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# THE ONE-MOTIF OF AN ALGEBRAIC SURFACE

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## 1. Introduction

One of the striking aspects of the theory of algebraic curves is that the Hodge structure on the cohomology has geometric meaning: Abel's theorem gives a natural isomorphism between the cohomologically defined Albanese variety and the geometrically defined Picard variety. The purpose of this paper is to discuss a partial analogue of Abel's theorem for sufficiently singular algebraic surfaces.

To place the problem in its proper context, recall that a complex one-motif [D, section 10] consists of:

(1) a compact complex torus $A$ which can be made into an abelian variety

(2) a complex multiplicative group $G$ (a group isomorphic to $(\mathbb{C}^*)^m$).

(3) an extension $J$ of $A$ by $G$:

$$1 \to G \to J \to A \to 0$$

(4) a lattice $L$ (a group isomorphic to $\mathbb{Z}^m$)

(5) a homomorphism

$$u\colon L \to J.$$

A morphism of one motifs $u \to u'$ is given by a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{u} & J \\ f_L \downarrow & & \downarrow f_J \\ L' & \xrightarrow[u']{} & J' \end{array}$$

where $f_J$ preserves the canonical extension. If both $f_L$ and $f_J$ are isomorphisms, then $u$ isomorphic to $u'$. A typical example of such an object is furnished by a stable curve $X$ with a distinguished finite point set $M \subset X_{\text{reg}}$. If one makes the definitions

$Z_M$ = {divisors of degree zero supported in M},
$P$ = generalized Picard of $X$,

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then the map

$$C\colon Z_M \to P$$

which sends a divisor to its linear equivalence class is a one-motif.

According to Deligne [D, section 10] the category of one-motifs is isomorphic to the category of torsion-free mixed Hodge structures of type $\{(p, p), (p, p-1), (p-1, p), (p-1, p-1)\}$. For a projective variety $X$ there is a unique largest such substructure of $H^2(X)$ with $p = 1$, and hence a unique "largest" one-motif, which we denote $\eta_X$ (the one-motif of Hodge). Our main result is that for an algebraic surface, $\eta_X$ is isomorphic to a geometrically defined object, the trace homomorphism $\tau_X$.

The general definition of the trace, which we shall give in section 4, requires the theory of semisimplicial resolutions. In simple cases, however, it is possible to give simple answers. Consider therefore a surface $X$ which consists of two smooth components $A$ and $B$ meeting transversely in a curve $C$. Assume further that $A$ and $B$ are simply connected. Let

$$\tilde{X} = A \amalg B$$

be the normalization, and let

$$Z = Z_A + Z_B$$

be a divisor on $X$ which meets each copy of $C$ properly. Define the trace of $Z$ on $C$ to be the zero-cycle obtained by difference of intersections:

$$\tau(Z) = Z_A \cdot C - Z_B \cdot C.$$

By analogy with the smooth case, define the Neron-Severi group of $X$ to be

$$\mathrm{NS}(X) = \frac{\{\text{divisors } Z \text{ on } \tilde{X} \mid \tau(X) \text{ is homologous to zero}\}}{\text{linear equivalence on } \tilde{X}}.$$

Then the trace induces a homomorphism

$$\tau_X\colon \mathrm{NS}(X) \to \mathrm{Pic}^0(C).$$

THEOREM A [C2]: *There is a natural isomorphism between the trace and the one-motif of* $H^2(X)$,

$$\eta_X \cong \tau_X,$$

*modulo the torsion of* $\mathrm{NS}(X)$.

The main result of the present paper is that Theorem A holds for any surface, once the trace has been properly defined.

We remark that when the second cohomology of $X$ is entirely one-motivic, as it is for $K3$ degenerations, that the geometric consequences are particularly strong. Typical is the following:

THEOREM G[C2]: *Let $X$ be the union of a smooth cubic surface and a plane, tangent in at most one point. Then $X$ is determined up to isomorphism by the polarized mixed Hodge structure on the primitive cohomology of $H^2(X)$.*

Although the proof of the theorem given in [C2] depends on analytic methods (integrals and currents), the proof given here relies entirely on the theory of sheaves and complexes of sheaves on a semisimplicial space [D]. This theory is reviewed briefly in sections 3, 6, and 8, as progressively larger segments of it are needed. Applications of the main theorem are given in sections 12, 13, and 14. They may be read independently of the proof of the main theorem, which is given in sections 6 through 11. We remark that as a dividend of our sheaf-theoretic method of proof, we obtain a second interpretation of the motif $\eta_X$: it measures, for normal crossing surfaces, the obstruction that a Weil divisor faces in becoming a Cartier divisor.

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## 2. One-motifs and mixed Hodge structures

In this section we recall a construction of the one-motif associated to a torsion-free mixed Hodge structure of level one [C1]. Since the level is the lenth of the Hodge filtration, a level one structure is of type $\{(p, p), (p, p-1), (p-1, p), (p-1, p-1)\}$ for a unique $p$. To begin the construction we define the Griffiths Jacobian [GP] of a mixed Hodge structure by

$$J^pH = W_{2p-1,\mathbf{C}}/\big(F^p \cap W_{2p-1,\mathbf{C}} + W_{2p-1,\mathbf{Z}}\big).$$

These Jacobians admit a canonical filtration by weight:

$$\begin{aligned} W_lJ^pH &= \frac{W_{l,\mathbf{C}} + F^p \cap W_{2p-1,\mathbf{C}} + W_{2p-1,\mathbf{Z}}}{\big(F^p \cap W_{2p-1,\mathbf{C}} + W_{2p-1,\mathbf{Z}}\big)} \\ &\cong W_{l,\mathbf{C}}/\big(F^p \cap W_{l,\mathbf{C}} + W_{l,\mathbf{Z}}\big) \\ &= J^p(W_lH) \end{aligned}$$

One verifies that

$$Gr_l^W(J^pH) = J^p\left(Gr_l^W H\right).$$

The weight filtration then induces a canonical filtration

$$0 \to J^p(W_{2p-2}H) \to J^p(W_{2p-1}H) \to J^p\left(Gr_{2p-1}^W H\right) \to 0,$$

where the right-hand term is compact. If $W_{2p-1}H$ has level one, then $J^pH$ is a semi-abelian variety: the right-hand term is polarizable and the left-hand term is of multiplicative type:

$$J^p(W_{2p-2}H) = \frac{W_{2p-2,\mathbf{C}}}{W_{2p-2,\mathbf{Z}}} \cong (\mathbf{C}^*)^m.$$

Define next a finitely generated abelian group

$$L_H^p = \left(Gr_{2p}^W\right)_{\mathbf{Z}}^{p,p},$$

When this group is torsion-free there is a canonical homomorphism

$$\eta^p\colon L^pH \to J^pH.$$

To construct the canonical homomorphism, consider the extension of mixed Hodge structures

$$0 \to W_{2p-1}H \to W_{2p}H \xrightarrow{\pi} Gr_{2p}^W H \to 0,$$

and choose sections $s_{\mathbf{Z}}$ and $s_F$ of $\pi$ over $L^pH$, where the first preserves the integer lattice and where the second preserves the Hodge filtration. Let $\psi = s_{\mathbf{Z}} - s_F$ be the difference homomorphism, and set

$$\eta^p(x) = \psi(x) + F^p \cap W_{2p-1,\mathbf{C}} + W_{2p-1,\mathbf{Z}}$$

for all $x$ in $L^pH$. The resulting map from $L^pH$ to $J^pH$ is a homomorphism independent of the choice of sections which gives the obstruction to representing $x$ by an integral element of $W_{2p}$ which also lies in $F^p$. When $W_{2p-1}$ has level one, $\eta^p$ carries the additional structure of a one-motif.

## 3. Semisimplicial spaces (geometry)

A *semisimplicial space* $X.$ is a sequence of spaces (simplices) $X_0, \ldots, X_n$ connected by morphisms (face operators)

$$\epsilon_i;\ X_p \to X_{p-1} \quad (i = 0, \ldots, p)$$

which satisfy the commutation relations

$$\delta_j \delta_i = \delta_i \delta_{j+1} \quad (i \leqslant j).$$

REMARK: We recall from [D], section 5, that a simplicial space has *both* face and degeneracy operators. Thus, a *semi*simplicial space is "half" of a simplicial space.

An *augmentation* of $X.$ toward $Y$ is given by a map

$$\epsilon \colon X_0 \to Y$$

such that $\epsilon \circ \delta_0 = \epsilon \circ \delta_1$. If all simplices and face maps of $X.$ are smooth (algebraic, etc.), we say that $X.$ is smooth (algebraic, etc.).

To every simplicial space is functorially associated a geometric realization, which is an ordinary topological space. To define it, let $\Delta_p$ be the standard geometric $p$-simplex, and define the realization by

$$|X.| = \left(\coprod X_p \times \Delta_p\right)/R$$

where $R$ is the relation generated by

$$\left(\delta_i x_p,\, t_{p-1}\right) \sim \left(x_p,\, \delta_i t_{p-1}\right),$$

where

$$\delta_i \colon \Delta_{p-1} \to \Delta_p \quad (i = 0, \dots, p)$$

is the linear inclusion of $\Delta_{p-1}$ in the $i$-th face of $\Delta_p$. Because the construction is functorial, a morphism

$$f. \colon X. \to X'.$$

induces a continuous map

$$|f.| \colon |X.| \to |X'.|$$

We say that $f$ is a *homotopy equivalence* of semisimplicial spaces if $|f.|$ is an ordinary homotopy equivalence of topological spaces.

If $X.$ is augmented towards $Y$, then there is an induced map

$$|\epsilon| \colon |X.| \to Y.$$

This map is a *resolution* of $Y$ when $|\epsilon|$ is a homotopy equivalence. Note that $X.$ resolves $Y$ if $|\epsilon|$ has contractible fibers, and note that

$$|\epsilon|^{-1}(y) = |\epsilon^{-1}(y)|.$$

EXAMPLE 1: Let

$$Y = \cup D_i$$

be a normal crossing variety, and let

$$X_p = \coprod D_{i_0} \cap \dots \cap D_{i_p} \quad (i_0 < \dots < i_p),$$

so that $X.$ is the nerve of the covering $\{D_i\}$. Let

$$Y^{(p)} = \{ y \in Y \mid y \text{ has multiplicity } p \}.$$

Since

$$|\epsilon|^{-1}(y) \cong \Delta_p$$

for all $y$ in $Y^{(p)}$, the given semisimplicial space resolves $Y$. The corresponding simplicial space is given by the disjoint union with $i_0 \leqslant \dots \leqslant i_p$. It therefore contains degenerate simplices, e.g. $D_k \cap D_k \cap \dots \cap D_k$.

EXAMPLE 2: Let $Y$ be a variety with singular locus $E$. Let $\tilde{Y}$ be a desingularization, and let $\tilde{E}$ be the pullback of $E$ to $Y$. Then the Mayer-Vietoris diagram

$$\left.\begin{array}{ccc} & \tilde{E} & \\ \swarrow & & \searrow \\ E & & \tilde{Y} \end{array}\right\} X. \qquad \begin{array}{ccc} \searrow & & \swarrow_{\pi} \\ & Y & \end{array}$$

defines a semisimplicial resolution of $Y$. If $y$ is a smooth point, then the fiber of $\epsilon$ over $y$ is a point. If $y$ is a singular point, then the fiber of $|\epsilon|$ over $y$ is a cone over $\pi^{-1}(y)$. Unlike the first example, this resolution is not necessarily smooth.

In the discussion below, it is useful to define a restricted class of semisimplicial spaces. First, we say that $X.$ has *natural dimension n* if

$$\dim X_p = n - p.$$

here dimension is taken as the maximum of the dimensions of the components. Second, we say that $X.$ is *nondegenerate* if for all $p$

(a) $\delta_i(X_p) \cap \delta_j(X_p) = \emptyset$ for $i \neq j$
(b) At least one $\delta_i$ is injective.

Finally, we say that $X.$ is *distinguished* if it is both nondegenerate and of natural dimension $n$. Both of the examples considered above are distinguished, and one can in fact show that every variety has distinguished resolutions [C3].

The technique is, roughly said, to resolve the singularities of the Mayer-Vietoris diagram, then to take the mapping cone of the resolution in the category of polyhedral spaces. Each time this basic step is applied, the dimension of the singular locus of the resolution drops by one. Repetition of the basic step at most $n$ times, where $n$ is the dimension of $Y$, results in a smooth resolution in the category of polyhedral objects.

