Compositio Mathematica

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Compositio Mathematica, tome 56, n° 2 (1985), p. 201-220
<http://www.numdam.org/item?id=CM_1985__56_2_201_0>

Compositio Mathematica 56 (1985) 201–220.

ISOCRYSTALS WITH ADDITIONAL STRUCTURE

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Let k be an algebraically closed field of characteristic $p > 0$, and let K be the fraction field of the Witt ring $W(k)$. The Frobenius automorphism of k induces an automorphism σ of K. An F-isocrystal is a finite dimensional vector space V over K together with a σ-semilinear bijection Φ: $V \to V$ (recall that Φ is said to be σ-semilinear if it is a group homomorphism such that $\Phi(\alpha v) = \sigma(\alpha)\Phi(v)$ for all $\alpha \in K$, $v \in V$). In this paper we will shorten "F-isocrystal" to "isocrystal".

Let V be an n-dimensional vector space over $\mathbb{Q}_p$ and let $G = GL(V)$. For any element $b \in G(K)$ we get an isocrystal (V_K, Φ), where $V_K = V \otimes_{\mathbb{Q}_p} K$ and $\Phi = b \circ (\mathrm{id}_V \otimes \sigma)$. If b' is σ-conjugate to b (in other words, if $b' = gb\sigma(g)^{-1}$ for some $g \in G(K)$), then the two isocrystals we get are isomorphic. This construction yields a bijection from the set of σ-conjugacy classes in $G(K)$ to the set of isomorphism classes of n-dimensional isocrystals. The Dieudonné-Manin description of the category of isocrystals (see §3 for a review of this theory) makes it possible to give a simple classification of n-dimensional isocrystals, which can then be translated into a classification of the σ-conjugacy classes in $G(K)$.

This paper studies the set $B(G)$ of σ-conjugacy classes in $G(K)$ for any connected reductive group G over $\mathbb{Q}_p$. In fact we work with a slightly more general situation, in which $\mathbb{Q}_p$ is replaced by a finite extension F, but in this introduction we discuss only the case $F = \mathbb{Q}_p$.

Let $\mathbb{D}$ be the diagonalizable pro-algebraic group over $\mathbb{Q}_p$ with character group $\mathbb{Q}$. Return for the moment to the case $G = GL(V)$. Let $b \in G(K)$ and let (V_K, Φ) be the corresponding isocrystal. The slope decomposition of V_K (see §3) gives us a homomorphism ν: $\mathbb{D} \to G$ defined over K. In this way we get a mapping $b \mapsto \nu$ from $G(K)$ to $\mathrm{Hom}_K(\mathbb{D}, G)$.

In §4 we construct a mapping $b \mapsto \nu$ for any connected linear group G. Roughly speaking, we consider simultaneously all representations ρ of G on finite dimensional vector spaces over $\mathbb{Q}_p$ in order to reduce to the case of $GL(V)$. We also show that ν is trivial if and only if the σ-conjugacy class of b is in the image of the canonical injection

$$H^1(\mathbb{Q}_p, G) \to B(G).$$

* Partially supported by the National Science Foundation under Grant MCS 82-00785.

We say that an element $b \in G(K)$ is basic if the corresponding homomorphism ν: $\mathbb{D} \to G$ factors through the center of G. For $G = GL(V)$ the element b is basic if and only if the corresponding isocrystal is isotypic. We write $B(G)_b$ for the set of σ-conjugacy classes of basic elements of $G(K)$. In §5 we construct a canonical bijection

$$B(G)_b \tilde{\to} X^*\big(Z(\hat{G})^\Gamma\big)$$

for any connected reductive group G over $\mathbb{Q}_p$. Here $Z(\hat{G})$ is the center of $\hat{G} = {}^LG^0$ (see [B] for the definition of ${}^LG^0$), and Γ is Gal($\overline{\mathbb{Q}}_p/\mathbb{Q}_p$). This bijection extends the bijection

$$H^1(\mathbb{Q}_p, G) \tilde{\to} \pi_0\big(Z(\hat{G})^\Gamma\big)^D$$

in §6 of [K1]. For $G = GL(V)$ we have $\hat{G} = GL_n(\mathbb{C})$ ($n = \dim V$) with trivial Γ-action, and therefore $X^*(Z(\hat{G})^\Gamma)$ is equal to $X^*(\mathbb{G}_m) = \mathbb{Z}$. The n-dimensional isocrystal corresponding to $m \in \mathbb{Z}$ is isotypic of slope m/n.

For tori T every element of $T(K)$ is basic, $B(T)$ is a group, and we get a group isomorphism

$$B(T) \tilde{\to} X^*(\hat{T}^\Gamma).$$

Since we prove the general case by reducing to the case of tori, we must handle tori directly. This is done in §2. Note that $X^*(\hat{T}^\Gamma) = X_*(T)_\Gamma$, the coinvariants of Γ on $X_*(T)$, and what we get in §2 is actually a canonical isomorphism

$$X_*(T)_\Gamma \tilde{\to} B(T).$$

This is proved by characterizing the functor $T \mapsto X_*(T)_\Gamma$ (Lemma 2.2) and showing that $B(T)$ satisfies this characterization (Proposition 2.3).

So far we have only discussed the subset $B(G)_b$ of $B(G)$. If G is quasisplit, we can describe all of $B(G)$ by using Levi subgroups M of G. For $G = GL(V)$ this just amounts to decomposing an isocrystal into a direct sum of isotypic subspaces. For a Levi subgroup M of any quasi-split group G we say that a basic element $b \in M(K)$ is G-regular if the centralizer in G of ν: $\mathbb{D} \to M$ is equal to M, and we write $B(M)_{br}$ for the subset of $B(M)_b$ consisting of G-regular elements. In §6 we show that every element of $B(G)$ arises from a pair (M, b), where M is a Levi subgroup of G and $b \in B(M)_{br}$, and that the pair (M, b) is uniquely determined up to $G(\mathbb{Q}_p)$-conjugacy.

These results about $B(G)$ are useful in studying the points mod p on Shimura varieties. In fact the appendix of [L] gives a construction of certain elements of $B(G)$, and this paper arose from an attempt to systematize Langlands's construction. However, the methods of the two papers are sufficiently different that it will require some patience on the part of the reader to appreciate the connection between the two.

1. Preliminaries

The following notation will be used throughout this paper.

k – an algebraically closed field of characteristic $p > 0$
K – the fraction field of the Witt ring $W(k)$
$\overline{K}$ – an algebraic closure of K
F – a finite extension of $\mathbb{Q}_p$ in $\overline{K}$
$\overline{F}$ – the algebraic closure of F in $\overline{K}$
L – the compositum of K and F in $\overline{K}$
Γ – the Galois group of $\overline{F}/F$

1.1. Let k_F denote the residue field of F and let M denote the fraction field of $W(k_F)$. The extension F/M is totally ramified of degree e, the absolute ramification index of F. The canonical homomorphism $K \otimes_M F \to L$ is an isomorphism, since an Eisenstein polynomial over M remains Eisenstein over K. In particular, the extension L/K is totally ramified of degree e, which means that e is also the absolute ramification index of L.

The Frobenius automorphism of k relative to k_F induces an automorphism of K over M, which in turn induces an automorphism σ of L over F.

