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A GENERAL NOTION OF EXTREME SUBSET

Marek Lassak

The purpose of this paper is a general view on various definitions of extreme subsets and extreme elements as used in several branches of mathematics. Using terms of set operations, we present a universal scheme common for many such definitions and we discuss it under some weak assumptions.

Let X be an arbitrary set and let $\mathscr{P}(X)$ denote the family of all subsets of X. Any function Φ mapping a subfamily $\mathscr{D}_\Phi$ of $\mathscr{P}(X)$ into $\mathscr{P}(X)$ is called a *set operation* in X or simply an *operation*. $\mathscr{D}_\Phi$ is the domain of Φ.

In [16] the following definition has been proposed

DEFINITION: If $A \subset B \subset X$ and if for any $K \in \mathscr{D}_\Phi \cap \mathscr{P}(B)$ and any $x \in A \cap \Phi(K)$ there exists $M \in \mathscr{D}_\Phi \cap \mathscr{P}(A \cap K)$ such that $x \in \Phi(M)$, then A is said to be a Φ-*extreme subset* of B.

In the case $A = \{a\}$ of the above definition, a will be called a Φ-*extreme element* of B.

Examples of Φ-extreme elements are extreme points of a set of a real linear space (Φ maps any set $\{x, y\}$ onto the segment joining x and y), isolated points of a set of any topological space (Φ is the closure operation), terminal points of a set of a metric space (Φ maps any set $\{x, y\}$ onto the set of all points lying metrically between x and y). For examples of Φ-extreme subsets and other examples of Φ-extreme elements see the last paragraph of this paper.

1. Set operations

This paragraph is of preparatory nature.

An operation Φ in X is called:

- *full*, if $\mathscr{D}_\Phi = \mathscr{P}(X)$,
- *isotonic*, if for any G, $H \in \mathscr{D}_\Phi$ from $G \subset H$ it follows $\Phi(G) \subset \Phi(H)$,
- *enlarging*, if $G \subset \Phi(G)$ for any $G \in \mathscr{D}_\Phi$,
- *idempotent*, if $\Phi(\Phi(A)) = \Phi(A)$ for any $A \in \mathscr{D}_\Phi$ such that $\Phi(A) \in \mathscr{D}_\Phi$,
- a *closure* operation, if it is full, isotonic, enlarging, and idempotent,

- *additive*, if it is full and $\Phi(G \cup H) = \Phi(G) \cup \Phi(H)$ for any $G, H \in \mathscr{P}(X)$,
- *cover finite*, if $\Phi(G) \subset \cup\{\Phi(F);\ F \in \mathscr{D}_\Phi \cap \mathscr{P}(G),\ |F| < \infty\}$ for any $G \in \mathscr{D}_\Phi$,
- *domain finite*, if $\Phi(G) = \cup\{\Phi(F);\ F \in D_\Phi \cap \mathscr{P}(G),\ |F| < \infty\}$ for any $G \in \mathscr{D}_\Phi$.

Full isotonic, full enlarging, full idempotent, full domain finite, and closure operations are considered by many authors (see [1],[6],[10],[11], [20],[21] for instance).

The smallest number $k \geqslant 0$ such that

$$\Phi(G) \subset \cup\{\Phi(F);\ F \in \mathscr{D}_\Phi \cap \mathscr{P}(G), |F| \leqslant k\} \quad \text{for any } G \in \mathscr{D}_\Phi$$

is called *Caratheodory's number* of Φ and it is denoted by $c(\Phi)$. If such k does not exist, then we put $c(\Phi) = \infty$.

Obviously, if Φ is isotonic, then the inclusion in the above definition can be replaced by the equality. Note that any operation with finite Caratheodory's number is cover finite.

If $\Phi(G) = \Phi(H)$, then G and H are called Φ-*equivalent*. If additionally $G \subset H$, then we call G a Φ-*equivalent subset* of H.

We call $K \in \mathscr{D}_\Phi$ a Φ-*stable* set if $\Phi(K) = K$.

Remeber that a closed under arbitrary intersections family $\mathscr{C} \subset \mathscr{P}(X)$ is called a *closure system* over X. If moreover $\phi \in \mathscr{C}$, then $(X, \mathscr{C})$ is usually called a *convexity structure*. Let the symbol $h_{\mathscr{C}}(A)$ denote the intersection of all sets of $\mathscr{C}$ which contain a given set $A \in \mathscr{P}(X)$.

