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JÁNOS KOLLÁR

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TOWARD MODULI OF SINGULAR VARIETIES

János Kollár

I. Introduction

The aim of this article is prove that in some large and natural classes of singular varieties a "good" moduli theory exists. It is well understood that even for smooth surfaces one cannot expect a good moduli theory, unless one endows the varieties with some projective data. It seems that the concept of polarization (i.e. declaring some ample divisor distinguished) is the right concept to remedy the situation. A short discussion is given in [M-F] Ch. 5 §1. Therefore in the sequel I shall consider polarized varieties.

The main results will be that for certain classes of varieties a moduli space exists which is a separated algebraic space of finite type. It is well understood how to construct moduli spaces that are algebraic spaces. The only problem is to guarantee that they are separated and of finite type. These will follow once one can answer the following two geometric questions:

Uniqueness of specializations: Given a family of varieties over the punctured disc, under what restrictions will it have at most one extension to a family over the disc?

Boundedness: Given a class of varieties, when can they be parametrized (not necessarily in a 1-1 way) by a scheme of finite type?

Chapter two is devoted to the question of boundedness. The main result is that polarized surfaces with given Hilbert polynomial form a bounded family (Theorem 2.1.2). For smooth surfaces this was proved by Matsusaka-Mumford [M-M] and for normal ones by Matsusaka [M4]. The general result yields boundedness for normal polarized threefolds (Theorem 2.1.3).

Uniqueness of specializations is considered in Chapter three. After some general remarks three different cases are discussed: irregular varieties (3.2), rational singularities (3.3) and "minimal" singularities (3.4). A singularity is minimal if it is Cohen-Macaulay, its multiplicity is the smallest possible, and the tangent cone is reduced. Their theory is developed in greater detail than is strictly necessary for the applications in Chapter four, but they seem to be of some independent interest.

The results of previous chapters are translated into statements about moduli spaces in Chapter four. The Main Theorem (Theorem 4.2.1) is in fact seven separate theorems put together concerning existence of moduli spaces under various conditions. For instance: normal, polarized, irregular, non-ruled surfaces have a moduli space which is a separated algebraic space of finite type. It is interesting to remark that for regular surfaces separatedness fails (examples 4.3.2-3). Another example shows that in general our methods lead to honest algebraic spaces (i.e. not schemes).

The end (or lack) of a proof will be denoted by $\square$.

The present article is an essentially unchanged version of part one of my doctoral dissertation completed under the supervision of Prof. T. Matsusaka. I am deeply indebted to him for his guidance and support.

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II. Boundedness of polarized surfaces

§2.1 Statement of the Main Theorem

DEFINITION 2.1.1:

(i) By a surface we mean a reduced, purely 2-dimensional, projective scheme over an algebraically closed field.

(ii) A pair (V, X) is called a polarized variety if V is a projective variety and X an ample Cartier divisor on V. Then $\chi(s)=\chi(V, X, s)=\chi(V, sX)$ is called the Hilbert polynomial of (V, X).

(iii) A family of polarized varieties $\{(V_\lambda, X_\lambda): \lambda \in \Lambda\}$ is called bounded, if there exists a map $f: A \to B$ between varieties and an f-ample Cartier divisor Y on A such that every (V_λ, X_λ) is isomorphic to some $(f^{-1}(b), Y \mid f^{-1}(b))$ for some $b \in B$. If the Hilbert polynomials of (V_λ, X_λ) are all the same, this is equivalent to the statement that for some fixed s $\mathcal{O}_{V_\lambda}(sX_\lambda)$ is very ample on V_λ for all $\lambda \in \Lambda$.

The aim of this chapter is to prove the following:

THEOREM 2.1.2: *The family of polarized surfaces with fixed Hilbert polynomial is bounded (arbitrary characteristic).*

In the last section we shall deduce the following two theorems as corollaries:

THEOREM 2.1.3: *The family of polarized normal 3-folds with fixed Hilbert polynomial is bounded (char. 0 only). In fact it is sufficient to know the two highest coefficients of the Hilbert polynomial.*

To formulate our result in characteristic p we need a definition:

DEFINITION 2.1.4: Let $\{(V_\lambda, X_\lambda): \lambda \in \Lambda\}$ be a family of polarized varieties. A subset $\Sigma \subset \Lambda$ is called a connected component of Λ, if it is closed under generalization and specialization over local rings and minimal among subsets satisfying this property (we exclude $\Sigma = \varnothing$).

THEOREM 2.1.5: *In the family of non-singular polarized 3-folds every connected component is bounded.*

Our starting point is the following result of Matsusaka:

THEOREM 2.1.6: [M4] *Let (V, X) be a normal polarized variety. Assume that $h^0(sX) \geqslant (X^n/n!)\, s^n - C\, s^{n-1}$. Then there is an s_0 depending only on X^n and C that for $s \geqslant s_0$ $|sX|$ contains a reduced, irreducible divisor W.*

This allows one to reduce the problem to lower dimensions, but W need not be normal. Still this allowed Matsusaka to conclude:

THEOREM 2.1.7: [M4] *The family of normal polarized surfaces with fixed Hilbert polynomial is bounded (arbitrary characteristic).* □

Our method of proving Theorem 2.1.2 will be to normalize the surface and analyze the conductor sufficiently to conclude boundedness. This will be carried out in section 5. In the preceeding sections auxiliary results will be discussed, some of which are probably well known, but we don't know of any convenient reference. Finally in the last section we derive the corollaries.

REMARK: 2.1.8: There is one unpleasant feature of polarization for singular varieties. Namely, if (V, X) is a polarized variety and $V \to V_0$ a specialization of V, then X might not specialize to a Cartier divisor X_0. But it can happen that mX specializes to an ample Cartier divisor mX_0 on V_0. It would be natural to include these limits in the moduli space. As a first step one would need boundedness. In general we might run into trouble: Let (V, X) be a normal polarized surface, $q(V) = 0$. Let ϕ_m: $V \to \mathbb{P}^N$ be the embedding given by $|mX|$ $(m \gg 0)$. We can deform V to a cone over a hyperplane section, to get a ruled surface V_m, with a singular vertex. mX will specialize to a Cartier divisor on V_m (the hyperplane section), but the V_m clearly form a non-bounded family.

There are some indications that such bad behaviour does not occur in general. For instance it cannot happen for normal subvarieties of Abelian varieties.

Note 2.1.9: Since this article has been written, Matsusaka succeeded in generalizing his results considerably. The full scope of this is not yet clear, but our Theorem 2.1.3 appears to be a special case of his results. So the original simple proof of Theorem 2.1.7 will probably never appear. Using the (rather easy) fact that irreducible curves with fixed p_a from a bounded family one can get it along the lines of section 2.6.

§2.2 Conductors of S_2 rings

Our standard reference to commulative algebra is the book of Matsumura [M.H], which can be consulted for all definitions.

CONVENTION 2.2.1: In this section all rings are supposed to have the following properties:

(i) reduced, noetherian;

(ii) the normalization is a finitely generated module.

Actually we could get along with non-reduced rings in most cases, but in the applications these conditions will be satisfied.

DEFINITION 2.2.2: (i) A ring R is called seminormal [Tr] if whenever $R \subset S$ is an overring such that

(a) the induced map $\operatorname{Spec} S \to \operatorname{Spec} R$ is a homeomorphism and

(b) $R/p \cap R \subset S/p$ is an equality for all $p \in \operatorname{Spec} S$

then in fact $R = S$.

(ii) Let R be a ring, $R \subset \overline{R}$ its normalization. The S_2-ification of R, denoted by $\tilde{R}$ is the smallest ring between R and $\overline{R}$ which is S_2. The seminormalization denoted by ${}^{+}R$ is the smallest ring between R and $\overline{R}$ which is seminormal. It always exists by [Tr].

(iii) Let $R \subset S$ be a ring extension. The conductor of S over R, $\operatorname{Cond}(S/R)$ is the annihilator of the R-module S/R. One can see that it is an ideal in R and S as well, and it is the largest such ideal.

The following lemma is very simple, but will be used repeatedly.

LEMMA 2.2.3: *Let $0 \to N \to M \to T \to 0$ be an exact sequence of modules. Assume that N is S_2 and* $\operatorname{codim}(\operatorname{supp} T) \geqslant 2$. *Then the sequence splits.*

PROOF: $\operatorname{Ext}^1(T, N) = 0$ by [M.H] Theorem 28. □

LEMMA 2.2.4: *The S_2-ification of a ring exists and is just $\tilde{R} = \bigcap_{htp=1} R_p$.*

PROOF: It is just [EGA] IV.5.10.16 and 17 put together. □

LEMMA 2.2.5: *Let R be an S_2 ring, T an S_1 overring of R. Then all associated primes of T/R and of* $\operatorname{Cond}(T/R)$ *in R have height $\leqslant 1$.*

PROOF: Let $ht\, p \geqslant 2$. Then

$$0 \to \operatorname{Hom}(R/p, R) \to \operatorname{Hom}(R/p, T) \to \operatorname{Hom}(R/p, T/R)$$
$$\to \operatorname{Ext}^1(R/p, R)$$

is exact. The second term is zero by assumption, the last one by [M.H]

Thm 28.So $\operatorname{Hom}(R/p,\ T/R) = 0$, which proves the first statement.
Let $p \in \operatorname{Ass} \operatorname{Cond}(T/R) = \operatorname{Ass} \operatorname{Ann}(T/R)$. Then

$$\operatorname{Hom}(R/p,\ R/\operatorname{Cond}(T/R)) \neq 0, \text{ so } \operatorname{Hom}(R/p,\ T/R) \neq 0.$$

Hence p is contained in an associated prime of T/R, and $ht\, p \leqslant 1$. □

COROLLARY 2.2.6: *With the above assumptions if $R \subset T \subset \overline{R}$ then T/R is unmixed of height* 1, $\operatorname{Cond}(T/R)$ *is unmixed of height* 1 *in R and T as well.*

PROOF: Clearly neither T/R nor $\operatorname{Cond}(T/R)$ can have height zero associated primes.
Let $q \in \operatorname{Ass}_T \operatorname{Cond}(T/R)$. Let $p = R \cap q$. So we have a monomorphism $R/p \to T/q \to T/\operatorname{Cond}(T/R)$. Hence $p \in \operatorname{Ass}_R$s2.2.8$\operatorname{Cond}(T/R)$. So $ht\, p \leqslant 1$, and therefore $ht\, q \leqslant 1$. □

PROPOSITION 2.2.7: *Let R and T as in Corollary* 2.2.6 *and assume that R is seminormal. Then* $\operatorname{Cond}(T/R)$ *is reduced of pure height* 1 *in R and in T as well.*

PROOF: It is of pure height 1 by Corollary 2.2.6 and reduced by [Tr] Lemma 1.3. □

PROPOSITION 2.2.8: *Let R be S_2, $I = \operatorname{Cond}(\overline{R}/R)$, $p \in \operatorname{Ass}_R(I)$. Then* $2\ \operatorname{length}_{R_p}(R/I) \leqslant \operatorname{length}_{R_p}(\overline{R}/I)$.