1.2. LEMMA: *The fixed field of σ on L is F.*

Let N denote this fixed field. It is clear that N contains F, has the same residue field as F, and is discretely valued. Therefore N is a finite totally ramified extension of F in L. Since L and F have the same absolute ramification index, N is equal to F.

1.3. We define the *Weil group* $W(\overline{K}/F)$ to be the group of continuous automorphisms of $\overline{K}$ which fix F pointwise and which induce on the residue field of $\overline{K}$ an integral power of the Frobenius automorphism. There is an exact sequence

$$1 \to \mathrm{Gal}(\overline{K}/L) \to W(\overline{K}/F) \to \langle\sigma\rangle \to 1, \tag{1.3.1}$$

where $\langle\sigma\rangle$ denotes the infinite cyclic group generated by σ. We turn $W(\overline{K}/F)$ into a topological group by requiring that the injection $\mathrm{Gal}(\overline{K}/L) \to W(\overline{K}/F)$ identify $\mathrm{Gal}(\overline{K}/L)$ with an open subgroup of $W(\overline{K}/F)$.

For any finite Galois extension E of F in $\overline{K}$ there is an exact sequence

$$1 \to W(\overline{K}/E) \to W(\overline{K}/F) \to \mathrm{Gal}(E/F) \to 1. \tag{1.3.2}$$

1.4. EXAMPLE. Suppose that k is an algebraic closure of $\mathbb{F}_p$. Then $\bar{F}$ is dense in $\bar{K}$, and the restriction homomorphism $W(\bar{K}/F) \to \mathrm{Gal}(\bar{F}/F)$ induces an isomorphism between $W(\bar{K}/F)$ and the usual absolute Weil group of the local field F.

1.5. EXAMPLE. The group $W(\bar{K}/\mathbb{Q}_p)$ is used in [B-O], where it is called the crystalline Weil group of $\bar{K}$.

1.6. Let A be a group on which $W(\bar{K}/F)$ acts. We say that A is *discrete* if the stabilizer of any element of A is open in $W(\bar{K}/F)$. For example, $G(\bar{K})$ is discrete for any algebraic group G over F. For any discrete $W(\bar{K}/F)$-group A we define $H^1(W(\bar{K}/F), A)$ to be the direct limit over N of the sets $H^1(W(\bar{K}/F)/N, A^N)$, where N runs through the directed set of open normal subgroups of $W(\bar{K}/F)$.

1.7. Let G be an algebraic group over F. We define $B(G)$ to be the pointed set $H^1(\langle\sigma\rangle, G(L))$. More concretely, $B(G)$ is the quotient of $G(L)$ by the equivalence relation σ-conjugacy: $x, y \in G(L)$ are said to be σ-conjugate if there exists $g \in G(L)$ such that $y = gx\sigma(g)^{-1}$. Note that x, y are σ-conjugate if and only if the elements $x\sigma$, $y\sigma$ of the semidirect product $G(L) \rtimes \langle\sigma\rangle$ are conjugate under $G(L)$.

1.8. Assume further that G is connected and linear. Then $H^1(L, G)$ is trivial [St] since (cohomological) dim $L \leqslant 1$, and thus we have a bijection

$$B(G) \to H^1(W(\bar{K}/F), G(\bar{K})) \tag{1.8.1}$$

(inflation for the quotient $\langle\sigma\rangle$ of $W(\bar{K}/F)$).

The restriction homomorphism $W(\bar{K}/F) \to \mathrm{Gal}(\bar{F}/F)$ and the inclusion $G(\bar{F}) \subset G(\bar{K})$ give us an injection

$$H^1(F, G) \to H^1(W(\bar{K}/F), G(\bar{K})). \tag{1.8.2}$$

To prove the injectivity it is enough to show that

$$H^1(E/F, G(E)) \to H^1(W(\bar{K}/F), G(\bar{K}))$$

is injective for every finite Galois extension E of F in $\bar{K}$. It follows from 1.2 and 1.3.2 that this map is the inflation map for $G(\bar{K})$ and the quotient $\mathrm{Gal}(E/F)$ of $W(\bar{K}/F)$, and inflation maps are always injective for H^1.

Combining (1.8.1) and (1.8.2), we get an injection

$$H^1(F, G) \to B(G). \tag{1.8.3}$$

1.9. Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of algebraic groups over F, and assume that G_1 is connected and linear. Since $H^1(L, G_1)$ is trivial, the sequence

$$1 \to G_1(L) \to G_2(L) \to G_3(L) \to 1$$

is exact. Taking cohomology with respect to the group $\langle \sigma \rangle$ and using 1.2, we get an exact sequence

$$1 \to G_1(F) \to G_2(F) \to G_3(F) \to B(G_1) \to B(G_2) \to B(G_3) \to 1. \tag{1.9.1}$$

The surjectivity of $B(G_2) \to B(G_3)$ is an immediate consequence of the surjectivity of $G_2(L) \to G_3(L)$.

1.10. Let F' be a finite extension of F contained in $\bar{K}$, and let L', σ' be the analogues of L, σ for F'. Let G be an algebraic group over F', and let RG denote the F-group obtained from G by restriction of scalars. Then the $\langle \sigma \rangle$-group $RG(L)$ is induced from the $\langle \sigma' \rangle$-group $G(L')$. Thus there is a Shapiro bijection

$$B(RG) \xrightarrow{\sim} B(G). \tag{1.10.1}$$

2. Tori

In this section we study $B(T)$ for F-tori T. Throughout this section E (and sometimes E' as well) denotes a finite Galois extension of F that is contained in $\bar{F}$. The set of all such E is a directed set, and it will be useful to consider the pro-object $\underset{E}{\overset{\text{“lim”}}{\leftarrow}} R_{E/F}\mathbb{G}_m$, where $R_{E/F}$ denotes restriction of scalars from E to F. For $E' \supset E$ the transition homomorphism is the norm homomorphism $N_{E'/E}$. The Γ-module $X_*(R_{E/F}\mathbb{G}_m)$ is canonically isomorphic to $\mathbb{Z}[\mathrm{Gal}(E/F)]$, with Γ acting by left translations. The group $\mathrm{Gal}(E/F)$ acts on $R_{E/F}\mathbb{G}_m$ by F-automorphisms; for $\tau \in \mathrm{Gal}(E/F)$ we use f_τ to denote the corresponding automorphism of $R_{E/F}\mathbb{G}_m$. For ρ, $\tau \in \mathrm{Gal}(E/F)$ the automorphism f_ρ sends the basis element τ of $X_*(R_{E/F}\mathbb{G}_m)$ to the basis element $\tau\rho^{-1}$. Applying X_* to the norm homomorphism $N_{E'/E}$: $R_{E'/F}\mathbb{G}_m \to R_{E/F}\mathbb{G}_m$, we get the homomorphism $\mathbb{Z}[\mathrm{Gal}(E'/F)] \to \mathbb{Z}[\mathrm{Gal}(E/F)]$ induced by the canonical surjection $\mathrm{Gal}(E'/F) \to \mathrm{Gal}(E/F)$.

2.1. Consider additive functors A: (F-tori) $\to$ (abelian groups) satisfying the following two conditions:

(2.1.1) $A(R_{E/F}\mathbb{G}_m)$ is isomorphic to $\mathbb{Z}$ for any E.

