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STARSHAPEDNESS IN CONVEXITY SPACES

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Abstract

In terms of the *kernel* of starshaped sets characterizations are given of the property that a set is convex if and only if the hull of each pair in the set is in the set too, and of the so-called join-hull commutativity property. The setting is that of a convexity space.

1. Introduction

A *convexity space* is a pair $(X, \mathscr{C})$ where X is a nonempty set and $\mathscr{C}$ is a family of subsets of X closed under arbitrary intersections and containing X and the empty set $\varnothing$. Members of $\mathscr{C}$ are called $\mathscr{C}$-convex sets. For any $S \subset X$ the *convex hull* of S is defined as $\mathscr{C}(S) = \cap\{A \in \mathscr{C} : S \subset A\}$. The concept of convexity spaces was introduced by Levi [10] and has been extensively studied by many authors (see, among others, [3,5,6,12]).

The following two classes of convexity spaces are well-known.

A convexity space $(X, \mathscr{C})$ is said to be *join-hull commutative* (JHC) iff for each $p \in X$ and $S \subset X$ we have $\mathscr{C}(p \cup S) = \cup\{\mathscr{C}(p, x) : x \in \mathscr{C}(S)\}$.

A convexity space $(X, \mathscr{C})$ is said to be *domain finite* (DF) iff for each $S \subset X$, $\mathscr{C}(S) = \cup\{\mathscr{C}(T) : T \subset S \text{ and card } T < \infty\}$.

In this note some new classes of convexity spaces are defined and their relationships are studied. It is shown that the known characterization of convex sets for the class of JD (= JHC and DF) convexity spaces is in fact true in a greater class. Using this we add a remark to [11] where the solution of the *linearization problem* of a convexity space has been given.

2. $\mathscr{C}$-starshaped sets

First we mention that $\mathscr{C}$-starshaped sets are already studied by Soltan [13] but our approach is somewhat different.

A set $S \subset X$ is called *$\mathscr{C}$-starshaped* relative to a point $p \in S$ iff $\mathscr{C}(p, x) \subset S$ for each $x \in S$. The set of points with respect to which S is a $\mathscr{C}$-starshaped set is called the *$\mathscr{C}$-kernel* of S and is denoted by $\mathscr{C}$-ker(S).

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Obviously each $\mathscr{C}$-convex set S is a $\mathscr{C}$-starshaped set relative to any point $p \in S$. A *$\mathscr{C}$-star* emanating from a point $q \in S$ is defined as the set of all points $y \in S$ such that $\mathscr{C}(q, y) \subset S$ and is denoted by $\mathscr{C}\text{-st}(q, S)$.

PROPERTY 2.1: $p \in \mathscr{C}\text{-ker}(S)$ iff $\cup\{\mathscr{C}(p, x) : x \in S\} \subset S$.

PROPERTY 2.2: For each set $S \subset X$

$$\mathscr{C}\text{-ker}(S) = \cap\{\mathscr{C}\text{-st}(q, S) : q \in S\}.$$

In general, in a convexity space singletons need not be convex. The next property shows that those points which are the $\mathscr{C}$-kernels of some $\mathscr{C}$-starshaped sets are $\mathscr{C}$-convex.

PROPERTY 2.3: If $\mathscr{C}\text{-ker}(S) = \{p\}$ for some $S \subset X$, then $\{p\} \in \mathscr{C}$.

PROOF: Suppose $\{p\} \notin \mathscr{C}$. Then there is an element $y \in \mathscr{C}(p)$ such that $p \neq y$. However we have

$$\cup\{\mathscr{C}(y, x) : x \in S\} \subset \cup\{\mathscr{C}(\mathscr{C}(p) \cup x) : x \in S\}$$
$$= \cup\{\mathscr{C}(p, x) : x \in S\} \subset S,$$

and this implies that $y \in \mathscr{C}\text{-ker}(S)$, which is a contradiction.

3. B-convexity spaces

In 1913 Brunn [1] showed that in the ordinary convexity space ($\mathbb{R}^n$, conv) the kernel of every subset of $\mathbb{R}^n$ is a convex set. This is also true for any linear space with the ordinary convex sets (cf. [15]), but is false for general convexity spaces, as the following example shows.

