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BEST APPROXIMANTS FROM NON-ARCHIMEDEAN STONE-WEIERSTRASS SUBSPACES

Maria Zoraide M. Costa Soares

0. Introduction

Let X be a locally compact Hausdorff space, let $(F, |\cdot|)$ be a non-archimedean non-trivially valued division ring and $(E, \|\cdot\|)$ a normed space over $(F, |\cdot|)$.

We say that g: $X \to E$ vanishes at infinity if, for each $\epsilon > 0$, the set $\{x \in X;\ \| g(x) \| \geqslant \epsilon\}$ is compact.

We denote by $\mathscr{C}(X;\ E)$ the vector space of all continuous functions from X into E. $\mathscr{C}_0(X;\ E)$ will denote the vector space of all continuous functions which vanish at infinity, equipped with the norm $f \mapsto \|f\| = \sup\{\| f(x) \|;\ x \in X\}$.

The vector subspace of $\mathscr{C}(X;\ F)$ consisting of all continuous functions f: $X \to F$ such that $f(X)$ has compact closure in F, is denoted by $\mathscr{C}^*(X;\ F)$.

If Δ is the equivalence relation determined by $A \subset \mathscr{C}(X;\ F)$, $\Delta(x) = \{y \in X;\ a(y) = a(x) \text{ for all } a \in A\}$ is the Δ-equivalence class containing x.

If $Y \subset X$ is any non-empty set, we denote by $f|_Y$ the mapping $y \in Y \to f(y)$. If $\mathscr{F}$ is any family of mappings f: $X \to S$, we denote by $\mathscr{F}|_Y$ the set $\{f|_Y;\ f \in \mathscr{F}\}$.

In this paper, we extend some results of Machado and Prolla [3] to the case of non-archimedean normed spaces, and other results of Prolla [4].

If $A \subset \mathscr{C}^*(X;\ F)$ is a subalgebra and $W \subset \mathscr{C}_0(X;\ E)$ is a vector subspace which is an A-module, we proved in [5] that for each $f \in \mathscr{C}_0(X;\ E)$,

$$\operatorname{dist}(f;\ W) = \sup_{x \in X} \operatorname{dist}\left(f|_{\Delta(x)};\ W|_{\Delta(x)}\right).$$

We extend this "localization formula" for set-valued mappings under an upper semicontinuity hypothesis (see Theorem 1.7 below) generalizing a result of Prolla [4].

In Approximation Theory, given a normed space $(N, \|\cdot\|)$ and a non-empty subset $W \subset N$, there are two main problems. The first one is to characterize the closure of W in N, i.e., the set of all $f \in N$ such that $\operatorname{dist}(f;\ W) = 0$. When N is a normed space of functions, this leads to

Stone-Weierstrass type theorems by choosing appropriate algebraic conditions on W. (For example, W is an A-module, etc.).

The second problem arises when $\text{dist}(f; W) > 0$. Does there exist $g \in W$ such that

$$\| f - g \| = \text{dist}(f; W)?$$

More generally, if instead of a single f one deals with a bounded set $B \subset N$, does there exist $g \in W$ such that

$$\sup_{f \in B} \| f - g \| = \inf_{w \in W} \sup_{f \in B} \| f - w \| ?$$

Such a g, when it exists is called a Chebyshev center of B in W. We present some results (see Theorems 3.8 and 3.9) when N is $\mathscr{C}_0(X; E)$ and W is a so-called Stone-Weierstrass subspace. (see Olech [2]).

When W is a $\mathscr{C}^*(X; F)$-module (or more generally an A-module, for some separating subalgebra $A \subset \mathscr{C}^*(X; F)$) it is natural to ask whether approximation properties of $W(x) = \{w(x); w \in W\}$ in E, for every $x \in X$, will ensure the same for W in $\mathscr{C}_0(X; E)$. Theorem 3.10 and 3.11 are along this line: in 3.10 one assumes that, for each $s \in X$, and $v \in E$ there is some element $w(x)$ such that $\| v - w(x) \| = \text{dist}(v; W(x))$. Theorem 3.11 deals with the analogous question for Chebyshev centers.

This work represents part of the author's dissertation at the Universidade de Campinas.

1. Stone-Weierstrass theorems

Let X, $(F, | \cdot |)$ and $(E, \| \cdot \|)$ be as in the introduction.

1.1. Definition: A *carrier* φ *from* X *to* E is a mapping from X into the non-empty subsets of E.

1.2. Definition: Let φ be a carrier from X into E. We define the *distance of* φ *from a function* $g \in \mathscr{C}_0(X; E)$ to be

$$\text{dist}(\varphi; g) = \sup_{x \in X} \left\{ \sup_{y \in \varphi(x)} \| y - g(x) \| \right\}$$

and *the distance of* φ *from a subset* $W \subset \mathscr{C}_0(X; E)$ to be

$$\text{dist}(\varphi; W) = \inf\{\text{dist}(\varphi; g); \ g \in W\}.$$

1.3. Definition: Let φ a carrier of X into E. We say that φ *is upper semicontinuous* (u.s.c.) *with respect to* $W \subset \mathscr{C}_0(X; E)$, if given $w \in W$ and

$r > 0$, for each $x \in X$ such that $\varphi(x) \in B(w(x); r)$ and each $\epsilon > 0$, there is a neighborhood U of x such that $\varphi(y) \subset B(w(y); r + \epsilon)$ for all $y \in U$. (If $v \in E$ and $s > 0$ we denote by $B(v; s)$ the set $\{u \in E; \|u - v\| < s\}$).

1.4. EXAMPLE: If $f \in \mathscr{C}_0(X; E)$, then $\varphi(x) = \{f(x)\}$, $x \in X$, is upper semicontinuous with respect to any $W \subset \mathscr{C}_0(X; E)$. Indeed, for each $w \in W$ and $r > 0$, the set

$$\{x \in X;\ \varphi(x) \subset B(w(x); r)\} = \{x \in X;\ \|f(x) - w(x)\| < r\}$$

is open.