(2.1.2) For any exact sequence $1 \to T_1 \to T_2 \to T_3 \to 1$ of F-tori, the sequence $A(T_1) \to A(T_2) \to A(T_3) \to 1$ is exact.

The following lemma is valid for any field F, not just the p-adic fields we are considering in this paper.

2.2. LEMMA. (a) *The functor* $T \mapsto X_*(T)_\Gamma$ *satisfies the conditions of* 2.1 (*we use a subscript* Γ *to denote the coinvariants of* Γ).

(b) *Assume that* A *satisfies the conditions of* 2.1. *Then* A *is isomorphic to the functor* $X_*(\ )_\Gamma$ *of* (a), *and the canonical homomorphism*

$$\mathrm{Hom}(X_*(\ )_\Gamma, A) \to \mathrm{Hom}(X_*(\mathbb{G}_m)_\Gamma, A(\mathbb{G}_m)) = A(\mathbb{G}_m) \text{ is an isomorphism.} \tag{2.2.1}$$

Part (a) is an easy exercise. As for part (b), we first note that

$$A(N_{E'/E}) \colon A(R_{E'/F}\mathbb{G}_m) \to A(R_{E/F}\mathbb{G}_m) \tag{2.2.2}$$

is an isomorphism for all E, E' with $E' \supset E$, since $A(N_{E'/E})$ is a surjection (use (2.1.2), which is applicable since the kernel of $N_{E'/E}$ is an F-torus) from one infinite cyclic group to another (use (2.1.1)).

Now we prove that (2.2.1) is an isomorphism. The functor X_* is pro-represented by $\underset{\overleftarrow{E}}{\text{“lim”}}\, R_{E/F}\mathbb{G}_m$ and the element $(\mu_E) \in \varprojlim_E X_*(R_{E/F}\mathbb{G}_m)$, where μ_E is the element of $X_*(R_{E/F}\mathbb{G}_m)$ corresponding to $1 \in \mathbb{Z}[\mathrm{Gal}(E/F)]$. Therefore we have an isomorphism

$$\mathrm{Hom}(X_*, A) \xrightarrow{\sim} \varprojlim_E A(R_{E/F}\mathbb{G}_m).$$

Since the transition homomorphisms (2.2.2) are isomorphisms, the projective limit is simply $A(\mathbb{G}_m)$, and thus we get an isomorphism

$$\mathrm{Hom}(X_*, A) \xrightarrow{\sim} A(\mathbb{G}_m). \tag{2.2.3}$$

The homomorphism (2.2.1) is the composition of (2.2.3) and the homomorphism

$$\mathrm{Hom}(X_*(\ )_\Gamma, A) \to \mathrm{Hom}(X_*, A) \tag{2.2.4}$$

obtained from the canonical homomorphism $X_* \to X_*(\ )_\Gamma$. We must show that (2.2.4) is an isomorphism.

The injectivity of (2.2.4) follows from the surjectivity of $X_*(T) \to X_*(T)_\Gamma$. To prove the surjectivity of (2.2.4), we consider $\alpha \in \mathrm{Hom}(X_*, A)$ and show that $\alpha_T \colon X_*(T) \to A(T)$ factors through $X_*(T) \to X_*(T)_\Gamma$ for any F-torus T. Choose E so that T splits over E. Let $\mu \in X_*(T)$ and let $\tau \in \mathrm{Gal}(E/F)$. There exists a (unique) F-homomorphism $R_{E/F}\mathbb{G}_m \to T$ that carries μ_E into μ. Thus, without loss of generality, we may assume that $T = R_{E/F}\mathbb{G}_m$ and $\mu = \mu_E$. In the beginning of this section we defined an F-automorphism f_τ of $R_{E/F}\mathbb{G}_m$. This F-automorphism carries $\tau\mu_E$ into μ_E. Thus it is enough to show that $A(f_\tau)$ is the identity, and this follows from the fact that $N_{E/F} \circ f_\tau = N_{E/F}$, since we have seen that $A(N_{E/F})$ is an isomorphism.

Now we prove the first part of (b). By (2.1.1) the group $A(\mathbb{G}_m)$ is isomorphic to $\mathbb{Z}$; we choose a generator of $A(\mathbb{G}_m)$ and let α: $X_*(\ )_\Gamma \to A$ be the corresponding homomorphism of functors (use (2.2.1)). Our plan is to show that α is an isomorphism.

By construction, α_T: $X_*(T)_\Gamma \to A(T)$ is an isomorphism for $T = \mathbb{G}_m$. Using (2.2.2) for the functors $X_*(\ )_\Gamma$ and A, we see that α_T is an isomorphism for $T = R_{E/F}\mathbb{G}_m$. Since A is additive, we have $A(T_1 \times T_2) = A(T_1) \times A(T_2)$, and therefore α_T is an isomorphism for any product of tori of the form $R_{E/F}\mathbb{G}_m$. Finally, for an arbitrary torus T we choose an exact sequence $1 \to U \to V \to T \to 1$ of tori, where V is a product of tori of the form $R_{E/F}\mathbb{G}_m$. We have a commutative diagram

$$\begin{array}{ccccccc} X_*(U)_\Gamma & \to & X_*(V)_\Gamma & \to & X_*(T)_\Gamma & \to & 1 \\ \downarrow \alpha_U & & \downarrow \alpha_V & & \downarrow \alpha_T & & \\ A(U) & \to & A(V) & \to & A(T) & \to & 1 \end{array}$$

with exact rows. Since α_V is an isomorphism, α_T is surjective. Applying this to U, we see that α_U is surjective. This is enough to show that α_T is an isomorphism.

2.3. PROPOSITION: *The functor* $T \mapsto B(T)$ *satisfies the two conditions of* 2.1.

First we check (2.1.1). Using 1.10, we see that it suffices to show that $B(\mathbb{G}_m)$ is isomorphic to $\mathbb{Z}$. For this it is enough to show that $x \in L^\times$ is of the form $y\sigma(y)^{-1}$ $(y \in L^\times)$ if and only if the valuation of x is 0. It is obvious that the valuation of $y\sigma(y)^{-1}$ is 0. We will prove the reverse implication by showing that the homomorphism β: $\mathfrak{o}_L^\times \to \mathfrak{o}_L^\times$ defined by $\beta(y) = y\sigma(y)^{-1}$ is surjective ($\mathfrak{o}_L$ denotes the valuation ring of L). Since β preserves the usual filtration of $\mathfrak{o}_L^\times$, and since $\mathfrak{o}_L^\times$ is complete and separated for the topology defined by this filtration, it is enough to check that β induces surjections on the pieces of the associated graded group. Thus it is enough to check that

$$\begin{array}{ll} \beta_1\colon k^\times \to k^\times & \beta_1(x) = x^{1-q} \\ \beta_2\colon k \to k & \beta_2(x) = x - x^q \end{array}$$

are surjective, where $q = \mathrm{Card}(k_F)$. This is obvious, since k is algebraically closed.

Condition (2.1.2) follows immediately from (1.9.1).