It is well known (see e.g. [6], p. 43) that there exists one-to-one correspondence between closure systems over X and closure operations in X: $(\rightarrow)$ if $\mathscr{C}$ is a closure system over X, then $h_{\mathscr{C}}$ is a closure operation, $(\leftarrow)$ if Φ is a closure operation, then the family $\mathscr{C}_\Phi$ of Φ-stable sets is a closure system over X and $h_{\mathscr{C}_\Phi} = \Phi$.

If Φ is a closure operation, then $\Phi(K)$ is called the *closure* of K.

For a given operation Φ in X we define an auxiliary operation

$$\Phi^{\cup}(G) = \cup\{\Phi(D);\ D \in \mathscr{D}_\Phi \cap \mathscr{P}(G)\}.$$

If Φ is full, then let (comp. [11], p. 311)

$$\Phi^{\omega}(G) = \bigcup_{n=0}^{\infty} \Phi^n(G),$$

where $\Phi^0(G) = G$, $\Phi^{i+1}(G) = \Phi(\Phi^i(G))$, $i = 0, 1, \ldots$.

Obviously, both of the operations $\Phi^{\cup}$, Φ^{ω} are full. Moreover, $\Phi^{\cup}$ is isotonic. If Φ is isotonic, then $\Phi^{\cup}(G) = \Phi(G)$ for any $G \in \mathscr{D}_\Phi$. If Φ is full and isotonic, then $\Phi^{\cup} = \Phi$.

2. General properties of Φ-extreme subsets

Observe that the definition of Φ-extreme subset A of B can be shortly expressed by the formula

$$A \cap \Phi(K) \subset \Phi^{\cup}(A \cap K) \quad \text{for any } K \in \mathscr{D}_{\Phi} \cap \mathscr{P}(B). \tag{1}$$

The following five statements of Theorem 1 generalize well known properties of the classic extreme subsets in real linear spaces.

THEOREM 1: *For each operation* Φ *in* X *we have*:

(a) *Any finite intersection of* Φ*-extreme subsets of a set* B *is a* Φ*-extreme subset of* B*. This is also true for arbitrary intersections provided* Φ *is cover finite.*

(b) *Any union of* Φ*-extreme subsets of a set* B *is also a* Φ*-extreme subset of* B*.*

(c) *If* B *is a* Φ*-extreme subset of* C *and* A *is a* Φ*-extreme subset of* B*, then* A *is a* Φ*-extreme subset of* C*.*

(d) *If* $A \subset B \subset C$ *and if* A *is a* Φ*-extreme subset of* C*, then* A *is a* Φ*-extreme subset of* B*.*

(e) *Sets* B*,* ϕ *and (if* $\phi \in D_{\Phi}$*) all subsets of* $B \cap \Phi(\phi)$ *are* Φ*-extreme subsets of* B*.*

PROOF: We prove only the first part. The other ones are left to the reader.

Let A_1, A_2 be Φ-extreme subsets of B. Suppose, $x \in (A_1 \cap A_2) \cap \Phi(K)$, where $K \in \mathscr{D}_{\Phi} \cap \mathscr{P}(B)$. Since A_1 is a Φ-extreme subset of B and $x \in A_1 \cap \Phi(K)$, there exists $M_1 \in \mathscr{D}_{\Phi} \cap \mathscr{P}(A_1 \cap K)$ such that $x \in \Phi(M_1)$. Similarly, from $x \in A_2 \cap \Phi(M_1)$ and $M_1 \in \mathscr{D}_{\Phi} \cap \mathscr{P}(B)$ we get that $x \in \Phi(M)$ for some $M \in \mathscr{D}_{\Phi} \cap \mathscr{P}(A_2 \cap M_1)$. Obviously, $M \in \mathscr{D}_{\Phi} \cap \mathscr{P}(A_1 \cap A_2 \cap K)$. Thus $A_1 \cap A_2$ is a Φ-extreme subset of B. Consequently, any finite intersection of Φ-extreme subsets of B is also Φ-extreme.