PROOF: We can localize everything at p and then this is just the classical $2(n_Q - \delta_Q) \leqslant n_Q$ inequality of Dedekind. (see e.g. [S] IV §11). □

COROLLARY 2.2.9: *Let R be S_2, $J = \operatorname{Cond}({}^+R/R)$, $p \in \operatorname{Ass}_R(J)$. Then*

$$2\ \operatorname{length}_{R_p}(R/J) \leqslant \operatorname{length}_{R_p}({}^+R/J) + k - 1,$$

where k is the number of branches of R_p. □

PROPOSITION 2.2.10: *Let R be a local ring, $R \subset S$ an overring such that the inclusion is an isomorphism outside the maximal ideal. Let $I =$* $\operatorname{Cond}(S/R)$. *Then* $\operatorname{length}_R(R/I) \leqslant \operatorname{length}_R(S/R)^2$.

PROOF: Let $t_1, \ldots, t_g$ be a minimal generating set for S/R. Then

$$I = \cap \operatorname{Ann}(\langle t_i \rangle),$$

and

$$\operatorname{length}_R(R/\operatorname{Ann}(\langle t_i \rangle)) = \operatorname{length}_R(\langle t_i \rangle) \leqslant \operatorname{length}_R(S/R).$$

Since $g \leqslant \operatorname{length}_R(S/R)$ we get

$$\operatorname{length}_R(R/I) \leqslant \operatorname{length}_R(S/R)^2. \qquad \square$$

PROPOSITION 2.2.11: *Let R be a finitely generated, 2-dimensional reduced S_2 algebra over an algebraically closed field. Then its seminormalization ${}^+R$ is S_2 again.*

PROOF: Let S be the S_2-ification of ${}^+R$. Then ${}^+R \to S$ is an isomorphism in codim 1, and since at closed points no residue field extension can occur, either ${}^+R = S$, or Spec $S \to$ Spec ${}^+R$ is not a homeomorphism; i.e. two closed points are "pinched together". But Spec R and Spec ${}^+R$ are homeomorphic, and an S_2 surface can not have closed points "pinched together" by a result of Hartshorne [H]. $\square$

REMARK 2.2.12: (i) In fact for seminormal surfaces S_2 is equivalent to not having points "pinched together".

(ii) It is reasonable to ask if the seminormalization of an S_2 ring is S_2 or not. The problem is that in char p seminormality is not a topological notion. Therefore an argument as above will not work.

Acknowledgement 2.2.13: My original version of this section was more complicated and less general. The present form was worked out following suggestions of D. Eisenbud. The proofs of 2.2.5 and 2.2.6 are due to him.

§2.3 Sheaves with many sections

DEFINITION 2.3.1: Let X be a quasiprojective scheme, $U \subset X$ an open set, $\mathscr{F}$ a sheaf on X. We say that $\mathscr{F}$ has many sections over U iff the following two conditions hold:

(i) For any two distinct closed points $p,\ q \in U$

$$H^0(X, \mathscr{F}) \to H^0(X, \mathscr{F}/m_p\mathscr{F}) + H^0(X, \mathscr{F}/m_q\mathscr{F}) \text{ is onto;}$$

(ii) For any closed $p \in U$

$$H^0(X, \mathscr{F}) \to H^0\left(X, \mathscr{F}/m_p^2\mathscr{F}\right) \text{ is onto.}$$

We shall say that $\mathscr{F}$ has many sections if it has many sections over $U = X$.

The following lemma lists basic properties of the notion.

LEMMA 2.3.2: (*i*) *Let $Z \subset X$ a closed subscheme, $U \subset X$ open, $\mathscr{F}$ a sheaf on Z. Then $\mathscr{F}$ has many sections over U (as a sheaf over X) iff $\mathscr{F}$ has* *many sections over* $U \cap Z$. *So the statement "has many sections" makes sense without specifying which scheme we have in mind.*

(ii) A linebundle on a reduced scheme has many sections iff it is very ample.

(iii) If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ *is exact,* $\mathscr{F}$ *and* $\mathscr{H}$ *have many sections and* $H^1(\mathscr{F}) = 0$ *then* $\mathscr{G}$ *has many sections as well.*

(iv) Let X *be a scheme,* I *an ideal sheaf. Let* J *be another ideal sheaf such that* $J \subset I^2$ *and* $\operatorname{supp}\mathcal{O}_X/I = \operatorname{supp}\mathcal{O}_X/J = Z$. *Finally, let* $\mathscr{L}$ *be a locally free sheaf on* X. *If* $J \otimes \mathscr{L}$ *has many sections over* $U = X - Z$, $(0_X/J) \otimes \mathscr{L}$ *has many sections and* $H^1(X, J \otimes \mathscr{L}) = 0$, *then* $\mathscr{L}$ *has many sections.*

PROOF: Straightforward and easy. We remark that in (iv) $J \subset I^2$ is necessary to assure that the second order behavior of $\mathscr{L}$ at Z is controlled by $(\mathcal{O}_X/J) \otimes \mathscr{L}$ alone, since we don't know much about sections of $J \otimes \mathscr{L}$ at Z. □

LEMMA 2.3.3: *Let* $\{(V_\lambda, X_\lambda, \mathscr{F}_\lambda)\colon \lambda \in \Lambda\}$ *be a bounded family of polarized varieties and a sheaf on them. There exists an* s_0, *such that for* $s \geqslant s_0$, $\mathscr{F}_\lambda \otimes \mathcal{O}(sX_\lambda)$ *has many sections and* $H^1(V_\lambda, \mathscr{F}_\lambda \otimes \mathcal{O}(sX_\lambda)) = 0$ *for all* $\lambda \in \Lambda$. □

The following lemma will allow us to get down from the normalization to the variety in some cases.

LEMMA 2.3.4: *Let* $\pi\colon V \to U$ *a finite, birational map,* $I \subset \mathcal{O}_V$ *an ideal sheaf such that* $I \subset Cond(V/U)$ *(so* I *is an ideal sheaf on* U *as well). Let* $\mathscr{F}_u$ *be a coherent sheaf on* U, $\mathscr{F}_v = \pi^*\mathscr{F}_u$. *Then*

$$H^i(U, I \otimes_U \mathscr{F}_u) \cong H^i(V, I \otimes_V \mathscr{F}_v).$$

PROOF: Let $\{U_i\}$ be an affine cover of U, $\{V_i = \pi^{-1}U_i\}$ be that of V. We compute the Čech complex of the sheaves. For V_i we get

$$I_{v_i} \otimes_{V_i} \mathscr{F}_{v_i} = I_{v_i} \otimes_{V_i} \mathcal{O}_{V_i} \otimes_{U_i} \mathscr{F}_{u_i} = I_{v_i} \otimes_{U_i} \mathscr{F}_{u_i} = I_{u_i} \otimes_{U_i} \mathscr{F}_{u_i}$$

(since $I_{v_i} = I_{u_i}$) and this is just the corresponding group over U_i. So the Čech complexes are the same, hence the cohomologies agree. □

The following theorem is the cornerstone of the proof in this chapter. It will be referred to as the "Conductor Principle".

THEOREM 2.3.5: *Let* $\{(U_\lambda, X_\lambda)\colon \lambda \in \Lambda\}$ *be a family of polarized varieties. Assume that for each* $\lambda \in \Lambda$ *we have a variety* V_λ, *a finite birational map* $\pi_\lambda\colon V_\lambda \to U_\lambda$, *an ideal sheaf* $J_\lambda \subset \mathcal{O}_{V_\lambda}$ *such that* $J_\lambda \subset \operatorname{Cond}(V_\lambda/U_\lambda)$. *Let* $C_\lambda = \operatorname{Spec}(\mathcal{O}_{U_\lambda}/J_\lambda)$, $D_\lambda = \operatorname{Spec}(\mathcal{O}_{V_\lambda}/J_\lambda)$. *Assume furthermore that* $\{(V_\lambda, \pi_\lambda^* X_\lambda, J_\lambda)\colon \lambda \in \Lambda\}$ *is a bounded family and either* *(i)* $\{(C_\lambda;\ X_\lambda)\colon \lambda \in \Lambda\}$ *is a bounded family; or*

(ii) $J_\lambda \subset \operatorname{Cond}(V_\lambda/U_\lambda)^2$ *and* $\mathcal{O}_{C_\lambda} \otimes \mathcal{O}_{U_\lambda}(sX_\lambda)$ *has many sections for* $s \geqslant s_0$, s_0 *independent of* λ.

Then $\{(U_\lambda,\ X_\lambda)\colon \lambda \in \Lambda\}$ *is a bounded family.*

PROOF: To have simpler notations we omit the index λ, and denote all pull-backs of X by X again.

First we prove that condition (i) implies (ii) if we replace the ideals J by their squares.

Clearly $(V,\ X,\ J_v^2)$ moves in a bounded family as well. Let $\mathcal{O}_E = \mathcal{O}_U/J_u^2$. We have a sequence $0 \to J_u/J_u^2 \to \mathcal{O}_E \to \mathcal{O}_C \to 0$. Now $J_u^2 = J_v^2$, so $J_u/J_u^2 = \pi_*(J_v/J_v^2)$. Since J_v moves in a bounded family and C moves in a bounded family, the $\mathcal{O}_C$ modules J_u/J_u^2 move in a bounded family. So by Lemma 2.3.3 for some s_0, if $s \geqslant s_0$ then $\mathcal{O}_C(sX)$ and $(J_u/J_u^2) \otimes \mathcal{O}_C(sX)$ have many sections. Thus by (iii) of Lemma 2.3.2 $\mathcal{O}_E(sX)$ has many sections, and this is just condition (ii).

Now we prove that (ii) implies the required statement. We have a sequence

$$O \to J_u \otimes \mathcal{O}_U(sX) \to \mathcal{O}_U(sX) \to \mathcal{O}_C \otimes \mathcal{O}_U(sX) \to 0.$$

We would like to apply (iv) of Lemma 2.3.2. By Lemma 2.3.4 $H^1(U,\ J_u \otimes \mathcal{O}_U(sX)) = H^1(V,\ J_v \otimes \mathcal{O}_v(sX))$ and since $(V,\ X,\ J_v)$ moves in a bounded family this group is zero for $s \geqslant s_0$.

By the same reasoning

$$H^0(U,\ J_u \otimes \mathcal{O}_U(sX)) = H^0(V,\ J_v \otimes \mathcal{O}_U(sX)),$$

and if $p \in U - C$ then π is a local isomorphism around $\pi^{-1}(p)$. So the map

$$H^0(U,\ J_u \otimes \mathcal{O}_U(sX)) \to H^0\big(U,\ J_u \otimes \mathcal{O}_U(sX)/m_p^2 J_u \otimes \mathcal{O}_U(sX)\big)$$

is the same as

$$\begin{aligned} &H^0(V,\ J_v \otimes \mathcal{O}_V(sX)) \\ &\quad \to H^0\big(V,\ J_v \otimes \mathcal{O}_V(sX)/m_{\pi^{-1}(p)}^2 J_v \otimes \mathcal{O}_V(sX)\big). \end{aligned}$$

Hence if $J_v \otimes \mathcal{O}_V(sX)$ has many sections over $V - D$, then $J_u \otimes \mathcal{O}_U(sX)$ has many sections over $U - C$. But the former moves in a bounded family; therefore $J_u \otimes \mathcal{O}_U(sX)$ has many sections over $U - C$ for $s \geqslant$ some s_0.

$\mathcal{O}_C \otimes \mathcal{O}_U(sX)$ has many sections for $s \geqslant s_0$ by assumption, so (iv) of Lemma 2.3.2 applies and we get that $\mathcal{O}_U(sX)$ has many sections for some $s \geqslant s_0$. Thus by (ii) of Lemma 2.3.2 it is very ample.

