0^{1-t_1}\cdots\int_0^{1-(t_1+\cdots+t_{n-2})} t_1^{-\langle a_1\rangle}\cdots t_{n-1}^{-\langle a_{n-1}\rangle} \times (1-(t_1+\cdots+t_{n-1}))^{-\langle a_n\rangle}\mathrm{d}t_1\cdots\mathrm{d}t_{n-1}$$

we have

$$\int_{\tilde{c}}\omega_a = \int_{2\pi\mathrm{i}\hat{\mathbf{Z}}^n} c(z)\left(\prod_{j=1}^{n}[-a_j, z_j]\Gamma(\langle -a_j\rangle)\right)\mathrm{d}z_1\cdots\mathrm{d}z_n \tag{10.6.4}$$

for all $a \in \Psi$ and $c \in C$. Let γ: $H_B^{n-1}(U) \to C$ be the unique $\mathbf{Q}$-linear map assigning to each cohomology class ω an element e of C such that for all elements c of C

$$\omega(\tilde{c}) = \int_{2\pi\mathrm{i}\hat{\mathbf{Z}}^n} c(z)e(z)\mathrm{d}z_1\cdots\mathrm{d}z_n,$$

where $\omega(\tilde{c})$ is the value taken on $\tilde{c}$ by any cochain representing the class ω. By (10.6.4) we see that γ is surjective. The commutatively of the diagrams (10.6.2) and (10.6.3) imply the commutatively of the diagrams

$$\begin{array}{ccc} H_B^{n-1}(U) & \xrightarrow{\gamma} & C \\ {\scriptstyle (-1)^e\sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ H_B^{n-1}(U) & \xrightarrow[\gamma]{} & C \end{array} \tag{10.6.5}$$

$$\begin{array}{ccc} H_B^{n-1}(U) & \xrightarrow{\gamma} & C \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\ H_B^{n-1}(U) & \xrightarrow[\gamma]{} & C \end{array} \tag{10.6.6}$$

for all $\sigma \in S_n$ and $\lambda \in \Lambda(\mathbf{C})$. Now there exists a unique $\Lambda(\mathbf{C})$-equivariant map β: $H_B^{n-2}(X) \to E_B$ rendering the diagram

$$\begin{array}{ccc} H_B^{n-1}(U) & \xrightarrow{\gamma} & C \\ \downarrow & & \downarrow{\scriptstyle \text{inclusion}} \\ H_B^{n-2}(X) & \xrightarrow[\beta]{} & E_B \end{array} \tag{10.6.7}$$

commutative where the left vertical arrow is extracted from the long exact sequence (10.4.1). Properties (I–III) of the map β so defined are immediate. In order to check (IV) we must verify that for all integers $p, q \in \mathbf{Z}$

$$(\beta \otimes 1) H^{(p,q)}(X) \subseteq E^{(p,q)} \tag{10.6.8}$$

and further that

$$(\beta \otimes 1)\big(I^{-1}\big(H_{DR}^{n-2}(X) \otimes 1\big) = E_{DR} \subseteq E_B \otimes \mathbf{C}. \tag{10.6.9}$$

Now (10.6.8) follows from (10.3.4), (10.5.2) and the $\Lambda(\mathbf{C})$-equivariance of β. The relation (10.6.9) follows from (10.6.4) and the commutativity of diagram (10.4.6). □

10.7. Fix $\boldsymbol{a} \in \mathbb{B}_p \cap \mathbb{B}^0 \cap \mathbb{B}^+$ such that $n = w(\boldsymbol{a})$ and $m(\boldsymbol{a}) \mid m$. Put

$$A \overset{\text{def}}{=} \left\{ (a_1, \ldots, a_n) \in (\mathbb{Q}/\mathbb{Z} \sim \{0\})^n \mid \sum_{j=1}^{n} [a_j] = \boldsymbol{a} \right\}.$$