Now, let Φ be cover finite and A_λ, $\lambda \in \Lambda$, be Φ-extreme subsets of B. Put $A = \cap\{A_\lambda, \lambda \in \Lambda\}$. Suppose, $K \in \mathscr{D}_{\Phi} \cap \mathscr{P}(B)$ and $x \in A \cap \Phi(K)$. Since Φ is cover finite, $x \in \Phi(M)$ for a finite $M \in \mathscr{D}_{\Phi} \cap \mathscr{P}(K)$. Without loss of generality, we can assume that M is minimum (in respect to inclusion) set of $\mathscr{D}_{\Phi} \cap \mathscr{P}(K)$ for which $x \in \Phi(M)$. Moreover, $x \in A_\lambda$ and A_λ is a Φ-extreme subset of B for any $\lambda \in \Lambda$. Thus for any $\lambda \in \Lambda$ there exists $M_\lambda \in \mathscr{D}_{\Phi} \cap \mathscr{P}(A_\lambda \cap M) \subset \mathscr{D}_{\Phi} \cap \mathscr{P}(K)$ such that $x \in \Phi(M_\lambda)$. Since M is minimal and $M_\lambda \subset M$, we have $M_\lambda = M$ for all $\lambda \in \Lambda$. From $M_\lambda \subset A_\lambda$, $\lambda \in \Lambda$, we get $M \subset A$. Hence $M \in \mathscr{D}_{\Phi} \cap \mathscr{P}(A \cap K)$. Thus A is a Φ-extreme subset of B.

Note that in general case the intersection of infinitely many Φ-extreme subsets may fail to be Φ-extreme (comp. e.g. Example 4).

PROPOSITION 1: *Let* Φ *be a full isotonic operation* [*respectively*: *a cover* *finite operation, an operation with finite Caratheodory's number, an additive operation*]. *A subset A of B is* Φ*-extreme if and only if the below condition* (2) [*respectively*: (3), (4), (5)] *holds*:

$$A \cap \Phi(K) \subset \Phi(A \cap K) \quad \textit{for any } K \in \mathscr{P}(B), \tag{2}$$

$$A \cap \Phi(K) \subset \Phi^{\cup}(A \cap K) \quad \textit{for any finite } K \in \mathscr{D}_{\Phi} \cap \mathscr{P}(B), \tag{3}$$

$$A \cap \Phi(K) \subset \Phi^{\cup}(A \cap K) \quad \textit{for any } K \in \mathscr{D}_{\Phi} \cap \mathscr{P}(B) \quad \textit{with } |K| \leqslant c(\Phi), \tag{4}$$

$$A \cap \Phi(B \setminus A) \subset \Phi(\phi). \tag{5}$$

PROOF: The first three statements result easily from (1), because $\Phi^{\cup} = \Phi$ for any full isotonic operation, and from the definitions of cover finite operation and Caratheodory's number.

We prove the last part. As additive, the operation Φ is isotonic and full. So it is sufficient to show that (2) and (5) are equivalent.

Immediately, (5) results from (2) putting $K = B \setminus A$.

Assume, (5) holds. Therefore $A \cap \Phi(B \setminus A) \subset A \cap \Phi(\phi)$. Let K be a subset of B. Since Φ is isotonic, $\Phi(K \setminus A) \subset \Phi(B \setminus A)$ and $\Phi(\phi) \subset \Phi(K \cap A)$. Moreover, Φ is additive. Thus we get that

$$\begin{aligned}
A \cap \Phi(K) &= A \cap \Phi[(K \cap A) \cup (K \setminus A)] \\
&= A \cap [\Phi(K \cap A) \cup \Phi(K \setminus A)] \\
&= [A \cap \Phi(K \cap A)] \cup [A \cap \Phi(K \setminus A)] \\
&\subset [A \cap \Phi(K \cap A)] \cup [A \cap \Phi(B \setminus A)] \\
&\subset [A \cap \Phi(K \cap A)] \cup [A \cap \Phi(\phi)] \\
&= A \cap \Phi(K \cap A) \subset \Phi(K \cap A),
\end{aligned}$$

which ends the proof.

Condition (2) is very useful and it can be applied to full isotonic operations, particularly to the operation $\Phi^{\cup}$ for any Φ. This is why we formulate the following proposition whose proof is left to the reader.

