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RELATIVE EXTREME SUBSETS

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Generalizing the notion of extreme point of a set in the real linear space L, Klee [2] introduced the following definition of *relative extreme point*. Let $B \subset L$ and $C \subset L$. If a point of B does not belong to any open segment $(b, c) = \{(1-\lambda)b + \lambda c;\ 0 < \lambda < 1\}$ determined by distinct points $b \in B$ and $c \in C$, then it is called an *extreme point in B relative to C*. Observe that the known notion of extreme subset can be generalized analogously:

DEFINITION: Let $A \subset B \subset L$ and $C \subset L$. We say that A is an *extreme subset of B relative to C* if, together with any point $a \in A$, the set A contains every point $b \in B$ such that $a \in (b, c)$ for some $c \in C$.

Let us note that the definition can be expressed more geometrically using the notion

$$P_C(A) = \{(1-\mu)c + \mu a;\ \mu \geqslant 1,\ c \in C,\ a \in A\}$$

of the penumbra ([5], p. 22) of A with respect to C. Namely, a subset A of $B \subset L$ is an extreme subset of B relative to a non-empty set $C \subset L$ if and only if

$$P_C(A) \cap B = A.$$

Obviously in the case $A = \{a\}$ of our definition we get the notion of extreme point a in B relative to C and in the case $B = C$ we obtain the usual notion of extreme subset A of B. On the other hand, the above definition is a special case of the notion (presented as Remark in [3]) of Φ-extreme subset, where $\Phi : \mathscr{D} \to 2^L$ is a function such that $\mathscr{D}$ consists of all one-point subsets of L and $\Phi(\{b\}) = \bigcup_{c \in C}(b, c)$. Let us observe also a connection of our definition with the notion of semi-extreme subset. Remember that a subset A of a convex set $B \subset L$ is called a *semi-extreme* subset of B if $B \setminus A$ is convex (comp. [1], p. 32). As in [6], pp. 186–187, this notion of semi-extreme subset can be extended to arbitrary (i.e. not necessary convex) set B: if $A \subset B$ and $A \cap \operatorname{conv}(B \setminus A) = \emptyset$, then we call A a *semi-extreme* subset of B. The above mentioned connection is expressed by the following easily provable:

PROPOSITION: *If A is a semi-extreme subset of B, then A is an extreme subset of B relative to $B \setminus A$. When B is convex, the inverse implication also holds.*

The reader can without difficulty verify six properties of relative extreme subsets presented in Theorem 1, the first five of which generalize well-known properties of extreme subsets in the usual sense.

THEOREM 1: *Relative extreme subsets have the following properties*

(a) *Any intersection of extreme subsets of B relative to C is an extreme subset of B relative to C.*
(b) *Any union of extreme subsets of B relative to C is an extreme subset of B relative to C.*
(c) *If A is an extreme subset of B relative to C and if A_1 is an extreme subset of A relative to C, then A_1 is an extreme subset of B relative to C.*
(d) *If $A \subset B_1 \subset B_2$ and if A is an extreme subset of B_2 relative to C, then A is an extreme subset of B_1 relative to C.*
(e) *Sets B and $\emptyset$ are extreme subsets of B relative to any set C.*
(f) *If $C_1 \subset C_2$ and if A is an extreme subset of B relative to C_2, then A is an extreme subset of B relative to C_1. Any subset of B is extreme in B relative to empty set.*

The notion of the usual extreme subset of a set B is considered mainly in the case when B is convex. Also the notion of extreme point of B relative to C plays an important part in the case when B is convex and $C \subset B$ (comp. [2] and [4]). This is why we now consider extreme subsets of a convex set B relative to a subset of B.

THEOREM 2: *Let B be a convex set of a real linear space L and let $A \subset B$, $C \subset B$. The set A is an extreme subset of B relative to C if and only if A is an extreme subset of B relative to the convex hull* conv C.

PROOF: Suppose that A is an extreme subset of B relative to C. To verify if A is an extreme subset of B relative to conv C we shall show that for any $a \in A$, $b \in B$ and $c \in \operatorname{conv} C$ such that $a \in (b, c)$ we have $b \in A$.