1.5. EXAMPLE: Let $N \subset \mathscr{C}_0(X; E)$ be a equicontinuous subset. Define a carrier φ from X into E by setting

$$\varphi(x) = \{f(x);\ f \in N\},$$

for all $x \in X$. We claim that φ is u.s.c. with respect to any $W \subset \mathscr{C}_0(X; E)$. Indeed, let $w \in W$, $r > 0$ and $x \in X$ with $\varphi(x) \subset B(w(x); r)$ be given. Let $\epsilon > 0$. If N is equicontinuous then $N - \{w\}$ is equicontinuous too, and there is a neighborhood U of x such that $\|f(y) - w(y) - (f(x) - w(x))\| < \epsilon$ for all $y \in U$.

Hence, for all $y \in U$

$$\begin{aligned}\|f(y) - w(y)\| &= \|f(y) - w(y) - (f(x) - w(x)) \\ &\quad + (f(x) - w(x))\| \\ &\leqslant \|f(y) - w(y) - (f(x) - w(x))\| \\ &\quad + \|f(x) - w(x)\| \\ &< \epsilon + r.\end{aligned}$$

1.6. DEFINITION: Let φ be a carrier of X into E and let $W \subset \mathscr{C}_0(X; E)$. We say that φ *vanishes at infinity with respect to* W, if for each $w \in W$ and $\epsilon > 0$ the set

$$\{x \in X;\ \varphi(x) \cap (E \setminus B(w(x); \epsilon)) \neq \emptyset\}$$

is relatively compact, i.e. has compact closure.

1.7. THEOREM: *Let $(E, \|\cdot\|)$ be a non-archimedean normed space over $(F, |\cdot|)$; let $A \subset \mathscr{C}^*(X; F)$ be a subalgebra and $W \subset \mathscr{C}_0(X; E)$ a vector subspace which is an A-module. For any carrier φ of X into E which is* *upper semicontinuous and vanishes at infinity with respect to W, we have*:

$$\operatorname{dist}(\varphi;\, W) = \sup_{x\in X} \operatorname{dist}\big(\varphi\,|_{\Delta(x)};\, W\,|_{\Delta(x)}\big).$$

PROOF: Let

$$\lambda = \sup_{x\in X} \operatorname{dist}\big(\varphi\,|_{\Delta(x)};\, W\,|_{\Delta(x)}\big).$$

We always have $\lambda \leqslant \operatorname{dist}(\varphi;\, W)$.

Let $\epsilon > 0$ and $x \in X$ be given; there exists $g_x \in W$ such that

$$\operatorname{dist}\big(\varphi\,|_{\Delta(x)};\, g_x\,|_{\Delta(x)}\big) < \lambda + \epsilon.$$

This implies that

$$\|t - g_x(y)\| < \lambda + \epsilon \qquad \text{for all } t \in \varphi(y) \text{ and } y \in \Delta(x).$$

Since φ is upper semicontinuous with respect to W, there is an open neighborhood U_x of x such that

$$\|t - g_x(z)\| < \lambda + \epsilon \qquad \text{for all } t \in \varphi(z) \text{ and } z \in U_x.$$

Clearly, $\Delta(x) \subset U_x$.

Since φ vanishes at infinity with respect to W, the closure K_x of

$$S_x = \{\, y \in X;\ \varphi(y) \cap \big(E \setminus B(g_x(y);\, \lambda + \epsilon)\big) \neq \emptyset \}$$

is compact. We claim that $\Delta(x) \cap K_x = \emptyset$. Indeed, assume $z \in \Delta(x) \cap K_x$. Since $\Delta(x) \subset U_x$ and K_x is the closure of S_x, there is some $y \in U_x \cap S_x$. But $\varphi(y) \subset B(g_x(y);\, \lambda + \epsilon)$ for all $y \in U_x$ and so y cannot be in S_x.

By Lemma 2.4, [5], there exists a finite set $\{x_1, x_2, \ldots, x_n\} \subset X$ such that for each $0 < \delta < 1$, there are functions $a_1, a_2, \ldots, a_n \in A_0$ satisfying:

(1) $|a_i(x)| \leqslant 1$ for all $x \in X$; $i = 1, \ldots, n$;

(2) $|a_i(t)| < \delta$ for all $t \in K_{x_i}$; $i = 1, \ldots, n$;

(3) $\sum_{i=1}^{n} a_i(x) = 1$ for all $x \in X$;

where A_0 is the subalgebra generated by A and the constant functions.

We choose $\delta > 0$ such that

$$\delta \cdot \max_{1 \leqslant i \leqslant n} \|t - g_{x_i}(x)\| < \lambda + \epsilon \qquad \text{for all } t \in \varphi(x)$$

and to this δ let $a_1, a_2, \ldots, a_n \in A$ be given satisfying (1) to (3).

Define

$$g = \sum_{i=1}^{n} a_i g_{x_i}.$$

Then $g \in W$, and for each $x \in X$ and $t \in \varphi(x)$, we have:

$$\|t - g(x)\| = \| \sum_{i=1}^{n} a_i(x) t - \sum_{i=1}^{n} a_i(x) g_{x_i}(x) \|$$

$$= \| \sum_{i=1}^{n} a_i(x)\big(t - g_{x_i}(x)\big) \|.$$

If $x \in K_{x_i}$, then

$$\| a_i(x)\big(t - g_{x_i}(x)\big) \| < \delta \cdot \| t - g_{x_i}(x) \|$$

$$\leqslant \delta \cdot \max_{1 \leqslant i \leqslant n} \| t - g_{x_i}(x) \| < \lambda + \epsilon.$$

If $x \notin K_{x_i}$, then

$$\| a_i(x)\big(t - g_{x_i}(x)\big) \| \leqslant 1 \cdot \| t - g_{x_i}(x) \| < \lambda + \epsilon.$$