PROPOSITION 2: *A is a* Φ*-extreme subset of B if and only if A is a* $\Phi^{\cup}$*-extreme subset of B.*

3. Φ-extreme and Φ^ω-extreme subsets

THEOREM 2: *Any Φ-extreme subset of B is also Φ^ω-extreme provided Φ is full domain finite and B is Φ^ω-stable (particularly: Φ-stable).*

PROOF: Let A be a Φ-extreme subset of B. Since Φ is full and (as domain finite) isotonic, Φ^ω is also full and isotonic. By the first part of Proposition 1, to verify that A is a Φ^ω-extreme subset of B it is sufficient to show the inclusion $A \cap \Phi^\omega(H) \subset \Phi^\omega(A \cap H)$ for any $H \dot{\in} \mathscr{P}(B)$.

Let $x \in A \cap \Phi^\omega(H)$. Let m be the smallest number such that $x \in \Phi^m(H)$. Recurrently, define finite sets $H_m, \ldots, H_0$ as follows. Put $H_m = \{x\}$. Obviously, $H_m \subset \Phi^m(H)$. Suppose, a finite subset H_n of $\Phi^n(H)$ is defined, where $m \geqslant n > 0$. Since Φ is full domain finite, there exists a finite set $G_{n-1} \subset \Phi^{n-1}(H)$ such that $H_n \subset \Phi(G_{n-1})$. Consequently, there exists a finite minimal (in respect to inclusion) subset H_{n-1} of $\Phi^{n-1}(H)$ such that $H_n \subset \Phi(H_{n-1})$. So $H_m, \ldots, H_0$ are defined. Of course, $H_k \subset \Phi(H_{k-1})$ for $k = 1, \ldots, n$ and $H_k \subset \Phi^k(H)$ for $k = 0, \ldots, m$.

Since Φ is isotonic, any Φ^k is also isotonic. Therefore

$$H_k \subset \Phi^k(H) \subset \Phi^k(B) \subset \Phi^\omega(B) = B \quad \text{for } k = 0, \ldots, m.$$

Obviously, $H_m \subset A$. Assume $H_n \subset A$, where $m \geqslant n > 0$. Putting $K = H_{n-1}$ in (2) we obtain $A \cap \Phi(H_{n-1}) \subset \Phi(A \cap H_{n-1})$. Since $H_n \subset A$ and $H_n \subset \Phi(H_{n-1})$, we have $H_n \subset \Phi(A \cap H_{n-1})$. Moreover, H_{n-1} is a minimal subset of $\Phi^{n-1}(H)$ such that $H_n \subset \Phi(H_{n-1})$. Therefore $H_{n-1} \subset A \cap H_{n-1}$ and consequently, $H_{n-1} \subset A$. Thus $H_m, \ldots, H_0$ are subsets of A. Particularly, $H_0 \subset A$.

From $H_k \subset \Phi(H_{k-1})$, $k = 1, \ldots, m$, from $H_0 \subset A$ and $H_0 \subset \Phi^0(H) = H$, thanks the isotonicity of Φ', we get

$$\{x\} = H_m \subset \Phi(H_{m-1}) \subset \Phi^2(H_{m-2}) \subset \ldots \subset \Phi^m(H_0)$$

$$= \Phi^m(H_0 \cap A) \subset \Phi^m(H \cap A) \subset \Phi^\omega(H \cap A).$$

PROPOSITION 3: *Any Φ^ω-extreme subset of B is also Φ-extreme provided: Φ is full, $c(\Phi) < \infty$, and $\Phi(F)$ is Φ-stable for $|F| \leqslant c(\Phi)$.*

PROOF: Let A be a Φ^ω-extreme subset of B. By (1) we obtain that

$$A \cap \Phi^\omega(K) \subset (\Phi^\omega)^\cup(A \cap K) \quad \text{for any } K \in \mathscr{P}(B).$$

If $|K| \leqslant c(\Phi)$, then $\Phi^\omega(K) = \Phi(K)$. Moreover, $\Phi^\omega(M) = \Phi(M)$ for $M \in \mathscr{P}(A \cap K)$. Consequently, $(\Phi^\omega)^\cup(A \cap K) = \Phi^\cup(A \cap K)$. Hence (4) is satisfied which, in virtue of Proposition 1, ends the proof.

4. Φ-extreme elements

If $\phi \in \mathscr{D}_\Phi$, then any element of $\Phi(\phi)$ is a Φ-extreme element of each set to which it belongs (comp. (e) in Theorem 1). Such elements are called *trivially Φ-extreme*. For an arbitrary operation Φ in X, we denote by $E_\Phi(B)$ the set of all *non-trivially Φ-extreme* elements, i.e. of all Φ-extreme elements of B which do not belong to $\Phi(\phi)$.