Trace Formula 10.7.1:

$$g_p(\boldsymbol{a}) = \frac{p}{m^n n!} \sum_{\sigma \in S_n} \sum_{\lambda \in \Lambda(\mathbb{C})} \sum_{a \in A} (-1)^{\sigma} (\lambda, a)^{-1}$$
$$\times \operatorname{tr}_{\mathbb{Q}_\ell}\left(F\sigma\lambda \mid H_\ell^{n-2}(X) \right).$$

Proof: Let the right hand side of the formula be denoted by $g_p'(\boldsymbol{a})$. Making the identification

$$\overline{\mathbb{F}}_p = \{ x \in \overline{\mathbb{Q}} \mid |x|_p \leqslant 1 \} / \{ x \in \overline{\mathbb{Q}} \mid |x|_p < 1 \}$$

let t: $\Lambda(\overline{\mathbb{F}}_p) \xrightarrow{\sim} \Lambda(\mathbb{C})$ denote the inverse of the isomorphism induced by reduction modulo p. Let $H_\ell^*(X_p)$ denote the cohomology of the constant ℓ-adic sheaf $\mathbb{Q}_\ell$ on the étale site of $X_p \times \operatorname{Spec}(\overline{\mathbb{F}}_p)$. We denote by F: $X_p \to X_p$ the absolute Frobenius (no confusion will result from using the same letter as for the geometric Frobenius). As X and X_p are fibers of one and the same smooth $\mathbb{Z}[1/m]$-scheme, it is possible to choose a $\mathbb{Q}_\ell$-linear isomorphism $H_\ell^{n-1}(X) \xrightarrow{\sim} H_\ell^{n-2}(X_p)$ rendering the diagrams

$$\begin{array}{ccc} H_\ell^{n-2}(X) & \xrightarrow{\sim} & H_\ell^{n-2}(X_p) \\ \sigma \downarrow & & \downarrow \sigma \\ H_\ell^{n-2}(X) & \xrightarrow{\sim} & H_\ell^{n-1}(X_p) \end{array} \qquad (10.7.2)$$

$$\begin{array}{ccc} H_\ell^{n-1}(X) & \xrightarrow{\sim} & H_\ell^{n-2}(X_p) \\ t(\lambda) \downarrow & & \downarrow \lambda \\ H_\ell^{n-1}(X) & \xrightarrow{\sim} & H_\ell^{n-2}(X_p) \end{array} \qquad (10.7.3)$$

$$\begin{array}{ccc} H_\ell^{n-2}(X) & \xrightarrow{\sim} & H_\ell^{n-2}(X_p) \\ F \downarrow & & \downarrow F^* \\ H_\ell^{n-2}(X) & \xrightarrow{\sim} & H_\ell^{n-2}(X_p) \end{array} \qquad (10.7.4)$$

simultaneously commutative for all $\lambda \in \Lambda(\overline{\mathbb{F}}_p)$ and $\sigma \in S_n$. Thus

$$g_p'(\boldsymbol{a}) = \frac{p}{m^n n!} \sum_{\sigma \in S_n} \sum_{\lambda \in \Lambda(\overline{\mathbb{F}}_p)} \sum_{a \in A} (-1)^\sigma (t(\lambda), a)^{-1}$$

$$\times \operatorname{tr}_{\mathbf{Q}_\ell}\left(F^*\sigma\lambda \mid H_\ell^{n-2}(X_p)\right). \tag{10.7.5}$$

For each $a \in A$ put

$W(a) \overset{\text{def}}{=}$ the subspace of $H_\ell^*(X_p) \otimes_{\mathbf{Q}_\ell} \mathbf{C}$ transforming under the action of $\Lambda(\overline{\mathbb{F}}_p)$ according to the character $\lambda \mapsto (t(\lambda), a)$.