Hence, for all $x \in X$ and $t \in \varphi(x)$,

$$\|t - g(x)\| = \| \sum_{i=1}^{n} a_i(x)\big(t - g_{x_i}(x)\big) \|$$

$$\leqslant \max_{1 \leqslant i \leqslant n} \| a_i(x)\big(t - g_{x_i}(x)\big) \|$$

$$< \lambda + \epsilon.$$

Then,

$$\text{dist}(\varphi;\ g) \leqslant \lambda + \epsilon.$$

A fortiori, $\text{dist}(\varphi;\ W) \leqslant \lambda + \epsilon$. Since $\epsilon > 0$ was arbitrary,

$$\text{dist}(\varphi;\ W) \leqslant \lambda = \sup_{x \in X} \text{dist}\big(\varphi|_{\Delta(x)};\ W|_{\Delta(x)}\big).$$

1.8. Definition: A family of functions $N \subset \mathscr{C}_0(X;\ E)$ is said to *vanish collectively at infinity* if, for each $\epsilon > 0$, there is a compact subset $K \subset X$ such that $\| f(x) \| < \epsilon$ for all $x \notin K$ and $f \in N$.

1.9. EXAMPLE: Let $N \subset \mathscr{C}_0(X; E)$ be a totally bounded subset. Then N vanishes collectively at infinity. Indeed, let $\epsilon > 0$ be given. There exists a finite set $\{f_1, f_2, \ldots, f_n\} \subset N$ such that, for each $f \in N$, there is $1 \leqslant i \leqslant n$ with $\| f - f_i \| < \epsilon/2$. For each $1 \leqslant i \leqslant n$, there is a compact subset $K_i \subset X$ such that $\| f_i(x) \| < \epsilon/2$ for all $x \notin K_i$. Let K be the union $K_1 \cup K_2 \cup \ldots \cup K_n$. Then for all $x \notin K$ and $f \in N$, $\| f(x) \| < \epsilon$.

1.10. PROPOSITION: *Let $N \subset \mathscr{C}_0(X; E)$ be a family which vanishes collectively at infinity and let $W \subset \mathscr{C}_0(X; E)$. The carrier*

$$\varphi(x) = \{ f(x); f \in N \}, \quad x \in X,$$

vanishes at infinity with respect to W.

PROOF: If $N \subset \mathscr{C}_0(X; E)$ vanishes collectively at infinity and $w \in \mathscr{C}_0(X; E)$, then $G = \{ f - w; f \in N \}$ vanishes collectively at infinity too.

Let $\epsilon > 0$ and $K \subset X$ be a compact set such that

$$\| f(x) - w(x) \| < \epsilon$$

for all $x \notin K$ and $f \in N$.

Then $\varphi(x) \subset B(w(x); \epsilon)$ for all $x \notin K$ and

$$X \setminus \{ x \in X; \varphi(x) \subset B(w(x); \epsilon) \} \subset K$$

and so the set

$$\{ x \in X; \varphi(x) \cap (E \setminus B(w(x); \epsilon)) \neq \emptyset \}$$

is relatively compact.

1.11. THEOREM: *Let $(E, \| \cdot \|)$ be a non-archimedean normed space over $(F, \| \cdot \|)$; let $A \subset \mathscr{C}^*(X; F)$ be a subalgebra; let $W \subset \mathscr{C}_0(X; E)$ be a vector subspace which is an A-module; and $N \subset \mathscr{C}_0(X; E)$ a totally bounded subset and define for all $x \in X$, $\varphi(x) = \{ f(x); f \in N \}$. Then,*

$$\operatorname{dist}(\varphi; W) = \sup_{x \in X} \operatorname{dist}\left(\varphi|_{\Delta(x)}; W|_{\Delta(x)}\right).$$

PROOF: By Example 1.5, φ is upper semicontinuous, and by Example 1.9, N vanishes collectively at infinity and by Proposition 1.10, φ vanishes at infinity with respect to any $W \subset \mathscr{C}_0(X; E)$. It remains to apply Theorem 1.7.

2. Chebyshev centers

2.1. DEFINITION: Let $(N, \| \cdot \|)$ be a normed space over $(F, | \cdot |)$, $W \subset N$ and B be a non-empty bounded subset of N. The *relative Chebyshev* *radius of* B (*with respect to* W) is, by definition, the number

$$\operatorname{rad}_W(B) = \inf\left\{ \sup_{f \in B} \| w - f \| ;\ w \in W \right\}.$$

If $W = N$, then we write

$$\operatorname{rad}_N(B) = \operatorname{rad}(B)$$

and call it the *Chebyshev radius of* B.

The elements $w_0 \in W$ where the infimum is attained are called *relative Chebyshev centers of* B (*with respect to* W), and we denote by $\operatorname{cent}_W(B)$ the set of all such $w_0 \in W$.

If $W = N$, there we write $\operatorname{cent}_N(B) = \operatorname{cent} B$ and call it the set of *Chebyshev centers of* B.

We say that W has the *relative Chebyshev center property in* N if $\operatorname{cent}_W(B) \neq \emptyset$ for all non-empty bounded sets $B \subset N$.

When $W = N$, and $\operatorname{cent}(B) \neq \emptyset$ for every non-empty bounded subset $B \subset N$, i.e. if N has the relative Chebyshev center property in N, we say that N *admits Chebyshev centers*.

Let $M \subset N$ be a closed linear subspace and $f \in N$. *A best approximant* of f in M is any element $g \in M$ such that

$$\| f - g \| = \inf_{h \in M} \| f - h \| = \operatorname{dist}(f;\ M).$$

We denote by $P_M(f)$ the set of all best approximants of f in M. If $P_M(f)$ contains at least one element for all $f \in N$, M is called *proximinal.*

The main problems of *best* (*simultaneous*) *approximation theory* are the following (in decreasing order of generality):

Problem I: Let $W \subset N$ be given. Determine if W has the relative Chebyshev center property in N. In particular, determine if N admits Chebyshev centers.