## 4. The trace

In this section we shall define a functorial trace homomorphism

$$\tau \colon \mathrm{NS}(X_.) \to P(X_.)$$

for an arbitrary semisimplicial manifold. When this manifold is the Mayer-Vietoris diagram

$$X_. = \begin{bmatrix} & C & \\ \swarrow & & \searrow \\ A & & B \end{bmatrix},$$

the construction agrees with that of the introduction. The proper generalization of Theorem A is therefore an assertion about semisimplicial spaces:

THEOREM A: *Let $X_.$ be a smooth projective semisimplicial space, let $\tau(X_.)$ be the trace, and let $\eta(X_.)$ be the one-motif associated to $H^2(X_.)$. Then there is a natural isomorphism between $\tau(X_.)$ and $\eta(X_.)$, modulo torsion in* NS($X_.$).

If $f$: $X_. \to X'_.$ is a morphism, one has a commutative diagram relating $\tau$ and $\eta$ on the two spaces:

$$\begin{array}{ccc} \tau(X'_.) & \to & \tau(X_.) \\ \downarrow & & \downarrow \\ \eta(X'_.) & \to & \eta(X_.). \end{array}$$

If in addition $f$ is a homotopy equivalence, then $\eta(X'_.)$ and $\eta(X_.)$ are isomorphic, so that $\tau(X'_.)$ and $\tau(X_.)$ are isomorphic modulo the torsion part of $NS(X'_.)$. Since the trace motif modulo torsion is a homotopy invariant, we are allowed to define the trace motif of a surface $Y$ to be the trace motif of any semisimplicial resolution. Furthermore, we are allowed, if we so desire, to assume that the resolution used for $Y$ is distinguished.

Let us begin with the construction of the torus $P(X.)$, which must specialize to $\mathrm{Pic}^0(C)$ in the Mayer-Vietoris case. Define the group of divisors in good position on $X_p$ by

$$\mathrm{Div}_{\mathrm{g}}(X_p) = \{\text{divisors } Z \text{ on } \mathrm{X_p} \mid \delta^{\mathrm{i}}\mathrm{Z} \textit{ is defined on } X_{p+1} \textit{ for all } i\},$$

where $\delta^i$ is the map determined by pullback of local defining equations. Thus we have homomorphisms

$$\delta^i\colon \mathrm{Div}_g(X_p) \to \mathrm{Div}_{\mathrm{g}}(X_{p+1})$$

$$\delta^* = \Sigma(-1)^i\delta^i.$$

Define a decreasing chain of groups of divisors on $X_p$ by

$$\mathrm{Div}_\delta(X_p) = \{Z \in \mathrm{Div}_{\mathrm{g}}(X_p) \mid \delta^* Z = 0 \text{ in homology}\}$$

$$\mathrm{Div}_h(X_p) = \{Z \in \mathrm{Div}_{\mathrm{g}}(X_p) \mid Z = 0 \text{ in homology}\}$$

$$\mathrm{Div}_a(X_p) = \{Z \in \mathrm{Div}_{\mathrm{g}}(X_p) \mid Z \text{ is algebraically equivalent to zero}\}.$$

Define also groups of meromorphic functions:

$$M^*(X_p) = \left\{\begin{array}{l}\text{meromorphic functions } f \text{ on } X_p \text{ not}\\ \text{identically zero or } \infty \text{ on any}\\ \text{component and with divisor (f)} \in \mathrm{Div_g(X_p)}\end{array}\right\}$$

$$\mathbb{C}^*(X_p) = \left\{\begin{array}{l}\text{locally constant nowhere vanishing functions}\\ \text{on } \mathrm{X_p}\end{array}\right\}$$

$$M_\delta(X_p) = \{f \in M^*(X_p) \mid \delta^* f \in \delta^*\mathbb{C}^*(X_p)\}.$$

In the last definition,

$$\delta^* f = \Pi(f \circ \delta_j)^{\epsilon_j}$$

$$\epsilon_j = \begin{cases} +1 & \text{if } j \text{ is even} \\ -1 & \text{if } j \text{ is odd} \end{cases}.$$

When $X_0$ has zero first Betti number, the generalized torus which we seek is then

$$\tilde{P}(X.) = \mathrm{Div}_h(X_1)/cM_\delta^*(X_1),$$

where $c$ is the function-to-divisor map. There is a natural subgroup

$$\tilde{P}_m(X.) = cM^*(X_1)/cM^*_\delta(X_1)$$

with compact quotient group

$$\tilde{P}_c(X.) = \mathrm{Div}_h(X_1)/cM^*(X_1) = \mathrm{Pic}^0(X_1).$$

PROPOSITION B: *If X. is a distinguished semisimplicial surface, then there is a natural isomorphism*

$$\tilde{P}_m(X.) \cong \frac{\mathbb{C}^*(X_2)}{\delta^*\mathbb{C}^*(X_1)}.$$

PROOF: Because $X.$ is a distinguished semisimplicial surface, dim $X_2 = 0$, and so $M^*(X_2) = \mathbb{C}^*(X_2)$. Because of the nondegeneracy assumption, the map

$$\delta^*\colon M^*(X_1) \to M^*(X_2)$$

is surjective. But then $\delta^*$ induces the required isomorphism, since the image of $M^*_\delta(X_1)$ is, by definition, $\delta^*\mathbb{C}^*(X_1)$.

COROLLARY C: *If X. is a distinguished semisimplicial surface, then* $\tilde{P}_m(X.)$ *is a multiplicative torus.*

For the general case in which $X_1$ has possibly nonzero first Betti number, we define

$$P(X.) = \tilde{P}(X.)/\delta^*\mathrm{Pic}^0(X_0),$$

where $\mathrm{Pic}^0(X_0)$ is the group of divisors on $X_0$ which are homologous to zero modulo those which are linearly equivalent to zero. Since $\delta^*\mathrm{Pic}^0(X_0)$ is compact and connected, $P_m(X.)$ and $\delta^*\mathrm{Pic}^0(X_0)$ intersect only in the identity element. Thus we may define

$$P_{\mathrm{m}}(X.) = \tilde{P}_m(X.)$$

$$P_{\mathrm{c}}(X.) = \tilde{P}_c(X.)/\delta^*\mathrm{Pic}^0(X_0)$$

to obtain a canonical extension

$$1 \to P_{\mathrm{m}} \to P \to P_{\mathrm{c}} \to 0,$$

where $P_{\mathrm{m}}$ is of multiplicative type, where $P_{\mathrm{c}}$ is a polarizable abelian variety.

Finally we define the Neron-Severi group of $X.$ by

$$NS(X.) = \frac{\mathrm{Div}_\delta(X_0)}{\mathrm{Div}_a(X_0)},$$

where $\mathrm{Div}_\delta$ and $\mathrm{Div}_a$ are as above, and we define the trace

$$\tau\colon NS(X.) \to P(X.)$$

to be the homomorphism induced by $\delta^*$.

## 5. The multiplicative group $P_m$

The multiplicative group $P_m$ has another interpretation, which we mention for its usefulness in the applications. Define a lattice (the "triple-point lattice")

$$T(X.) = \mathrm{kernel}\{\delta_*\colon H_0(X_2, \mathbb{Z}) \to H_0(X_1, \mathbb{Z})\},$$

and observe that the standard spectral sequence for the homology of a semisimplicial space defines a natural isomorphism

$$Gr_0^W H_2(X, \mathbb{Z}) \to T(X.).$$

Elements of this lattice are given by special zero-cycles

$$\zeta = \sum_{y \in X_2} n_y y.$$

Dual to $T(X.)$ is the multiplicative group

$$\hat{T}(X.) = \mathrm{Hom}(T(X.), \mathbb{C}^*).$$

We claim that there is a natural isomorphism

$$P_m(X.) \to \hat{T}(X.).$$

To construct it, consider a variety $Y$, a meromorphic function $f$, an a zero-cycle $\xi$ whose support is disjoint from the support of $(f)$. Define a partial pairing by

$$\langle f, \xi\rangle_Y = \prod f(y)^{n_y}$$

where $n_y$ is the multiplicity of $y$ in $\xi$. If $f$ is constant on components and $\xi$ has degree zero on each component, then $\langle f, \xi\rangle = 1$. Now consider the pairing

$$M_\delta^*(X_1) \otimes T(X.) \to \mathbb{C}^*$$

given by

$$\langle f, \xi \rangle = \langle \delta^* f, \xi \rangle_{X_2}.$$

Because

$$\langle \delta^* f, \xi \rangle_{X_2} = \langle f, \delta_* \xi \rangle_{X_1},$$

$M_\delta^*(X_1)$ is annihilated by $T(X.)$. Indeed, if $\delta^* f = \delta^* g$ with $g \in \mathbb{C}^*(X_1)$, then

$$\langle \delta^* f, \xi \rangle_{X_2} = \langle \delta^* g, \xi \rangle_{X_2} = \langle g, \delta_* \xi \rangle_{X_1} = 1.$$

The natural pairing between functions on $X_1$ and zero-cycles on $X_2$ therefore descends to a pairing

$$P_{\mathrm{m}}(X.) \otimes T(X.) \to \mathbb{C}^*,$$

and this pairing induces the required map.

That this map is an isomorphism follows by composition of the following sequence of isomorphisms

$$\begin{aligned} \frac{cM^*(X_1)}{cM_\delta^*(X_1)} &\cong \frac{\mathbb{C}^*(X_2)}{\delta^*\mathbb{C}^*(X_1)} \\ &\cong Gr_0^W H^2(X_2, \mathbb{C}^*) \ (\text{spectral sequence for } H^{\cdot}(X.)) \\ &\cong \operatorname{Hom}\big(Gr_0^W H_2(X_2, \mathbb{Z}), \mathbb{C}^*\big) \\ &\cong \operatorname{Hom}(T(X.), \mathbb{C}^*). \end{aligned}$$

## 6. Semisimplicial Spaces (Cohomology)

To compare the trace homomorphism with the Hodge motif, we shall use the cohomology theory of sheaves on a semisimplicial space. In this section we review the rudiments of the theory, which is essentially the same as for simplicial spaces [D, section 5].

To begin, recall that a semisimplicial object in the category $K_A$ of complexes of $A$-modules is by definition a sequence of complexes $(K^p, d_p)$ connected by morphisms

$$\delta^i\colon K^p \to K^{p+1} \quad (i = 0, \ldots, p)$$

which satisfy

$$\delta^i \delta^j = \delta^{j+1} \delta^i \quad (i \leqslant j). \qquad (*)$$

Define

$$\delta^* = \Sigma(-1)^i\delta^i$$

and note that $\delta^* \circ \delta^* = 0$. Denote the $q$-th graded piece of $K^p$ by $(K^p)^q$, and define the shifted complex $K^p[r]$ by

$$(K^p[r])^s = (K^p)^{r+s}.$$

Set

$$sK = \oplus K^p[p]$$

and define

$$D = \bigoplus_p (-1)^p d_p + \delta^*.$$

The object $(sK, D)$ is the single complex associated to the simplicial object $K^{\cdot}$. The simplicial structure defines a decreasing filtration

$$W^l sK = \bigoplus_{p \geqslant l} K^p[p]$$

and a spectral sequence

$$E_1^{p,q} = H^q(K^p) \Rightarrow H^{p+q}(sK).$$

On $H^n(sK)$ there is a corresponding increasing filtration

$$W_l H^n(sK) = W^{n-l} H^n(sK)$$

for which

$$Gr_l^W H^n(sK) \text{ is a subquotient of } H^l(K^{n-l}).$$

This construction suffices to define the singular cohomology of a semisimplicial space: If $C^{\cdot}$ is the singular cochain functor, then the object

$$C^{\cdot}(X_{\cdot}) = [C^{\cdot}(X_0) \rightrightarrows C^{\cdot}(X_1) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots]$$

is a simplicial object in $K_A$, so that one defines

$$H^{\cdot}_{\text{sing}}(X_{\cdot}, A) = H^{\cdot}(sC^{\cdot}(X_{\cdot})).$$

In the same way one defines the homology of $X_{\cdot}$, and one obtains the analogues of the usual universal coefficient theorems. In either case, the cohomology comes equipped with the simplicial filtration, where

$$Gr_l^W H^n(X_{\cdot}) \text{ is a subquotient of } H^l(X_{n-l}).$$

If $X.$ is augmented towards $Y$, there is an induced map

$$\epsilon^*: H_{\text{sing}}(Y, \mathbb{Z}) \to H_{\text{sing}}(X., \mathbb{Z})$$

which is an isomorphism whenever $\epsilon$ is a resolution.