The commutativity of diagram (10.7.3) together with (10.3.2) insures that

$$W(a) \subseteq H_\ell^{n-2}(X_p) \otimes_{\mathbf{Q}_\ell} \mathbf{C}. \tag{10.7.6}$$

Further, for all $a = (a_1, \ldots, a_n) \in A$ and $\sigma \in S_n$

$$\sigma W(a) = W(\sigma a), \quad F^* W(a) = W(a') \tag{10.7.7}$$

where

$$\sigma^{-1} a \overset{\text{def}}{=} (a_{\sigma(1)}, \ldots, a_{\sigma(n)}), \quad a' \overset{\text{def}}{=} (pa_1, \ldots, pa_n).$$

Thus

$$g_p'(\boldsymbol{a}) = \frac{p}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \operatorname{tr}_{\mathbf{C}}\left(F^*\sigma \mid \bigoplus_{a \in A} W(a)\right)$$

$$= \frac{p}{n!} \sum_{\substack{\sigma \in S_n \\ a \in A \\ a = (\sigma a)'}} (-1)^\sigma \operatorname{tr}_{\mathbf{C}}(F^*\sigma \mid W(a)).$$

Now fix $a \in A$ and $\sigma \in S_n$ satisfying $(\sigma a)' = a$, i.e., satisfying the relation $a_{\sigma(j)} = pa_j$ for $j = 1, \ldots, n$. It will be enough to prove that

$$g_p(\boldsymbol{a}) = (-1)^\sigma p \operatorname{tr}_{\mathbf{C}}(F^*\sigma \mid W(a)). \tag{10.7.8}$$

Let g'' denote the righthand side of (10.7.8). Combining the Lefschetz trace formula with (10.7.6) we have

$$g'' = \frac{p(-1)^n(-1)^\sigma}{m^n} \times \sum_{\lambda\in\Lambda(\bar{\mathbb{F}}_p)} (t(\lambda), a)^{-1} \# \{ x \in X_p(\bar{\mathbb{F}}_p) \mid Fx = \sigma\lambda x \}.$$

Let

$X'_p \stackrel{\text{def}}{=}$ the complement in X_p of the coordinate hyperplanes $x_1 = 0, \ldots, x_n = 0$.

Then

$$g'' = \frac{p(-1)^n(-1)^\sigma}{m^n} \times \sum_{\lambda\in\Lambda(\bar{\mathbb{F}}_p)} (t(\lambda), a)^{-1} \# \{ x \in X'_p(\bar{\mathbb{F}}_p) \mid Fx = \sigma\lambda x \}.$$

Since every element of $\bar{\mathbb{F}}_p^*$ has a $(p-1)^{\text{st}}$ root,

$$g'' = \frac{p(-1)^n(-1)^\sigma}{m^n(p-1)} \times \sum_{\lambda\in\Lambda(\bar{\mathbb{F}}_p)} (t(\lambda), a)^{-1} \# \{ x \in (\bar{\mathbb{F}}_p^*)^n \mid x^p_{\sigma(j)} = \lambda_j x_j, \textstyle\sum x_j^m = 0 \}.$$

Put

$G_p \stackrel{\text{def}}{=}$ the group of complex roots of unity of order prime to p,

$D \stackrel{\text{def}}{=} \{ y \in G_p^n \mid y^p_{\sigma(j)} = y_j \}$,

$D_0 \stackrel{\text{def}}{=} \{ y \in D \mid |\sum y_j|_p < 1 \}$.

Then

$$g'' = \frac{p(-1)^n(-1)^\sigma}{m^n(p-1)} \sum_{y\in D_0} \sum_{x^m = y} \prod_{j=1}^n \left(x^p_{\sigma(j)}/x_j \right)^{-(m\langle a_j \rangle)}. \tag{10.7.9}$$

Put

$\mathrm{R} \stackrel{\text{def}}{=}$ a set of representatives for the orbits in $\{1, \ldots, n\}$ of the subgroup of S_n generated by σ.