Problem II: Let $W \subset N$ be given. Determine the class B of all non-empty bounded sets $B \subset N$ such that $\operatorname{cent}_W(B) \neq \emptyset$.

Problem III: Let $W \subset N$ be given. Determine if W is proximinal in N, i.e., determine if the class B of Problem II contains all sets of the form $B = \{ f \}$, $f \in N$.

Suppose that N is $\mathscr{C}_0(X;\ E)$ equipped with the sup-norm an let $W \subset \mathscr{C}_0(X;\ E)$. To each non-empty and bounded set $B \subset \mathscr{C}_0(X;\ E)$, we define the carrier

$$\varphi_B(x) = \{ f(x);\ f \in B \}$$

for all $x \in X$. It follows that

$$\operatorname{dist}(\varphi_B;\ W) = \operatorname{rad}_W(B).$$

Consequently, by Theorem 1.11, we have the following formula of localizability for the Chebyshev radius.

2.2. THEOREM: *Let $(E, \|\cdot\|)$ be a non-archimedean normed space over $(F, |\cdot|)$; let $A \subset \mathscr{C}^*(X;\ F)$ be a subalgebra and $W \subset \mathscr{C}_0(X;\ E)$ a vector subspace which is an A-module. For each non-empty and totally bounded subset $B \subset \mathscr{C}_0(X;\ E)$ we have*

$$\operatorname{rad}_W(B) = \sup_{x \in X} \operatorname{rad}_{W|\Delta(x)}\left(B|_{\Delta(x)}\right).$$

2.3. DEFINITION: Let Δ be an equivalence relation in X. We say that a carrier φ from X into E is *Δ-bounded* if

$$\varphi(\Delta(x)) = \cup \{ \varphi(t);\ t \in \Delta(x) \}$$

is a bounded subset of E, for all $x \in X$. Let us define

$$\delta(\varphi) = \sup_{x \in X} \operatorname{rad}(\varphi(\Delta(x)))$$

2.4. THEOREM: *Let $(E, \|\cdot\|)$ be a non-archimedean normed space over $(F, |\cdot|)$ and $A \subset \mathscr{C}^*(X;\ F)$ a subalgebra. Let $W \subset \mathscr{C}_0(X;\ E)$ be an A-module such that for each $x \in X$ and $z \in E$, there is some $w \in W$ such that $w(t) = z$ for all $t \in \Delta(x)$. Then for any Δ-bounded carrier φ from X into E which is upper semicontinuous and vanishes at infinity with respect to W, we have*:

$$\operatorname{dist}(\varphi;\ W) \leqslant \delta(\varphi).$$

PROOF: By Theorem 1.7, we have:

$$\begin{aligned} \operatorname{dist}(\varphi;\ W) &= \sup_{x \in X} \operatorname{dist}\left(\varphi|_{\Delta(x)};\ W|_{\Delta(x)}\right) \\ &= \sup_{x \in X} \inf_{w \in W} \operatorname{dist}\left(\varphi|_{\Delta(x)};\ w\right) \\ &= \sup_{x \in X} \inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \| y - w(t) \|. \end{aligned}$$

Let $x \in X$. For each $z \in E$, choose $w_z \in W$ such that $w_z(t) = z$ for all $t \in \Delta(x)$. Then

$$\inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \| y - w(t) \|$$

$$\leqslant \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \| y - w_z(t) \|$$

$$= \sup_{y \in \varphi(\Delta(x))} \| y - z \|.$$

Since $z \in E$ was arbitrary, we have

$$\inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \| y - w(t) \| \leqslant \inf_{z \in E} \sup_{y \in \varphi(\Delta(x))} \| y - z \|.$$

Hence,

$$\operatorname{dist}(\varphi; W) \leqslant \delta(\varphi).$$

3. Stone-Weierstrass subspaces

3.1. DEFINITION: A vector subspace $W \subset \mathscr{C}_0(X; E)$ is said to be a *Stone-Weierstrass subspace* if there is a locally compact Hausdorff space Y and a proper continuous surjection π: $X \to Y$ such that

$$W = \{ g \circ \pi;\ g \in \mathscr{C}_0(Y; E) \}.$$

We denote by W_π the Stone-Weierstrass subspace determined by π.
If $W_\pi \subset \mathscr{C}_0(X; E)$ is a Stone-Weierstrass subspace, then

$$A_\pi = \{ \varphi \circ \pi;\ \varphi \in \mathscr{C}^*(X; F) \}$$

is a subalgebra of $\mathscr{C}^*(X; F)$ which contains the constants and

$$\{ \pi^{-1}(y);\ y \in Y \}$$

is the set of equivalence classes modulo A_π. Therefore, W_π is an A_π-module.

Clearly W_π is closed in $\mathscr{C}_0(X; E)$.

We will prove that this definition of Stone-Weierstrass subspace is the same as Definition 3.5, [5], by proving that $\Delta(W_\pi) \subset W_\pi$, where $\Delta(W_\pi)$ is the Stone-Weierstrass hull of W_π in $\mathscr{C}_0(X; E)$.

Let $f \in \Delta(W_\pi)$. We will prove that f is constant on the sets $\pi^{-1}(y)$ for all $y \in Y$.

Let t and t' be in X such that $\pi(t)=\pi(t')$. Then $g(t)=g(t')$ for all $g\in W_\pi$. Then, the pair $(t, t')\in\Delta_{W_\pi}$.

If $\delta(t, t')=0$ then $\delta_{t|W_\pi}=\delta_{t'|W_\pi}=0$ and by hypothesis $f\in\Delta(W_\pi)$, then we have $f(t)=0\cdot f(t')=0$.

If $\delta(t, t')=1$ then $0\neq\delta_{t|W_\pi}=\delta_{t'|W_\pi}$ and since $f\in\Delta(W_\pi)$ we have $f(t)=1\cdot f(t')=f(t')$.