When $X.$ is smooth, the same construction applies with $C^{\cdot}$ replaced by $A^{\cdot}$, the de Rham functor. Thus de Rham cohomology is defined by

$$H^{\cdot}_{DR}(X.) = H^{\cdot}(sA^{\cdot}(X.)).$$

If $X.$ is a semisimplicial object in the category of complex manifolds, then each complex $A^{\cdot}(X_p)$ carries a Hodge filtration $F^{\cdot}$. As a result, there are induced filtrations on both the de Rham complex and the de Rham cohomology of $X.$. If in addition $X.$ is compact and Kähler, the simplicial (weight) spectral sequence degenerates at $E_2$ and the Hodge filtration induces a Hodge structure of weight $l$ on $Gr_l^W H^n_{DR}$. Consequently, the complex and semisimplicial structures jointly define a mixed Hodge structure [D, sections 8.12 through 8.15].

A *sheaf* on a semisimplicial space consists of a sheaf $S^p$ on each $X_p$ and of morphisms

$$\delta^i: S^p \to S^{p+1}$$

satisfying the commutation relations (*). Note that a semisimplicial sheaf is filtered by weight: the objects

$$W^l S^{\cdot} = \bigoplus_{p \geq l} S^p$$

define subsheaves, and the graded quotients

$$Gr^l_W S^{\cdot} = S^p$$

are ordinary sheaves. Typically examples are the sheaves $\mathbb{Z}^{\cdot}$, $\mathcal{O}^{\cdot}$, and $\mathcal{O}^{*\cdot}$ which fit together to give an exponential sheaf sequence for semisimplicial spaces:

$$0 \to \mathbb{Z}^{\cdot} \to \mathcal{O}^{\cdot} \overset{\exp 2\pi i}{\to} \mathcal{O}^{*\cdot} \to 0.$$

Sheaf cohomology in the semisimplicial setting is defined by choosing $\Gamma$-acyclic resolutions $C^{\cdot}(S^p)$ which support morphisms

$$\delta^i: C^{\cdot}(S^p) \to C^{\cdot}(S^{p+1})$$

satisfying the normal commutation rules. The object $\Gamma C^{\cdot}(S^{\cdot})$ is then semisimplicial in $K_A$, so that one may set

$$\mathbb{H}(S^{\cdot}) = H(s\Gamma C^{\cdot}(S^{\cdot})),$$

with

$$W^l\mathbb{H}(S^{\cdot}) = \text{image}\{\mathbb{H}(W^lS^{\cdot})) \to \mathbb{H}(S^{\cdot})\}.$$

Thus semisimplicial sheaf cohomology comes equipped with a weight filtration and a spectral sequence

$$E_1^{p,q} = H^q(X_p, S^p) \Rightarrow \mathbb{H}^{p+q}(X_{\cdot}, S^{\cdot})$$

which relates it to ordinary sheaf cohomology.

We remark that if $G^{\cdot}$ denotes the constant sheaf of $G$-valued functions, then

$$H_{\text{sing}}(X_{\cdot}, G) \cong \mathbb{H}(X_{\cdot}, G^{\cdot}).$$

Conventions: If $S^{\cdot}$ is a sheaf on $X_{\cdot}$, we shall use the notations

$$H^n(X_{\cdot}, S^{\cdot}) = H^n(S^0 \to S^1 \to \dots)$$

$$H^n(X_{\cdot}, W^lS^{\cdot}) = H^n(0 \to \dots \to 0 \to S^l \to S^{l+1} \to \dots)$$

$$= H^{n-l}(S^l \to S^{l+1} \to \dots)$$

$$H^n(X_{\cdot}, Gr_W^lS^{\cdot}) = H^{n-l}(X_l, S^l),$$

where the last group is ordinary sheaf cohomology. The explicit calculations of later sections will be carried out using Čech cohomology for semisimplicial sheaves. The Čech complex is then

$$(C^{\cdot}(S^{\cdot}))^n = \bigoplus_p C^{n-p}(U(X_p), S^p)$$

with differential

$$D = \bigoplus_p (-1)^p\delta_p + \delta^*$$

on $X_p$. Here $U(X_p)$ refers to a suitable cover of $X_p$.

## 7. The Cartier motif

To show that the trace and the Hodge motif are naturally isomorphic, we introduce two versions of what we shall call the Cartier motif,

$$k\colon \mathrm{NS}(X.) \to \mathrm{CA}(X.).$$

This motif is of independent interest because of the following:

DEFINITION: A *line bundle* on $X.$ is a sheaf $S^{\cdot}$ such that each $S^p$ is invertible.

REMARK: If $X.$ is augmented towards $Y$ and if $L$ is a line bundle on $Y$, then $L^{\cdot} = \epsilon^* L$ is a line bundle on $X.$.

THEOREM D: *An element $Z \in \mathrm{NS}(X.)$ arises from a line bunlde on $X.$ if and only if $k(Z)$ vanishes in* $\mathrm{CA}(X.)$.

(Thus $k(Z)$ is the obstruction which distinguishes Cartier divisors from Weil divisors.)

THEOREM E: *Let $X$ be a normal crossing surface, $X.$ its canonical resolution. An element $Z \in \mathrm{NS}(X.)$ comes from a line bundle on $X$ if and only if $k(Z)$ vanishes in* $\mathrm{CA}(X.)$.

PROOF (E): Because of theorem D it suffices to show that the map below is an isomorphism:

$$\epsilon^*\colon H^1(X, \mathcal{O}^*) \to H^1(X., \mathcal{O}^{\cdot *}).$$

To this end, consider the exponential sheaf sequences for both $X$ and $X.$:

$$\begin{array}{ccccccc} 0 \to \mathbb{Z}^{\cdot} & \to & \mathcal{O}^{\cdot} & \to & \mathcal{O}^{*\cdot} & \to & 1 \\ \uparrow \epsilon^* & & \uparrow \epsilon^* & & \uparrow \epsilon^* & & \\ 0 \to \mathbb{Z} & \to & \mathcal{O} & \to & \mathcal{O}^* & \to & 1 \end{array}$$

The induced map on $\mathbb{Z}$-cohomology is an isomorphism because the augmentation is a resolution (section 6). For the analogous statement for $\mathcal{O}$-cohomology, we apply theorem 3 of [S]. The five-lemma then implies the required isomorphism for $\mathcal{O}^*$-cohomology. Q.E.D.

To construct the first version of the cartier motif, consider once again the exponential sheaf sequence for $X.$. Because this sequence is filtered by weight, one can define

$$\mathrm{CA}'(X.) = W_1 H^2(X., \mathcal{O}^{\cdot}) / W_1 H^2(X., \mathbb{Z}^{\cdot}).$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathbf{Z}^0 & \to & \mathcal{O}^0 & \to & \mathcal{O}^{*0} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathbf{Z}^{\cdot} & \to & \mathcal{O}^{\cdot} & \to & \mathcal{O}^{*\cdot} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & W^1\mathbf{Z}^{\cdot} & \to & W^1\mathcal{O}^{\cdot} & \to & W^1\mathcal{O}^{*\cdot} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Figure 1

To construct the motivic homomorphism

$$k': \mathrm{NS}(X_.) \to \mathrm{CA}'(X_.),$$

consider the commutative grid obtained by filtering the exponential sheaf sequence by weight (Figure 1), and consider its resulting cohomology grid (Figure 2). Make the identification (justified below)

$$\mathrm{NS}(X_.) \cong \ker A \cap \ker B,$$

and observe that $DC^{-1}$ defines a correspondence from $\mathrm{NS}(X_.)$ to $H^2(\mathcal{O}^0)$ whose image lies in the image of $E$. Since the set $DC^{-1}(x)$ is a coset of the image of $DF$, the correspondence defines a homomorphism $k'$, as desired.

To justify the identification of the Neron-Severi group of $X_.$ with $\ker A \cap \ker B$, we note first that the kernel of $B$ is the Neron-Severi group of $X_0$: it is the group of integral cohomology classes of dimension $H^3(W^1\mathbf{Z}^{\cdot})$

$$
\begin{array}{ccccccccc}
H^3(W^1\mathbf{Z}^{\cdot}) & & & & & & & & \\
\uparrow A & & & & & & & & \\
H^2(\mathbf{Z}^0) & \xrightarrow{B} & H^2(\mathcal{O}^0) & & & & & & \\
\uparrow C & & \uparrow & & & & & & \\
H^2(\mathbf{Z}^{\cdot}) & \xrightarrow{D} & H^2(\mathcal{O}^{\cdot}) & \to & H^2(\mathcal{O}^{*\cdot}) & \to & H^3(\mathbf{Z}^{\cdot}) & & \\
\uparrow F & & E\uparrow & & \uparrow & & \uparrow & & \\
H^2(W^1\mathbf{Z}^{\cdot}) & \to & H^2(W^1\mathcal{O}^{\cdot}) & \xrightarrow{f} & H^2(W^1\mathcal{O}^{*\cdot}) & \xrightarrow{e} & H^3(W^1\mathbf{Z}^{\cdot}) & & \\
\uparrow & & \uparrow g & & \uparrow d & & \uparrow b & & \\
H^1(\mathbf{Z}^0) & \to & H^1(\mathcal{O}^0) & \xrightarrow{h} & H^1(\mathcal{O}^{*0}) & \xrightarrow{c} & H^2(\mathbf{Z}^0) & \xrightarrow{a} & H^2(\mathcal{O}^0) \\
 & & & & \uparrow & & & & \\
 & & & & H^1(\mathcal{O}^{*\cdot}) & & & &
\end{array}
$$

Figure 2.

2 and type (1, 1). Next, we claim that $H^3(W^1\mathbb{Z}^\cdot)$ is just $H^2(X_1, \mathbb{Z})$. Begin with the relations

$$H^3(W^1\mathbb{Z}^\cdot) = H^3(0 \to \mathbb{Z}^1 \to \mathbb{Z}^2) = H^2(\mathbb{Z}^1 \to \mathbb{Z}^2)$$

and then apply the spectral sequence for the last group:

$$E_1^{0,2} = H^2(\mathbb{Z}^1) = H^2(X_1, \mathbb{Z})$$

$$E_1^{1,1} = H^1(\mathbb{Z}^2) = H^1(X_2, \mathbb{Z}) = 0,$$

where the last equality requires the fact that $X.$ is distinguished. One therefore has

$$\ker A \cap \ker B = \ker\left[\mathrm{NS}(X_0) \xrightarrow{A} H^2(X_1, \mathbb{Z})\right],$$

where $A$, as the coboundary in the filtration sequence for $\mathbb{Z}^\cdot$, is identified with $\delta^*$, the difference of the two maps induced by the inclusions of $X_1$ in $X_0$. This completes the required justification.

To construct the second Cartier motif, make the identification

$$\mathrm{NS}(X.) \cong \ker a \cap \ker b,$$

and observe that $dc^{-1}$ defines a correspondence from $\mathrm{NS}(X.)$ to $H^2(W^1\mathcal{O}^{*\cdot})$ whose image is in the kernel of $e$. Since the set $dc^{-1}(x)$ is a coset of the image of $fg$, there is a well-defined homomorphism

$$k'': \mathrm{NS}(X.) \to \mathrm{CA}''(X.),$$

where

$$\mathrm{CA}''(X.) = \frac{\ker\{H^2(W^1\mathcal{O}^{*\cdot}) \to H^3(W^1\mathbb{Z}^\cdot)\}}{\mathrm{image}\{H^1(\mathcal{O}^0) \to H^2(W^1\mathcal{O}^{*\cdot})\}}.$$

As we show in section 10, the two versions of the Cariter motif are in fact the same:

PROPOSITION F: *There is a canonical and functorial isomorphism from $k'$ to $k''$.*

The proof of Theorem D follows directly from the definitions: on the one hand, $c^{-1}(x)$ lifts to $H^1(\mathcal{O}^{*\cdot})$ modulo the image of $H^1(\mathcal{O}^0)$ if and only if its image under $d$ vanishes, but on the other hand, $dc^{-1}$ defines $k''$.