EXAMPLE 3.1: Let $X = \mathbb{R}^2$ and suppose $\mathscr{C}$ consists of all ordinary convex closed subsets of X. $(X, \mathscr{C})$ is a convexity space. Now take the set $S = \{(x, y) : x^2 + y^2 < 1\}$. It is clear that S is a $\mathscr{C}$-starshaped set and $\mathscr{C}\text{-ker}(S) = S$, but S not belongs to $\mathscr{C}$.

A convexity space $(X, \mathscr{C})$ is said to be a B-*convexity space* iff for each $S \subset X$, $\mathscr{C}\text{-ker}(S)$ is a $\mathscr{C}$-convex set.

As already mentioned, ($\mathbb{R}^n$, conv) is a B-convexity space. A Bryant-Webster convexity space is also a B-space, this follows from [2] Theorem 13.

We are interested in the following question: What is the class of convexity spaces for which Brunn's theorem holds?

THEOREM 3.2: *Each* B-*convexity space is* DF.

PROOF: Let us recall that $(X, \mathscr{C})$ is a DF-convexity space if and only if $\cup\mathscr{F}\in\mathscr{C}$ for each chain $\mathscr{F}\subset\mathscr{C}$ [4] (a chain being a nonempty family of sets totally ordered by inclusion). Let $(X, \mathscr{C})$ be a B-convexity space and suppose that $(X, \mathscr{C})$ is not DF. Then there is a chain $\mathscr{F}\subset\mathscr{C}$ such that $\cup\mathscr{F}\notin\mathscr{C}$. Clearly, $\cup\mathscr{F}$ is a $\mathscr{C}$-starshaped set and $\mathscr{C}\text{-ker}(\cup\mathscr{F})=\cup\mathscr{F}$. Hence, $\mathscr{C}\text{-ker}(\cup\mathscr{F})\notin\mathscr{C}$, which is a contradiction.

THEOREM 3.3: *Any* JD-*convexity space is a* B-*convexity space.*

PROOF: Let $(X, \mathscr{C})$ be a JD-convexity space and take any $S\subset X$. If S is not $\mathscr{C}$-starshaped, then $\mathscr{C}\text{-ker}(S)=\varnothing$ and we are done. So we may assume that S is $\mathscr{C}$-starshaped. Take any $x_1, x_2\in\mathscr{C}\text{-ker}(S)$, $p\in\mathscr{C}(x_1, x_2)$, and $z\in S$. Always we have $\mathscr{C}(z, p)\subset\mathscr{C}(z\cup\mathscr{C}(x_1, x_2))=\mathscr{C}(z, x_1, x_2)=\mathscr{C}(x_1, z, x_2)=\mathscr{C}(x_1\cup\mathscr{C}(z, x_2))$. The starshapedness of S implies that $\mathscr{C}(x_1, q)\subset S$ for each $q\in\mathscr{C}(x_2, z)$ (since $\mathscr{C}(x_2, z)\subset S$). As $(X, \mathscr{C})$ is JHC it follows that

$$\mathscr{C}(x_1\cup\mathscr{C}(x_2, z))=\cup\{\mathscr{C}(x_1, q): q\in\mathscr{C}(x_2, z)\}\subset S.$$

Hence, $\mathscr{C}(z, p)\subset S$. Therefore, $\mathscr{C}(x_1, x_2)\subset\mathscr{C}\text{-ker}(S)$. Applying [6] Theorem 2 gives $\mathscr{C}\text{-ker}(S)\in\mathscr{C}$.

The example below shows that the converse of Theorem 3.3 is not true in general.

EXAMPLE 3.4: Let $X=\{(x, y)\in\mathbb{R}^2: x^2+y^2\geqslant 1\}$ and we define $\mathscr{C}$ as the family of the sets of the form $A\cap X$ where A is an ordinary convex set in $\mathbb{R}^2$. We show that $(X, \mathscr{C})$ is a B-convexity space. Take any $S\subset X$. We may of course suppose that $\mathscr{C}\text{-ker}(S)\neq\varnothing$. In $\mathbb{R}^2$ we define the set $S^*=\cup\{\text{conv}(x\cup\mathscr{C}\text{-ker}(S)): x\in S\}$. It is easy to see that $S=S^*\cap X$ and $\mathscr{C}\text{-ker}(S)=\text{ker}(S^*)\cap X$. Now Brunn's theorem (in $\mathbb{R}^2$) and the definition of $\mathscr{C}$ imply that $\mathscr{C}\text{-ker}(S)\in\mathscr{C}$ and, consequently, that $(X, \mathscr{C})$ is a B-convexity space. However, it is easy to verify that $(X, \mathscr{C})$ does not satisfy JD because it does not satisfy JHC.