Therefore, $f\in W_\pi$.

Let $f\in\mathscr{C}_0(X;\ E)$ be given. Since π is proper, $\pi^{-1}(y)$ is compact and then $f(\pi^{-1}(y))$ is compact, hence bounded in E, for each $y\in Y$. Let us define

$$\delta(f)=\sup_{y\in Y}\operatorname{rad}\big(f(\pi^{-1}(y))\big).$$

If $w\in W_\pi$ then

$$\|f-w\|=\sup_{y\in Y}\ \sup_{t\in\pi^{-1}(y)}\|f(t)-w(t)\|\geqslant\delta(f).$$

Hence

$$\delta(f)\leqslant\operatorname{dist}(f;\ W_\pi).$$

3.2. THEOREM: *Let* $(E, \|\cdot\|)$ *be a non-archimedean normed space over* $(F,|\cdot|)$ *and* $W_\pi\subset\mathscr{C}_0(X;\ E)$ *a Stone-Weierstrass subspace. Then, for all* $f\in\mathscr{C}_0(X;\ E)$

$$\operatorname{dist}(f;\ W_\pi)=\delta(f).$$

PROOF: By Theorem 2.4, $\operatorname{dist}(f;\ W_\pi)\leqslant\delta(f)$ and by remarks made before we have $\delta(f)\leqslant\operatorname{dist}(f;\ W_\pi)$.

Let us now generalize the above results for the case of Chebyshev centers. Consider then a bounded and equicontinuous subset $B\subset\mathscr{C}_0(X;\ E)$ and the associated carrier φ_B from X into E defined by

$$\varphi_B(x)=\{f(x);\ f\in B\}\qquad\text{for all } x\in X.$$

Since B is bounded, it follows that φ_B is Δ-bounded for any equivalence relation Δ on X.

For each $y\in Y$ define

$$B(\pi^{-1}(y))=\cup\{f(\pi^{-1}(y));\ f\in B\}$$

and

$$\delta(B)=\sup\{\operatorname{rad}\big(B(\pi^{-1}(y))\big);\ y\in Y\}.$$

then $\delta(B)=\delta(\varphi_B)$, and by Theorem 2.4,

$$\mathrm{rad}_{W_\pi}(B)\leqslant\delta(B)$$

because W_π is a Stone-Weierstrass subspace.

Conversely, each $w\in W_\pi$ is constant on $\pi^{-1}(y)$ for every $y\in Y$. Thus

$$\begin{aligned}\mathrm{dist}(\varphi_B;\, w) &= \sup_{y\in Y}\ \sup_{t\in\pi^{-1}(y)}\ \sup_{z\in\varphi_B(t)} \|z-w(t)\|\\ &\geqslant \sup_{y\in Y}\ \inf_{v\in E}\ \sup_{t\in\pi^{-1}(y)}\ \sup_{z\in\varphi_B(t)} \|z-v\|\\ &= \sup_{y\in Y}\ \inf_{v\in E}\ \sup_{t\in\pi^{-1}(y)}\ \sup_{f\in B} \|f(t)-v\|\\ &= \sup_{y\in Y}\mathrm{rad}\big(B(\pi^{-1}(y))\big)=\delta(B).\end{aligned}$$

Hence

$$\delta(B)\leqslant\mathrm{dist}(\varphi_B;\, W_\pi)=\mathrm{rad}_{W_\pi}(B).$$

We have thus proved the following.

3.3. THEOREM: *Let* $(E,\|\cdot\|)$ *be a non-archimedean normed space over* $(F,|\cdot|)$ *and* $W_\pi\subset\mathscr{C}_0(X;\,E)$ *a Stone-Weierstrass subspace. Then, for any bounded and equicontinuous subset* $B\subset\mathscr{C}_0(X;\,E)$, we have

$$\mathrm{rad}_{W_\pi}(B)=\sup_{y\in Y}\mathrm{rad}\big(B(\pi^{-1}(y))\big).$$

3.4. DEFINITION: Let X and Z be two topological spaces. A set valued mapping φ from X into Z is said to be *lower semicontinuous* if $\{x\in X;\ \varphi(x)\cap G\neq\emptyset\}$ is open in X for every open subset $G\subset Z$.

A continuous mapping $f\colon X\to Z$ is called a *continuous selection for* a carrier φ if $f(x)\in\varphi(x)$ for all $x\in X$.

The following result is a consequence of Michael [1], Theorem 2, page 233.

3.5. THEOREM: *Let* X *be a* 0*-dimensional compact* T_1*-space and let* $(E,\|\cdot\|)$ *be a Banach space over a non-trivially valued division ring* $(F,|\cdot|)$. *Every lower semicontinuous carrier* φ *from* X *into the non-empty, closed subsets of* E *admits a continuous selection.*

3.6. REMARK: Let X be a 0-dimensional, Hausdorff and locally compact space. The Alexandroff compactification, X_ω, of X is 0-dimensional and Hausdorff space. There is a linear isometry of $\mathscr{C}_0(X; E)$ into $\mathscr{C}(X_\omega; E)$.

Let X be a locally compact T_1-space, and π a proper continuous surjection of X onto another locally compact T_1-space Y. Let $(E, \|\cdot\|)$ be a non-archimedean normed space over $(F, |\cdot|)$. Let $B \subset \mathscr{C}_0(X; E)$ be a bounded non-empty subset which is equicontinuous and vanishes collectively at infinity. For each $x \in E$ let be given a closed vector subspace $W(x) \subset E$. Let $\delta > 0$ be given.