For each $r \in R$ put

$$c(r) \stackrel{\text{def}}{=} \text{cardinality of the orbit of which } r \text{ is representative},$$

$$q(r) \stackrel{\text{def}}{=} p^{c(r)}.$$

Then for all $y \in D$ and $x \in G$ such that $x^m = y$,

$$\prod_{j=1}^{n} \left(x_{\sigma(j)}^{p}/x_j\right)^{-(m\langle a_j\rangle)} = \prod_{r\in R} \left(x_r^{q(r)-1}\right)^{-(m\langle a_j\rangle)}$$

$$= \prod_{r\in R} y_r^{(-\langle a_r\rangle(q(r)-1))}. \tag{10.7.10}$$

Here we have used the assumption $(\sigma a)' = a$ in order to "telescope" the product appearing as the summand in the right hand side of (10.7.9). Let $b\colon D \to \frac{1}{p}\mathbb{Z}/\mathbb{Z}$ be the unique function verifying

$$|\, p\langle b(y)\rangle - \sum_{j=1}^{n} y_j \,|_p < 1$$

for all $y \in D$, and let $\psi\colon D \to \mu_p(\mathbb{C})$ be defined by the relation

$$\psi(y) = e(\langle b(y)\rangle).$$

Combining (10.7.9) and (10.7.10) with the evident character identity,

$$g'' = \frac{(-1)^n(-1)^\sigma}{p\text{-}1} \sum_{j=0}^{p-1} \sum_{y\in D} \psi(y)^j \prod_{r\in R} y_r^{(-\langle a_r\rangle(q(r)-1))}.$$

Since

$$\sum_{y\in D} \prod_{r\in R} y_r^{(-\langle a_r\rangle(q(r)-1))} = 0,$$

and for all integers j not divisible by p and $y \in D$,

$$\prod_{r\in R} y_r^{((j-1)\langle a_r\rangle(q(r)-1))} = 1$$

as a consequence of the assumption $\boldsymbol{a} \in \mathbb{B}^0$, it follows that

$$g'' = (-1)^n(-1)^\sigma \sum_{y\in D} \psi(y) \prod_{r\in R} y_r^{(-\langle a_r\rangle(q(r)-1))}.$$

Finally, appealing to the definitions,

$$g'' = (-1)^n(-1)^\sigma(-1)^{\#(R)} g_p(\boldsymbol{a}) = g_p(\boldsymbol{a}). \quad \square$$

§11. The Proof of Theorem 8

11.0. We shall assemble the proof of Theorem 8. Most of the hard work has already been done.

11.1. We begin by establishing the claim that for all integers m, $n > 1$, $H^*_{\text{prim}}(X^n_m)$ belongs to $\mathscr{PCM}$. By Theorem 9 it suffices to show that $H^1(X^3_m)$ belong to $\mathscr{PCM}$ for all $m > 1$. Now let J_m denote for each integers $m > 1$ the Jacobian variety of X^3_m. The map $H^1(J_m) \to H^1(X^3_m)$ induced by some embedding $X^3_m \to J_m$ is an isomorphism of motives. Now as a consequence of Theorem 10, the Hodge structure underlying $H^1(X^3_m)$, hence also that underlying $H^1(J_m)$, is of $\mathscr{CM}$ type. Thus $H^1(X^3_m)$ belongs to $\mathscr{PCM}$ and the claim is established.