Since $h^0(U, \mathcal{O}_U(sX)) \leqslant h^0(V, \mathcal{O}_V(sX))$ the dimensions are uniformly bounded, so $|sX|$ embedds U into a fixed projective space $\mathbb{P}^N$. If $h = \dim U$ then the degree of the image can be bounded by $s^h X_U^h + h + 1$ (see e.g. [L-M]) which is equal to $s^h X_V^h + h + 1$, so is bounded. Hence the U_λ's are parametrized by a bounded part of the Hilbert scheme of $\mathbb{P}^N$, and this was to be proved. □

§2.4 Some remarks on quot schemes

DEFINITION 2.4.2: (i) In this section all sheaves will be coherent over a fixed projective space $\mathbb{P}$.

(ii) If $H = \{\mathscr{H}_\lambda\colon \lambda \in \Lambda\}$, $F = \{\mathscr{F}_\mu\colon \mu \in \mathscr{M}\}$ are two families of sheaves we say that F is a family of quotients of H if every $\mathscr{F}_\mu$ is the quotient of some $\mathscr{H}_\lambda$.

(iii) For a sheaf $\mathscr{K}$, $\chi(\mathscr{K}) = \chi(\mathscr{K}, s) = \chi(\mathscr{K} \otimes \mathcal{O}(s))$ will be called the Hilbert polynomial of $\mathscr{K}$. The coefficient of s^j will be denoted by $a_j(\mathscr{K})$ or simply a_j.

(iv) A family of sheaves $H = \{\mathscr{H}_\lambda\colon \lambda \in \Lambda\}$ will be called bounded, if there is a quasi-projective scheme X and a sheaf $\mathscr{H}$ on $X \times \mathbb{P}$ such that each $\mathscr{H}_\lambda$ is isomorphic to some $\mathscr{H} \otimes \mathcal{O}_{\mathbb{P}_x}$, where $\mathbb{P}_x$ denotes the fibre of pr_2 over $x \in X$.

(v) a family of sheaves $F = \{\mathscr{F}_\mu\colon \mu \in \mathscr{M}\}$ is called χ-bounded if $\{\chi(\mathscr{F}_\mu)\colon \mu \in \mathscr{M}\}$ is a finite set of polynomials.

The following is just a re-formulation of a theorem of Grothendieck:

THEOREM 2.4.2: [G1] *Let H be a bounded family of sheaves, F be a χ-bounded family of quotients. Then F is bounded.* □

Now we shall prove two statements that follow easily from the results and methods of [G1] but are not mentioned there.

CONVENTION 2.4.3: $\phi(a, b, \ldots)$ will stand for some function which depends only on the variables explicity listed. Whenever we write ϕ in a statement it means that there is a function for which the statement is true.

LEMMA 2.4.4: *Let H be a bounded family, F the family of quotients, $\mathscr{F} \in F$, $\chi(\mathscr{F}) = \Sigma a_i s^i$. Then $a_j \geqslant \phi(a_N, \ldots, a_{j+1}, H)$.*

PROOF: By taking generic hyperplane sections we can reduce the problem to proving $a_0 \geqslant \phi(a_N, \ldots, a_1, H)$. We prove this by induction on dim(supp $\mathscr{F}$). If it is zero, then $a_0 \geqslant 0$.

Let $0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{G} \to 0$. Then $a_i(\mathscr{G}) = \phi(a_N, \ldots, a_1)$, so if we fix $a_N, \ldots, a_1$, $\mathscr{G}$ moves in a bounded family. Hence for $s \geqslant \phi(a_N, \ldots, a_1, H)$ we have $H^i(\mathscr{G}(s)) = 0$, $i > 0$ and so $H^i(\mathscr{F}(s)) = 0$ for $i > 1$.

On the other hand, let I be the family of kernels $0 \to \mathcal{J}_{\lambda\mu} \to \mathcal{H}_\lambda \to \mathcal{F}_\mu \to 0$. By doing the same induction on I we get $H^i(\mathcal{J}(s)) = 0$ for $s > \phi(a_N, \ldots, a_1, H)$ and $i > 1$. from $H^i(\mathcal{H}_\lambda(s)) \to H^1(\mathcal{F}_\mu(s)) \to H^2(\mathcal{J}_{\lambda\mu}(s))$ we get that $H^1(\mathcal{F}(s)) = 0$ for $s > \phi(a_N, \ldots, a_1, H)$. So $\chi(\mathcal{F}, s) = \mathrm{H}^0(\mathcal{F}(s)) \geqslant 0$; hence $a_0 \geqslant -\Sigma_1^N a_i s^i$, which is the desired bound. □

THEOREM 2.4.5: *Let H be a bounded family, F be a family of quotients. Assume that every $\mathcal{F} \in F$ is unmixed of pure dimension n, and $a_n(\mathcal{F}) \leqslant c$ for some constant c. Then there exists a bounded family G of quotients of H, such that each $\mathcal{G} \in G$ is unmixed of pure dimension n and each $\mathcal{F}$ is the quotient of some $\mathcal{G}$ for which* supp $\mathcal{G} =$ supp $\mathcal{F}$.

REMARK 2.4.6: It is of course *not* true that $\mathcal{F}$ is a bounded family. The example one should keep in mind: the family of double lines in $\mathbb{P}^3$ is not bounded, but they are all contained in one of the simplest triple lines, given locally by (x^2, y^2).

PROOF OF THE THEOREM: Let $\mathcal{J}(\mathcal{F})$ be the ideal sheaf of supp $\mathcal{F}$. Then $\mathcal{O}_{\mathbb{P}^N}/\mathcal{J}(\mathcal{F})$ is reduced of pure dimension n, and $\deg(\mathcal{O}_{\mathbb{P}^N}/\mathcal{J}(\mathcal{F})) \leqslant n! \cdot c$, so by the theory of Chow forms ([G1]L.25), $\{\mathcal{J}(\mathcal{F}): \mathcal{F} \in F\}$ is a bounded family. If $q: \mathcal{H} \to \mathcal{F}$ is a quotient map then it factors through $\mathcal{H} \to \mathcal{H}/\mathcal{J}(\mathcal{F})^c \cdot \mathcal{H}$. The family $\{\mathcal{H}/\mathcal{J}(\mathcal{F})^c\ \mathcal{H}: \mathcal{F} \in F, \mathcal{H} \in H\}$ is a bounded family, it satisfies all requirements except that these sheaves might not be unmixed. But from [G1] Theorem 2.2 it follows that if we take the quotient by the subsheaf generated by local sections whose support has dimension $< n$, then we get a bounded family again. This is our family G. □

§2.5 *Proof of the Main Theorem*

2.5.1 *Step 1*: General set-up

The local notions introduced in §2.2 (seminormalization, conductor, etc.) glue together to global ones. So for a surface V let $\tilde{V}$, $({}^+V, \bar{V})$ be the S_2-ification (seminormalization, normalization). For simplicity all pull-backs of the ample divisor X will be denoted by X again. For geometric reasons the coefficients of $\chi(V, X)$ will be denoted by $\mathrm{d}/2s^2 - \xi/2s + \chi$, that of $\chi(\tilde{V}, \tilde{X})$ by $\tilde{d}/2s^2 - \tilde{\xi}/2s + \tilde{\chi}$ etc.

LEMMA 2.5.2: (*i*) $d = \tilde{d} = {}^+d = \bar{d}$.

(*ii*) $\xi = \tilde{\xi} \geqslant {}^+\xi \geqslant \bar{\xi} \geqslant -3\bar{d} = -3d$.

(*iii*) $\chi \leqslant \tilde{\chi}$; $\tilde{\chi}$, ${}^+\chi$, $\bar{\chi} \leqslant 3 + \xi^2$; $\bar{\chi} \geqslant \phi(d, \xi)$ *for some* ϕ.

PROOF: (i) is clear, and so is (ii) except the last inequality, which follows from [K-M] Lemma 2.1.

As for (iii) $\chi(\mathcal{O}_{\tilde{V}}) = \chi(\mathcal{O}_V) + \chi(\mathcal{O}_{\tilde{V}}/\mathcal{O}_V) = \chi(\mathcal{O}_V) + \text{length } \mathcal{O}_{\tilde{V}}/\mathcal{O}_V$. If W is an S_2-surface then $\chi(\mathcal{O}_W) \leqslant h^0(\omega_W) + h^0(\mathcal{O}_W) \leqslant 2 + (X \cdot K)^2 + 1$ by [L-M] Lemma 2.1.

If $V' \to \bar{V}$ is a desingularization then $\chi(\mathcal{O}_{V'}) \leqslant \chi(\mathcal{O}_{\bar{V}})$ from the Leray spectral sequence, and on V' we can use [K-M] Lemma 5.2 to estimate $\chi(\mathcal{O}_{V'})$ from below. □

2.5.3 *Step 2*: $(\bar{V}, \bar{X}, \bar{C})$ is bounded.
$(\bar{V}, \bar{X})$ is a normal surface with Hilbert polynomial $\bar{d}/2s^2 - \bar{\xi}/2s + \bar{\chi}$. From Lemma 2.5.2 we see that we can bound all coefficients in terms of (d, ξ, χ), so we have only finitely many possibilities for $\chi(\bar{V}, \bar{X})$. We could apply Theorem 2.1.7 of Matsusaka to conclude that $(\bar{V}, \bar{X})$ moves in a bounded family, but $\bar{V}$ is not necessarily irreducible. But if $V = \cup V_i$, d_i, ξ_i, χ_i the corresponding quantities then $\Sigma d_i = d$ so we have only finitely many possibilities. $\Sigma \xi_i = \xi$ and $\xi_i \geqslant -3d_i \geqslant -3d$ by Lemma 2.5.2 so this is again finite in number. Finally $3 + \xi_i^2 \geqslant \chi_i \geqslant \phi(d_i, \xi_i)$ is bounded, so the irreducible components move in a bounded family and so does $(\bar{V}, \bar{X})$.

Now let $\mathcal{O}_{\bar{C}} = \mathcal{O}_{\bar{V}}/\text{Cond}(\bar{V}/{}^+V)$, $\mathcal{O}_{+C} = \mathcal{O}_{+V}/\text{Cond}(\bar{V}/{}^+V)$. Using the sequences

$$0 \to \mathcal{O}_{+V} \to \mathcal{O}_{\bar{V}} \to \mathcal{H} \to 0$$

$$0 \to \mathcal{O}_{+C} \to \mathcal{O}_{\bar{C}} \to \mathcal{H} \to 0$$

we get $I({}^+C, X) - I(\bar{C}, X) = (\bar{\xi} - {}^+\xi)/2$, where $I(\ ,\ )$ denotes the intersection number. Furhermore we have $I({}^+C, X) \leqslant \frac{1}{2} I(\bar{C}, X)$ (this is just a global version of Proposition 2.2.8), so $I(\bar{C}, X) \leqslant {}^+\xi - \bar{\xi} \leqslant \xi + 3d$.