The strategy of the proof of Theorem A is now to establish the sequence of isomorphisms

$$\eta \cong k' \cong k'' \cong \tau.$$

## 8. The isomorphism of $\eta$ and $k'$

Grothendieck noted in [GA] that since the Poincaré lemma holds in the Zariski topology for the map

$$\mathbb{C} \to \Omega^{\cdot}$$

of sheaf complexes, that one has an algebraic de Rham theorem on the level of hypercohomology:

$$H(X, \mathbb{C}) \overset{\cong}{\to} \mathbb{H}(X, \Omega^{\cdot}).$$

Deligne shows that an analogous algebraic de Rham theorem holds in the semisimplicial setting [D]. Thus, the object

$$\Omega^{\cdot}(X_{.}) = \left[\Omega^{\cdot}(X_0) \rightrightarrows \Omega^{\cdot}(X_1) \mathrel{\substack{\to\\\to\\\to}} \dots\right]$$

is a semisimplicial complex of sheaves, and it admits a compatible resolution (Čech, injective, etc.):

$$sC^{\cdot}\Omega^{\cdot}(X_{.}) = \left[sC^{\cdot}\Omega^{\cdot}(X_0) \overset{\delta^*}{\to} sC^{\cdot}\Omega^{\cdot}(X_1) \overset{\delta^*}{\to} \dots\right].$$

Taking global sections one obtains a semisimplicial object in $K_A$:

$$\Gamma sC^{\cdot}\Omega^{\cdot}(X_{.}) = \left[\Gamma sC^{\cdot}\Omega^{\cdot}(X_0) \overset{\delta^*}{\to} \Gamma sC^{\cdot}\Omega^{\cdot}(X_1) \overset{\delta^*}{\to} \dots\right].$$

Thus one can define algebraic de Rham cohomology for $X_{.}$ by

$$\mathbb{H}(X_{.}, \Omega^{\cdot}) = H(s\Gamma sC^{\cdot}\Omega^{\cdot}(X_{.})).$$

The underlying semisimplicial complex of sheaves supports two filtrations, one coming from the algebraic and one coming from the simplicial structure:

$$F^p\Omega^{\cdot}(X_{.}) = \bigoplus_{i \geqslant p} \Omega^i(X_{.})$$

$$W^p\Omega^{\cdot}(X_{.}) = \bigoplus_{i \geqslant p} \Omega^{\cdot}(X_i).$$

These filtrations give a mixed Hodge structure on $H^\cdot(X_\cdot)$ which is isomorphic to the one defined using forms [D].

We now claim that $Gr_F$ commutes with $\mathbb{H}$:

$$\text{(a)} \quad Gr_F^p \mathbb{H}^\cdot(X_\cdot, \Omega^\cdot) \xrightarrow{\cong} \mathbb{H}^\cdot(X_\cdot, Gr_F^p \Omega^\cdot) = \mathbb{H}^\cdot(X_\cdot, \Omega^p).$$

To see that this is so, observe first that by [D, 8.1.9.v], that the spectral sequence for $\mathbb{H}(X_\cdot, \Omega^\cdot)$ defined by the Hodge filtration degenerates at $E_1$. By Deligne's degeneration criterion [D, 1.3.2] this is equivalent to the fact that the differential for the complex $K = s\Gamma s C^\cdot \Omega^\cdot(X_\cdot)$ is strictly compatible with the Hodge filtration. Strictness implies that the exact sequence of complexes

$$0 \to F^p K \to K \to K/F^p K \to 0$$

passes to an exact sequence in cohomology:

$$0 \to H(F^p K) \to H(K) \to H(K/F^p K) \to 0$$

It follows that $F^p$ and $H$ commute:

$$F^p H(K) \underset{\text{def}}{=} \text{image}[H(F^p K) \to H(K)] \cong H(F^p K).$$

To complete the argument, consider the diagram below:

$$\begin{array}{ccccccccc}
0 \to & F^{p+1}H(K) & \to & F^p H(K) & \to & Gr_F^p H(K) & \to 0 \\
& \uparrow \cong & & \uparrow \cong & & \uparrow & \\
0 \to & H(F^{p+1}K) & \to & H(F^p K) & \to & H(Gr_F^p K) & \to 0
\end{array}$$

The top row is exact by definition and the bottom row is exact by strictness of the differential with respect to $F$. Commutativity of the diagram and the existence of the indicated isomorphisms forces the desired isomorphism on the level of $Gr_F$.

Since the source and the target of the map (a) are filtered by weight, there are induced maps

$$\text{(b)} \quad W^l Gr_F^p \mathbb{H}(X_\cdot, \Omega^\cdot) \to W^l \mathbb{H}^\cdot(X_\cdot, \Omega^p)$$

$$\text{(c)} \quad Gr_W^l Gr_F^p \mathbb{H}(X_\cdot, \Omega^\cdot) \to Gr_W^l \mathbb{H}(X_\cdot, \Omega^p).$$

Proposition 7.2.8 of [D] asserts that (c) is an isomorphism, from which it follows that (b) is as well.

To construct the asserted isomorphism between $\eta$ and $k'$, we must construct a commutative diagram

$$\begin{array}{ccc} \mathrm{NS}(X.)/\text{torsion} & \xrightarrow{k'} & \mathrm{CA}'(X.) \\ \downarrow \cong & & \downarrow \cong \\ L^1H^2(X.) & \xrightarrow[\eta]{} & J^1H^2(X.). \end{array}$$

To construct the left-hand isomorphism, consider the following diagram, induced from the homomorphism $\mathbb{Z}^{\cdot} \to \Omega^{\cdot}$:

$$\begin{array}{ccc} H^3(W^1\mathbb{Z}^{\cdot}) & \xrightarrow{\mu} & H^3(W^1\Omega^{\cdot}(X.)) \\ B\uparrow & & \uparrow B' \\ H^2(\mathbb{Z}^0) & \xrightarrow{\nu} & H^2(\Omega^{\cdot}(X_0)) \\ A\searrow & & \swarrow A' \\ & H^2(\mathcal{O}^0) & \end{array}$$

Because of the identifications

$$L^1H^2(X.) = \left(Gr_2^W H^2(X.)\right)_{\mathbb{Z}}^{1,1} \cong \ker A' \cap \ker B' \cap \text{image } \nu$$

and

$$\mathrm{NS}(X.) = \ker A \cap \ker B,$$

and because $\mu$ and $\nu$ are essentially the maps "tensor with $\mathbb{C}$", $\nu$ induces a surjective map from NS to $L^1H^2$ whose kernel is the torsion subgroup.

To construct the right-hand isomorphism, recall that

$$J^1H^2(X) = \frac{W_1H^2(X, \mathbb{C})}{F^1 \cap W_1H^2(X, \mathbb{C}) + W_1H^2(X, \mathbb{Z})}$$

$$= \frac{Gr_F^0 W_1 H^2(X, \mathbb{C})}{W_1H^2(X, \mathbb{Z})}.$$

Now

$$Gr_F^0 W_1 = W_1/\left(W_1 \cap F^1\right) \cong \left(W_1 + F^1\right)/F^1 = W_1 Gr_F^0,$$

so that

$$J^1H^2(X) \cong \frac{W_1 Gr_F^0 H^2(X, \mathbb{C})}{W_1H^2(X, \mathbb{Z})}.$$

The required isomorphism now results from the commutative diagram

$$\begin{array}{ccc} Gr_F^0 W_1 H^2(X_., \Omega^{\cdot}) & \xrightarrow{\cong} & W_1 H^2(X_., \mathcal{O}) \\ \nwarrow & & \nearrow \\ & W_1 H^2(X_., \mathbb{Z}) & \end{array}$$

To show that the diagram which compares $\eta$ and $k$ is commutative, recall that the motivic homomorphism $\eta$ is constructed from sections $s_{\mathbb{Z}}$ and $s_F$ of

$$H^2(X) \to Gr_2^W H^2(X)$$

over $L^1H^2(X)$ which preserve, respectively, the lattice and Hodge filtration. Consider the exact commutative diagram

$$\begin{array}{ccccc} H^2(\mathbb{Z}^0) & \to & H^2(\Omega^1(X_0)) & \to & H^2(\mathcal{O}^0) \\ \uparrow C & & \uparrow & & \uparrow \\ H^2(\mathbb{Z}^{\cdot}) & \xrightarrow{D'} & H^2(\Omega^{\cdot}) & \xrightarrow{D''} & H^2(\mathcal{O}^{\cdot}) \\ & & & & \uparrow \\ & & & & H^2(W^1\mathcal{O}^{\cdot}). \end{array}$$

Let $C^{-1}$ be a section of $C$ over $L^1H^2(X)$, set $s_{\mathbb{Z}} = D' \circ C^{-1}$, and write

$$\eta = D'C^{-1} - s_F.$$

Projection to $H^2(\mathcal{O}^{\cdot})$ yields

$$D'' \circ \eta = D''D'C^{-1} = DC^{-1} = k',$$

as required.

## 9. The isomorphism of $k'$ and $k''$

Referring once again to Figure 2, we see that the path $Ef^{-1}$ induces a map from CA″ to CA′:

$$\begin{array}{ccc} & & \mathrm{CA}''(X.) \\ & k'' \nearrow & \\ \mathrm{NS}(X^{\cdot}) & & \downarrow Ef^{-1} \\ & k' \searrow & \\ & & \mathrm{CA}'(X.). \end{array}$$

The exactness of Figure 2 implies that $Ef^{-1}$ is an isomorphism, so it remains only to show that the above diagram commutes.

To this end, let $\xi$ be an arbitrary element of NS( $X_{\cdot}$), and let $\{c_{\alpha\beta\gamma}\}$ be a cocycle in $C^2(\mathbb{Z}^0)$ which represents $\xi$. By hypothesis, $\{c_{\alpha\beta\gamma}\}$ lifts to a cocycle

$$c = \{c_{\alpha\beta\gamma}\} + \{c_{\alpha\beta}\} + \{c_{\alpha}\} \in C^2(\mathbb{Z}^{\cdot})$$

where

$$C^2(\mathbb{Z}^{\cdot}) = C^2(\mathbb{Z}^0) \oplus C^1(\mathbb{Z}^1) \oplus C^0(\mathbb{Z}^2)$$

Therefore

$$k'(\xi) = \text{class}(c) \text{ in } H^2(\mathcal{O}^{\cdot}).$$

To compute a representative of $k''(\xi)$, note that $\xi$ is the Chern class of a line bundle $L$ on $X_0$. Let $\{g_{\alpha\beta}\}$ be a cocycle in $C^1(\mathcal{O}^{*0})$ representing this line bundle and observe that

$$\delta\left\{\frac{1}{2\pi i} \log g_{\alpha\beta}\right\} = \{c_{\alpha\beta\gamma}\},$$

perhaps after modifying $\{c_{\alpha\beta\gamma}\}$ by a coboundary. Thus

$$k''(\xi) = \text{class of } \delta^*\{g_{\alpha\beta}\} \text{ in } H^2(W^1\mathcal{O}^{*\cdot}).$$

As a cochain $\delta^*\{g_{\alpha\beta}\}$ lifts to $H^2(W^1\mathcal{O}^{\cdot})$ according to the formula

$$f^{-1}k''(\xi) = \delta^*\left\{\frac{1}{2\pi i} \log g_{\alpha\beta}\right\}.$$

Note, however, that

$$D\delta^*\left\{\frac{1}{2\pi i} \log g_{\alpha\beta}\right\} = \delta\delta^*\left\{\frac{1}{2\pi i} \log g_{\alpha\beta}\right\}$$

$$= \delta^*\delta\left\{\frac{1}{2\pi i} \log g_{\alpha\beta}\right\}$$

$$= \delta^*\{c_{\alpha\beta\gamma}\},$$

so that this lift is not necessarily a cocycle. Nevertheless, the image $\delta^*\{c_{\alpha\beta\gamma}\}$ in $H^2(W^1\mathbb{Z}^{\cdot})$ vanishes by hypothesis, so there is a cochain

$$c' = \{c'_{\alpha\beta}\} + \{c'_{\alpha}\} \in C^1(\mathbb{Z}^1) \oplus C^0(\mathbb{Z}^2)$$

such that

$$Dc' = \delta^*\{c_{\alpha\beta\gamma}\}.$$

The element

$$\delta^*\left\{\frac{1}{2\pi i}\log g_{\alpha\beta}\right\}-c'$$

is then the desired lifting to a cocycle. Since

$$Ef^{-1}k''(\xi)=\operatorname{class}\left(\delta^*\left\{\frac{1}{2\pi i}\log g_{\alpha\beta}\right\}-c'\right),$$

it remains to show that

$$k'(\xi)-Ef^{-1}k''(\xi)=\operatorname{class}\left[c-\delta^*\left\{\frac{1}{2\pi i}\log g_{\alpha\beta}\right\}+c'\right] \qquad (*)$$

vanishes in $H^2(\mathcal{O}^\cdot)$ modulo the image of $H^2(W^1\mathbb{Z}^\cdot)$. Because of the relation

$$D\left\{\frac{1}{2\pi i}\log g_{\alpha\beta}\right\}=\delta^*\left\{\frac{1}{2\pi i}\log g_{\alpha\beta}\right\}-\delta\left\{\frac{1}{2\pi i}\log g_{\alpha\beta}\right\}$$
$$=\delta^*\left\{\frac{1}{2\pi i}\log g_{\alpha\beta}\right\}-\{c_{\alpha\beta\gamma}\},$$

the right hand side of (*) can be expressed as

$$\operatorname{class}\left(D\left\{\frac{1}{2\pi i}\log g_{\alpha\beta}\right\}+\{c_{\alpha\beta}+c'_{\alpha\beta}\}+\{c_\alpha+c'_\alpha\}\right),$$

which is manifestly in the image of $H^1(W^1\mathbb{Z}^\cdot)$. This completes the proof of the commutativity.