2.4. The results in 2.2 and 2.3 show that the functors $X_*(\ )_\Gamma$ and B are isomorphic, and that choosing an isomorphism

$$X_*(\ )_\Gamma \to B \tag{2.4.1}$$

is the same as choosing a generator of the infinite cyclic group $B(\mathbb{G}_m)$. We normalize (2.4.1) by choosing as generator of $B(\mathbb{G}_m)$ the σ-conjugacy class in $L^\times$ consisting of elements with normalized valuation 1.

2.5. There is an explicit formula for the isomorphism $X_*(T)_\Gamma \xrightarrow{\sim} B(T)$. Choose E so that T splits over E. Let E_0 be the largest field between E and F that is unramified over F. Consider $\mu \in X_*(T)$. Under $X_*(T)_\Gamma \xrightarrow{\sim} B(T)$, the class of μ maps to the σ-conjugacy class containing $Nm_{E/E_0}(\mu(\pi_E))$, where π_E is a uniformizing element for E and Nm_{E/E_0} is the norm homomorphism $T(E) \to T(E_0)$ (note that $E_0 \subset L$). By functoriality it is enough to prove this formula in the universal case $T = R_{E/F}\mathbb{G}_m$, $\mu = \mu_E$, which can be handled easily by using the fact that

$$B(N_{E/F})\colon B(T) \to B(\mathbb{G}_m)$$

is an isomorphism.

2.6. The isomorphism $X_*(T)_\Gamma \xrightarrow{\sim} B(T)$ of 2.4 is compatible with the Shapiro isomorphism of 1.10. In other words, for any finite extension F' of F in $\bar{K}$ and any F'-torus T, the diagram

$$\begin{array}{ccc} X_*(T)_{\Gamma'} & \xrightarrow{\sim} & B(T) \\ \wr\downarrow & & \wr\uparrow \\ X_*(RT)_\Gamma & \xrightarrow{\sim} & B(RT) \end{array} \tag{2.6.1}$$

commutes, where the right vertical arrow is the Shapiro isomorphism, Γ' is the Galois group of $\bar{F}/F'$, and the left vertical arrow is induced by the Γ'-equivariant homomorphism

$$X_*(T) \to X_*(T) \otimes_{\mathbb{Z}[\Gamma']} \mathbb{Z}[\Gamma] = X_*(RT)$$

which sends $\mu \in X_*(T)$ to $\mu \otimes 1$.

To prove this, we note that the diagram (2.6.1) yields two functorial homomorphisms $X_*(T)_{\Gamma'} \to B(T)$; we must prove that these homomorphisms are equal. Using Lemma 2.2 (for the field E), we reduce to the case $T = \mathbb{G}_m$, which is easy to handle.

2.7. Consider a torus T and a field E that splits T. Then the isomorphism $X_*(T)_\Gamma \xrightarrow{\sim} B(T)$ of 2.4 is compatible with the Tate-Nakayama isomorphism

$$\tilde{H}^{-1}(E/F, X_*(T)) \xrightarrow{\sim} H^1(E/F, T(E)).$$

In other words, the diagram

$$\begin{array}{ccc} \tilde{H}^{-1}(E/F,\ X_*(T)) & \xrightarrow{\sim} & H^1(E/F,\ T(E)) \\ \downarrow & & \downarrow \\ X_*(T)_\Gamma & \xrightarrow{\sim} & B(T) \end{array} \tag{2.7.1}$$

commutes, where the right vertical arrow is the composition of the inflation map $H^1(E/F, T(E)) \to H^1(F, T)$ (an isomorphism in this case) and the injection (1.8.3) $H^1(F, T) \to B(T)$, and where the left vertical arrow is the canonical injection

$$\tilde{H}^{-1}(E/F,\ X_*(T)) \to H_0(E/F,\ X_*(T)) = X_*(T)_\Gamma.$$

We now prove that the diagram (2.7.1) commutes. For any $\mu \in X_*(T)$ such that $\sum_{\tau \in \mathrm{Gal}(E/F)} \tau\mu = 0$, we must show that the two homomorphisms $\tilde{H}^{-1}(E/F, X_*(T)) \to B(T)$ obtained from (2.7.1) carry the class of μ into the same element of $B(T)$. Both of these homomorphisms are functorial in T, and therefore it is enough to consider the following universal case: $T = R_{E/F}\mathbb{G}_m/\nu(\mathbb{G}_m)$, $\mu = \beta \circ \mu_E$, where $\nu = \sum_{\tau \in \mathrm{Gal}(E/F)} \tau\mu_E$ and β is the canonical homomorphism $R_{E/F}\mathbb{G}_m \to T$.

We write c_1 for the element of $B(T)$ obtained from μ by going first across and then down in diagram (2.7.1) and write c_2 for the element of $B(T)$ obtained by going the other way around (2.7.1). We must show that $c_1 = c_2$. We will accomplish this by showing that c_1, c_2 both lie in the subset $H^1(F, T)$ of $B(T)$ and that they both map to $[E : F]^{-1} \in \mathbb{Q}/\mathbb{Z} = H^2(F, \mathbb{G}_m)$ under the injection

$$H^1(F, T) \to H^2(F, \mathbb{G}_m) \tag{2.7.2}$$

obtained as a connecting homomorphism for the exact sequence

$$1 \to \mathbb{G}_m \to R_{E/F}\mathbb{G}_m \to T \to 1. \tag{2.7.3}$$

For c_1 this can be seen easily using the commutative diagram

$$\begin{array}{ccc} \tilde{H}^{-1}(E/F,\ X_*(T) & \xrightarrow{\sim} & H^1(E/F,\ T(E)) \\ \downarrow & & \downarrow \\ \tilde{H}^0(E/F,\ X_*(\mathbb{G}_m)) & \xrightarrow{\sim} & H^2(E/F,\ \mathbb{G}_m(E)), \end{array}$$

in which the horizontal arrows are Tate-Nakayama isomorphisms and the vertical arrows are connecting isomorphisms coming from (2.7.3) and the corresponding sequence of cocharacter groups.

It requires more work to handle c_2. Let E_0 be the largest field between E and F such that E_0 is unramified over F, and let $e_0 = [E : E_0]$. Let N be the unique unramified extension of E_0 in $\bar{K}$ such that

$[N: E_0] = e_0$. Let π_F be a uniformizing element of F. Since EN/E is unramified of degree e_0, and since π_F has normalized valuation e_0 in E, there exists a uniformizing element π_{EN} of EN such that $Nm_{EN/E}\pi_{EN} = \pi_F$. From 2.5 we know that the image of the class of μ_E under the isomorphism

$$X_*(R_{E/F}\mathbb{G}_m)_\Gamma \tilde{\to} B(R_{E/F}\mathbb{G}_m)$$

is the σ-conjugacy class containing the element $Nm_{EN/N}\mu_E(\pi_{EN}) \in T(N)$; we will use b to denote this element of $T(N)$. The image of the class of μ under the isomorphism $X_*(T)_\Gamma \tilde{\to} B(T)$ is the σ-conjugacy class of $\beta(b)$. An easy calculation shows that $Nm_{N/F}b = \nu(\pi_F)$, which means that $\beta(b)$ gives us a 1-cocycle of the cyclic group Gal(N/F) with values in $T(N)$. The class of this 1-cocycle in $H^1(F, T)$ is c_2. Of course we are using the canonical generator of Gal(N/F), namely the restriction of σ to N, to regard $\beta(b)$ as a 1-cocycle of Gal(N/F) in $T(N)$. This choice of generator determines isomorphisms

$$\tilde{H}^i(\mathrm{Gal}(N/F), M) \tilde{\to} \tilde{H}^{i+2}(\mathrm{Gal}(N/F), M)$$

for any Gal(N/F)-module M and any $i \in \mathbb{Z}$. The equation $Nm_{N/F}b = \nu(\pi_F)$ has the further consequence that the image of c_2 under (2.7.2) is equal to the imge of the class of π_F under

$$F^\times/Nm_{N/F}N^\times = \tilde{H}^0(N/F, \mathbb{G}_m(N)) \tilde{\to} \tilde{H}^2(N/F, \mathbb{G}_m(N)),$$

namely the fundamental class of N/F. Since $[N: F] = [E: F]$, this completes the proof.