By Proposition 2.2.7 $\bar{C}$ is reduced, hence by the theory of Chow forms (see e.g. [G1] Lemma 2.4) the triplets $(\bar{V}, \bar{X}, \bar{C})$ move in a bounded family. □

2.5.4 *Step 3*: $({}^+V, X)$ is bounded.
Using our earlier notation we again look at the sequence $0 \to \mathcal{O}_{+C} \to \mathcal{O}_{\bar{C}} \to \mathcal{H} \to 0$. Here ${}^+C$ and $\bar{C}$ are reduced curves by Proposition 2.2.7 and $\mathcal{H}$ is unmixed of pure dimension 1 by Corollary 2.2.6. So if $\bar{\mathcal{O}}_{+C}$ (resp. $\bar{\mathcal{O}}_{\bar{C}}$) denotes the coordinate ring of the normalization of ${}^+C$ (resp. $\bar{C}$), then $\bar{\mathcal{O}}_{+C}/\mathcal{O}_{+C} \to \bar{\mathcal{O}}_{\bar{C}}/\mathcal{O}_{\bar{C}}$ is an injection. So the singularities of ${}^+C$ are "not worse" than the "sum" of the singularities of $\bar{C}$ lying above the given point. Since $\bar{C}$ moves in a bounded family, we can estimate $p_a({}^+C)$ and the number of components; hence ${}^+C$ moves in a bounded family. (This is an easy and well-known fact, but I don't know of any references. It is of course an easy special case of the Conductor Principle: Theorem 2.3.5.)

Now we can use (i) of the Conductor Principle to conclude the boundedness of $({}^+V, X)$. □

As a preparation for the next step, let $E = \Sigma\, a_i A_i$ be a cycle on ${}^+V$ where A_i are the multiple curves of ${}^+V$ and $a_i = \mathrm{mult}(A_i) - 1$. It is easy to see that the triplets $({}^+V,\ X,\ E)$ move in a bounded family as well.

2.5.5 *Step 4*: $(\tilde{V},\ X)$ is bounded.
This is the most complicated step and the higher dimensional generalizations broke down here.

Let $\mathcal{O}_{+D} = \mathcal{O}_{+V}/\mathrm{Cond}({}^+V/\tilde{V})$, $\mathcal{O}_{\tilde{D}} = \mathcal{O}_{\tilde{V}}/\mathrm{Cond}({}^+V/\tilde{V})$. By Corollary 2.2.6 these are unmixed of pure dimension one. As in Step 2 we get $I(\tilde{D},\ X) - I({}^+D,\ X) = ({}^+\xi - \tilde{\xi})/2$ and Corollary 2.2.9 gives $2I(\tilde{D},\ X) \leqslant I({}^+D,\ X) + I(E,\ X)$. Therefore $I({}^+D,\ X) \leqslant \xi + 3d + 2I(E,\ X)$ and so it is bounded. Now in general ${}^+D$ is not reduced, so we cannot conclude that ${}^+D$ moves in a bounded family, but by Theorem 2.4.5 there is a family of curves D', such that the triplets $({}^+V,\ X,\ D')$ move in a bounded family and ${}^+D$ is a closed subscheme of D', satisfying $\mathrm{supp}\ {}^+D = \mathrm{supp}\ D'$. Let D be the image of D' in $\tilde{V}$. Here we get $0 \to \mathcal{O}_D \to \mathcal{O}_{D'} \to \mathcal{O}_{+V}/\mathcal{O}_{\tilde{V}} \to 0$. In this sequence $\chi(\mathcal{O}_{D'},\ X)$ and $\chi(\mathcal{O}_{+V}/\mathcal{O}_{\tilde{V}},\ X)$ are known up to finite ambiguity, so $\chi(\mathcal{O}_D,\ X)$ is bounded.

$\mathcal{O}_{D'}$ has a natural filtration by successive socles so let $\mathrm{gr}\mathcal{O}_{D'}$ be the corresponding $\mathcal{O}_{\mathrm{red}\ D'}$ module. If we look at $\mathcal{O}_{D'}$ as an $\mathcal{O}_D$ module, then this is a filtering of $\mathcal{O}_D$ modules so we get a sequence of $\mathcal{O}_{\mathrm{red}\ D}$ modules $0 \to \mathrm{gr}\ \mathcal{O}_D \to \mathrm{gr}\ \mathcal{O}_{D'} \to \mathrm{gr}\ \mathcal{Q} \to 0$. ($\mathrm{gr}\ \mathcal{Q}$ is just the quotient filtering on $\mathcal{O}_{+V}/\mathcal{O}_{\tilde{V}}$.)

Since ${}^+V \to \tilde{V}$ is a homeomorphism, $\mathrm{red}\ D' \to \mathrm{red}\ D$ is an isomorphism at the generic points (but not necessarily an isomorphism, see Example 2.5.7).

Let $\mathcal{F}$ be the $\mathcal{O}_{\mathrm{red}\ D'}$ submodule of $\mathrm{gr}\ \mathcal{O}_{D'}$ generated by $\mathrm{gr}\ \mathcal{O}_D$. Since $\mathcal{O}_{\mathrm{red}\ D}$ and $\mathcal{O}_{\mathrm{red}\ D'}$ are generically equal, $\mathcal{F}/\mathrm{gr}\ \mathcal{O}_D$ has finite length; let this be $l \geqslant 0$. So $\chi(\mathrm{gr}\ \mathcal{O}_{D'}/\mathcal{F}) = \chi(\mathrm{gr}\ \mathcal{O}) - l = ax + b - l$. But $\mathrm{gr}\ \mathcal{O}_{D'}$ moves in a bounded family of $\mathcal{O}_{\mathrm{red}\ D'}$ modules and $\mathrm{gr}\ \mathcal{O}_{D'}/\mathcal{F}$ is a family of quotients, so by Lemma 2.4.4 $b - l$ is bounded from below. Thus l is bounded from above. Hence $\mathcal{F}$ moves in a bounded family of $\mathcal{O}_{\mathrm{red}\ D'}$ modules.

Now $\mathrm{length}(\mathcal{O}_{\mathrm{red}\ D'}/\mathcal{O}_{\mathrm{red}\ D}) \leqslant l$, so the Conductor Principle applied to red D shows that red D moves in a bounded family. Since $\mathrm{length}\ (\mathcal{F}/\mathrm{gr}\ \mathcal{O}_D) = l$ and $\mathcal{F}$ moves in a bounded family of $\mathcal{O}_{\mathrm{red}\ D}$ modules as well, we conclude that $\mathrm{gr}\ \mathcal{O}_D$ moves in a bounded family of $\mathcal{O}_{\mathrm{red}\ D}$ modules.

Now by Lemma 2.3.3 $\mathrm{gr}\ \mathcal{O}_D \otimes \mathcal{O}_{\mathrm{red}\ D}(sX)$ has many sections for $s \geqslant s_0$, and its H^1 is zero. So the same holds if we look at it as an $\mathcal{O}_D$ module by (i) of Lemma 2.3.2. Thus a successive application of (iii) Lemma 2.3.2 gives that $\mathcal{O}_D(sX)$ has many sections for $s \geqslant s_0$. (Note that we can *not* claim yet that $\mathcal{O}_D$ moves in a bounded family.)

At last we are in the situation (ii) of the Conductor Principle, so we get that $(\tilde{V},\ \tilde{X})$ moves in a bounded family. □

2.5.6 *Step 5*: (V, X) is bounded.
Let $I = \mathrm{Cond}(\tilde{V}/V)$. Then $\mathcal{O}_{\tilde{V}}/I$ is a module of finite length. Furthermore $\mathrm{length}(\mathcal{O}_{\tilde{V}}/I) - \mathrm{length}(\mathcal{O}_V/I) = \tilde{\chi} - \chi$. From Proposition 2.2.10 we get that $\mathrm{length}(\mathcal{O}_V/I) \leqslant (\tilde{\chi} - \chi)^2$, so $\mathrm{length}(\mathcal{O}_{\tilde{V}}/I) \leqslant (\tilde{\chi} - \chi)^2 + (\tilde{\chi} - \chi)$. So I moves in a bounded family of $\mathcal{O}_{\tilde{V}}$ modules and the form (ii) of the Conductor Principle applies again to finish the proof of the Main Theorem. □

EXAMPLE 2.5.7: Let $R = k[y^j x^i\colon\ j \geqslant 2,\ i \geqslant 0,\ x^n + nx^{n-2}y\colon\ n \geqslant 2]$ be a subring of $k[x, y]$ given by a linear basis. It is easy to check that R is generated by its elements of degree 2, 3, and 4, so it is finitely generated. Let $S = k[x, y]$. Then $\mathrm{Cond}(S/R) = y^2 S$ so $R/R \cap y^2 S = k[x^2, x^3]$ is a cuspidal curve and $S/y^2 S \cong k[x, \varepsilon]$. But $\phi\colon S/R \to k[x]$: $\phi(x) = -1$, $\phi(y) = x$ is an isomorphism, so R is S_2, and S is its (semi)normalization. The reduced conductor is S in the affine line and not the cuspidal line.

§2.6 *Consequences for polarized threefolds*

Now we shall prove Theorem 2.1.3 and 2.1.5 together. Let (V, X) be a polarized variety as there.

LEMMA 2.6.1: *Assumptions as in those theorems. Then there exists a fixed* s, *such that* $|sX|$ *contains a reduced, irreducible surface* W.

PROOF: Of course we want to use Theorem 2.1.6. For normal varieties in characteristic 0 the required estimates are proved in [K-M], for characteristic p in [M3]. □

2.6.2 Now we choose W to be general. Since $\chi(W, X) = \chi(V, X, t) - \chi(V, X, t\text{-}s)$, the polarized surfaces (W, X) move in a bounded family by Theorem 2.1.2. So for some s_0, $|sX|$ is very ample on W and $h^i(W, sX) = 0$ $(i > 0)$ for $s \geqslant s_0$.

From $0 \to \mathcal{O}_V((k-1)sX) \to \mathcal{O}_V(ksX) \to \mathcal{O}_W(ksX) \to 0$ we deduce that for $ks > s_0$ we have $H^i(V, ksX) = 0$ for $i \geqslant 2$ and

$$H^0(V, ksX) \to H^0(W, ksX) \to H^1(V, (k-1)sX)$$
$$\to H^1(V, ksX) \to 0 \qquad (*)$$

is exact. For a fixed k_0 we get

$$H^1(V, k_0 sX) = H^0(V, k_0 sX) - \chi(V, X, k_0 s) \leqslant$$
$$\leqslant k_0^3 s^3 X^3 + 4 - \chi(V, X, k_0 s) = N$$

by [L-M] Lemma 2.1.

So for some $k_0 \leqslant k_1 \leqslant k_0 + N + 1$ we have that $H^0(V, k_1 sX) \to H^0(W, k_1 sX)$ is onto, and thus $H^0(V, k_1 sX)$ is base point free. Since W was general $|k_1 sX|$ defines a birational map on V, and as V runs through our family the images form a bounded family. So the family of normalizations is bounded as well, which proves our theorems, except the remark in Theorem 2.1.3 about the two coefficients.

2.6.3 To prove this we remark that if we use the inequality

$$H^1(V, k_0 sX) \leqslant H^1(\mathcal{O}_V) + \Sigma_0^{k_0-1} H^1(W, isX)$$

$$\leqslant H^1(\mathcal{O}_W) + \Sigma_0^{k_0-1} H^1(W, isX)$$

which we get from repeated use of the sequence (*) then we get boundedness without using the constant term of the Hilbert polynomial.

So all we need is to estimate the linear coefficient in $\chi(V, X)$ by the two highest ones. Let $\chi(V, X) = \Sigma d_i n^i$, $d_0 = \chi(\mathcal{O}_V)$. Then $\chi(W, X) = (3sd_3)n^2 + (2sd_2 - 3s^2 d_3)n + s^3 d_3 - s^2 d_2 + sd_1$. By Lemma 2.5.2, $\chi(\mathcal{O}_W) \leqslant 3 + (2sd_2 - 3s^2 d_3)^2$ and this gives an estimate $d_1 \leqslant \phi(d_3, d_2)$.