Let us define two set valued mappings φ_ω and ψ_ω on Y_ω and X_ω respectively, by setting for any $y \in Y$

$$\varphi_\omega(y) = \left\{ s \in E;\ \sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \| f(x) - s \| \leqslant \delta \right\}$$

and

$$\varphi_\omega(\omega) = \{0\};$$

and for any $x \in X$

$$\psi_\omega(x) = W(x) \cap \left\{ s \in E;\ \sup_{f \in B} | f(x) - s | \leqslant \delta \right\}$$

$$\psi_\omega(\omega) = \{0\}.$$

3.7. LEMMA: *Under the preceding hypothesis, the set valued mappings φ_ω and ψ_ω are lower semicontinuous on Y_ω and X_ω respectively.*

PROOF: a) Let $g \subset E$ be open such that $\varphi_\omega(y_0) \cap G \neq \emptyset$. If $y_0 \in Y$, we choose $s_0 \in \varphi_\omega(y_0) \cap G$, then

$$\sup_{f \in B} \sup_{x \in \pi^{-1}(y_0)} | f(x) - s_0 | \leqslant \delta.$$

Since $\pi^{-1}(y_0)$ is a compact subset of X, there exists a finite open covering $V_1, V_2, \ldots, V_n$ of $\pi^{-1}(y_0)$, with

$$V_i \cap \pi^{-1}(y_0) \neq \emptyset, \quad 1 \leqslant i \leqslant n,$$

such that

$$x, x' \in V_i \Rightarrow \| f(x) - f(x') \| < \delta$$

for all $f \in B$. This is possible because the set $B \subset \mathscr{C}_0(X; E)$ is equicontinuous.

Let $x \in X$. For each $z \in E$, choose $w_z \in W$ such that $w_z(t) = z$ for all $t \in \Delta(x)$. Then

$$\inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \| y - w(t) \|$$

$$\leqslant \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \| y - w_z(t) \|$$

$$= \sup_{y \in \varphi(\Delta(x))} \| y - z \|.$$

Since $z \in E$ was arbitrary, we have

$$\inf_{w \in W} \sup_{t \in \Delta(x)} \sup_{y \in \varphi(t)} \| y - w(t) \| \leqslant \inf_{z \in E} \sup_{y \in \varphi(\Delta(x))} \| y - z \|.$$

Hence,

$$\operatorname{dist}(\varphi; W) \leqslant \delta(\varphi).$$

3. Stone-Weierstrass subspaces

3.1. Definition: A vector subspace $W \subset \mathscr{C}_0(X; E)$ is said to be a *Stone-Weierstrass subspace* if there is a locally compact Hausdorff space Y and a proper continuous surjection π: $X \to Y$ such that

$$W = \{ g \circ \pi;\ g \in \mathscr{C}_0(Y; E) \}.$$

We denote by W_π the Stone-Weierstrass subspace determined by π.

If $W_\pi \subset {}_0(X; E)$ is a Stone-Weierstrass subspace, then

$$A_\pi = \{ \varphi \circ \pi;\ \varphi \in \mathscr{C}^*(X; F) \}$$

is a subalgebra of $\mathscr{C}^*(X; F)$ which contains the constants and

$$\{ \pi^{-1}(y);\ y \in Y \}$$

is the set of equivalence classes modulo A_π. Therefore, W_π is an A_π-module.

Clearly W_π is closed in $\mathscr{C}_0(X; E)$.

We will prove that this definition of Stone-Weierstrass subspace is the same as Definition 3.5, [5], by proving that $\Delta(W_\pi) \subset W_\pi$, where $\Delta(W_\pi)$ is the Stone-Weierstrass hull of W_π in $\mathscr{C}_0(X; E)$.

Let $f \in \Delta(W_\pi)$. We will prove that f is constant on the sets $\pi^{-1}(y)$ for all $y \in Y$.

Let t and t' be in X such that $\pi(t)=\pi(t')$. Then $g(t)=g(t')$ for all $g \in W_\pi$. Then, the pair $(t, t') \in \Delta_{W_\pi}$.

If $\delta(t, t')=0$ then $\delta_{t|W_\pi}=\delta_{t'|W_\pi}=0$ and by hypothesis $f \in \Delta(W_\pi)$, then we have $f(t)=0 \cdot f(t')=0$.

If $\delta(t, t')=1$ then $0 \neq \delta_{t|W_\pi}=\delta_{t'|W_\pi}$ and since $f \in \Delta(W_\pi)$ we have $f(t)=1 \cdot f(t')=f(t')$.

Therefore, $f \in W_\pi$.

Let $f \in \mathscr{C}_0(X; E)$ be given. Since π is proper, $\pi^{-1}(y)$ is compact and then $f(\pi^{-1}(y))$ is compact, hence bounded in E, for each $y \in Y$. Let us define

$$\delta(f)=\sup_{y \in Y} \operatorname{rad}\big(f(\pi^{-1}(y))\big).$$

If $w \in W_\pi$ then

$$\|f-w\| = \sup_{y \in Y} \sup_{t \in \pi^{-1}(y)} \|f(t)-w(t)\| \geqslant \delta(f).$$

Hence

$$\delta(f) \leqslant \operatorname{dist}(f; W_\pi).$$

3.2. Theorem: *Let $(E, \|\cdot\|)$ be a non-archimedean normed space over $(F, |\cdot|)$ and $W_\pi \subset \mathscr{C}_0(X; E)$ a Stone-Weierstrass subspace. Then, for all $f \in \mathscr{C}_0(X; E)$*

$$\operatorname{dist}(f; W_\pi)=\delta(f).$$

Proof: By Theorem 2.4, $\operatorname{dist}(f; W_\pi) \leqslant \delta(f)$ and by remarks made before we have $\delta(f) \leqslant \operatorname{dist}(f; W_\pi)$.

Let us now generalize the above results for the case of Chebyshev centers. Consider then a bounded and equicontinuous subset $B \subset \mathscr{C}_0(X; E)$ and the associated carrier φ_B from X into E defined by

$$\varphi_B(x)=\{f(x);\, f \in B\} \qquad \text{for all } x \in X.$$

Since B is bounded, it follows that φ_B is Δ-bounded for any equivalence relation Δ on X.