2.8. There is a functorial homomorphism $B(T) \to X_*(T)^\Gamma \otimes \mathbb{Q}$ defined as follows. Let $b \in T(L)$. Then there exists a unique element $\nu \in X_*(T)^\Gamma \otimes \mathbb{Q}$ such that

$$\mathrm{val}(\lambda(b)) = \langle \lambda, \nu \rangle$$

for all $\lambda \in X^*(T)^\Gamma$, where val is the normalized valuation on L. The element ν depends only on the σ-conjugacy class of b, and $b \mapsto \nu$ induces the desired homomorphism.

The composed map

$$X_*(T) \to X_*(T)_\Gamma \tilde{\to} B(T) \to X_*(T)^\Gamma \otimes \mathbb{Q} \tag{2.8.1}$$

is given by $\mu \mapsto [\Gamma: \Gamma_\mu]^{-1}\Sigma_{\tau \in \Gamma/\Gamma_\mu} \tau\mu$, where Γ_μ denotes the stabilizer of μ in Γ. This can be seen easily from 2.5. It now follows from 2.7 that the sequence

$$0 \to H^1(F, T) \to B(T) \to X_*(T)^\Gamma \otimes \mathbb{Q} \tag{2.8.2}$$

is exact.

2.9. Suppose that T is an unramified F-torus. Then T splits over L, and by tensoring the normalized valuation $L^{\times} \to \mathbb{Z}$ with $X_*(T)$ we get a canonical surjection $T(L) \to X_*(T)$. Applying the functor $H^1(\langle\sigma\rangle, \,)$ to this surjection, we get a functorial homomorphism

$$B(T) \to X_*(T)_\Gamma. \tag{2.9.1}$$

We claim that (2.9.1) is the inverse of the isomorphism $X_*(T)_\Gamma \to B(T)$ of 2.4. It is enough to show that the composition $X_*(T)_\Gamma \to B(T) \to X_*(T)_\Gamma$ is the identity. This follows from the explicit version of $X_*(T)_\Gamma \to B(T)$ given in 2.5 (take E to be unramified).

3. σ-L-spaces

To understand $B(G)$ for groups other than tori we need the following definition. A *σ-L-space* is a pair (V, Φ) consisting of a finite dimensional vector space V over L and a σ-semilinear bijection Φ: $V \to V$. For $F = \mathbb{Q}_p$ such a space is simply an isocrystal. The category of σ-L-spaces is equivalent to the category of pairs (W, i) where W is an isocrystal and i is a homomorphism $F \to \mathrm{End}(W)$ of $\mathbb{Q}_p$-algebras.

The category of isocrystals is Tannakian over $\mathbb{Q}_p$. Its gerb $\mathcal{G}$ is bound by the diagonalizable pro-algebraic group $\mathbb{D}$ over $\mathbb{Q}_p$ whose character group is $\mathbb{Q}$. Giving such a gerb is the same as giving a homomorphism $\mathbb{Q} = X^*(\mathbb{D}) \to \mathrm{Br}(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$. For our gerb the homomorphism is the canonical projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ [Sa].

The description of the category of σ-L-spaces in terms of pairs (W, i) shows that this category is Tannakian over F and that its gerb is $\mathcal{G}_F$, the gerb over F obtained from $\mathcal{G}$ by extending scalars from $\mathbb{Q}_p$ to F. In particular $\mathcal{G}_F$ is bound by $\mathbb{D}_F$ and corresponds to the homomorphism $X^*(\mathbb{D}) \to \mathrm{Br}(F)$ obtained as the composition

$$X^*(\mathbb{D}) \to \mathrm{Br}(\mathbb{Q}_p) \to \mathrm{Br}(F),$$

which is simply the homomorphism $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ that sends $r \in \mathbb{Q}$ to the class of $[F:\mathbb{Q}_p]\cdot r \bmod \mathbb{Z}$.

From [Sa] we see that the category of σ-L-spaces is semisimple, that the simple objects are parameterized by rational numbers, and that the endomorphism ring of the simple object corresponding to $r \in \mathbb{Q}$ is the division algebra with center F and invariant $[F:\mathbb{Q}_p]\cdot r$. By using the number $[F:\mathbb{Q}_p]$ to renormalize the parameterization (in other words, by renormalizing the isomorphism between $\mathbb{D}$ and the band of $\mathcal{G}_F$), we obtain a parametrization for which $\mathrm{End}(V_r)$ has invariant r, where V_r is a simple object corresponding to $r \in \mathbb{Q}$. We refer to r as the *slope* of V_r (just as for isocrystals).

Let π be a uniformizing element for F. Suppose that V is a σ-L-space which is isotypic of slope r. Write $r = m/n$ with $m, n \in \mathbb{Z}$. Let E be the fixed field of σ^n on L. Then the E-vector space $V^{\pi^{-m}\Phi^n}$ (the elements of V fixed by $\pi^{-m}\Phi^n$) generates V over L; in other words,

$$V^{\pi^{-m}\Phi^n} \otimes_E L \xrightarrow{\sim} V.$$

More generally, let V be any σ-L-space and let n be a non-zero integer such that $nr \in \mathbb{Z}$ for every slope r of V. Let ν: $\mathbb{D} \to GL(V)$ be the homomorphism (defined over L) corresponding to the slope decomposition of V. Then $n\nu$ factors through $\mathbb{D} \to \mathbb{G}_m$ (dual to $\mathbb{Z} \to \mathbb{Q}$), yielding ν': $\mathbb{G}_m \to GL(V)$. Again let E be the fixed field of σ^n on L. Then we have

$$V^{\nu'(\pi)^{-1}\Phi^n} \otimes_E L \xrightarrow{\sim} V.$$

4. Construction of ν in general

In this section G is a connected linear algebraic group over F. In 2.8 we constructed a functorial homomorphism $B(T) \to X_*(T)^\Gamma \otimes \mathbb{Q}$. We now wish to generalize this construction.

4.1. The generalization is best stated in terms of the diagonalizable group $\mathbb{D}$ of §3. The usual inclusion $\mathbb{Z} \subset \mathbb{Q}$ corresponds to an F-homomorphism $\mathbb{D} \to \mathbb{G}_m$, which gives us an inclusion $\mathrm{Hom}_F(\mathbb{G}_m, G) \subset \mathrm{Hom}_F(\mathbb{D}, G)$. Furthermore, for any $\nu \in \mathrm{Hom}_F(\mathbb{D}, G)$ there exists a positive integer n such that $n\nu \in \mathrm{Hom}_F(\mathbb{G}_m, G)$. For an F-torus T we have $\mathrm{Hom}_F(\mathbb{D}, T) = X_*(T)^\Gamma \otimes \mathbb{Q}$.