## 10. The isomorphism of $k''$ and $\tau$

To compare $k''$ and $\tau$ we use the divisor sequence

$$0\to\mathcal{O}^{*\cdot}\to\mathcal{M}^{*\cdot}\to\mathcal{D}^\cdot\to 0$$

for a semisimplicial space. This makes sense when the local sections of the multiplicative sheaf $\mathcal{M}^{*p}$ on $X_p$ are in general position with respect to the images $\delta_i(X_{p+1})$.

To begin, we claim that $\tilde{P}(X.)$ has the following sheaf cohomology description:

$$\tilde{P}(X.)\cong H^1(W^1\mathcal{D}^\cdot)/H^1(W^1\mathcal{M}^{*\cdot}).$$

To see this, consider the exact commutative grid of sheaves analogous to that of Figure 1 which comes from the divisor sequence and the filtration sequence

$$0 \to Gr_W^2 \to W^1 \to Gr_W^1 \to 0.$$

There results an exact commutative grid of cohomology groups, the relevant portion of which is displayed below:

$$\begin{array}{ccccc} 0 \to H^0(\mathscr{D}^1 \to \mathscr{D}^2) & \to & H^0(\mathscr{D}^1) \to 0 & & \\ \uparrow & & \uparrow & & \\ 0 \to H^0(\mathscr{M}^{*1} \to \mathscr{M}^{*2}) & \to & H^0(\mathscr{M}^{*1}) & \to & H^0(\mathscr{M}^{*2}) \\ & & \uparrow & & \\ & & H^0(\mathscr{O}^{*1}) & & \end{array}$$

Here we have used the reductions

$$H^n(X_., W^l S^{\cdot}) = H^{n-l}(S^l \to S^{l+1} \to \dots)$$

$$H^n(X_., Gr_W^l S^{\cdot}) \to H^{n-l}(X_l, S^l),$$

as well as the fact that $\mathscr{D}^2 = 0$ for surfaces. The diagram then yields the isomorphisms

$$\frac{H^1(W^1\mathscr{D}^{\cdot})}{H^1(W^1\mathscr{M}^{*\cdot})} \cong [H^0(\mathscr{D}^0)]\Big[\Big(\ker\Big\{H^0(\mathscr{M}^{*1}) \overset{\delta^*}{\to} H^0(\mathscr{M}^{*2})\Big\}\Big)$$

$$\cdot\big(\text{image}\{H^0(\mathscr{O}^{*1}) \to H^0(\mathscr{M}^{*1})\}\big)\Big]^{-1}$$

$$= \tilde{P}(X_.).$$

Under the above identification, the subgroup $\tilde{P}$ is simply the kernel of the composition

$$H^1(W^1\mathscr{D}^1) \to H^2(W^1\mathscr{O}^*) \to H^3(W^1\mathbb{Z}^{\cdot}),$$

and the trace is induced by the coboundary map

$$H^0(\mathscr{D}^0) \to H^1(W^1\mathscr{D}^{\cdot})$$

which comes from the weight filtration of the divisor sheaf. We can therefore compare $\tau$ with $k''$ using the diagram below:

$$\begin{array}{ccc}
H^0(\mathscr{D}^0) & \overset{\tilde{\tau}}{\to} & H^1(W^1\mathscr{D}^{\cdot}) \\
\text{div}\downarrow & & \downarrow\text{div} \\
H^1(\mathcal{O}^{*0}) & \overset{\delta^*}{\to} & H^2(W^1\mathcal{O}^{*\cdot}) \\
\downarrow & & \uparrow \\
H^2(\mathbb{Z}^0) & & \Big| \exp 2\pi\mathrm{i} \\
\uparrow & & \Big| \\
\mathrm{NS}(X.) & \overset{k''}{\to} & H^2(W^1\mathcal{O}^{\cdot})
\end{array}$$

It then suffices to show that

$$\text{div} \circ \tilde{\tau} = \exp 2\pi\mathrm{i} \circ \tilde{k}''$$

modulo

$$\text{image}\{ H^1(\mathcal{O}^0) + H^2(W^1\mathbb{Z}) \to H^2(W^1\mathcal{O}^{*\cdot})\}$$

$$= \exp 2\pi\mathrm{i}\big(\delta^* H^1(\mathcal{O}^0)\big).$$

We will show separately that the upper and lower squares are commutative. First, let $\{f_\alpha\}$ be a system of meromorphic functions representing a divisor $\xi \in H^0(\mathscr{D}^0)$. The composition $\delta^* \circ$ div is then represented on $\xi$ by the cocycle

$$\delta^*\delta\{f_\alpha\},$$

while the composition $\tilde{\tau} \circ$ div is represented on $\xi$ by

$$\delta\delta^*\{f_\alpha\}.$$

Since $\delta$ and $\delta^*$ commute, so does the upper square.

Second, let $\xi$ be a class in NS($X.$), and let $c = \{c_{\alpha\beta\gamma}\}$ be a representative cocycle in $C^2(\mathbb{Z}^0)$. Since its image in $H^2(\mathcal{O}^0)$ is zero by definition of the Neron-Severi group, there is a cochain $\{f_{\alpha\beta}\} \in C^1(\mathcal{O}^0)$ such that

$$\delta\{f_{\alpha\beta}\} = \{c_{\alpha\beta\gamma}\}. \qquad (*)$$

The cocycle

$$\{c_{\alpha\beta\gamma}\} - D\{f_{\alpha\beta}\} = \delta^*\{f_{\alpha\beta}\}$$

represents $k'(\xi)$ in $H^2(W^1\mathcal{O}^{\cdot})$, and so

$$\exp 2\pi i \cdot k''(\xi) = \text{class}\left(\delta^*\{\exp 2\pi i f_{\alpha\beta}\}\right) \in H^2(W^1\mathcal{O}^{*\cdot}).$$

Since $\xi = [\{c_{\alpha\beta\gamma}\}]$ is also in the image of $H^1(\mathcal{O}^{*0})$, there is a cocycle $\{g_{\alpha\beta}\} \in C^1(\mathcal{O}^{*0})$ such that

$$\frac{1}{2\pi i}\delta\{\log g_{\alpha\beta}\} = \{c_{\alpha\beta\gamma}\} + \delta\{k_{\alpha\beta}\} \tag{**}$$

where $k_{\alpha\beta}$ is in $C^1(\mathbb{Z}^0)$. Then

$$\delta^*(\xi) = \text{class}\left(\delta^*\{g_{\alpha\beta}\}\right) \in H^2(W^1\mathcal{O}^{*\cdot}).$$

Combining (*) and (**), we have

$$\delta\left(\{f_{\alpha\beta}\} - \frac{1}{2\pi i}\{\log g_{\alpha\beta}\} + \{k_{\alpha\beta}\}\right) = 0.$$

Therefore

$$\{f_{\alpha\beta}\} = \frac{1}{2\pi i}\{\log g_{\alpha\beta}\} + \{A_{\alpha\beta}\} + \{k_{\alpha\beta}\}$$

where

$$\{A_{\alpha\beta}\} \in Z^1(\mathcal{O}^0).$$

Exponentiating and applying $\delta^*$, we find that

$$\delta^*\{\exp 2\pi i f_{\alpha\beta}\} = \delta^*\{g_{\alpha\beta}\}\delta^*\{\exp 2\pi i A_{\alpha\beta}\}.$$

Thus the lower square is commutative up to the action of $\exp 2\pi i \delta^* H^1(\mathcal{O}^0)$, as desired.

## 11. Polarizations and primitive cohomology

We shall now construct bilinear forms on each of the graded pieces of $H^2(X_{\cdot})$, where $\tilde{X}_{\cdot}$ is a distinguished semisimplicial resolution. These bilinear forms play a crucial role in the applications of our theory.

WEIGHT 2: Since

$$Gr_2^W H^2(X_{\cdot}) \cong \text{kernel}\{\delta^*:\ H^2(X_0) \to H^2(X_1)\},$$

the second graded piece inherits the form on $H^2(X_0)$ given by cup-product followed by evaluation on the fundamental class. The latter is by definition the sum of the fundamental classes of the components.

WEIGHT 1: Because the resolution $X.$ is distinguished, $X_2$ is zero-dimensional, and so the weight (simplicial) spectral sequence gives the first graded piece as the quotient of a Hodge structure by a sub-Hodge structure:

$$Gr_1^W H^2(X) \cong H^1(X_1)/\delta^* H^1(X_0).$$

Cup-product defines a nondegenerate skew form on $H^1(X_1)$ which remains nondegenerate upon restriction to $\delta^* H^1(X_0)$. Restriction of this form to the orthogonal complement of $\delta^* H^1(X_0)$ defines the required form on the first graded piece.

WEIGHT 0: Since the zeroth graded pieces of homology and cohomology are dual by the universal coefficient theorem modulo torsion, a bilinear form on one induces a bilinear form on the other. Now the homological object has a natural geometric interpretation as the "triple-point lattice" (section 5):

$$Gr_0^W H_2(X.) \cong \ker\{\delta_*\colon H_0(X_2) \to H_0(X_1)\} \underset{\text{def}}{=} T(X.).$$

Since $X_2$ is a finite set of points, its zeroth homology carries the obvious bilinear form defined by

$$\langle \Sigma m_p p, \Sigma n_p p \rangle = \Sigma m_p n_p.$$

Since this form is positive definite on $H_0(X_2)$, so is the induced form on $T(X.)$.

Primitive cohomology is defined in the usual way: Let $Y.$ be a subobject of $X.$ such that $Y_p$ is an ample divisor on every positive-dimensional component of $X_p$. The divisor $Y.$ will necessarily be empty on zero-dimensional components. Let

$$H^2(X.)_0 = \text{kernel}\{H^2(X.) \to H^2(Y.)\}$$

be the primitive subspace. We claim that the bilinear forms defined above polarize the primitive cohomology in the sense that relative to them the graded pieces are polarized Hodge structures: the primitive cohomology is a *polarized mixed Hodge structure*.

To see this, it suffices to establish the relations

$$Gr_2^W\big(H^2(X.)_0\big) \cong \text{kernel}\{H^2(X_0)_0 \to H^2(X_1)\} \quad (*)$$

$$W_1\big(H^2(X.)_0\big) \cong W_1 H^2(X.). \quad (**)$$

Consider therefore the diagram below, which relates the "Mayer Vietoris sequence" of $X.$ to that of $Y.$:

$$\begin{array}{ccccccc} W_1H^2(X.) & \to & H^2(X.) & \overset{p}{\to} & H^2(X_0) & \overset{\delta^*}{\to} & H^2(X_1) \\ \downarrow & & \downarrow & & \downarrow i^* & & \downarrow \\ W_1H^2(Y.) & \to & H^2(Y.) & \overset{q}{\to} & H^2(Y_0) & \overset{\delta^*}{\to} & H^2(Y_1) \end{array}$$

Since $\dim Y_p = 1 - p$, $W_1H^2(Y.)$, which is a subquotient of $H^1(Y_1) \oplus H^0(Y_2)$, must be zero. This fact, coupled with the exactness of the diagram implies that (**) is true. Moreover, since $p$ induces an injection of the left-hand side of (*) into the right-hand side, it suffices to establish surjectivity. Thus, let $\alpha$ be a primitive element of $H^2(X_0)$ which is annihilated by $\delta^*$, and let $\tilde{\alpha}$ be any lift to $H^2(X.)$. Then $i^*\tilde{\alpha}$ is annihilated by $q$, and so is in the image of $W_1H^2(Y.)$. Since this latter group vanishes, $i^*\tilde{\alpha} = 0$, and so $\tilde{\alpha}$ is primitive, as desired.