11.2. In order to abbreviate notation, put

$$G_{\text{ahod}} \stackrel{\text{def}}{=} \mathscr{AUT}_{\mathbf{Q}}(\mathscr{AHOD}, \otimes, \omega_{\text{ahod}}).$$

The morphism $G_{\text{ahod}} \to \mathfrak{g}$ dual to the functor $\mathscr{ART} \to \mathscr{AHOD}$ obtained by restricting ω_∞: $\mathscr{M} \to \mathscr{AHOD}$ to $\mathscr{ART}$ is, so we claim, faithfully flat. Now an arithmetic Hodge structure W is isomorphic to the arithmetic Hodge structure underlying an artinian motive precisely, when the following hold, as the discussion of paragraph 4.8 shows:

$$W_B \otimes \mathbf{C} = W^{(0,0)} \tag{11.2.1}$$

There exists a $\mathbf{Q}$-linear admissible action of $\mathfrak{g}$ upon W_B such that for all $\sigma \in \mathfrak{g}$ and $s \in \operatorname{Aut}(\mathbf{C})$ extending σ the diagram

$$\begin{array}{ccc} W_{DR} \subseteq & W_B \otimes \mathbf{C} \\ 1 \downarrow & \downarrow \sigma \otimes s \\ W_{DR} \subseteq & W_B \otimes \mathbf{C} \end{array}$$

commutes. (Note that the action $\mathfrak{g} \to \mathrm{GL}_{\mathbf{Q}}(\mathrm{W_B})$ is unique.)

(11.2.2)

It is then not difficult to check that the criterion for faithful flatness given by Prop. 3.7.4 holds, thereby establishing the claim.

11.3. To abbreviate notation, put

$$G_{\text{hod}} = \mathcal{AUT}_{\mathbf{Q}}(\mathcal{HOD}, \otimes, \omega_{\text{hod}}).$$

The inclusion $\mathcal{HCM} \to \mathcal{HOD}$ induces a morphism $G_{\text{hod}} \to S$ which, by another application of Prop. 3.7.4, is faithfully flat.

11.4. The commutative diagram of categories and functors

$$\begin{array}{ccccc} \mathcal{HOD} & \xleftarrow{\text{forgetting}} & \mathcal{AHOD} & & \\ \text{inclusion}\uparrow & & \uparrow \omega_\infty & & \\ \mathcal{HCM} & \xleftarrow[\text{forgetting}]{} & \mathcal{PCM} & \xleftarrow[\text{inclusion}]{} & \mathcal{ART} \end{array} \tag{11.4.1}$$

gives rise to a commutative diagram

$$\begin{array}{ccccccc} & & G_{\text{hod}} & \to & G_{\text{ahod}} & & \\ & & \downarrow & & \downarrow & & \\ 1 & \to & S & \xrightarrow{i} & T & \xrightarrow{\varphi} \mathfrak{g} & \to 1 \end{array} \tag{11.4.2}$$

in which, by Theorem 5, the bottom row is exact and in which the arrows $G_{\text{hod}} \to S$ and $B_{\text{ahod}} \to \mathfrak{g}$ as we have seen are faithfully flat. It follows that the morphism $G_{\text{ahod}} \to T$ dual to the functor ω_∞: $\mathcal{PCM} \to \mathcal{AHOD}$ is faithfully flat.

11.5. Yet another application of Prop. 3.7.4 shows that the morphism $G_{\text{ahod}} \to T'$ induced by the inclusion $\mathcal{PCM}^{\sim} \to \mathcal{AHOD}$ is faithfully flat. Now since the diagram

$$\begin{array}{ccc} & G_{\text{ahod}} & \\ \swarrow & & \searrow \\ T' & \xrightarrow{\psi} & T \end{array} \tag{11.5.1}$$

commutes and the two arrows with source G_{ahod} are faithfully flat, the morphism ψ: $T' \to T$ is faithfuly flat.

11.6. By Prop. 3.7.3 it follows that the morphism j: $2\pi i\hat{\mathbb{Z}} \to \tilde{T}$ is a closed immersion. By definition, $2\pi i\hat{\mathbb{Z}}$ operates trivially via j on the $\mathbb{Q}$-vectorspace underlying an integral object of $\mathscr{PCM}^{\sim}$; it follows that the image of j: $2\pi i\hat{\mathbb{Z}} \to \tilde{T}$ lies in the kernel of ψ: $\tilde{T} \to T$.