PROPOSITION 4: *If $a \in E_\Phi(B)$, then*

$$a \in \Phi(K) \Rightarrow a \in K \quad \text{for any } K \in \mathscr{D}_\Phi \cap P(B). \tag{6}$$

For a full enlarging operation Φ, the conditions $a \in E_\Phi(B)$ and (6) are equivalent.

Let us observe (for use in Example 2) that if Φ is cover finite, than (6) is equivalent to the condition

$$a \in \Phi(K) \Rightarrow a \in K \quad \text{for any finite } K \in \mathscr{D}_\Phi \cap \mathscr{P}(B). \tag{7}$$

PROPOSITION 5: *If Φ is a full operation and $a \in E_\Phi(B)$, then*

$$a \in B \setminus \Phi(B \setminus \{a\}). \tag{8}$$

For full enlarging isotonic operations, the conditions $a \in E_\Phi(B)$ and (8) are equivalent.

PROPOSITION 6: *If Φ is full and enlarging, then $a \in E_\Phi(B)$ implies*

$$\Phi(B \setminus \{a\}) \neq \Phi(B). \tag{9}$$

If Φ is a closure operation, $a \in E_\Phi(B)$ is equivalent to (9).

PROPOSITION 7: *Let Φ be a full, enlarging, isotonic operation and let a belong to a Φ-stable set B. Then $a \in E_\Phi(B)$ if and only if*

$$B \setminus \{a\} \text{ is } \Phi\text{-stable}. \tag{10}$$

We omit the proofs of Propositions 4–7 as rather tedious. Simply examples show that all assumptions about Φ are necessary there. In the case of closure operations, an additional characterization of non-trivially Φ-extreme elements is given in the first part of Proposition 9.

With the help of Proposition 4 one can easily obtain the following

PROPOSITION 8: *Let Φ be an enlarging operation and $B \in \mathscr{D}_\Phi$. Then $E_\Phi(B)$ is contained in each Φ-equivalent subset of B. Moreover, if $E_\Phi(B)$*

is a Φ-equivalent subset of B, then it is the smallest Φ-equivalent subset of B.

PROPOSITION 9: *For any closure operation Φ and any $B \in \mathscr{P}(X)$ we have*:

(a) *$E_\Phi(B)$ coincides with the intersection of all Φ-equivalent subsets of B.*

(b) *If there exists the smallest Φ-equivalent subset of B, then it is equal to $E_\Phi(B)$.*

(c) *$\Phi(E_\Phi(B)) = \Phi(B)$ if and only if the family of all Φ-equivalent subsets of B is a closure system over B.*

(d) *$E_\Phi(\Phi(B)) \subset E_\Phi(B)$.*

PROOF: (a) Let $a \notin E_\Phi(B)$. In virtue of Proposition 8 it is sufficient to show that a does not belong to some Φ-equivalent subset of B. The searched subset is $B \setminus \{a\}$ because (9) does not hold.

(b) If there exists the smallest Φ-equivalent subset of B, then it coincides with the intersection of all Φ-equivalent subsets of B. Consequently, it is equal to $E_\Phi(B)$ as we have shown in (a).

(c) This statement easy results from (b) and from Proposition 8.

(d) Let $a \in E_\Phi(\Phi(B))$. By (a) we obtain that a belongs to any Φ-equivalent subset $\Phi(B)$. Particularly, $a \in B$. Since a is a Φ-extreme element of $\Phi(B)$ and $B \subset \Phi(B)$, we infer from part (d) of Theorem 1 that a is also a Φ-extreme element of B. From $a \in E_\Phi(\Phi(B))$ it follows $a \notin \Phi(\phi)$. Consequently, $a \in E_\Phi(B)$.

For B being Φ-stable, the reader can observe a connection of statement (a) of Proposition 9 with the operation j_h considered in [24] and consequently, a connection of Examples 1 and 4 presented at the end of this paper with Corollaries 2.3 and 2.4 of [24].