## 12. A Torelli theorem

We shall now prove a slightly weakened version of the Torelli theorem mentioned in the introduction.

THEOREM G: *Let $X$ be the union in $\mathbb{P}^3$ of a smooth cubic surface $A$ and a plane $B$ meeting transversely in a curve $C$. Then $X$ is determined up to isomorphism by the polarized mixed Hodge structure on $H^2(X)_0$.*

The essence of the proof is to show that both $X$ and its mixed Hodge structure correspond to a plane elliptic curve with six marked points (denoted $(\mathscr{P}, C, \mathbb{P}_2)$ below).

We begin by recalling [M] that the lattice $L^1H_0^2$ is isomorphic to that of the Dynkin diagram $E_6$, with choices of positive primitive root systems in one-to-one correspondence with choice of a skew-six of lines on $A$. Observe first that because the hyperplane section of $X$ is homologous to $C$ on $A$, that

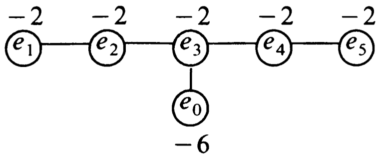
$$L^1H_0^2(X) = Gr_2^W H_0^2(X) = H_0^2(A).$$

View $A$ as $\mathbb{P}^2$-blown up at six points $p_1, \ldots, p_6$, let $L_1, \ldots, L_6$ be the corresponding exceptional curves, and let $\lambda_1, \ldots, \lambda_6$ be their cohomology classes in $H^2(A)$. Let $h$ be the cohomology-class of a hyperplane section of $A$, and set

$$e_0 = h - \lambda_1 - \lambda_2 - \lambda_3$$

$$e_i = \lambda_{i+1} - \lambda_i \quad (i = 1, \ldots, 5).$$

The intersections of these vectors follow the diagram below.




The weights attached to the nodes represent the self-intersection number, while distinct vectors have intersection one or zero according to whether or not the corresponding nodes are joined by an edge.

Observe next that a choice of a skew six on $X$ determines a triple

$$(\mathscr{P}, C, \mathbb{P}_2)$$

where $C$ is a cubic curve in $P_2$ and where $\mathscr{P}$ is a set $\{p_1, \ldots, p_6\}$ of points on $C$. The pair $(C, \mathbb{P}_2)$ is the pair $(C, B)$ defined by $X$, while the pair $(\mathscr{P}, C)$ is that defined by $p_i = L_i \cap C$, with the intersection taken in $A$.

Conversely, such a triple determines $X$: Blow up $\mathbb{P}_2$ along $\mathscr{P}$ to obtain a cubic surface $A'$ with a distinguished plane section $C'$, namely that obtained as the proper transform of $C$. Let $B'$ be the unique plane which cuts out $C'$ on $A'$, and construct the quartic surface

$$X' = A' \cup B'.$$

The given surface $X$ and the constructed surface $X'$ are clearly isomorphic.

The proof of the theorem therefore rests on the construction from the Hodge-theoretic data $H^2(X_0)$ of a triple $(\mathscr{P}_h, C_h, \mathbb{P}_2)$ which is isomorphic to some natural triple $(\mathscr{P}, C, \mathbb{P}_2)$. To construct the abstract curve $C_h$ simply take

$$C_h = J^1 H_0^2(X).$$

and observe that $C_h \cong C$ (use a Mayer-Vietoris sequence). To construct the correct projective imbedding, use a basis of sections for $L_0^3$, where $L_0$ is the line bundle defined by the divisor $D =$ zero element of the group structure.

To construct the point set $\mathscr{P}_h$ in $C_h$, fix an ordered set of primitive positive roots, $\{e_0, \ldots, e_5\}$ and consider the system of equations on $C_h$ defined in the Jacobian by

$$-(p_1 + p_2 + p_3) = \eta(e_0)$$

$$p_{i+1} - p_i = \eta(e_i) \quad (i = 1, \ldots, 5).$$

Elimination of $p_2$ and $p_3$ from the first three equations yields

$$3p_1 = (\eta(e_1) + 2\eta(e_2) + \eta(e_3)),$$

so that $p_1$ is determined up to translation by points of order three on $C_h$. The equations for $e_1, \ldots, e_5$ determine the remaining points modulo the same translation. Choose such a solution set $\mathscr{P}_h$, and consider the resulting Hodge-theoretic triple $(\mathscr{P}_h, C_h, \mathbb{P}^2)$. Because the group of three-division points on $C_h$ acts on $L_0^3$, all triples constructed from a fixed root system are projectively equivalent.

Finally, consider the skew-six $\{L_1 \ldots L_6\}$ defined by the given root system. Then the points $p_i = L_i \cdot C$ which occur in the natural triple solve the equations in $\mathrm{Pic}^0(C)$ defined by

$$\tau(e_0) = H \cdot C - (p_1 + p_2 + p_3)$$

$$\tau(e_i) = p_{i+1} - p_i.$$

From the isomorphism of $u$ with $\tau$, we see that the natural triple arises as a solution set to the same set of equations as does the Hodge-theoretic triple. This completes the proof.

## 13. Two surfaces meeting in a curve

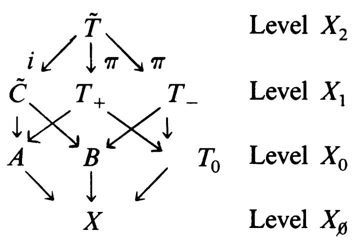
The preceding theorem holds even when the components of the quartic surface are tangent. To generalize the proof, it suffices to identify the group $P(X)$.

PROPOSITION H: *Let $X$ be the union of two smooth surfaces $A$ and $B$. Suppose that the intersection curve $C$ has at most ordinary singularities. Then*

$$P(X) \cong \mathrm{Pic}^0(C),$$

*where the right-hand group is the generalized Picard variety of Rosenlicht.*

PROOF: Repetition of the sequence of operations "desingularize, then form the mapping cone over the nonhomeomorphism locus" leads to a polyhedral resolution of an $n$-dimensional variety in at most $n$ steps ([C3]). For surfaces of the type considered, such a resolution is given by the semisimplicial space below:



Here $T_+$, $T_-$, and $T_0$ denote copies of $T$, the singular locus of $C$, while $\tilde{T}$ is the lift of $T$ to $\tilde{C}$, the normalization of $C$. Because $X_1$ equals $\tilde{C}$ up to spaces of dimension zero, $\mathrm{Div}_h(X_1)$ is the group of divisors on $\tilde{C}$ which has degree zero on each irreducible component:

$$\mathrm{Div}_h(X_1) = \mathrm{Div}_h(\tilde{C}).$$

The compact part of $P(X)$ is therefore the Picard variety of divisors of degree zero on each component of $\tilde{C}$:

$$P_c(X) = \mathrm{Pic}^0(\tilde{C}).$$

To determine the structure of $P$ itself, write a meromorphic function $f$ on $X_1$ as

$$f = f_{\tilde{C}} + f_{T_+} + f_{T_-}$$

and observe that $f$ is in $M_\delta^*$ if and only if

(1) support $(f_C) \cap \tilde{T} = \emptyset$
(2) $T_{T_\pm} \in \mathbb{C}^*(T_\pm)$.

Such a function lies in the subgroup $M_\delta^*$ if and only if the relation

$$\delta^* f \in \delta^* \mathbb{C}^*(X_1)$$

holds as well. But

$$\delta^* f = (i^* f_{\tilde{C}})(\pi^* f_{T_+})^{-1}(\tau^* f_{T_-})$$

and

$$\delta^* \mathbb{C}(X_1) = i^* \mathbb{C}^*(\tilde{C}) \cdot \pi^* \mathbb{C}^*(T).$$

so that the defining condition for $M_\delta^*$ reduces to

$$i^* f_{\tilde{C}} \in i^* \mathbb{C}^*(\tilde{C}) \cdot \pi^* \mathbb{C}^*(T). \qquad \text{(GP)}$$

If $\tilde{C}$ is connected, this reduces still further to

$$i^* f_{\tilde{C}} \in \pi^* \mathbb{C}(T), \qquad \text{(GP}'\text{)}$$

so that $f$ is constant along the fibers of the normalization map. If $C$ has ordinary singularities, these last two relations (GP or GP′) are the defining conditions for the special meromorphic functions used in the construction of the generalized Picard variety of $C$.

REMARK: According to the results of section 5, the multiplicative part of $P(X)$ is given by the group of characters on the lattice

$$\begin{aligned} T(X.) &= \ker\{\delta_*: H_0(X_2, \mathbb{Z}) \to H_0(X_1, \mathbb{Z})\} \\ &= \ker \pi_* \cap \ker i_*. \end{aligned}$$

In the present case, this lattice is given by divisors $\xi$ on $\tilde{C}$ subject to the following restrictions:

1) $\xi$ is supported in $\tilde{T}$
2) $\xi$ is of degree zero on each component of $\tilde{C}$
3) $\xi$ is of degree zero on each fiber of $\pi$, the normalization map.

*Proof that Theorem G holds in the tangent case*

Let $A$ be a smooth cubic surface, and let $B$ be a plane which is tangent to $A$ at a single point $q_0$. Thus $q_0$ is a nodal point of the intersection curve $C$. Set

$$T = \{q_0\}$$

$$\tilde{T} = \{q_+, q_-\}$$

and let $v = q_+ - q_-$ generate $T(X)$. Let $e_i = l_{i+1} - l_i$ be a primitive root vector for $L^1H^2(X)$, and consider its motivic value

$$\tau(e_j) \in P(X).$$

Since $\tilde{C}$ is rational, $P(X)$ has trivial compact part, so that $\tau(e_i)$ takes values in $P_m(X)$.

To compute this value, observe that the divisor

$$\tau(e_i) = p_{i+1} - p_i$$

is the divisor of a meromorphic function $f$ on $\tilde{C}$. Via the simplicial coboundary, $f$ yields a character on the lattice $T(X.)$, and this character is the motivic value of $e_i$ viewed under the identification of $P_m(X)$ with $\hat{T}(X)$. In the case at hand, $T(X)$ is generated by $v$, so that the motivic value is determined by the single complex number

$$\langle \tau(e_i), v \rangle = \langle \delta^* f, v \rangle = f(q_+)/f(q_-).$$

If we choose a coordinate $t$ on $\tilde{C} \cong \mathbb{P}^1$ so that $t(p_i) = 0$, $t(p_{i-1}) = \infty$, then we can take $f$ to be $t$, in which case

$$\langle \tau(e_i), v \rangle = q_+/q_-$$

is revealed as the cross ratio $(p_l, p_{l+1}, q_-, q_+)$. The reconstruction of the degenerate quartic surface $X$ from Hodge-theoretic data now proceeds exactly as before, except that the abelian equations (*) of the last section are replaced by equations in multiplicative form.

## 14. The infinitesimal base locus

Let $R$ be a ring of polynomials in variables $\xi_0, \ldots, \xi_n$, each with weights $w_0, \ldots, w_n$, and let $R^d$ be the component of total weight $d$. Choose $F$ and $G$ in $R^d$, set $H(\xi, t) = F + tG$, and let

$$\pi = \{(\xi, t) \mid H(\xi, t) = 0\}$$

be the resulting pencil of hypersurfaces with fibers $X_t = \{\xi \mid H(\xi, t) = 0\}$. The *infinitesimal base locus* $B_\pi$ is the part of the singular locus of the total space which lies on the central fiber, and is defined by the homogeneous ideal

$$I_\pi = \text{Radical}(J_H, t)$$

where

$$J_H = \left( \frac{\partial H}{\partial \xi_0}, \ldots \frac{\partial H}{\partial \xi_n}, \frac{\partial H}{\partial t} \right)$$

is the Jacobian ideal of $H$. The infinitesimal base locus is a projective invariant of pencils, or, more generally, of first order deformations.

EXAMPLE 1: Consider a family of double planes,

$$y^2 = F_d^2 + tG_{2d},$$

where the subscript indicates the weight. If the branch locus $\{F_d = 0\}$ is smooth, then

$$I_\pi = (t, F_d, G_{2d}),$$

so that $B_\pi$ is a variety of $2d^2$ points on the double curve of the "central fiber".