For each $y \in Y$ define

$$B(\pi^{-1}(y))=\cup\{f(\pi^{-1}(y));\, f \in B\}$$

and

$$\delta(B)=\sup\{\operatorname{rad}(B(\pi^{-1}(y)));\, y \in Y\}.$$

We claim that $\text{rad}(K) \leqslant \delta$. Indeed, let $g \in W_\pi$ be given. Then

$$\text{rad}(K) = \inf_{z \in E} \sup_{x \in \pi^{-1}(y)} \| f(x) - z \|$$

$$\leqslant \sup_{x \in \pi^{-1}(y)} \| f(x) - g(x) \| \leqslant \| f - g \|.$$

Since g was arbitrary,

$$\text{rad}(K) \leqslant \inf_{g \in W_\pi} \| f - g \|.$$

It follows that $s_0 \in \varphi_\omega(y)$ and hence $\varphi_\omega(y) \neq \emptyset$ for all $y \in Y$.

By Lemma 3.7 applied to $B = \{f\}$, φ_ω is lower semicontinuous.

By Theorem 3.5, there is $g_\omega \in \mathscr{C}(Y_\omega; E)$ with $g_\omega(y) \in \varphi_\omega(y)$ for all $y \in Y_\omega$, furthermore $g_\omega(\omega) = 0$. Let $g \in \mathscr{C}_0(X; E)$ be the restriction of g_ω to Y. Then $g(y) \in \varphi(y)$ for all $y \in Y$. Let $w = g \circ \pi$. Then $w \in W_\pi$ and, for any $x \in X$ let $y = \pi(x)$. Then

$$\| f(x) - w(x) \| = \| f(x) - g(y) \| \leqslant \delta.$$

Hence

$$\| f - w \| \leqslant \text{dist}(f; W_\pi).$$

This ends the proof that W_π is proximinal in $\mathscr{C}_0(X; E)$.

3.9. THEOREM: *Let X be a 0-dimensional, locally compact T_1-space. Let $(E, \|\cdot\|)$ be a non-archimedean Banach space over $(F, |\cdot|)$. If E admits Chebyshev centers, and $W_\pi \subset \mathscr{C}_0(X; E)$ is a Stone-Weierstrass subspace, then* $\text{cent}_{W_\pi}(B) \neq \emptyset$ *for every non-empty bounded subset $B \subset \mathscr{C}_0(X; E)$ which is equicontinuous and vanishes collectively at infinity.*

PROOF: Let π: $X \to Y$ be the continuous and proper mapping of X onto a locally compact Hausdorff space Y such that

$$W_\pi = \{ g \circ \pi;\ g \in \mathscr{C}_0(Y; E) \}.$$

Let $B \subset \mathscr{C}_0(X; E)$ be a non-empty bounded subset which is equicontinuous.

Let $\delta = \text{rad}_{W_\pi}(B)$:

CASE I: $\delta > 0$. Consider $Y_\omega = Y \cup \{\omega\}$ the compactification of Alexandroff of Y.

For each $y \in Y$, let

$$\varphi_\omega(y) = \left\{ s \in E;\ \sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \| f(x) - s \| \leqslant \delta \right\}$$

and

$$\varphi_\omega(\omega) = \{0\}.$$

Let us prove that φ_ω is a carrier from Y_ω into the non-empty closed subsets of E. Let $y \in Y_\omega$ be given. If $y = \omega$ then $\varphi_\omega(y) = \{0\}$ and hence $\varphi_\omega(y)$ is non-empty and closed. If $y \in Y$ then $\varphi_\omega(y)$ is closed in E. Since $B \subset \mathscr{C}_0(X;\ E)$ is bounded,

$$B(y) = \{ f(x);\ x \in \pi^{-1}(y),\ f \in B \}$$

is bounded in E, and by hypothesis cent$(B(y)) \neq \emptyset$, i.e., there exists $s_0 \in E$ such that

$$\sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \| f(x) - s_0 \| = \operatorname{rad}(B(y)).$$

To each $g \in W_\pi$, we have

$$\operatorname{rad}(B(y)) \leqslant \sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \| f(x) - g(x) \|$$

because g is constant on $\pi^{-1}(y)$. Hence

$$\begin{aligned} \operatorname{rad}(B(y)) &\leqslant \sup_{y \in Y} \sup_{f \in B} \sup_{x \in \pi^{-1}(y)} \| f(x) - g(x) \| \\ &= \sup_{f \in B} \sup_{y \in Y} \sup_{x \in \pi^{-1}(y)} \| f(x) - g(x) \| \\ &= \sup_{f \in B} \| f - g \|. \end{aligned}$$

Since g was arbitrary,

$$\operatorname{rad}(B(y)) \leqslant \inf_{g \in W_\pi} \sup_{f \in B} \| f - g \| = \operatorname{rad}_{W_\pi}(B) = \delta.$$

Therefore, $s_0 \in \varphi_\omega(y)$ and $\varphi_\omega(y)$ is non-empty.

By Lemma 3.7, φ_ω is lower semicontinuous.

By Theorem 3.5, there is $g_\omega \in \mathscr{C}(Y_\omega;\ E)$ with $g_\omega(y) \in \varphi_\omega(y)$ for all $y \in Y_\omega$. Notice that $g_\omega(\omega) = 0$. Let $g \in \mathscr{C}_0(X;\ E)$ be the restriction of g_ω

to Y. Then $g(y) \in \varphi_\omega(y)$ for all $y \in Y$. Let $w = g \circ \pi$. Then $w \in W_\pi$ and for any $x \in X$, let $y = \pi(x)$. Then for any $f \in B$ we have

$$\| f(x) - w(x) \| = \| f(x) - g(y) \| \leqslant \sup_{t \in \pi^{-1}(y)} \| f(t) - g(y) \| \leqslant \delta.$$

Hence

$$\sup_{f \in B} \| f - w \| \leqslant \delta, \qquad \text{and so } w \in \mathrm{cent}_{W_\pi}(B).$$

CASE II: $\delta = 0$.