THEOREM 3.5: *Let* $(X, \mathscr{C})$ *be a* B-*convexity space. The set* $S\subset X$ *is* $\mathscr{C}$-*convex if and only if* $\mathscr{C}(x, y)\subset S$ *for every* $x, y\in S$.

PROOF: The necessity is obvious. To prove sufficiency take any $x\in S$. Then by our assumption we have $\cup\{\mathscr{C}(x, y): y\in S\}\subset S$. This means that $x\in\mathscr{C}\text{-ker}(S)$ and consequently that $S\subset\mathscr{C}\text{-ker}(S)$. The reverse inclusion always holds, hence $S=\mathscr{C}\text{-ker}(S)$. Now the definition of B-convexity space implies that $S\in\mathscr{C}$. This completes the proof.

COROLLARY 3.6: *Let* $(X, \mathscr{C})$ *be a* B*-convexity space. Then for each* $S \subset X$ *we have* $\mathscr{C}\text{-ker}(\mathscr{C}\text{-ker}(S)) = \mathscr{C}\text{-ker}(S)$.

Our Theorem 3.5 and Corollary 3.6 are extensions (from JD to B-convexity space) of Theorem 2 in [6] and Corollary 1 in [13], respectively.

4. T-convexity spaces

Toranzos [14] has characterized ker(S) for ordinary convexity space as the intersection of all maximal (in the sense of inclusion) convex subsets of S. Such a characterization of the $\mathscr{C}$-kernel is true for a wider class of convexity spaces.

EXAMPLE 4.1: Let $X = \{1, \dots, 7\}$ and let $\mathscr{C}$ consists of all singletons and the system of Steiner triples on X (i.e. a family $\mathscr{F}$ of 3-element subsets of X such that each pair in X belongs to exactly one triple; for instance $\mathscr{F} = \{\{1, 2, 3\}, \{1, 5, 7\}, \{1, 4, 6\}, \{2, 4, 5\}, \{2, 6, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}$). It is easy to verify that for any set S, $\mathscr{C}$-ker(S) is equal to the intersection of all maximal $\mathscr{C}$-convex subsets of S.

A convexity space $(X, \mathscr{C})$ is said to be a T*-convexity space* iff for each $S \subset X$, $\mathscr{C}$-ker(S) is the intersection of all maximal $\mathscr{C}$-convex subsets of S.

Obviously any T-convexity space is a B-space. The Example 3.4 shows that the converse is not true.

The answer to the question: Characterize the class of T-convexity spaces, is the following theorem. A similar result can be found in Soltan [13]. A $\mathbb{T}_1$ *convexity space* has all singletons convex.

THEOREM 4.2: *Let* $(X, \mathscr{C})$ *be a* $\mathbb{T}_1$ *convexity space.* $(X, \mathscr{C})$ *is a* T*-convexity space if and only if it is a* JD*-convexity space.*

PROOF: Suppose first that $(X, \mathscr{C})$ is a JD-space. Take any $S \subset X$. Our assumptions (with $\mathbb{T}_1$) imply that S can be described as the union of the maximal $\mathscr{C}$-convex subsets of S:

$$S = \cup \{ M_a : a \in I_S \}.$$

It sufficies to show that

$$\mathscr{C}\text{-ker}(S) = \cap \{ M_a : a \in I_S \}. \tag{4.3}$$

The inclusion $\cap \{ M_a : a \in I_S \} \subset \mathscr{C}$-ker$(S)$ can be shown as follows. Take any $p \in \cap \{ M_a : a \in I_S \}$ and any $x \in S$. Then there is an $a' \in I_S$ such that $x \in M_{a'}$. Hence, $\{p, x\} \subset M_{a'}$, and so $\mathscr{C}(p, x) \subset M_{a'} \subset S$, which implies that $p \in \mathscr{C}\text{-ker}(S)$. To prove the reverse inclusion take any $p \in \mathscr{C}\text{-ker}(S)$. Then $\mathscr{C}(p, x) \subset S$ for each $x \in S$. As $(X, \mathscr{C})$ is JHC it follows for each $a \in I_S$ that $\mathscr{C}(p \cup M_a) = \cup\{\mathscr{C}(p, x) : x \in M_a\} \subset S$. The maximality of the M_a's implies that $\mathscr{C}(p \cup M_a) = M_a$. Hence $p \in M_a$ for each $a \in I_S$, and we are done.