As an element of conv C, the point c belongs to the convex hull of a finite number of points of C. Consequently, there exists a minimal finite collection of points $c_1, \dots, c_k \in C$ such that

$$c \in \operatorname{conv}\{b, c_1, \dots, c_k\}.$$

In other words

$$c = \alpha_0 b + \alpha_1 c_1 + \dots + \alpha_k c_k,$$

where $\alpha_0 \geqslant 0$, $\alpha_1 > 0, \dots, \alpha_k > 0$ and $\alpha_0 + \alpha_1 + \dots + \alpha_k = 1$. Since $a = \beta b + \gamma c$ for some $\beta > 0$ and $\gamma > 0$ such that $\beta + \gamma = 1$, we have

$$a = (1 - \delta_1 - \dots - \delta_k)b + \delta_1 c_1 + \dots + \delta_k c_k,$$

where $\delta_1 = \gamma\alpha_1 > 0, \dots, \delta_k = \gamma\alpha_k > 0$ and $1 - \delta_1 - \dots - \delta_k = 1 - \gamma(\alpha_1 + \dots + \alpha_k) = 1 - \gamma(1 - \alpha_0) = \beta + \gamma\alpha_0 > 0$.

Now, we recurrently define points $b_k, b_{k-1}, \dots, b_1$ as follows

$$b_k = b,$$

$$b_i = \frac{\delta_{i+1}}{1 - \delta_1 - \dots - \delta_i} c_{i+1} + \frac{1 - \delta_1 - \dots - \delta_{i+1}}{1 - \delta_1 - \dots - \delta_i} b_{i+1}, \quad i = k-1, \dots, 1.$$

Since the coefficients

$$\delta_{i+1}/(1 - \delta_1 - \dots - \delta_i),\ (1 - \delta_1 - \dots - \delta_{i+1})/(1 - \delta_1 - \dots - \delta_i)$$

are positive and since the sum of them is equal to 1, the definition of b_i implies that

$$b_i \in (c_{i+1}, b_{i+1}), \quad i = 1, \dots, k-1. \tag{1}$$

By the definition of b_i, the equality

$$\delta_{i+1}c_{i+1} + (1 - \delta_1 - \dots - \delta_{i+1})b_{i+1} = (1 - \delta_1 - \dots - \delta_i)b_i$$

holds for $i = k-1, \dots, 1$ and consequently

$$\begin{aligned} a &= \delta_1 c_1 + \dots + \delta_k c_k + (1 - \delta_1 - \dots - \delta_k)b_k \\ &= \delta_1 c_1 + \dots + \delta_{k-1}c_{k-1} + [\delta_k c_k + (1 - \delta_1 - \dots - \delta_k)b_k] \\ &= \delta_1 c_1 + \dots + \delta_{k-1}c_{k-1} + (1 - \delta_1 - \dots - \delta_{k-1})b_{k-1} \\ &= \dots = \delta_1 c_1 + (1 - \delta_1)b_1. \end{aligned}$$

Thus in virtue of $\delta_1 > 0$ and $1 - \delta_1 > 0$ we have

$$a \in (c_1, b_1). \tag{2}$$

Since B is convex, from $b_k \in B$ and $c_k, \dots, c_1 \in B$ and also from $b_i \in (c_{i+1}, b_{i+1})$ for $i = k-1, \dots, 1$ we get in turn that $b_i \in B$ for $i = k-1, \dots, 1$.

Since A is an extreme subset of B relative to C and since $a \in A$, $b_i \in B$ and $c_i \in C$ for $i = 1, \dots, k$, we first obtain from (2) that $b_1 \in A$ and next (if $k \geqslant 2$), applying $(k-1)$-times (1) we get in turn that

$b_2 \in A, \ldots, b_k \in A$. Thus $b = b_k \in A$. Hence A is an extreme subset of B relative to C.

The inverse implication of our theorem results immediately from the inclusion $C \subset \operatorname{conv} C$ and from property (f) of Theorem 1.

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