11.7. Let H denotes the kernel of ψ: $\tilde{T} \to T$. We claim that the map $2\pi i\hat{\mathbb{Z}} \to H$ through which j: $2\pi i\hat{\mathbb{Z}} \to T$ factors is an isomorphism; at any rate we may identify $2\pi i\hat{\mathbb{Z}}$ with a closed subgroup of H under the map $2\pi i\hat{\mathbb{Z}} \to H$. By Theorem 10, for all positive integers m and n, the arithmetic Hodge structure $E_m^{\otimes n}$ contains a copy of the arithmetic Hodge structure $\omega_\infty(H^*_{\text{prim}}(X_m^n)(-1))$; it follows that

$$\left(\mathbb{E}_m^{\otimes n}\right)^{2\pi i\hat{\mathbb{Z}}} = \left(\mathbb{E}_m^{\otimes n}\right)^{H}. \tag{11.7.1}$$

Now since the functor $V \mapsto V^{2\pi i\hat{\mathbb{Z}}}$: $\mathscr{REP}_0(2\pi i\hat{\mathbb{Z}}/\mathbb{Q}) \to \mathscr{MOD}(\mathbb{Q})$ is exact and $\{\mathbb{E}_m\}_{m=1}^{\infty}$ is a faithful family of representations of H, it follows without difficulty from (11.7.1) that for all finite-dimensional representations V of H,

$$V^{2\pi i\hat{\mathbb{Z}}} = V^{H}. \tag{11.7.2}$$

By Prop. 3.7.4 together with the Remark 3.7.6, we deduce that the closed immersion $2\pi i\hat{\mathbb{Z}} \to H$ is faithfully flat, hence an isomorphism. This completes the proof of statement (I) of Theorem 8.

11.8. In the course of proving Thm. 8(I) we observed that by Theorem 10, $E_m^{\otimes n}(1)$ contains a copy of $\omega_\infty(H^*_{\text{prim}}(X_m^n))$. This is also the observation that proves statement (II) of Theorem 8.

11.9. Let $\tilde{S}$ denote the kernel of the composite morphism $\varphi \circ \psi$: $\tilde{T} \to \mathfrak{g}$. The action of $\tilde{S}$ upon $2\pi i\hat{\mathbb{Z}}$ deduced from the exact sequence

$$1 \to 2\pi i\hat{\mathbb{Z}} \xrightarrow{j} \tilde{S} \xrightarrow{\psi} S \to 1$$

must be *trivial* because S is connected, $2\pi i\hat{\mathbb{Z}}$ totally disconnected. Consequently $\tilde{S}$ is a *central* extension of S by $2\pi i\hat{\mathbb{Z}}$.

11.10. Let $\sigma \in \mathfrak{g}$ and $s \in \mathrm{Aut}(\mathbb{C})$ extending σ be chosen. For each object W of $\mathscr{PCM}^{\sim}$ let $\tilde{\imath}_0(W) \in \mathrm{Aut}_{\mathbb{C}}(W_B \otimes \mathbb{C})$ be the unique map rendering the diagram

$$\begin{array}{ccccc} W_B\otimes\mathbb{C} & \overset{I}{\text{———}} & W_{DR}\otimes\mathbb{C} & \overset{1\otimes s}{\text{—}} & W_{DR}\otimes\mathbb{C}\\ & \searrow & & & \uparrow\, \mathrm{I} \\ & & & & W_B\otimes\mathbb{C} \\ & \tilde{t}_0(W) \searrow & & & \uparrow\, \otimes s\\ & & & & W_B\otimes\mathbb{C}\end{array}$$

commutative. Then $\tilde{t}_0$ is a point of $\tilde{T}$ defined over $\mathbb{C}$ which satisfies $\varphi(\psi(\tilde{t}_0)) = \sigma$. Further, for all $0 \neq a \in \mathbb{Q}/\mathbb{Z}$ one has

$$\tilde{t}_0\mathbb{E}(a) = \mathbb{E}\big(\chi(\sigma)^{-1}a\big)$$

be an easy computation. Since any $\tilde{t} \in \tilde{T}(\mathbb{C})$ such that $\varphi(\psi(\tilde{t})) = \sigma$ is the product of $\tilde{t}_0$ and an element of $\tilde{S}(\mathbb{C})$, statement (III) of Theorem 8 is proven.