The purpose of this section is to show how the infinitesimal base locus influences the limit mixed Hodge structure $H_\pi^2$ which is determined by the associated degeneration of Hodge structures. In essence, the result is that $B_\pi$ determines a lattice $\Lambda_\pi$ of Hodge classes in $H_\pi^2/W_1$, and that the natural one-motif

$$u_\pi \colon \Lambda_\pi \to J^1 W_1 H_\pi^2$$

reflects the position of $B_\pi$ on the singular locus of the central fiber. In favorable cases, the Hodge theory and the arithmetic of $u_\pi$ cooperate to determine the imbedding of the base locus in the singular locus.

We shall assume henceforth that $X_0$ has normal crossing singularities and that $G = 0$ meets the singular locus of $X_0$ transversely, hence in double points of $X$. Let $\tilde{X}$ be the family obtained by resolution of singularities of $X$, and note that $\tilde{X}_0$ consists of the proper transforms of $X_0$ and one copy $Q_p$ of $\mathbb{P}_1 \times \mathbb{P}_1$, for each point $p$ of $B_\pi$. Let $X'$ be the family obtained by blowing each $Q_p$ down along one of its rulings, and note that $X_0'$ is obtained from $X_0$ by blowing up each point of $p$ in one or the other components of $X_0$ which meet at $p$. Let $\mathscr{E}_p$ be the resulting exceptional curve, and let $\Lambda$ be the sublattice of $H^2(X_0', \mathbb{Z})$ consisting of classes supported in the union of the $\mathscr{E}_p$.

Since there is a morphism of mixed Hodge structures

$$k^*\colon H^2(X_0') \to H_\pi^2,$$

one may define a "topological marking" of the limit cohomology by the infinitesimal base locus via

$$\Lambda_\pi = k^*\Lambda.$$

One can in fact show that $\Lambda$ and $\Lambda_\pi$ are isomorphic. To do so, let $\alpha_0$ be an element of $H^2(X_0', \mathbb{Z})$ and let $\alpha = k^*\alpha_0$. Denote by $\alpha_t$ the restriction of $\alpha$ to $X_t$, and note that the $\alpha_0$ defined by restriction is the same as the original $\alpha_0$ since $k$ is a retraction. Treat $\beta_0$ in the same way and consider the function

$$\langle \alpha_t, \beta_t \rangle_t = \int_{X_t} \alpha_t \wedge \beta_t.$$

Since this function is continuous and integer valued, it is constant. Since it restricts to the bilinear form defined by the sum of cup products on components when $t = 0$, it is nondegenerate on $\Lambda$. Now $k^*$, as an isometry, must also be an isomorphism, as required.

We note that $k^*$ also induces an isomorphism on the weight one parts of the mixed Hodge structures. Indeed, the kernel of $k^*$ is the image of $H^2(X', X' - X_0')$ in $H^2(X_0)$. Since the relative cohomology group is generated by Chern classes of components of $X_0$, it is of pure type (1, 1), and so one has

$$W_1 H^2(X_0') \cong W_1 H_\pi^2.$$

It follows that the canonical one-motif for $H^2(X_0')$ and $H_\pi^2$, restricted to $\Lambda$ and $\Lambda_\pi$ are isomorphic, and so we can calculate $u_\pi$ from $u$.

EXAMPLE 1 (continued): When $\pi$ is a pencil of double planes, $y^2 = F_d^2 + tG_{2d}$, $\Lambda_\pi$ is the lattice of rank $r = 2d^2 - 1$ generated by the classes $\mathscr{E}_p - \mathscr{E}_q$. Since

$$\langle \mathscr{E}_p - \mathscr{E}_q, \mathscr{E}_p - \mathscr{E}_q \rangle = -2$$

$$\langle \mathscr{E}_p - \mathscr{E}_q, \mathscr{E}_q - \mathscr{E}_r \rangle = 1$$

$$\langle \mathscr{E}_p - \mathscr{E}_q, \mathscr{E}_r - \mathscr{E}_s \rangle = 0,$$

$\Lambda_\pi$ is the lattice of the Dynkin diagram $A_r$.

Let $D = \{F_d = 0, G_{2d} = 0\}$ be the double curve. Then the limit motif $u_\pi$ is isomorphic to the trace

$$\tau: \Lambda_\pi \to \mathrm{Pic}^0(D).$$

An argument analogous to that of section (12) shows that $\tau$ together with the arithmetic of $\Lambda_\pi$ determines the imbedding

$$B_\pi \subset D.$$

Note that if $d = 3$, then $X/\Delta$ is a $K-3$ degeneration and $B_\pi$ is a set of 18 points on an cubic curve which lies on a sextic. As such $B_\pi$ has 17 moduli which are given Hodge-theoretically as the motivic values of vectors in a root basis of $\Lambda_\pi$. The number of moduli of $u_\pi$ itself is 18, where one comes from the modulus of $D$, and seventeen come from the values of the root vectors.

EXAMPLE 2: Let

$$X_0 X_1 X_2 X_3 + tG = 0$$

be a degeneration of a quartic surface to the "tetrahedron" of coordinate planes. The infinitesimal base locus consists of 24 points, four on each of the six "edges" of the tetrahedron. The lattice is

$$\Lambda_\pi \cong \bigoplus_{i<j} A_3^{ij},$$

where $i$ and $j$ are indices for the components $D_k = \{X_k = 0\}$, and where $A_3^{ij}$ is a copy of the lattice of the Dynkin diagram $A_3$. In this case the one-motif takes values in a torus isomorphic to $\mathbb{C}^*$, namely that given by $\hat{T}_X$, where, following section 5,

$$T_X = \ker\{\delta_*: H_0(X_2, \mathbb{Z}) \to H_0(X_1, \mathbb{Z})\}.$$

To describe a generator of this group let

$[i]$ be a formal symbol for $D_i$
$[ij]$ be a formal symbol for $D_{ij}$
$[ijk]$ be a formal symbol for $D_{ijk}$.

Then

$$\partial[0123] = [123] - [023] + [013] - [012]$$

is a combination of triple points which generates $T_X$.

Now let $p$ and $q$ be infinitesimal base points on $D_{01}$, and let $\xi = \mathscr{E}_p - \mathscr{E}_q$ be the corresponding class in $\Lambda_\pi$. The motivic image of $\xi$ in $T_X^*$ is given by the divisor class of the trace of $\mathscr{E}_p - \mathscr{E}_q$, i.e., by $p - q$. Since all components of $X_1$ are rational, this divisor class is principal, and so is given by a function $f_{pq}$ which

(i) on $D_{01}$ has only a simple zero at $p$ and a simple pole at $q$,
(ii) is a nonzero constant on the other $D_{ij}$.

Thus

$$\langle f_{pq}, \partial[0123]\rangle = \frac{\langle f_{pq}, [123]\rangle\langle f_{pq}, [013]\rangle}{\langle f_{pq}, [023]\rangle\langle f_{pq}, [012]\rangle}$$

$$= \frac{\langle f_{pq}, [013]\rangle}{\langle f_{pq}, [012]\rangle}$$

$$= \text{cross-ratio}\,(p, q, [013], [012]) \quad \text{on } D_{01}.$$

The preceding examples can be summarized by the following general result. Let $\pi$ be a pencil with central fiber $Y$. Assume $Y$ to be a normal crossing variety with simplicial resolution $Y_.$. Let $Z_\pi$ be the group of divisors on $Y_1$ which are supported in $B_\pi$ and which map to zero in the homology of $Y_0$ under $\delta_*$. Let $\tau_\pi$ be the natural one-motif given by the trace:

$$\tau_\pi\colon Z_\pi \to P(Y_.).$$

THEOREM I: *There is a natural isomorphism between $\tau_\pi$ and $u_\pi$.*

REMARK: The ideal $I_\pi$ is determined by a vector in the quotient space $R^d/\text{Radical}(J_F)^d$. Since $R^d/J_F^d$ is the tangent space to $F$ in moduli, we may view this as a vector in $R^d/J_F^d$ modulo the subspace $\text{Radical}(J_F)^d/J_F^d$. In the simple case where

$$F = \prod F_i$$

is a product of "smooth factors", this subspace is the tangent space of deformations obtained by deforming the factors:

$$F + tG \equiv \prod(F_i + tG_i) \quad (\text{modulo } t^2).$$

Consequently $I_\pi$ is determined by a vector in the normal space of $F$ relative to its "equisingular stratum" in moduli. Since $F + tG$ and $F + t\lambda G$ have the same infinitesimal base locus, there is a one-to-one correspondence between such base loci at $F$ and points of the projectivized normal bundle to the equisingular stratum of $F$ in moduli. The infinitesimal base locus thus determines a point in a suitable blow-up of the moduli space along a boundary component.

## 15. Relative cohomology and open varieties

As a final consequence of our main result, we interpret the one-motive for $H^2(X, Y)$, where $X$ is a smooth projective surface and where $Y$ is a singular curve. Take the canonical resolution of $Y$,

$$\begin{array}{ccc} & \tilde{\Sigma} & \\ {\scriptstyle i}\swarrow & & \searrow{\scriptstyle \pi} \\ \tilde{Y} & & \Sigma \\ \searrow & & \swarrow \\ & Y & \end{array}$$

and define the generalized Picard of $Y$ by

$$\mathrm{Pic}^{\#}(Y) = \frac{\left\{ \begin{array}{l} \text{divisors homologous to zero on } \tilde{Y} \\ \text{whose support is disjoint from } \tilde{\Sigma} \end{array} \right\}}{\{\text{divisor}(f) \mid i^*f \in i^*\mathbb{C}^*(\tilde{Y}) \cdot \pi^*(\Sigma)\}}.$$

When $Y$ has ordinary singularities, this group agrees with that of Rosenlicht (see section 13, condition GP).

Define the Neron-Severi group of $X$ relative to $Y$ by

$$\mathrm{NS}(X, Y) = \frac{\left\{ \begin{array}{l} \text{divisors on } X \text{ whose trace} \\ \text{on } \tilde{Y} \text{ is homologous to zero} \end{array} \right\}}{\text{algebraic equivalence}}.$$

Then the trace defines a one-motif

$$\tau\colon \mathrm{NS}(X, Y) \to \mathrm{Pic}^{\#}(Y)/\text{algebraic equivalence on } X$$

which is related to the Hodge-theoretic motif in the expected way:

THEOREM J: *There is a natural isomorphism between the Hodge-theoretic motif u for $H^2(X, Y)$ and the geometric motif $\tau$ for the pair $(X, Y)$.*

Note that if $H^1(X, \mathbb{Q})$ vanishes, then the target torus is just $\mathrm{Pic}^{\#}(Y)$.

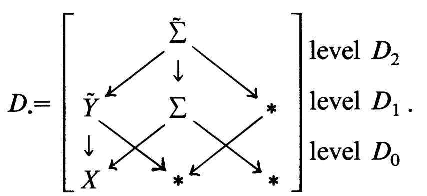
PROOF: Begin with the mapping cone of the inclusion:

$$C(X, Y) = \left[\begin{array}{ccc} & Y & \\ \swarrow & & \searrow \\ X & & * \end{array}\right]$$

If $Y$ is smooth, then the main result applies to give the theorem. If $Y$ is singular, then consider the map of diagrams

$$\left[\begin{array}{ccc} & \tilde{Y} & \\ \swarrow & & \searrow \\ X & & * \end{array}\right] \leftarrow \left[\begin{array}{ccc} & \tilde{\Sigma} & \\ \swarrow & & \searrow \\ \Sigma & & * \end{array}\right]$$

where range is a "desingularization" of $C(X, Y)$, and where the domain is the "exceptional locus" of the desingularization. The mapping cone of this map of diagrams is homotopy equivalent to $C(X, Y)$, and is given by the diagram below:




Since the cohomology of $D.$ is the cohomology of the pair (check directly or see [C3]), we may apply the main result to achieve the present theorem.

As an added dividend, duality gives us a comparison result for the cohomology of open varieties:

THEOREM K: *Let u be the one-motif of $H^2(X - Y)$. Then the trace homomorphism defined above is naturally isomorphic to u.*

## 16. Cartier duals

Cartier duality generalizes to the category of one-motifs the usual duality in the category of abelian varieties. We recall the basic notions [D, section 10.2] of this theory and then describe the Cartier dual of $\eta_X$, the one-motif of Hodge for an algebraic surface.