To get a lower estimate let π: $V' \to V$ be a resolution, $\chi(V', \pi^* X) = \Sigma d_i' n^i$. The Leray spectral sequence gives $d_3' = d_3$, $d_2' = d_2$ and $d_1' \leqslant d_1$. But Lemma 5.2 in [K-M] gives $d_1' \geqslant \phi(d_2', d_2')$, so we can bound d_1 in terms of d_3 and d_2. Hence the proof of Theorem 2.1.3 is complete. □

Proposition 2.6.4: *The family of polarized, seminormal S_2 3-folds with fixed Hilbert polynomial is bounded.*

Proof: Let (V, X) be as above, $(\overline{V}, X)$ be the normalization, $\mathcal{O}_{\overline{C}} = \mathcal{O}_{\overline{V}}/\mathrm{Cond}(\overline{V}/V)$, $\mathcal{O}_C = \mathcal{O}_V/\mathrm{Cond}(\overline{V}/V)$. By Proposition 2.2.7 $\overline{C}$ and C are reduced of pure dimension 2. As in 2.5.3 we can prove that $\overline{C}$ moves in a bounded family. From $0 \to \mathcal{O}_C \to \mathcal{O}_{\overline{C}} \to \mathcal{O}_{\overline{V}}/\mathcal{O}_V \to 0$ we can compute the Hilbert polynomial of (C, X) up to finite ambiguity. Thus by Theorem 2.1.3 we see that (C, X) moves in a bounded family. Hence we can apply the Conductor Principle to conclude the proof of the proposition. □

III. Uniqueness of specializations

§*3.1 General results*

In this chapter we address the following problem: Given a family of varieties over the punctured disc, when can this be extended in a unique way to a family over the disc. In this generality the answer is never, so we must restrict the possible central fibres. The first result is due to Matsusaka-Mumford [M-M].

THEOREM 3.1.1: [M-M] *Let $X^* \to T^*$ be a family of smooth projective polarized varieties. There is at most one extension of it to $X \to T$ if we require the central fibre to be smooth and nonruled.* □

Very often the extensions do not exist in this class, so it is natural to consider certain singular central fibres. In the following sections we shall prove theorems of this kind. But first we recall two statements that are implicit in the mentioned article [M-M].

PROPOSITION 3.1.2: *Let $X^* \to T^*$ be a family of projective, irreducible, reduced polarized varieties and $X^1 \to T$, $X^2 \to T$ two extensions, X_0^1 (resp. X_0^2) the central fibres. The identity map on X^* specializes to a rational map ϕ: $X_0^1 \to X_0^2$. Then, if ϕ is birational, it is in fact an isomorphism.*

PROOF: For a family o smooth varieties this is just Theorem 2 in [M-M]. In fact the proof given there works for normal varieties as well. To cover the general case let $Z \subset X^1 \times_1 X^2$ be the closure of the graph of the isomorphism over the generic point. By assumption the special fibre of Z has a decomposition $Z_0 = Z_0' \cup Z_0''$ where Z_0' is the graph of the birational isomorphism ϕ. Let A^i be the polarizing divisor on X^i, and $B = \pi_1^*A^1 + \pi_2^*A^2$ a relatively ample divisor on $X^1 \times_T X^2$. Over the generic point of T we have $I(Z_t, B_t^{(n)}) = 2^n A^{(n)}$, so $I(Z_0, B_0^{(n)}) = 2^n A^{(n)}$. But since $\phi(A^1) = A^2$ it is easy to compute that $I(Z_0', B_0^{(n)}) = 2^n A^{(n)}$. So $Z_0'' = \emptyset$, and hence Z_0 is irreducible. Now I claim that π_i: $Z_0 \to X_0^i$ are finite. Indeed, let $C \subset Z_0$ be a curve such that $\pi_1(C)$ is a point. Then dim $\pi_2(C) = 1$, so deg $\pi_1^*(A^1 0\,|_C) = 0$ whereas deg $\pi_2^*(A_0^2\,|_C) > 0$. But $\pi_1^*A_0^1$ and $\pi_2^*A_0^2$ are linearly equivalent since they are specialization of $\pi_1^*A_t^1 \cong \pi_2^*A_t^2$ and $\mathrm{Pic}(Z/T)$ is separated since the fibres are irreducible and reduced (cf. [G2]). So π_i: $Z_0 \to X_0^i$ are finite and birational.

Now A_0^i and $\pi_i^*A_0^i$ have the same Hilbert polynomial since they are specializations of $A_t^i \cong \pi_i^*A_t^i$. So $\pi_{i*}\mathcal{O}_{Z_0} = \mathcal{O}_{X_0^i}$ hence Z_0 is isomorphic to X_0^i. Therefore X_0^1 is isomorphic to X_0^2 via ϕ. This is what we wanted to prove. □

PROPOSITION 3.1.3: [M-M] *Let Σ be a class of singularities, and Π be a class of varieties. Assume that whenever ϕ: $X \to T$ is a 1-par. deformation of a singularity in Σ and f: $X' \to X$ any birational regular map, $E \subset X'$ an exceptional divisor such that X' is smooth at the generic point of E then E belongs to the class Π.*

Let Φ be the class of varieties that are not birational to any variety in Π and have all their singularities in Σ.

Under these conditions, if $X^* \to T^*$ is a family of polarized varieties, it has at most one extension where the central fibre is in Φ.

PROOF: This is just a complicated formulation of the first part of Theorem 1 in [M-M], and our Proposition 3.1.2. □

DEFINITION 3.1.4: [Mu]. A singularity X is called an insignificant limit singularity if $\Sigma = \{X\}$ and $\Pi = \{\text{ruled varieties}\}$ satisfy the conditions of Proposition 3.1.3.

REMARK 3.1.5: It is proved in [M-M] that smooth points are insignificant limit singularities.

§3.2 *Irregular varieties*

THEOREM 3.2.1: *Let $X^* \to T^*$ be a family of polarized varieties. Then it has only one extension to $X \to T$ where the central fibre is normal with isolated singularities only, is not ruled and has positive irregularity. (dimension of the Picard variety is positive in* char. p).

PROOF: Let f^1: $X^1 \to T$ and f^2: $X^2 \to T$ be two such extensions. The identity map of X* extends to a rational map ϕ: $X^1 \to X^2$. Let $\overline{X} \subset X^1 \times_T X^2$ be the graph of ϕ, ψ^i: $\overline{X} \to X^i$ the projections. Let X_0^i be the central fibre of X^i and $E = (\psi^2)^{-1} X_0^2$. Then ϕ_0: $X_0^1 \to X_0^2$ is birational iff $\psi^1(E) = X_0^1$. At any rate $\psi^1(E) \subset X_0^1$. If dim $\psi^1(E) > 0$ then its generic point dominates a smooth point of X^1, so by Remark 3.1.5 E is ruled. But E is birational to X_0^2, a contradiction.

If dim $\psi^1(E) = 0$ then we first consider the case of char. 0. Then by [G2] the relative Albanese maps exist: α^i: $X^i \to \mathrm{Alb}(X^i/T)$. For abelian schemes specialization is unique, so $\mathrm{Alb}(X^1/T) \cong \mathrm{Alb}(X^2/T)$. Now $\alpha^1 \circ \psi^1(E)$ is a point, but $\alpha^2 \circ \psi^2(E) = \alpha^2(X_0^2)$ has positive dimension so they cannot be equal, a contradiction.

In characteristic p we proceed as follows. Let $\mathrm{Pic}^\circ X^1/T$ (resp. $\mathrm{Pic}^\circ X^2/T$) be the connected components of the Picard schemes. These are projective group schemes but might not be smooth. But by Koizumi [K.S] there are abelian schemes A^i/T and maps ϕ_i: $A^i/T \to \mathrm{Pic}^\circ X^i/T$ such that over the generic fibre ϕ_i is an isomorphism onto the reduced induced subvariety of $\mathrm{Pic}^\circ X^i/T$. Now A^1/T and A^2/T are abelian schemes, isomorphic over the generic point, so they are isomorphic.

Let P^i denote the Poincaré bundle on $X^i/T \times_T \mathrm{Pic}^\circ(X^i/T)$. Let $\phi^* P^i$ denote its pull-back to $X^i/T \times_T A^i/T$. This $\phi^* P^i$ defines a map α^i: $X^i/T \to \mathrm{Pic}^\circ(A^i/T)$. But since $A^1/T \cong A^2/T$ the targets of α^1 and α^2 are the same. Once we have these substitutes for the Albanese map the proof is the same as in char. 0. □

We remark that this proof works in mixed characteristic as well.

REMARK 3.2.2: In the applications it is sometimes necessary to weaken the assumption that the singularities are isolated. For instance, if we assume that a general hyperplane section of the central fibre has only insignificant singularities, then the present proof applies verbatim.

§3.3 Rational singularities

PROPOSITION 3.3.1: *The classes* $\Sigma = \{$*rational singularities*$\}$ *and* $\Pi = \{$*varieties* X, *such that for a smooth projective model* X' *we have* $H^0(X', \omega_{X'}) = 0\}$ *satisfy the conditions of Proposition* 3.1.3. (*Only in char.* 0).

We shall prove a more general statement but first we need a definition.

DEFINITION 3.3.2: Let $y \in Y$ be a normal singularity, dim $Y = n$. Let f: $Y' \to Y$ be a resolution. Then $R^{n-1}f_*\mathcal{O}_{Y'}$ is a skyscraper sheaf at y. Its length will be denoted by $p_a(y, Y)$ or just $p_a(y)$. This is independent of f (at least in char. 0).

LEMMA 3.3.3: *Let* $y \in Y$ *be as in the definition. Assume that* $Y - y$ *has rational singularities only. Then the natural map* $f_*\omega_{y'} \to \omega_y$ *is an isomorphism outside* y, *and the length of the cokernel is* $p_a(y, Y)$.

PROOF: Let $E = f^{-1}(y)$. We have the following diagram

$$\begin{array}{ccccccc} 0 \to \mathrm{H}^0_{\mathrm{E}}(\omega_{\mathrm{Y}'}) \to & \mathrm{H}^0(\mathrm{Y}', \omega_{\mathrm{Y}'}) & \to & \mathrm{H}^0(\mathrm{Y}' - \mathrm{E}, \omega_{\mathrm{Y}'}) & \to \mathrm{H}^1_{\mathrm{E}}(\omega_{\mathrm{Y}'}) \\ & & & & \to \mathrm{H}^1(\mathrm{Y}', \omega_{\mathrm{Y}'}) \\ & \downarrow \mathrm{a} & & \downarrow \mathrm{b} & \\ 0 \to \mathrm{H}^0_{\mathrm{y}}(\omega_{\mathrm{Y}}) \to & \mathrm{H}^0(\mathrm{Y}, \omega_{\mathrm{Y}}) & \xrightarrow{c} & \mathrm{H}^0(\mathrm{Y} - \mathrm{y}, \omega_{\mathrm{Y}}) & \to \mathrm{H}^1_{\mathrm{y}}(\omega_{\mathrm{Y}}) \end{array}$$

Here $H^0_y(\omega_Y) = H^1_y(\omega_Y) = 0$ since ω_Y is S_2, so c is an isomorphism. $H^0_E(\omega_{Y'}) = 0$ since $\omega_{Y'}$ is torsion-free and $H^1(Y', \omega_{Y'}) = 0$ by Grauert-Riemenschneider vanishing [G-R]. We are interested in coker a, which is just $H^1_E(\omega_{Y'})$ (easy diagram chasing). By [SGAII] II.6, $H^1_E(\omega_{Y'}) = \lim_{\to} \mathrm{Ext}^1(\mathcal{O}_{nE}, \omega_{Y'})$. This is dual to $\lim_{\leftarrow} H^{n-1}(Y', \mathcal{O}_{nE})$ by Serre duality (both groups are of finite length, so the limits commute with duality). By the Formal Function Theorem this latter group is just $R^{n-1}f_*\mathcal{O}_{Y'}$, which proves the lemma. □

Now we can formulate our promised generalization of Proposition 3.3.1.