Now assume that $(X, \mathscr{C})$ is a T-convexity space. Then $(X, \mathscr{C})$ is a B-convexity space and by Theorem 3.2 it is a DF-space. We must show that $(X, \mathscr{C})$ satisfies JHC. Let $p \in X$ and $A \subset X$. Consider the set $S = \cup\{\mathscr{C}(p, x) : x \in \mathscr{C}(A)\}$. It is clear that S is a $\mathscr{C}$-starshaped set with $p \in \mathscr{C}\text{-ker}(S)$. As $\mathscr{C}(A) \subset S$, there is a maximal convex subset $M_{a'} \subset S$ containing $\mathscr{C}(A)$ and p (since (4.3) holds). So we get

$$\begin{aligned}\mathscr{C}(p \cup A) &= \mathscr{C}(p \cup \mathscr{C}(A)) \\ &\subset M_{a'} \\ &\subset \cup\{M_a : a \in I_S\} = S \\ &= \cup\{\mathscr{C}(p, x) : x \in \mathscr{C}(A)\}.\end{aligned}$$

So $(X, \mathscr{C})$ is in fact JHC and the proof is complete.

5. Remarks

R-1. A well-known theorem of Krasnosel'skii [9,15] gives necessary and sufficient conditions for a compact set S in $\mathbb{R}^n$ to be starshaped. In the proof of that theorem the following additional property of the kernel of a compact set S is established.

$$\mathscr{C}\text{-ker}(S) = \cap\{\mathscr{C}(\mathscr{C}\text{-st}(x, S)) : x \in S\}. \tag{5.1}$$

By means of Krasnosel'skii's lemma (see [15]) it can be shown that any closed set also satisfies (5.1). Using (5.1) we define the following class of sets in an arbitrary convexity space.

A set S of X is called a K-set if and only if it satisfies the equality (5.1).

Obviously and $\mathscr{C}$-convex set is a K-set.

Note that K-sets play an important part in [8] where, using the ideas of [7], the "starshapedness number" of a convexity space is introduced.

R-2. A convexity space $(X, \mathscr{C})$ is said to be a K-*convexity space* iff any subset of X is a K-set.

EXAMPLE 5.2: $(X, 2^X)$ is a K-convexity space, but the space considered in Example 4.1 does not satisfy K.

EXAMPLE 5.3: Let D be a nonempty subset of a set X. Following [12] consider the family of sets $\mathscr{C}=\{X\}\cup\{A: D\not\subset A\}$. Clearly $(X, \mathscr{C})$ is a convexity space. It is easy to verify that $(X, \mathscr{C})$ is a K-space if and only if card $D \leqslant 2$.

Obviously any K-convexity space is also a B-space but not conversely as the example of the space $(\mathbb{R}^n$, conv) shows. Moreover, we remark that conditions K and T are independent.

R-3. In [11] necessary and sufficient conditions for the existence of a real linear structure for X such that the set of all convex sets of the resulting linear space is precisely $\mathscr{C}$, are given. This is a solution of the linearization problem of a convexity space.

Our approach enables us to give the solution for the larger class of convexity spaces. Namely, the main theorem in [11] can be formulated as follows (for definitions see [11]).

THEOREM 5.4: *Let $(X, \mathscr{C})$ be a* B*-convexity space with the property that for all $x, y, z \in X$, $\mathscr{C}(x, y)=\mathscr{C}(z, y)$ implies $x=z$. A necessary and sufficient condition that $\mathscr{C}$ is the family of all convex sets generated by a real linear structure for X is that X has a linearization family X^*.*

PROOF: The proof goes just as in [11] but we apply Theorem 3.5 instead of Theorem 2 in [6].

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