11.11. We turn to the proof of statement (IV) of Theorem 8. Fix distinct primes p and ℓ. Let $\tilde{F} \in \tilde{T}(\mathbb{C})$ be any point such that $\alpha_\ell(F(p)) = \psi(\tilde{F})$. We employ the notation introduced in paragraph 7.2 for the discussion of the symmetric tensor algebra on $\mathbb{E}$. Let g_p': $\mathbb{B}_p \to \mathbb{C}^*$ be the unique homomorphism such that for all $\boldsymbol{a} \in \mathbb{B}_p \cap \mathbb{B}^+$

$$g_p'(\boldsymbol{a}) \overset{\text{def}}{=} \operatorname{tr}_{\mathbb{C}}(\tilde{F} \,|\, \mathbb{E}(\boldsymbol{a})); \tag{11.11.1}$$

the right hand side of (11.11.1) makes sense because by Thm. 8(III)

$$\tilde{F}\mathbb{E}(\boldsymbol{a}) = \mathbb{E}(\boldsymbol{a}). \tag{11.11.2}$$

By Theorem 10 combined with Trace Formula 10.7.1, for all $\boldsymbol{a} \in \mathbb{B}_p \cap \mathbb{B}^0$,

$$g_p'(\boldsymbol{a}) = g_p(\boldsymbol{a}). \tag{11.11.3}$$

Let γ_0 denote the topological generator $2\pi\mathrm{i}$ of $2\pi\mathrm{i}\hat{\mathbb{Z}}$, and let γ: $\mathbb{B}_p \to \mathbb{C}^*$ denote the unique homomorphism such that for all $\boldsymbol{a} \in \mathbb{B}_p \cap \mathbb{B}^+$

$$\gamma(\boldsymbol{a}) \overset{\text{def}}{=} \operatorname{tr}_{\mathbb{C}}(\gamma_0 \,|\, \mathbb{E}(\boldsymbol{a})).$$

Then the sequence

$$0 \to \mathbb{B}_p \cap \mathbb{B}_0 \to \mathbb{B}_p \overset{\gamma}{\to} \mu_{p-1}(\mathbb{C}) \to 1 \tag{11.11.4}$$

is exact. By (11.11.3,4) there exists an integer ν such that for all $\boldsymbol{a} \in \mathbb{B}_p \cap \mathbb{B}^+$

$$g_p(\boldsymbol{a}) = \mathrm{tr}_{\mathbf{C}}\left(\gamma_0^\nu \tilde{F} \mid \mathbb{E}(\boldsymbol{a})\right). \tag{11.11.5}$$

Put

$$\tilde{F}_{p,\ell} \overset{\mathrm{def}}{=} \gamma_0^\nu \tilde{F}.$$

Then for all positive integers f and $0 \neq a \in \mathbb{Q}/\mathbb{Z}$ annihilated by $p^f - 1$, an application of Lemma 7.1.2 gives

$$\mathrm{tr}_{\mathbf{C}}\left(\tilde{F}_{p,\ell}^f \mid \mathbb{E}(a)\right) = \mathrm{tr}_{\mathbf{C}}\left(\tilde{F}_{p,\ell} \mid \mathbb{E}\left(\sum_{j=1}^{f} [p^j a]\right)\right)$$
$$= g_p\left(\sum_{j=1}^{f} [p^j a]\right).$$

This completes the proof of Theorem 8 (and the paper).

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