4.2. The generalization of $B(T) \to X_*(T)^\Gamma \otimes \mathbb{Q}$ is a mapping $b \mapsto \nu$ from $G(L)$ to $\mathrm{Hom}_L(D, G)$. To get ν from b, we start by noticing that b turns representations of G into σ-L-spaces: for any representation ρ: $G \to GL(V)$ of G on a finite dimensional vector space V over F, the pair (V_L, Φ) is a σ-L-space, where $V_L = V \otimes_F L$ and $\Phi = \rho(b) \circ (\mathrm{id}_V \otimes \sigma)$. The slope decomposition of (V_L, Φ) gives us an element $\nu_\rho \in \mathrm{Hom}_L(\mathbb{D}, GL(V))$ (for each $r \in \mathbb{Q}$ the group $\mathbb{D}$ acts on the corresponding subspace of V_L by the character $r \in \mathbb{Q} = X^*(\mathbb{D})$). Let R be an L-algebra and let $x \in \mathbb{D}(R)$. We write $\mathrm{Rep}(G)$ for the category of finite dimensional representations ρ: $G \to GL(V)$. Then the elements $\nu_\rho(x)$ $(\rho \in \mathrm{Rep}(G))$ give an automorphism of the standard fiber functor of $\mathrm{Rep}(G)$, and therefore there exists a unique element $y \in G(R)$ such that $\rho(y) = \nu_\rho(x)$ for all ρ. The homomorphism $x \mapsto y$ is functorial in R and thus defines an element $\nu \in \mathrm{Hom}_L(\mathbb{D}, G)$ such that $\rho \circ \nu = \nu_\rho$ for all ρ. This completes the construction of $b \mapsto \nu$.

4.3. The element ν obtained from $b \in G(L)$ can also be characterized in a way that avoids any reference to σ-L-spaces. It is the unique element $\nu \in \mathrm{Hom}_L(\mathbb{D}, G)$ for which there exist an integer $n > 0$, an element $c \in G(L)$ and a uniformizing element π of F such that the following three conditions hold:

$$n\nu \in \mathrm{Hom}_L(\mathbb{G}_m, G). \tag{4.3.1}$$

$$\text{Int(c)} \circ (\text{n}\nu) \text{ is defined over the fixed field of } \sigma^n \text{ on } L. \tag{4.3.2}$$

$$c(b\sigma)^n c^{-1} = c \cdot (n\nu)(\pi) \cdot c^{-1} \cdot \sigma^n. \tag{4.3.3}$$

Here, as in the rest of the paper, $\mathrm{Int}(c)$ denotes the inner automorphism $x \mapsto cxc^{-1}$. The equality in (4.3.3) is between elements of $G(L) \rtimes \langle\sigma\rangle$, as in 1.7.

Let ν be the element of $\mathrm{Hom}_L(\mathbb{D}, G)$ constructed in 4.2. We need to show that there exist n, c, π such that (4.3.1)–(4.3.3) hold. For π we take any uniformizing element of F. As a first choice for n we take any positive integer such that $n\nu \in \mathrm{Hom}_L(\mathbb{G}_m, G)$. We write ν' for $n\nu$ and E for the fixed field of σ^n on L. For any $(\rho, V) \in \mathrm{Rep}(G)$ the σ-L-space V_L has the property that $nr \in \mathbb{Z}$ for any slope of V_L. From the discussion at the end of §3 we see that

$$V_L^{\rho(x)} \otimes_E L \xrightarrow{\sim} V_L,$$

where $x = \nu'(\pi)^{-1}(b\sigma)^n \in G(L) \rtimes \langle\sigma\rangle$ (we have extended V_L to a representation of $G(L) \rtimes \langle\sigma\rangle$ in the obvious way). We now have two fiber functors for $\mathrm{Rep}(G)$ with values in E-vector spaces: ω_1: $V \mapsto V_E$ and ω_2: $V \mapsto V_L^{\rho(x)}$. The difference between ω_1 and ω_2 is measured by an E-torsor X under G for the f.p.q.c. topology on $\mathrm{Spec}(E)$. Since G_E is an affine scheme of finite type over E, the same is true for X. Therefore X has a point over a finite extension E' of E, and the difference between ω_1 and ω_2 can be measured by an element of the Galois cohomology set $H^1(E, G)$. Since G is connected and linear, the result of Steinberg used in 1.8 implies that ω_1 and ω_2 become isomorphic over a finite extension E' of E in L. Let $m = [E' : E]$. Then the inclusion $V_L^{\rho(x)} \subset V_L^{\rho(x)^m}$ induces an isomorphism

$$V_L^{\rho(x)} \otimes_E E' \xrightarrow{\sim} V_L^{\rho(x)^m}.$$

Thus, replacing our original n by nm, we may assume that the fiber functors ω_1 and ω_2 are isomorphic over E. Choose an isomorphism α: $\omega_1 \xrightarrow{\sim} \omega_2$ over E. By extending scalars from E to L we get an isomorphism α_L: $\omega_1 \to \omega_2$ over L. But over L we also have the obvious isomorphism β: $\omega_2 \to \omega_1$ given by

$$V_L^{\rho(x)} \otimes_E L = V_L = V_E \otimes_E L.$$

The composition $\beta \circ \alpha_L$ is an automorphism of ω_1 over L, which gives us an element of $G(L)$. Using this element of $G(L)$ to σ-conjugate our original b, we reduce to the case in which $\beta \circ \alpha_L = \mathrm{id}$. In this case (4.3.1)–(4.3.3) hold with $c = 1$, as we will now check.

The equality $\beta \circ \alpha_L = \mathrm{id}$ tells us that $V_L^{\rho(x)} = V_L^{\rho(\sigma^n)}$ for all ρ. The first consequence of this is that the slope decomposition of V_L is defined over E, which means that (4.3.2) holds. The second consequence is that the L-linear automorphism $\rho(x\sigma^{-n})$ of V_L fixes $V_L^{\rho(x)}$ pointwise, and since $V_L^{\rho(x)}$ generates V_L over L, we find that $\rho(x\sigma^{-n})$ acts trivially on V_L for all ρ. Therefore $x = \sigma^n$, which means that (4.3.3) holds.

We have now finished the proof that ν has the property stated at the beginning of this section. Using any faithful representation ρ of G, one sees easily that ν is the only homomorphism $\mathbb{D} \to G$ over L satisfying that property.

4.4. Using the description of ν given in 4.3, we find that the mapping $b \mapsto \nu$ satisfies

$$\sigma(b) \mapsto \sigma(\nu), \tag{4.4.1}$$

$$gb\sigma(g)^{-1} \mapsto \mathrm{Int}(g) \circ \nu \ (g \in G(L)). \tag{4.4.2}$$

In particular the $G(L)$-conjugacy class of ν depends only on the σ-conjugacy class of b. Furthermore, the equation $b = b\sigma(b)\sigma(b)^{-1}$ shows that

$$\nu = \mathrm{Int}(b) \circ \sigma(\nu), \tag{4.4.3}$$

which means that the $G(L)$-conjugacy class of ν is fixed by σ.