*a. Basic notions*

The simplest definition of the Cartier dual is in terms of Hodge theory. Let $M = (G, J, A, L, u)$ be a one-motif with extension $[1 \to G \to J \to A \to 0]$ and homomorphism $[u\colon L \to J]$, and let $T(M)$ be the corresponding mixed Hodge structure of type $\{(p, p), (p, p-1), (p-1, p), (p-1, p-1)\}$. Let $\mathbb{Z}(-p)$ be the trivial Hodge structure of type $(p, p)$ with lattice $(2\pi i)^{-p}\mathbb{Z}$, and let

$$T(M)^* = \operatorname{Hom}(T(M), \mathbb{Z}(-p))$$

be the dual mixed Hodge structure, of type $\{(1-p, 1-p), (1-p, -p), (-p, 1-p), (-p, -p)\}$. By the inverse of the correspondence just used, there is a unique one-motif $M^*$ such that $T(M^*) = T(M)^*$. This is the Cartier dual.

It is also possible, although more difficult, to give an algebraic description of the Cartier dual: To begin, denote $M^*$ by $(G', J', A', L', u')$, an observe that

$$G' = \check{L} \quad \text{and} \quad L' = \check{G}$$

are Pontryagin duals, while

$$A' = \check{A}$$

is the dual abelian variety.

To go further, we show how an extension $[1 \to G \to J \to A \to 0]$ defines a one motif $[u_0'\colon \check{G} \to \check{A}]$. To this end, recall that there is a canonical isomorphism of $\check{A}$ with $\operatorname{Pic}^0(A)$. Thus, given a principal $G$-bundle $J$ over $A$ and a character $\chi$ of $G$, one may form the $\mathbb{C}^*$-bundle $J_\chi$ defined by the homomorphism of structure groups $\chi\colon G \to \mathbb{C}^*$. The map

$$\chi \mapsto \text{class of } J_\chi$$

defines the required homomorphism

$$u_0'\colon \check{G} \to \check{A}.$$

The inverse of the construction just given applied to the natural quotient motif

$$[u_0\colon L \to A]$$

then defines an extension

$$1 \to \check{L} \to \check{J} \to \check{A} \to 0$$

This is the extension $(G', J', A')$ of the Cartier dual.

The data on hand suffice to define a diagram

$$\begin{array}{ccccccccc} 1 & \to & \check{L} & \to & \check{J} & \to & \check{A} & \to & 0 \\ & & u' & \updownarrow & \nearrow u_0' & & & & \\ & & & & \check{G} & & & & \end{array}$$

where the lifting $u'$ remains to be determined. To this end consider the Poincaré bundle $P$ over $A \times \check{A}$, and recall that

$$P_{A \times \{\xi\}} = \mathbb{C}^*\text{-bundle of class} \qquad \xi \in \check{A} \cong \operatorname{Pic}^0(A)$$
$$P_{\{\xi\} \times \check{A}} = \mathbb{C}^*\text{-bundle of class} \qquad \xi \in A \cong \operatorname{Pic}^0(\check{A}).$$

If $\check{\mu}$ is an element of the lattice $\check{G}$, then $u_0'(\check{\mu})$ in $\check{A}$ is the class of $\check{\mu}_* J$, so that

$$\text{(a)} \quad P_{A \times \{u_0'(\mu')\}} \cong \check{\mu}_* J.$$

The same argument shows that

$$\text{(b)} \quad P_{\{u_0(\lambda)\} \times \check{A}} \cong \lambda_* \check{J}.$$

From (a) we see that

$$\text{(a}') \quad P_{(u_0(\lambda),\, u'_0(\check{\mu}))} \cong (\check{\mu}_* J)_{u_0(\lambda)} = \check{\mu}_*\left(J_{u_0(\lambda)}\right),$$

and from (b), that

$$\text{(b}') \quad P_{(u_0(\lambda),\, u_0'(\check{\mu}))} \cong (\lambda_* \check{J})_{u_0'(\check{\mu})} = \lambda_*\left(J_{u_0'(\check{\mu})}\right).$$

Since the homomorphism $u$ determines a point $u(\lambda)$ in $J_{u_0(\lambda)}$, it associates to each pair $(\lambda, \check{\mu})$ in $L \times \check{G}$ a point in

$$\check{\mu}_*\left(J_{u_0(\lambda)}\right) = \left(P_{u_0(\lambda),\, u_0'(\check{\mu})}\right).$$

Thus, a lifting $u$ of $u_0$ determines a section

$$\psi_u \colon L \times \check{G} \to (u_0, u_0')^* P,$$

and, conversely, a section $\psi$ determines a lifting $u_\psi$. To define $u'$, form $\psi_u$ and then apply the converse correspondence just defined with the roles of $L$ and $\check{G}$ reversed. This completes the algebraic description of the Cartier dual.

A useful conclusion to be drawn from the preceding discussion is that to give a one-motif $M$ is to give the data

(i) a pair of lattices $L$ and $L'$

(ii) two abelian varieties $A$ and $A'$ in duality

(iii) a pair of homomorphisms

$$u_0\colon L \to A$$

$$u_0'\colon L' \to A'$$

(iv) a section $\psi$ of $(u_0, u_0')^*P$.

The object $sM = (L, L', A, A', u_0, u_0', \psi)$ so defined is the *symmetric incarnation* of $M$ defined by Deligne [D, 10.2.13].

*b. The Cartier dual of $\eta$*

We now propose to identify the geometric counterpart of the Cartier dual of the one-motif of Hodge. Because $M$ has a symmetric incarnation, it suffices to identify the geometric objects corresponding to $\eta_0'$ and to section of the pullback of the Poincare bundle. To this end, consider the dual structure on homology modulo $W_{-2}$. The canonical extension is

$$0 \to Gr_{-1}^W H_2 \to H_2/W_{-2} \to Gr_0^W H_2 \to 0,$$

with associated one-motif

$$\eta_0'\colon L^0 H_2 \to J^0 Gr_{-1}^W H_{2\cdot}.$$

Definition: Let $X.$ be a semisimplicial surface, and recall that an element $\xi$ of $T(X.)$ is a zero-cycle on $X_2$ such that $\delta_*(\xi)$ is homologous to zero on $X_1$. Let

$$\mathscr{A}\colon T(X.) \to \operatorname{Pic}^0(X_1)$$

be the homomorphism defined by

$$\xi \mapsto \text{divisor class of } \delta_*(\xi).$$

Theorem L: *The one-motif $\eta_0'$ is isomorphic to $\mathscr{A}$ modulo the image of algebraic equivalence on $X_0$.*

Remarks:

(1). The cup product on $X_1$ makes $A$ self-dual.

(2). In the normal crossing case $\eta_0'$ reflects the position of the triple points on the double curve.

PROOF:

To calculate $\eta'_0$, consider sections

$$s_{\mathbf{Z}}\colon L^0H_2 \to H_{2,\mathbf{Z}}$$

$$s_F\colon L^0H_{2,\mathbf{C}} \to H_{2,\mathbf{C}},$$

and recall that

$$\eta'_0(\gamma) = \text{class of } (s_{\mathbf{Z}} - s_F)(\gamma) \text{ in } J^0Gr^W_{-1}H_2.$$

Since $Gr^W_{-1}H_2$ and $Gr^W_1H^2$ are dual, we may identify

$$\frac{Gr^W_{-1}H_{2,\mathbf{C}}}{F^0Gr^W_{-1}H_2} \quad \text{with } \left(F^1Gr^W_1H^2\right)^*$$

and

$$J^0Gr^W_0H_2 \quad \text{with} \frac{\left(F^1Gr^W_1H^2\right)^*}{Gr^W_1H_{2,\mathbf{Z}}}.$$

Via this identification $\eta'_0$ is defined by the functional

$$[\omega] \mapsto \langle (s_{\mathbf{Z}} - s_F)(\gamma), [\omega] \rangle$$

where $[\omega] \in F^1Gr^W_1H^2$.

It remains to identify the preceding functional. To this end, we remark that the integral homology of $X.$ can be calculated from the double complex defined by

$$C.(X.) = \left[C.(X_0) \overset{\delta_*}{\leftarrow} C.(X_1) \overset{\delta_*}{\leftarrow} \dots\right],$$

where $C.$ is the singular chain functor. Thus, a $k$-cycle on $X.$ consists of a sum

$$\Gamma = \sum_P \Gamma_p$$

where

$$\Gamma_p \in C_{k-p}(X_p)$$

and

$$D\Gamma = 0,$$

i.e.,

$$\delta_* \Gamma_p + (-1)^{p-1} \partial \Gamma_{p-1} = 0.$$

In addition we note that $[\omega]$ in $H^2$ is represented by

$$\omega = \omega_0 + \omega_1 + \omega_2$$

with

$$\omega_p \in F^1 A^{2-p}(X_p).$$

We may choose $\omega_0 = \omega_2 = 0$ and $\omega_1$ in $H^0(X_1, \Omega^1)$. Since $s_F([\omega])$ is in $F^0 H_2 = (F^1 H^2)^\perp$, we have

$$\langle (s_{\mathbf{Z}} - s_F)(\gamma), [\omega] \rangle = \langle s_{\mathbf{Z}}(\gamma), [\omega] \rangle = \langle s_{\mathbf{Z}}(\gamma), [\omega_1] \rangle$$

$$= \int_{\Gamma_1} \omega_1$$

But $D\Gamma = 0$ implies that

$$\partial \Gamma_1 = \delta_* \Gamma_2 = \delta_* \gamma,$$

so that the integral in question is the abelian integral associated to the zero-cycle $\delta_*(\gamma)$. Q.E.D.

We close with a discussion of the section $\psi_\eta$ of $(\eta_0, \eta_0')^* P$. By relation (a′) above, such a section corresponds to the lifting $\eta$ of $\eta_0$ and is determined by the "twisted pairing"

$$(\lambda, \check{\mu}) \mapsto \check{\mu}_*(\eta(\lambda)) \in \check{\mu}_* J_{\eta_0(\lambda)} \cong \mathbf{C}^*.$$

Now the canonical extension for $J = J^1 W_1 H^2(X.)$ is

$$1 \to \frac{W_0 H^2_{\mathbf{C}}}{W_0 H^2_{\mathbf{Z}}} \to \frac{W_1 H^2_{\mathbf{C}}}{F^1 W_1 H^2_{\mathbf{C}} + W_1 H^2_{\mathbf{Z}}} \to \frac{Gr_1^W H^2_{\mathbf{C}}}{F^1 Gr_1^W H^2_{\mathbf{C}} + Gr_1^W H^2_{\mathbf{Z}}} \to 0$$

so that the fiber over zero,

$$J_0 = \frac{W_0 H^2_{\mathbf{C}}}{W_0 H^2_{\mathbf{Z}}},$$

has a natural identification $\varphi$ with $(Gr_0^W H_{2,\mathbf{Z}})^\vee$ given by

$$\psi(z)(w) = \exp 2\pi i z(w)$$

where $z \in W_0 H^2_{\mathbb{C}}$ and $w \in Gr^W_0 H_{2,\mathbb{Z}}$. Thus, the twisted pairing specializes, for $\lambda$ with $\eta_0(\lambda) = 0$, to the untwisted homomorphism

$$\check{G} \to \mathbb{C}^*$$

defined by

$$\check{\mu} \mapsto \check{\mu}_*(\eta(\lambda)) = \check{\mu}(\varphi \circ \eta(\lambda)) \in \mathbb{C}^*.$$

To describe the geometric counterpart of this homomorphism suppose given $Z \in \mathrm{NS}(X.)$, and suppose that the "primary obstruction" $\tau_0(Z)$ vanishes, so that $Z$ is represented by a line bundle on the truncated space $[X_0 \leftleftarrows X_1]$. Then the section $\psi_\eta$ corresponds to a section $\psi_\tau$ which in turn corresponds to a homomorphism

$$T(X.) \to \mathbb{C}^*,$$

namely that given by

$$\xi \mapsto \langle f_Z, \delta_* \xi \rangle$$

where $f$ is a meromorphic function on $X_1$ with divisor $(f) = \tau_0(Z)$. This homomorphism is the (secondary)obstruction to extending the line bundle on $[X_0 \leftleftarrows X_1]$ to one on $[X_0 \leftleftarrows X_1 \leftleftarrows X_2]$. The preceding correspondence, whose verification we leave to the reader, generalizes theorem $H$ of section 14: the secondary obstruction corresponds to a set of generalized cross-ratios formed from the trace of $Z$ on the singular curve and from the singular points of the singular curve.

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