THEOREM 3.3.4: *Let* $y \in Y_0$ *be a normal singularity, rational outside* y. *Let* $Y_0 \hookrightarrow Y$ *be a* 1-*parameter deformation of* Y_0, $f: Y' \to Y$ *a resolution*, $E \subset Y'$ *a smooth exceptional divisor such that* $f(E) = y$. *Then* $H^0(E, \omega_E) \leqslant p_a(y, Y_0)$.

PROOF: Further blowing-up Y' we may assume that the total transform Y'_0 of Y_0 is nonsingular. Let the deformation be π: $Y \to T$, $T =$ Spec R, $Y_0 = \pi^{-1}(0)$. Let $Z = f^{-1} \circ \pi^{-1}(0)$. So both E and Y'_0 are components of Z. From $0 \to \omega_{Y'} \to \omega_{Y'}(E) \to \omega_E \to 0$ we get $0 \to f_*\omega_{Y'} \to f_*\omega_{Y'}(E) \to H^0(\omega_E) \to R^1f_*\omega_{Y'} = 0$, the latter by [G-R]. Now both $f_*\omega_{Y'}$ and

$f_*\omega_{Y'}(E)$ are torsion-free sheaves, equal to ω_Y outside a codimension two set, so their double duals are all the same. But $\omega_Y^{**} = \omega_Y$ since it is S_2, so we get natural injections $f_*\omega_{Y'} \to f_*\omega_{Y'}(E) \to \omega_Y$. Let $Q = \omega_Y / f_*\omega_{Y'}$ as an R-module; then $H^0(\omega_E)$ is a submodule of Q, supported at $0 \in T$. Now let $t \in R$ be a local parameter. Then $\omega_Y \otimes R/t \cong \omega_{Y_0}$ and $im[f_*\omega_{Y'} \otimes R/t \to \omega_Y \otimes R/t]$ contains $im[f_*\omega_{Y_0'} \to \omega_{Y_0}]$ by ([E1] diagram on p. 146). So length $(Q \otimes R/t) \leqslant$ length $(\omega_{Y_0}/f_*\omega_{Y_0'}) \leqslant p_a(y, Y_0)$. Now $H^0(\omega_E)$ is a submodule of Q, supported at $0 \in T$. Moreover, from the definition it is clear that multiplication by t is trivial on $H^0(\omega_E)$, so it is contained in the socle of Q. We saw that Q is a direct sum of at most $p_a(y, Y_0)$ cyclic modules (since over a discrete valuation ring any coherent module is such), so its socle is at most $p_a(y, Y_0)$ dimensional. This proves the theorem. □

§3.4 *Singularities with minimal multiplicities*

DEFINITION 3.4.1: We shall call a singularity (x, X) minimal if it is reduced, Cohen-Macaulay, $\operatorname{mult}_x X = \operatorname{emdim}_x X - \dim_x X + 1$ and the tangent cone of X at x is geometrically reduced.

REMARK 3.4.2: In general $\operatorname{mult}_x X \geqslant \operatorname{emdim}_x X - \dim_x X + 1$ for Cohen-Macaulay singularities, hence the name "minimal".

LEMMA 3.4.3: (*i*) *A curve singularity is minimal iff it has smooth branches with independent tangencies.*

(*ii*) *Let $x \in X$ be a singularity, $x \in H$ a hyperplane section. Then*

(*a*) *if $x \in H$ is minimal, so is $x \in X$;*

(*b*) *if $x \in X$ is minimal and H is general then $x \in H$ is minimal;*

(*iii*) *A scheme Y with minimal singularities is seminormal* (*see* 2.2.2. (*i*)).

PROOF: (i) and (ii) are straightforward. To prove (iii) let f: $Y' \to Y$ be a finite homeomorphism. If dim $Y = 1$ then Y has seminormal singularities by (i), so f is an isomorphism. If dim $Y > 1$ let $H \subset Y$ be a general hyperplane section through a point $y \in Y$. Then $y \in H$ is minimal by (ii/a) so f is an isomorphism above H at y. Thus f is an isomorphism at y. □

THEOREM 3.4.4: *Let f: $X \to T$ be a flat deformation of X_0 over the spectrum of a* DVR. *If X_0 has only minimal singularities then the general fibre, X_t, has only minimal singularities as well.*

PROOF: The problem is clearly local on X_0. Assume that we know the statement for dim $X_0 = 1$. Then by a generic projection we can view X as a family of curves over $\mathbb{A}_T^{n-1}$, such that the central fibre has minimal singularities by Lemma 3.4.3 (ii/a). So by the assumption all fibres have minimal singularities. Therefore by Lemma 3.4.3 (ii/b) X_t has minimal singularities.

Now to prove the curve case first I claim that X is seminormal. Indeed, let f: $X' \to X$ be a finite homeomorphism. The locus where f is not an isomorphism is closed and does not intersect X_0, so it must be empty. Hence X is seminormal.

This implies that X_t is seminormal. Indeed, if f_t: $X_t' \to X_t$ is a homeomorphism then f_t: $X_t' \to X$ is quasi-finite, so it can be factored as $X_t' \xrightarrow{i} X' \xrightarrow{f} X$, i an open immersion and f finite. If $z \in X_0$ is a nonclosed point then $z \in X$ is regular, so by Zariski connectedness f is a homeomorphism above z. If $x \in X_0$ is closed and $f^{-1}(x)$ is not a single point, then the completion of X at x has at least two components meeting at x only. So by Hartshorne [H] it is not S_2, which is a contradiction. So f: $X' \to X$ is a homeomorphism and therefore an isomorphism by the previous claim; hence X_t is seminormal.

Now we are nearly done. Indeed, by a result of Davis ([D], especially Corollary 4) if Y is a curve defined over L, and $Y \otimes_L K$ is seminormal for all finite extensions $L \subset K$ then Y is minimal. By changing our T suitably we get the required property for X_t, so X_t has minimal singularities. □

Note that over a field of characteristic zero the base change is not necessary. In the case where the residue characteristic of T is zero we could have used [B-G] Proposition 7.2.6.

COROLLARY 3.4.5: (*of proof*) *Minimality is an open condition.* □

The following statement is a combination of results of Sally [S.J.] and classical results (see e.g. [X]).

PROPOSITION 3.4.6: *Let $x \in X$ be a minimal singularity. Then the projectivized tangent cone has only minimal singularities and rational components.*

PROOF: By [S.J] $\Sigma\, m_x^i/m_x^{i+1}$ (m_x: ideal of $x \in X$) is Cohen-Macaulay and has the same multiplicity and embedding dimension as $x \in X$. Its tangent cone is itself, so it is minimal. Its proj is the projectivized tangent cone, so Corollary 3.4.5 implies that it has only minimal singularities. Furthermore it is a variety of minimal degree in $\mathbb{P}^n$, hence its irreducible components are all rational (see e.g. [X]). □

COROLLARY 3.4.7: *Let $V \subset \mathbb{P}^n$ be a reduced nondegenerate variety of minimal degree, connected in codimension 1. If (x, X) is a small deformation of the vertex of the cone over V then (x, X) is minimal. Conversely, every minimal singularity arises this way.*

PROOF: By Xambo [X] V is arithmetically Cohen-Macaulay, so we have the first statement by Theorem 3.4.4. Since every singularity deforms to its tangent cone, Proposition 3.4.6 implies the converse. □

COROLLARY 3.4.8: *Let $x \in X$ be a minimal singularity and let $B_x X$ be the blow up. Then $B_x X$ has minimal singularities only and all exceptional divisors are rational.*

PROOF: Let $T \subset B_x X$ be the exceptional divisor: It is just the projectivized tangent cone, so T has minimal singularities by Proposition 3.4.6 and so $B_x X$ has minimal singularities by Lemma 3.4.3 (ii) and Corollary 3.4.5. □

THEOREM 3.4.9: *The normalization of a minimal singularity is minimal again. A normal minimal singularity is rational.* (*Characteristic zero only*).

PROOF: My proof of the first statement is rather complicated. Since I think that a simple proof should exist and the statement shall not be used in the sequel, only the second part will be proved here.

Since a singularity is rational if a hyperplane section of it is rational (Elkik, [E1]), it is sufficient to prove the second statement for surfaces. We do induction on the number of blow-ups that are needed to resolve the singularity. Let $x \in X$ be normal, minimal, dim $X = 2$. Then $B_x X$ has minimal singularities by Corollary 3.4.8 and is normal outside the singularities of the exceptional divisor T, hence normal. Hence $B_x X$ has rational singularities by induction. So all we need is that $R^1\sigma_* \mathcal{O}_{B_x X} = 0$. By the Formal Function Theorem this is just $\lim_{\to} H^1(T, \mathcal{O}_{B_x X}/I^k)$, where I is the ideal of T. Now T is just a connected curve of minimal degree in $\mathbb{P}^n$ and $I/I^2 \cong \mathcal{O}_T(1)$. So we get short exact sequences $0 \to \mathcal{O}_T(n) \to \mathcal{O}_{B_x X}/I^{n+1} \to \mathcal{O}_{B_x X}/I^n \to 0$. Taking cohomology we see that all we need is that $H^1(T, \mathcal{O}(k)) = 0$ for $k \geqslant 0$. Now $T \subset \mathbb{P}^n$ is 2-regular (see [S.P] for the definition and the result) so $H^1(T, \mathcal{O}(k)) = 0$ for $k > 0$. $H^1(T, \mathcal{O}) = 0$ is readily computable from the normalization. This finishes the proof of the theorem. □

REMARK 3.4.10: It is not difficult to see that a rational surface singularity is minimal iff the fundamental cycle (cf. [A1]) is reduced.

Now we come to the main theorem of this section, but first we need a definition.

DEFINITION 3.4.11: A variety E will be called a relative Severi-Brauer variety, if there exists a rational map π: $E \to X$ such that the fibre of π over the generic point of X is birational to a Severi-Brauer variety (i.e. after a separable base field extension becomes birational to $\mathbb{P}^k$, $k \geqslant 1$).

THEOREM 3.4.12: *Let X be a scheme with minimal singularities $\pi_1\colon Z \to X$ a birational map, $E \subset Z$ an exceptional divisor, Z normal. Then E is a relative Severi-Brauer variety.*

PROOF: Let $X_1 = X$. If we have $\pi_i\colon Z \to X_i$ and $\pi_i(E)$ is a closed point $x_i \in X_i$ then let X_{i+1} be the blow-up of x_i, $\pi_{i+1}\colon Z \to X_{i+1}$ resulting map. Then for some $k \geqslant 1$ $\pi_k(E) \subset X_k$ is not closed. The proof of this is postponed until the end of the argument.