PROPOSITION 10: *For any closure operation Φ and any set $B \in \mathscr{P}(X)$ the following conditions are equivalent*:

$$E(\Phi(B)) = E_\Phi(B), \tag{11}$$

$$\text{for any } a \in B \setminus \Phi(B \setminus \{a\}) \text{ the set } \Phi(B) \setminus \{a\} \text{ is } \Phi\text{-stable}, \tag{12}$$

$$\text{for any } a \in B \setminus \Phi(B \setminus \{a\}) \text{ there exists a } \Phi\text{-stable set } S$$
$$\text{such that } S \cup \{a\} \text{ is } \Phi\text{-stable}, \ a \notin S \text{ and } \Phi(B \setminus \{a\}) \subset S. \tag{13}$$

PROOF: (11) $\Rightarrow$ (12). Let $a \in B \setminus \Phi(B \setminus \{a\})$ in (12). From Proposition 5 we obtain that $a \in E_\Phi(B) = E_\Phi(\Phi(B))$. Since $\Phi(B)$ is Φ-stable, by Proposition 7 we get that the set $\Phi(B) \setminus \{a\}$ is Φ-stable.

(12) $\Rightarrow$ (11). Let $a \in E_\Phi(B)$. By Proposition 5 and by our assumption (12), the set $\Phi(B) \setminus \{a\}$ is Φ-stable. From Proposition 7 we infer that $a \in E_\Phi(\Phi(B))$. Thus $E_\Phi(B) \subset E_\Phi(\Phi(B))$. The inverse inclusion has been shown in Proposition 9.

(12) $\Rightarrow$ (13). Put $S = \Phi(B)\setminus\{a\}$ in (12). Since Φ is enlarging, $a \in \Phi(B)$. Consequently, $S \cup \{a\} = \Phi(B)$. By the idempotence of Φ, the set $S \cup \{a\}$ is Φ-stable. From $a \notin \Phi(B\setminus\{a\})$ and from the isotonicity of Φ we get that $\Phi(B\setminus\{a\}) \subset \Phi(B)\setminus\{a\} = S$.

(13) $\Rightarrow$ (12). Thanks to the correspondence between closure operations and closure systems, it is sufficient to show that $\Phi(B)\setminus\{a\}$ is an intersection of Φ-stable sets, i.e. that for any $c \notin \Phi(B)\setminus\{a\}$ there exists a Φ-stable set K_c such that $c \notin K_c$ and $\Phi(B)\setminus\{a\} \subset K_c$. Obviously, if our c is different from a, then one can put $K_c = \Phi(B)$. In the case $c = a$ put $K_a = S$. Since $S \cup \{a\}$ is Φ-stable and since Φ is enlarging and isotonic, from $\Phi(B\setminus\{a\}) \subset S$ we infer that

$$\Phi(B) \subset \Phi[\Phi(B\setminus\{a\}) \cup \{a\}] \subset S \cup \{a\}.$$

Consequently, $\Phi(B)\setminus\{a\} \subset S = K_a$. Of course, $a \notin K_a$.

The proof is complete.

Note that any Φ-stable set C such that $a \notin C$ can be presented in the form $\Phi(B\setminus\{a\})$, where $B = C \cup \{a\}$. Consequently, using standard techniques, from Proposition 10 we obtain

THEOREM 3: *If Φ is a closure operation, then the following conditions are equivalent:*

any set and its closure have identical non-trivially Φ-extreme elements, (14)

for any Φ-stable set C and $a \notin C$ the set $\Phi(C \cup \{a\})\setminus\{a\}$ is Φ-stable, (15)

for any Φ-stable set C and $a \notin C$ there exists a Φ-stable set S such that $S \cup \{a\}$ is Φ-stable, $a \notin S$ and $C \subset S$, (16)

Φ-equivalent sets have identical non-trivially Φ-extreme elements. (17)

From Theorem 1.4 of [6], p. 46, and from the correspondence between closure operations and closure systems it results that if Φ is a domain finite closure operation, then any Φ-stable set not containing an element $x \in X$ can be enlarged to a maximal Φ-stable set not containing x. Consequently, the equivalence of (14) and (16) implies

THEOREM 4: *Let Φ be a domain finite closure operation. Any set and its closure have identical non-trivially Φ-extreme elements if and only if for any*

$x \in X$ and any maximal Φ-stable set C not containing x, the set $C \cup \{x\}$ is Φ-stable.

5. Some examples

Usually, Φ-extreme subsets and elements are considered for Φ defined on pairs of elements of X. This way one defines the notions of extreme subset and extreme point of a set of a real linear space, extreme support hyperplanes (comp. [3], p. 15), terminal points ([2], p. 53) and subsets [16] of a set of a metric space, extreme points of a subset of a partially ordered set ([7], comp. also Example 3 below), extreme points and subsets in various axiomatic convexity spaces (see e.g. [5]). Also extreme rays of a cone can be defined on this way. The equivalence of some formulas used in such definitions is presented in Proposition 11, where instead $\Phi(\{x, y\})$ we simply write $\Phi\{x, y\}$.