Thus, if G is a torus T, the element ν depends only on the σ-conjugacy class of b and belongs to $\mathrm{Hom}_F(\mathbb{D}, T) = X_*(T)^\Gamma \otimes \mathbb{Q}$. To see that this mapping $b \mapsto \nu$ agrees with that of 2.8, one uses (4.3.3) to verify that $\mathrm{val}(\lambda(b)) = \langle \lambda, \nu \rangle$ for all $\lambda \in X^*(T)^\Gamma$.

4.5. The description of ν given in 4.3 shows that ν is trivial if and only if the element of $B(G)$ obtained from b is in the image of $H^1(F, G) \to B(G)$ (see 1.8 for the definition of this map).

5. Connected reductive groups

In this section G is a connected reductive group over F. We write Z for the center $Z(G)$ of G.

5.1. We say that an element $b \in G(L)$ is *basic* if the corresponding element $\nu \in \mathrm{Hom}_L(\mathbb{D}, G)$ factors through Z. Since the $G(L)$-conjugacy class of ν is fixed by σ (see 4.4), the homomorphism ν: $\mathbb{D} \to Z$ is defined over F. We say that a σ-conjugacy class is basic if it consists of basic elements, and we write $B(G)_b$ for the set of basic σ-conjugacy classes in $G(L)$.

5.2. Let b be a basic element of $G(L)$. Then there exists a pair (J, u) satisfying the following three conditions.

$$J \text{ is an inner twist of } G. \tag{5.2.1}$$

$$u \text{ is an } L\text{-isomorphism } J_L \xrightarrow{\sim} G_L. \tag{5.2.2}$$

$$u(\sigma(x)) = b \cdot \sigma(u(x)) \cdot b^{-1} \qquad \text{for all } x \in J(L). \tag{5.2.3}$$

Furthermore, this pair is unique up to a unique isomorphism. Indeed, suppose that (J, u), (K, v) satisfy the conditions above. Then $w = v^{-1} \circ u$ is an L-isomorphism $J_L \xrightarrow{\sim} K_L$. Using (5.2.3) and the fact that $J(L)$ is Zariski dense in J, we see that w commutes with σ. Since J, K are connected reductive F-groups, the functor $A \mapsto \mathrm{Isom}_A(J_A, K_A)$ from the category of F-algebras A to the category of sets is representable by an algebraic variety over F. It follows from Lemma 1.2 that w comes from an F-isomorphism $J \to K$.

To prove the existence of (J, u), we first note that if (J, u) works for b, then $(J, \mathrm{Int}(c) \circ u)$ works for $cb\sigma(c)^{-1}$, where c is any element of $G(L)$. Thus, using 4.3, we may assume that $(b\sigma)^n = z\sigma^n$ for some $z \in Z(L)$ and some positive integer n. Let b_{ad} denote the image of b in G_{ad}. The equation $(b\sigma)^n = z\sigma^n$ shows that $b_{ad} \in G(E)$, where E is the fixed field of σ^n on L, and that $(b_{ad}\sigma)^n = \sigma^n$. Since E/F is cyclic of degree n and the restriction of σ to E generates $\mathrm{Gal}(E/F)$, we can use b_{ad} to get an inner twist J of G and an isomorphism $J_E \xrightarrow{\sim} G_E$. Extending scalars to L gives us u: $J_L \xrightarrow{\sim} G_L$, and (J, u) is the desired pair.

5.3. PROPOSITION: *Let T be an elliptic maximal F-torus of G. Then the image of $B(T)$ in $B(G)$ is equal to $B(G)_b$.*

First we show that the image of $B(T)$ is contained in $B(G)_b$. Let $b \in T(L)$ and let ν be the corresponding element of $X_*(T)^\Gamma \otimes \mathbb{Q}$. Since T is elliptic, we have $X_*(T)^\Gamma \subset X_*(Z)$. Therefore b is basic in $G(L)$.

Next we show that if b is basic in $G(L)$, then b is σ-conjugate to an element of $T(L)$. After replacing b by a σ-conjugate we get z, n, E, b_{ad} as in 5.2:

(i) $b_{ad} \in G_{ad}(E)$
(ii) the σ-conjugacy class of b_{ad} in $G_{ad}(L)$ belongs to the image of $H^1(E/F, G_{ad}(E))$ in $B(G_{ad})$.

Since T_{ad}, the image of T in G_{ad}, is anisotropic, the map $H^1(F, T_{ad}) \to H^1(F, G_{ad})$ is surjective (see [Kn]). Therefore, there exists $h \in G_{ad}(L)$ such that $hb_{ad}\sigma(h)^{-1} \in T_{ad}(L)$. Choose $g \in G(\bar{K})$ such that $g_{ad} = h$. Then $(g \cdot \tau(g)^{-1})_{\tau \in \mathrm{Gal}(\bar{K}/L)}$ is a 1-cocycle of $\mathrm{Gal}(\bar{K}/L)$ in $Z(\bar{K})$. Since $H^1(L, T)$ is trivial (see 1.8), there exists $t \in T(\bar{K})$ such that $g \cdot \tau(g)^{-1} = t \cdot \tau(t)^{-1}$ for all $\tau \in \mathrm{Gal}(\bar{K}/L)$. It follows immediately that $t^{-1}g$ belongs to $G(L)$ and that the σ-conjugate of b by $t^{-1}g$ belongs to $T(L)$.

5.4. PROPOSITION: *Assume that the derived group G_{der} of G is simply connected, and let $D = G/G_{\text{der}}$. Then $G \to D$ induces a bijection $B(G)_b \tilde{\to} B(D)$. In particular, $B(G)_b$ is trivial if G is semisimple and simply connected.*

We prove the last statement first. Assume that G is semisimple and simply connected, and choose an anisotropic maximal F-torus T of G (see [Kn] for the existence of T). Then by the previous proposition $B(T)$ maps onto $B(G)_b$. From 2.7 we know that $H^1(F, T) \to B(T)$ is an isomorphism. Since $H^1(F, G)$ is trivial [Kn], we conclude that $B(G)_b$ is trivial.

Now we prove the first statement, starting with the surjectivity of $B(G)_b \to B(D)$. Choose an elliptic maximal F-torus T of G. By the previous proposition it is enough to show that $B(T)$ maps onto $B(D)$, and this follows immediately from 1.9 since the kernel T_{der} of $T \to D$ is connected.

Finally, we prove the injectivity of $B(G)_b \to B(D)$. Let b, b' be basic elements of $G(L)$, and assume that b, b' map to the same element of $B(D)$. Since $G(L)$ maps onto $D(L)$, we may replace b' by a σ-conjugate so that $b' = xb$ with $x \in G_{\text{der}}(L)$. We now use b to get a pair (J, u) as in 5.2. It is not hard to check that the inverse image y of x under u is a basic element of $J_{\text{der}}(L)$. We already know that $B(J_{\text{der}})_b$ is trivial, which means that $y = c\sigma(c)^{-1}$ for some $c \in J_{\text{der}}(L)$. Let $d = u(c)$. An easy calculation shows that $b' = db\sigma(d)^{-1}$.