Now $\mathrm{rad}_{W_\pi}(B) = 0$ implies $B = \{ f \}$ and $\mathrm{dist}(f;\ W_\pi) = \mathrm{rad}_{W_\pi}(B) = 0$. therefore $f \in W_\pi$ and there is nothing to prove.

3.10. THEOREM: *Let X be a 0-dimensional, locally compact T_1-space. Let $(E, \| \cdot \|)$ be a non-archimedean Banach space over $(F, | \cdot |)$. Let $A \subset \mathscr{C}^*(X;\ F)$ be a separating subalgebra and let $W \subset \mathscr{C}_0(X;\ E)$ be a closed vector subspace which is an A-module such that $W(x)$ is proximinal in E for every $x \in X$. Then, W is proximinal in $\mathscr{C}_0(X;\ E)$.*

PROOF: Let $f \in \mathscr{C}_0(X;\ E)$ be given with $f \notin W$. Then

$$\delta = \mathrm{dist}(f;\ W) > 0,$$

because W is closed. Consider $X_\omega = X \cup \{ \omega \}$ the compactification of Alexandroff of X. For each $x \in X$, let

$$\psi_\omega(x) = W(x) \cap \{ s \in E;\ \| f(x) - s \| < \delta \}$$

and

$$\psi_\omega(\omega) = \{0\}.$$

Let us prove that ψ_ω is a carrier from X_ω into the non-empty closed subset of E. Indeed, let $x \in X_\omega$. If $x = \omega$ then $\psi_\omega(x) = \{0\}$ and then $\psi_\omega(x)$ is non-empty and closed. If $x \in X$, there exists $w \in W$ such that

$$\| w(x) - f(x) \| \leqslant \mathrm{dist}(f(x);\ W(x)) \leqslant \delta$$

and hence $\psi_\omega(x) \neq \emptyset$ and closed since $W(x)$ is proximinal.

By Lemma 3.7 applied with $B = \{ f \}$, ψ_ω is lower semicontinuous.

By Theorem 3.5, there exists $g_\omega \in \mathscr{C}(X_\omega;\ E)$ such that $g_\omega(x) \in \psi_\omega(x)$ for all $x \in X_\omega$, furthermore $g_\omega(\omega) = 0$.

Let $g \in \mathscr{C}_0(X;\ E)$ be the restriction of g_ω to X. Hence $g(x) \in W(x)$. By Theorem 2.5 [5], $g \in \overline{W}$. Since W is closed, $g \in W$. On the other hand

$$\| f(x) - g(x) \| \leqslant \delta = \mathrm{dist}(f;\ W)$$

for all $x \in X$, and therefore

$$\| f - g \| \leqslant \operatorname{dist}(f; W),$$

i.e., W is proximinal in $\mathscr{C}_0(X; E)$.

3.11. THEOREM: *Let X and E as Theorem 3.10. Let $A \subset \mathscr{C}^*(X; F)$ be a separating subalgebra and let $W \subset \mathscr{C}_0(X; E)$ be a closed vector subspace which is an A-module and such that $W(x)$ has the relative Chebyshev center property in E, for every $x \in X$. Then*

$$\operatorname{cent}_W(B) \neq \emptyset,$$

for every non-empty equicontinuous and bounded $B \subset \mathscr{C}_0(X; E)$ which vanishes collectively at infinity.

PROOF: Let $B \subset \mathscr{C}_0(X; E)$ be a non-empty bounded subset which is equicontinuous at every point of X and vanishes at infinity. Let $\delta = \operatorname{rad}_W(B)$. If $\delta = 0$, then B is a singleton $\{f\}$ with $f \in W$ and there is nothing to prove. We may assume that $\delta > 0$.

Let X_ω be the compactification of Alexandroff of X. To each $x \in X$,

$$\psi_\omega(x) = W(x) \cap \left\{ s \in E;\ \sup_{f \in B} \| f(x) - s \| \leqslant \delta \right\}$$

and

$$\psi_\omega(\omega) = \{0\}.$$

We will prove that ψ_ω is a carrier from X_ω into the nonempty closed subsets of E. Indeed. Let $x \in X_\omega$. If $x = \omega$ then $\psi_\omega(x) = \{0\} \neq \emptyset$ and $\psi_\omega(x)$ is closed in E. If $x \neq \omega$, we define $B(x) = \{f(x);\ f \in B\}$, then $B(x)$ is bounded in E and by hypothesis there is some $w \in W$ such that

$$\sup_{f \in B} \| f(x) - w(x) \| \leqslant \operatorname{rad}_{W(x)}(B(x)).$$

Now

$$\operatorname{rad}_{W(x)}(B(x)) = \inf_{w \in W} \sup_{f \in B} \| f(x) - w(x) \|$$

$$\leqslant \inf_{w \in W} \sup_{f \in B} \| f - w \| = \delta.$$

Hence $\psi_\omega(x) \neq \emptyset$. Clearly, $\psi_\omega(x)$ is closed.

By Lemma 3.7, ψ_ω is lower semicontinuous.

By Theorem 3.5, there exists $g_\omega \in \mathscr{C}(X_\omega; E)$ such that $g_\omega(x) \in \psi_\omega(x)$ an $g_\omega(\omega) = 0$.

Let $g \in \mathscr{C}_0(X; E)$ be the restriction of g_ω to X. Hence $g(x) \in W(x)$ for all $x \in X$. By Theorem 2.5 [5], $g \in \overline{W}$. Since W is closed, $g \in W$. On the other hand,

$$\sup_{f \in B} \| f(x) - g(x) \| \leqslant \delta$$

for all $x \in X$, and hence

$$\sup_{f \in B} \| f - g \| \leqslant \delta = \mathrm{rad}_W(B) \quad \text{and} \quad g \in \mathrm{cent}_W(B).$$

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