PROPOSITION 11: *Let Φ and $\overline{\Phi}$ be two operations defined on all pairs of (not necessarily different) elements of X such that $x, y \notin \Phi\{x, y\}$, $\Phi\{x, x\} = \phi$ and $\overline{\Phi}\{x, y\} = \{x, y\} \cup \Phi\{x, y\}$ for any $x, y \in X$. For any subset A of B the following conditions are equivalent:*

(a) *A is a Φ-extreme subset of B,*
(b) *A is a $\overline{\Phi}$-extreme subset of B,*
(c) *for every $x, y \in B$ from $A \cap \Phi\{x, y\} \neq \phi$ it results $x, y \in A$,*
(d) *for every $x, y \in B$ and $a \in A \cap \overline{\Phi}\{x, y\}$ it is $x = a$, or $y = a$, or $x, y \in A$,*
(e) *$A \cap \Phi\{x, y\} = \phi$ for each $x \in B \backslash A$ and each $y \in B$,*
(f) *$A \cap \overline{\Phi}\{x, y\} \subset \{y\}$ for each $x \in B \backslash A$ and each $y \in B$.*

EXAMPLE 1: Let L be a real liner space and let $\text{segm}\{x, y\} = \{(1-\alpha)x + \alpha y;\ 0 < \alpha < 1\}$. Obviously, the classical notion of extreme subset of a set is just the notion of segm-extreme subset. Consider other kinds of extremeness in L. Let conv denote the operation of convex hull. From the equality $(\overline{\text{segm}^{\cup}})^{\omega} = \text{conv}$ (comp. [4]) and by Propositions 2, 3, 5 and Theorems 2, 4 (as it results from Theorem 3.1 of [11], for any maximal convex set C not containing a gien point a, the set $C \cup \{a\}$ is also convex) we conclude that conv-extreme subsets of any convex set coincide with extreme subsets and that conv-extreme points of any set B are identical with extreme points of conv B. A generalization of conv-extremeness is presented in Example 2. A subset A of a convex set B is called a *semi-extreme subset* of B if $B \backslash A$ is convex (comp. [12], p. 32). As in [22], pp. 186–187, this notion can be extended to arbitrary (i.e. not necessary convex) set B: if $A \subset B$ and $A \cap \text{conv}(B \backslash A) = \phi$, then we call A a *semi-extreme subset* of B. Simply examples show that the intersection of two semi-extreme subsets may not be a semi-extreme subset. From (a) in Theorem 1 we infer that the notion of semi-extreme subset is not a case of our scheme of Φ-extremeness. One can extend the definition of semi-extreme subsets for operations as follows. If $A \subset B \subset X$ and if for any $K \in \mathscr{D}_\Phi \cap \mathscr{P}(B \setminus A)$ and any $x \in A \cap \Phi(K)$ there exists $M \in \mathscr{D}_\Phi \cap \mathscr{P}(A \cap K)$ such that $x \in \Phi(M)$, then A is called a *Φ-semi-extreme subset* of B. The reader can easily check up that for Φ-semi-extreme subsets there hold analogical properties as (b), (d) and (e) of Theorem 1 and that conv-semi-extreme subsets are just semi-extreme ones. Also relative extreme points [15] and relative extreme subsets [17] are studied. Remember that a subset A of B is said to be an *extreme subset* of B *relative* to C if for any $x \in B$, $y \in C$, from $A \cap \text{segm}\{x, y\} \neq \phi$ it follows $x \in A$. This is a special case of our notion of Φ-extreme subset, where $\mathscr{D}_\Phi$ consists of all one-point sets and $\Phi(\{x\}) = \bigcup_{y \in C} \text{segm}\{x, y\}$.