By Corollary 3.4.8 X_k has minimal singularities. If $\dim \pi_k(E) = \dim X_k - 1$ then $\pi_k(E)$ is an exceptional divisor of $X_k \to X_{k-1}$, hence rational by Corollary 3.4.8. But E is birational to $\pi_k(E)$ since X_k is smooth at the generic point of $\pi_k(E)$, hence E is rational. If $\dim \pi_k(E) < \dim X_k - 1$ then we can localize at $\pi_k(E)$ and by induction on $\dim X_k$ we get that the generic fibre of $E \to \pi_k(E)$ is a relative Severi-Brauer variety. This proves the theorem. □

What remains is to see that the process of blowing-up stops in finitely many steps. This is probably not new. I worked out the following proof following some suggestions of M. Spivakovsky.

PROPOSITION 3.4.13: *Let R_0 be a complete local ring, ν a discrete rank 1 valuation centered at m_R. We blow up m_R and obtain the scheme BR_0. If the centre of ν on BR_0 is closed then let R_1 be its local ring. Iterating this we get $R_2, R_3, \ldots$. We claim that this sequence is finite, i.e. for some k the centre of ν on BR_k is not closed.*

PROOF: Let t be a local parameter for ν and let $k \subset R_0$ denote either the residue field of R_0 or the Witt vectors over the residue field. Let $x_i = t^{b_i}v_i$ $(\nu(v_i) = 0)$ be a generating set for m_R. We perform a k-linear change on the set $\{x_i\}$ such that for the resulting basis $z_i = t^{a_i}w_i$ the sequence $a_1 \leqslant a_2 \leqslant \ldots$ is lexicographically the largest possible (i.e. first maximize a_1, then a_2, then $a_3 \ldots$) If $a_1 = a_2 = \ldots = a_i \neq a_{i+1}$ then after blowing-up we get the ring

$$k\left[y^{a_1}w_1, \frac{w_2}{w_1}, \ldots, \frac{w_i}{w_1}, y^{a_{i+1}-a_1}\frac{w_{i+1}}{w_1}, \ldots\right]$$

We claim that the centre of ν is not closed. Indeed, if it is, then for some $c \in k$, $w_j/w_1 - c = tw'$ for some $2 \leqslant j \leqslant i$. But then

$$t^{a_j}w_j - ct^{a_1}w_1 = t^{a_j+1}w'w_1 \quad \text{so the sequence } \{a_i\} \text{ was not maximal.}$$

So we may assume $a_1 < a_2$. Now we keep blowing-up until we reach the point where a_1 is the smallest possible. Again let $z_i = t^{a_i}w_i$ denote a system of generators. Here we normalize $\{z_i\}$ using $k[[z_1]]$ linear expressions. (Note that since $a_1 < a_2$ z_1 is unique). Now at each blow-up a_2 gets smaller, until at one point we get $a_2 = 2a_1$. In the next blow-up we get

$$k\left[t^{a_1}w_1,\ t^{a_1}\frac{w_2}{w_1},\ldots,t^{a_n-a_1}\frac{w_n}{w_1}\right].$$

We are in the situation considered earlier except that we might not have a lexicographically maximal sequence $a_1 \leqslant a_1 \leqslant a_3 - a_1 \leqslant \ldots$. But I claim that this problem cannot rise. Indeed, if

$$t^{a_1}\frac{w_2}{w_1} - ct^{a_1}w_1 = t^{a_1+1}w' \qquad \text{then} \quad t^{a_2}w_2 - c\left(t^{a_2}w_1\right)^2 = t^{a_2+1}w'w_1,$$

contradicting the fact that $a_1 < a_2 \leqslant \ldots$ was a lexicographically maximal sequence for $k[[z_1]]$ linear coordinate changes. So after one more blow-up the centre of ν will not be closed. □

Remark 3.4.14: We could expect something similar to hold if $(x,\ X)$ is a Gorenstein singularity with $\operatorname{mult}_x X = \operatorname{emdim}_x X - \dim_x X + 2$ and reduced tangent cone. But here in general we can run into trouble-e.g. if the tangent cone is a cuspidal cubic. So let us define a minimal Gorenstein singularity by requiring the above conditions and the following: If we cut down $(x,\ X)$ to a surface singularity by a general linear section then we get either a simple elliptic or a cusp singularity (for definitions see e.g. [S.K], [Ka]).

Then if $x \in X$ has a minimal Gorenstein singularity and $f\colon Z \to X$ is a birational map, $E \subset Z$ an exceptional divisor then E is covered by elliptic curves. If dim $X \geqslant 3$ and X is nonsingular in codimension two, then E is uniruled.

This can be proved as in the case of minimal singularities with two changes:

(i) In some cases instead of a blow-up a weighted blow-up should be used.

(ii) We can get some singularities with nonreduced tangent cones in the process. These should be taken care of with ad hoc methods.

These make the proof rather cumbersome. Since the main idea is the same as earlier, and the above changes are explained carefully in [R11], [S-B], [Sh] in a similar context we do not give the proof here.

IV. Construction of moduli spaces

§4.1 *Generalities*

In this chapter we will use the theorems we proved so far to construct some moduli spaces. We proceed in the usual way: given a family of polarized varieties we embed them into a big projective space via a large multiple of the polarizing line bundle. The images will be parametrized by a subset of the Hilbert scheme. Then we want to take the quotient by the equivalence relation "isomorphism". To do this we need some quotient theorem. The following one is more or less a special case of a result of Artin [A3] 6.3 Corollary. The present formulation is taken from [M-F] p. 172 with slight modifications.

THEOREM 4.1.1: *Let H be a separated scheme (or algebraic space) of finite type, and let f: $R \to H \times H$ a map. Assume that*

(*i*) *$f(R) \subset H \times H$ is an equivalence relation,*

(*ii*) *$pr_i \circ f$: $R \to H$ are smooth,*

(*iii*) *f is proper,*

(*iv*) *f as a map into its image is equidimensional and for every $h \in H$ f_h: $R \times_H h \to H \times \{h\}$ is smooth onto its image.*

Then H/R is represented by a separated algebraic space of finite type.

PROOF: Compared to the proof in [M-F] p. 172 the only change is that there instead of (iv) it is assumed that f is finite and unramified, and the fibres of pr_i: $f(R) \to H$ are smooth. If we replace our f with its Stein factorization then we get a finite and unramified map and (ii) and (iv) imply that the fibres of pr_i: $f(R) \to H$ are smooth. □

DEFINITION 4.1.2: Two polarized varieties (V_1, X_1) and (V_2, X_2) are called (numerically) isomorphic if there exists an isomorphism g: $V_1 \to V_2$ such that X_1 and $g^*(X_2)$ are numerically equivalent, i.e. $\mathcal{O}(X_1 - g^*(X_2)) \in \mathrm{Pic}^{\tau}(V_1)$. This is clearly an equivalence relation.

BASIC SETUP 4.1.3.: Let $\{(V_\lambda, X_\lambda)$: $\lambda \in \Lambda\}$ be a family of polarized varieties with fixed Hilbert polynomial χ. Assume that Λ is a bounded family. Pick a large number s, such that $\mathcal{O}(sX_\lambda)$ is very ample and $H^i(V_\lambda, \mathcal{O}(sX_\lambda)) = 0$ for $i > 0$. Then the linear system $|sX_\lambda|$ maps V_λ into a given projective space $\mathbb{P}^?$, and they form a subset H of the Hilbert scheme. Assume that H is open, so it determines a subscheme, denoted by H as well. Let R be the scheme representing the equivalence relation "numerical isomorphism", and f: $R \to H \times H$ the natural map. The moduli space we want is just the quotient H/R if it exists. We have to check conditions of Theorem 4.1.1.

LEMMA 4.1.4: *Condition (i) is satisfied by definition. Condition (ii) is satisfied in* char. 0 *if* dim Pic V *is locally constant. Condition (iv) is satisfied in* char. 0 *if conditions (ii) and (iii) are satisfied.*

PROOF: The first statement is clear. The second is the same as [M-F] p. 172 (i). The last statement is due to Matsusaka [M2] p. 217 if the V_λ are smooth. We sketch the proof in general.

The fibres of f: $R \to H \times H$ are just the automorphism groups of the varieties corresponding to the image point. In characteristic zero these are smooth, so all we have to check is that they have locally constant dimension. Since (iii) is satisfied this dimension is upper semicontinuous. On the other hand the fibres of pr_1: $f(R) \to H$ have dimension $h^0(V_\lambda, \mathcal{O}(sX_\lambda)) + h^1(V_\lambda, \mathcal{O}_\lambda)$-dim Aut (V_λ). By properness of f, $f(R)$ is a variety, so the semicontinuity of fibre dimensions applies and we get that dim Aut(V_λ) is lower semicontinuous as well, hence locally constant. This proves the lemma. □

LEMMA 4.1.5: *If the family of polarized varieties of* 5.1.3 *has the property that specializations are unique* (*i.e. if* g^*: $Y^* \to T^*$ *is a family over the punctured spectrum of a* DVR, *then it has at most one extension to* g: $Y \to T$ *where the central fibre is in our family*), *then condition* (*iii*) *of Theorem* 4.1.1 *is satisfied.*

PROOF: This is just the concrete form of the valuative criterion of properness. □

In all our theorems in chapter three we had to assume that the varieties we consider are nonruled. Unfortunately, as shown by examples of M. Levine [L2] being ruled is *not* a closed condition. So if we just throw away the ruled varieties we might end up with something strange. But at least in char. 0 there is a satisfactory remedy, as in [F.A], [L1]:

LEMMA 4.1.6: (char 0.) *Let* f: $X \to Y$ *be a flat family of irreducible varieties. Then those* $y \in Y$ *such that* $f^{-1}(y)$ *is uniruled, form a closed subset of* Y.

PROOF: Recall that a variety is uniruled if it is covered by rational curves. So the specialization of a uniruled variety is uniruled again.

By results of [L1], [F.A] uniruledness generalizes for smooth morphisms in char. 0. So using simultaneous resolution over a stratification of Y we get that uniruledness is a constructible condition. But it specializes, so it is closed. □

REMARK 4.1.7: Another approach to this problem is the one chosen by Matsusaka [M2]. He shows that it is sufficient to throw away *some* ruled varieties. So we are left with all nonruled varieties and some ruled ones. Unfortunately it is not clear which ruled varieties should be thrown away. This question should be investigated.

§4.2 *Main Theorem*

The following theorem is the culmination point of chapters 2 and 3.

THEOREM 4.2.1: *Let us consider the family of polarized varieties* (V, X) *with a fixed Hilbert polynomial. The moduli functor is coarsely represented by a separated algebraic space of finite type if we restrict* V *to any of the following classes* (Char. 0 *always assumed*):

(*i*) *irregular, normal nonruled surfaces,*
(*ii*) *nonruled surfaces for which* $q > 0$ *or* $p_g > 0$, *and only rational singularities,*
(*iii*) *nonruled surfaces with minimal singularities only,*
(*iv*) *irregular, normal, nonuniruled* 3*-folds with isolated singularities only,*
(*v*) *nonuniruled* 3*-folds with minimal singularities only,*
(*vi*) 3*-folds with* $p_g > 0$ *having rational singularities only*; *or*
(*vii*) *nonuniruled smooth varieties.*

REMARKS 5.2.2:

Concerning (i) Among normal surfaces being irregular and nonruled is deformation invariant.