EXAMPLE 2: Consider two kinds of extremeness in a convexity structure $(X, \mathscr{C})$. As in many papers concerning convexity structures, the hull operation (i.e. the closure operation) $h_{\mathscr{C}}$ generated by $\mathscr{C}$ will be simply denoted by the same symbol $\mathscr{C}$. In [22], pp. 186–187, there was introduced a notion of extreme subset for convexity structures for which, as for a special case of the definition of Φ-semi-extreme subset presented in Example 1, we use the term $\mathscr{C}$-semi-extreme subset or semi-extreme subset for short. Let $A \subset B \subset X$. We call A a *$\mathscr{C}$-semi-extreme* subset of B if

$$A \cap \mathscr{C}(B \setminus A) = \phi.$$

Simultaneously, it is natural to consider also $\mathscr{C}$-extreme (shortly: extreme) subsets. By Proposition 1 it is clear that (2) may be used as a definition: if

$$A \cap \mathscr{C}(K) \subset \mathscr{C}(A \cap K) \quad \text{for any } K \subset B,$$

then A is called a *$\mathscr{C}$-extreme subset* of B. In the case $A = \{a\}$, a is called a *$\mathscr{C}$-extreme point* of B. In two special cases, the notion of $\mathscr{C}$-extreme point $a \in B$ has been introduced earlier: using (7) for $\mathscr{C}$ being domain finite ([8], p. 151) and with the help of (10) when $B \in \mathscr{C}$ (see [13], p. 127 and [14], p. 119). From Proposition 5 we obtain a simply characterization of $\mathscr{C}$-extremeness of a point a of B, namely

$$a \notin \mathscr{C}(B \setminus \{a\}).$$

It enables us to observe a surprising connection of the notions of $\mathscr{C}$-extreme point and $\mathscr{C}$-independent set ([21], p. 38, [18], p. 174, [9], p. 27, [13], p. 120) which can be simply expressed by defining a $\mathscr{C}$-independent set as one with all points $\mathscr{C}$-extreme. Other properties of $\mathscr{C}$-extreme subsets and points are given in Theorems 1, 3, 4 and Propositions 4, 6, 8, 9, 10. Moreover, from (3), (4) and from the equality $\mathscr{C}^{\cup} = \mathscr{C}$ they result characterizations of $\mathscr{C}$-extreme subsets for $\mathscr{C}$ being domain finite and for $\mathscr{C}$ with finite Caratheodory's number. Obviously, any $\mathscr{C}$-extreme subset of B is a $\mathscr{C}$-semi-extreme one of B. Moreover, $\mathscr{C}$-semi-extreme and $\mathscr{C}$-extreme points of arbitrary set coincide.

EXAMPLE 3: Let P be a set partially ordered by a relation $\leqslant$. For any x, $z \in P$ put $\bar{\Sigma}\{x, z\} = \{y; x \leqslant y \leqslant z$ or $z \leqslant y \leqslant x\}$ and $\Sigma\{x, z\} = \bar{\Sigma}\{x, z\} \setminus \{x, z\}$. A set $C \subset P$ is called *order convex* [7] if $\bar{\Sigma}\{x, z\} \subset C$ for any x, $z \in C$. Since the family of order convex subsets is a closure system over P, there is defined the corresponding closure (hull) operation Ω. By *order extreme subsets* we understand $\bar{\Sigma}$-extreme subsets, i.e., Σ-extreme subsets (see Proposition 11). Order extreme elements have been introduced and discussed in [7]. From the equality $\bar{\Sigma}^{\cup} = \Omega$ (see [7]) and from Proposition 2 and Theorem 4 we obtain that order extreme and Ω-extreme subsets of arbitrary set coincide and that any set and its order convex hull have identical order extreme elements. As an example of order extreme elements, one can take $\leqslant_a$-maximal elements (see [23], pp. 18 and 30) in a semi-regular topological convexity structure and, more general, in any convexity structure such that for any a, b, c from $c \in \mathscr{C}\{a, b\}$ and $b \in \mathscr{C}\{a, c\}$ it results $b = c$. Similarly as in Examples 1 and 2 we can consider *relative order extreme subsets* (i.e. relative Σ-extreme subsets) and *Ω-semi-extreme subsets*. It is easy to test that order extreme subsets relative a set and relative its order convex hull are identical and that A is an Ω-semi-extreme subset of B if and only if A is an order extreme subset of B relative to $B \setminus A$.

EXAMPLE 4: Denote by cl the closure operation in a topological space T. From the last part of Proposition 1 we obtain that a subset A of B is cl-extreme if and only if $A \cap cl(B \setminus A) = \phi$. Particularly, cl-extreme points of B coincide with isolated points of B. Note that cl-extreme subsets of T are just open sets.

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