Concerning (ii) There are very few surfaces for which $q = p_g = 0$. Namely: Enriques surfaces, certain elliptic surfaces over $\mathbb{P}^1$, and finitely many families of general type. Furthermore, being non-ruled is deformation invariant.

Concerning (iv) For irregular, normal 3-folds with isolated singularities nonuniruledness is deformation invariant.

Concerning (vi) If $p_g > 0$ then the varieties are automatically nonuniruled. Furthermore, the condition $p_g > 0$ is deformation invariant (assuming rational singularities).

Concerning (vii) This result was proved by Matsuska [M2]. We quote it here for two reasons.

First, in [M-F] p. 171 a superfluous condition is added to the formulation of the theorem (namely that $H^0(T_V) = 0$). Secondly, in [P] p. 13 a counterexample is claimed. The problem with it is as follows (we use the notation there). $(1, 1) \notin Y(n, \mathbb{P}^1)$ hence condition (iii) is violated. If we compactify the family then as (P, Q) degenerates to (1, 1) the special fibre will be the surface corresponding to (1, 1) plus a $(6n$-$3)$-fold hyperplane at $t = \infty$.

PROOF OF THEOREM 4.2.1 We have to check the conditions of Theorem 4.1.1: The families are all bounded by Theorem 2.1.2, Theorem 2.1.3, Proposition 2.6.4 (using Lemma 3.4.3 (iii)) and [M5]. The families form an open subvariety of the Hilbert scheme by Lemma 4.1.6. For condition (iii) of Theorem 4.1.1 we use Lemma 4.1.5 and Theorem 3.2.1 in the cases

(i) and (iv), Theorem 3.4.12 in the cases (iii) and (v), and Remark 3.1.5 for (vii). In case (ii) and (vi) one can use Theorem 3.2.1 if $q > 0$ and Proposition 3.3.1 if $p_g > 0$.

By [G2] for a family of irreducible and reduced varieties dim Pic V_t is locally constant if none of the Pic V_t has an additive component. If the V_t are normal then Pic V_t is projective so we are done. If V_t has minimal singularities then it is S_2, so if C_t is a sufficiently ample generic curve section then $\mathrm{Pic}^\circ V_t$ injects into $\mathrm{Pic}^\circ C_t$. Now by Lemma 3.4.3 (i) C_t has only normal crossings with independent tangencies, so $\mathrm{Pic}^\circ C_t$ has no additive components. (see e.g. [S] V. 14)

So the proof of Theorem 4.2.1 is finished. □

REMARK 4.2.3: Here we discuss the problems that arise in char. p. The main theoretical problem is that $\mathrm{Pic}^\circ(V)$ and $\mathrm{Aut}^\circ(V)$ can be nonreduced, so conditions (ii) and (iv) will not be satisfied in general. I do not know how to overcome this problem.

Another problem was pointed out before Lemma 4.1.6 about the closedness of the ruled locus. For surfaces this is true even in char p. In general, if a variety is ruled then any small deformation is separably uniruled (see [Ll]), but I don't know if this holds for the specializations.

§4.3 *Remarks and further examples*

There is one question concerning our results that naturally comes to mind. Are these moduli spaces (quasi-projective) varieties or not? Here we address this question. First we investigate the standard approach of geometric invariant theory, say in the case of normal surfaces. This would go as follows: we embed V with high muliples of X, and try to decide whether the corresponding Chow or Hilbert points are (semi) stable or not. A pair (V, X) is called asymptotically (semi-, un-) stable if for all high enough embeddings we get a (semi-, un-) stable surface. (see [Mu] and the references there).

Mumford proved ([Mu] 3.20) that if (V, X) is asymptotically semistable, then the singularities of V are rational or elliptic at worst. So in the majority of our cases the standard approach of geometric invariant theory would not work.

A similar remark applies to the case of minimal singularities as well. By (ibid. 3.19) they are asymptotically unstable if the multiplicity is at least 7. Hence the connection between semistability and insignificance is not as close as the hypersurface case suggested (ibid. Remark 2 on p. 81).

We could not decide in any of the cases in Theorem 4.2.1 whether the resulting moduli space can be an honest algebraic space (i.e. not a scheme). But here we present a similar example where we do get an algebraic space which is not a scheme.

THEOREM 4.3.1: *Let $S(n)$ be the family of normal surfaces X of degree n in $\mathbb{P}^3$ for which $\Sigma_{x \in X} p_a(x, X) < \frac{1}{2}\binom{n-1}{3}$. Then the quotient $S(n)/SL(4)$ is a separated algebraic space which is not a scheme for large n.*

PROOF: $\binom{n-1}{3} = h^0(X, \omega_X)$, so the condition says that if f: $X' \to X$ is a desingularization then $h^0(X', \omega_{X'}) > \frac{1}{2}h^0(X, \omega_X)$. To prove separatedness we must show that specializations are unique. If $X \hookrightarrow \mathscr{X}$ is a 1-parameter deformation of a surface in $S(n)$ and f: $\mathscr{X}' \to \mathscr{X}$ is a resolution, then by Theorem 4.3.4 all exceptional divisors E of f have $h^0(E, \omega_E) \leqslant p_a(f(E), X) < \frac{1}{2}h^0(X, \omega_X)$, so it cannot be birational to a surface in $S(n)$. Hence Proposition 3.1.3 implies that the action of $SL(4)$ on $S(n)$ is proper. Hence the quotient $S(n)/SL(4)$ is a separated algebraic space.

What remains to prove is that it is not a scheme. By [M-F] Converse 1.13, if U is an open family of degree n surfaces, such that the operation of $SL(4)$ on U is proper and $U/SL(4)$ is a scheme, then all surfaces in U are stable. So to prove our last statement we have to show the existence of an unstable surface in $S(n)$.

Let us look at surfaces of the form $\Sigma_I^k a_I x^I$ where $I = (i_0, i_1, i_2, i_3)$, and the summation is over those I's satisfying $\Sigma i_j = n$, $i_0 \leqslant k$. We choose the coefficients a_I to be general. The resulting surface F will have only one singular point at $(1:0:0:0)$, which is an $(n\text{-}k)$-fold point. It is easy to compute that its p_a is $\binom{n-k}{3}$. $h^0(F, \omega_F) = \binom{n-1}{3}$, so direct computation gives that if $k > 0.21n$ and n is large enough then F is in $S(n)$. On the other hand if $k < 0.25n$ then the one parameter subgroup given by $\lambda(t)x_0 = t^{-3}x_0$, $\lambda(t)x_i = tx_i$ $(i = 1, 2, 3)$ shows that F is unstable. This proves our theorem. □

Another natural question is: what is the situation with regular surfaces? The preceding theorem provides a partial answer. But in general the specializations are not unique, as shown by the following examples.

EXAMPLE 4.3.2: Let $F = (f = 0)$, $G = (g = 0)$, $H = (h = 0)$ be smooth surfaces in $\mathbb{P}^3$ intersecting transversally, and assume that deg $h =$ deg $f +$ deg g. Consider the family of surfaces in $\mathbb{P}^3 \times \mathbb{A}^1$ given by $(fg + th = 0)$. Along the special fibre the total space has some ordinary double points where $f = g = h = 0$. To resolve these singularities we can blow-up the ideal sheaf of G.

Outside the special fibre the new family is unchanged and the new special fibre is $\tilde{F} \cup \tilde{G}$ where $\tilde{F} = (F$ blown-up at the points $f = g = h = 0)$ and $\tilde{G} \cong G$, $\tilde{F} \cap \tilde{G}$ is the proper transform of $F \cap G$. Now clearly the normal bundle of $\tilde{G}$ is $\mathcal{O}_{\tilde{G}}(-\tilde{F} \cap \tilde{G}) \cong \mathcal{O}_G(-\deg f)$. This is negative, so $\tilde{G}$ is contractible to a point π: $X' \to X$. It is easy to see that X is in fact quasi-projective. I claim that X is Cohen-Macaulay. To prove this let $x = \pi(\tilde{G})$ be the only nonsmooth point of X. $\tilde{G}$ is the exceptional divisor of π, its ideal sheaf is I, $I/I^2 \cong (N_G)^* = \mathcal{O}_G(\deg f)$. Now (x, X) is Cohen-Macaulay iff $H^i_x(\mathcal{O}) = 0$ for $i < 3$ ([SGAII] III.4.3). Since X is normal the only interesting case is $i = 2$. Now $H^i_x(\mathcal{O}) \cong R^{i-1}\ \pi_*(\mathcal{O}_{X'})$, so all I need is $R^1\pi_*(\mathcal{O}_{X'}) = 0$. By the Formal Function Theorem this is just $\lim_{\to}\ H^1(\mathcal{O}/I^n)$. From $0 \to I^n/I^{n+1} \to \mathcal{O}/I^{n+1} \to \mathcal{O}/I^n \to 0$ we get

$$H^1(G, \mathcal{O}(n \deg f)) \to H^1(\mathcal{O}/I^{n+1}) \to H^1(\mathcal{O}/I^n)$$

But $H^1(\mathcal{O}/I) \cong H^1(G, \mathcal{O}) = 0$ and $H^1(G, \mathcal{O}(n \deg f)) = 0$ for $n \geqslant 1$ so $H^1(\mathcal{O}/I^{n+1}) = 0$ for all n. Thus $H^2_x(\mathcal{O}_X) = 0$ and X is Cohen-Macaulay, hence after contracting $\tilde{G}$ we get a family of normal surfaces, where the special fibre is birational to F.

Of course the role of F and G can be interchanged, and then we get a family where the special fibre is birational to G. Outside the special fibres they are isomorphic to the original family. □

A similar example with actual equations is the following:

EXAMPLE 4.3.3: Let $\mathbb{P}^4$ have coordinates $(x : y : z : u : v)$ and $\mathbb{A}^1$ the coordinate t. Let $\phi(x, y, z, u, v)$ be a general homogeneous polynomial of degree $\geqslant 4$, $Q(x, y, z)$ a general quadratic form. We consider two families of complete intersections in $\mathbb{P}^4$, parametrized by t:

$$\begin{array}{lll} F^1 \to \mathbb{A}^1 & \text{defined by} & \phi(x, y, z, tu, v) = 0 \\ & & \text{and} \quad Q(x, y, z) + uv = 0 \\ F^2 \to \mathbb{A}^1 & \text{defined by} & \phi(x, y, z, u, tv) = 0 \\ & & \text{and} \quad Q(x, y, z) + uv = 0 \end{array}$$

The map $\sigma(x, y, z, u, v, t) = (x, y, z, t_u^{-1}, tv, t)$ gives an isomorphism outside the special fibres. The special fibres are birational to the $u = 0$ (resp. $v = 0$) hyperplane section of $\phi = 0$, so they are not isomorphic. □

It is interesting to compute the basic numerical invariants. The general surface has $p_a(F_t) = 2\binom{n-1}{3} + \binom{n-1}{2}$, and the contribution of the singularity is $p_a(0, F_0) = \binom{n-1}{3} + \binom{n-1}{2}$ so $p_a(0, F_0) > \frac{1}{2}p_a(F_t)$ as it should be by Theorem 4.3.1.

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János Kollár
Dept. of Mathematics
Brandeis University
Waltham, MA 02254
USA

current adress:
Dept. of Mathematics
Harvard University
Cambridge, MA 02138
USA