5.5. We follow the conventions of [K1] regarding L-groups; in particular, we write $\hat{G}$ instead of ${}^LG^0$. For any F-torus T we have defined a canonical isomorphism $X_*(T)_\Gamma \tilde{\to} B(T)$ (see 2.4). The inverse of this isomorphism gives us an isomorphism

$$B(T) \tilde{\to} X_*(T)_\Gamma. \tag{5.5.1}$$

We have $X_*(T) = X^*(\hat{T})$, and the restriction map $X^*(\hat{T}) \to X^*(\hat{T}^\Gamma)$ induces an isomorphism $X^*(\hat{T})_\Gamma \tilde{\to} X^*(\hat{T}^\Gamma)$. Thus we have a canonical isomorphism

$$X_*(T)_\Gamma \tilde{\to} X^*(\hat{T}^\Gamma). \tag{5.5.2}$$

Combining (5.5.1) and (5.5.2), we get a canonical isomorphism

$$B(T) \tilde{\to} X^*(\hat{T}^\Gamma). \tag{5.5.3}$$

We want to give a generalization of (5.5.3), in which T is replaced by an arbitrary connected reductive group G.

As in [K1], we say that a homomorphism α: $G \to H$ is normal if $\alpha(G)$ is normal in H. We consider the functors $G \mapsto B(G)_b$ and $G \mapsto$

$X^*(Z(\hat{G})^\Gamma)$ from the category of connected reductive groups and normal homomorphisms to the category of sets. To see that $B(G)_b$ is indeed functorial for a normal homomorphism $\alpha: G \to H$, it is enough to notice that α carries $Z(G)$ into $Z(H)$.

5.6. PROPOSITION: *There is a unique functorial isomorphism*

$$B(G)_b \xrightarrow{\sim} X^*\big(Z(\hat{G})^\Gamma\big) \tag{5.6.1}$$

that agrees with (5.5.3) *for tori.*

We extend the isomorphism of functors in two stages. At the first stage we extend it to groups G such that G_{der} is simply connected. In view of the isomorphism $Z(\hat{G}) = \hat{D}$, where $D = G/G_{\mathrm{der}}$, the existence and uniqueness in the first stage of the extension is a consequence of Proposition 5.4.

At the second stage we extend (5.6.1) to all connected reductive groups. Given such a group G, we choose a central extension H of G such that

(a) the kernel C of $H \to G$ is a torus,

(b) H_{der} is simply connected.

Consider the diagram

$$\begin{array}{ccc} B(H)_b & \to & B(G)_b \\ \wr\downarrow & & \\ X^*(Z(\hat{H})^\Gamma) & \to & X^*(Z(\hat{G})^\Gamma). \end{array} \tag{5.6.2}$$

The group $B(C)$ acts on $B(H)_b$, and the map $B(H)_b \to B(G)_b$ induces a bijection from the quotient set to $B(G)_b$. The exact sequence

$$1 \to Z(\hat{G})^\Gamma \to Z(\hat{H})^\Gamma \to \hat{C}^\Gamma$$

gives us an exact sequence

$$X^*(\hat{C}^\Gamma) \to X^*\big(Z(\hat{H})^\Gamma\big) \to X^*\big(Z(\hat{G})^\Gamma\big) \to 1.$$

Therefore $X^*(\hat{C}^\Gamma)$ acts on $X^*(Z(\hat{H})^\Gamma)$, and the map $X^*(Z(\hat{H})^\Gamma) \to X^*(Z(\hat{G})^\Gamma)$ induces a bijection from the quotient set to $X^*(Z(\hat{G})^\Gamma)$. Since $B(C) = X^*(\hat{C}^\Gamma)$, we see that there is a unique bijection $B(G)_b \to X^*(Z(\hat{G})^\Gamma)$ making diagram (5.6.2) commute. A variant of Lemma 2.4.4 of [K1] shows that the bijection $B(G)_b \to X^*(Z(\hat{G})^\Gamma)$ is independent of the choice of $H \to G$ and is functorial for normal homomorphisms.

5.7. REMARK: The following diagram commutes.

$$\begin{array}{ccc} H^1(F, G) & \xrightarrow{\sim} & \pi_0(Z(\hat{G})^\Gamma)^D \\ \downarrow & & \downarrow \\ B(G)_b & \xrightarrow{\sim} & X^*(Z(\hat{G})^\Gamma) \end{array} \tag{5.7.1}$$

The upper horizontal arrow is defined in Proposition 6.4 of [K1]. The left vertical arrow is induced by (1.8.3); we know from 4.5 that the image of $H^1(F, G)$ in $B(G)$ is contained in $B(G)_b$.

In the case of tori the commutativity of (5.7.1) is a consequence of 2.7. In the case of a group G such that G_{der} is simply connected, we reduce to the case of tori by considering $D = G/G_{\mathrm{der}}$. In the general case we choose a central extension $H \to G$ as in the proof of Proposition 5.6 in order to reduce to the case in which G_{der} is simply connected.

5.8. REMARK: Let T be an elliptic maximal F-torus in G. Then the following diagram commutes.

$$\begin{array}{ccc} B(T) & \xrightarrow{\sim} & X^*(\hat{T}^\Gamma) \\ \downarrow & & \downarrow \\ B(G)_b & \xrightarrow{\sim} & X^*(Z(\hat{G})^\Gamma) \end{array} \tag{5.8.1}$$

The right vertical arrow is the restriction homomorphism obtained from the canonical injection $Z(\hat{G})^\Gamma \to \hat{T}^\Gamma$. It is easy to use (5.8.1) to show that the fibers of $B(T) \to B(G)_b$ are the cosets of $\mathrm{im}[B(T_{\mathrm{sc}}) \to B(T)]$ in $B(T)$.

Since $T \to G$ is not a normal homomorphism (unless G is a torus) the commutativity of (5.8.1) is not a consequence of the functoriality of (5.6.1). However, if G_{der} is simply connected, we let $D = G/G_{\mathrm{der}}$ and obtain the desired commutativity from the functoriality of (5.6.1) for $T \to D$. Then, in the general case, we choose a central extension $H \to G$ as before and reduce to the case just treated.

6. Quasisplit groups

In this section we assume that G is a quasisplit connected reductive group over F. This assumption allows us to obtain a description of the whole set $B(G)$, using Levi subgroups of G (by which we mean Levi components of parabolic F-subgroups of G).

6.1. Let M be a Levi subgroup of G, and let S be the maximal F-split torus in the center of M. Consider an element $b \in B(M)_b$ and let ν be the element of $\mathrm{Hom}_F(\mathbb{D}, Z(M))$ associated to b (see 5.1). Of course ν factors through S and gives us an element of $\mathrm{Hom}_F(\mathbb{D}, S)$, which we also denote by ν. The centralizer $\mathrm{Cent}_G(\nu)$ of ν in G is a Levi subgroup of G containing M. We say that b is *G-regular* if $\mathrm{Cent}_G(\nu)$ is equal to M. We write $B(M)_{br}$ for the set of G-regular basic elements of $B(M)$.

6.2. PROPOSITION: *Let $b \in G(L)$. Then b is σ-conjugate to a basic G-regular element of some Levi subgroup of G.*

Let ν be the element of $\mathrm{Hom}_L(\mathbb{D}, G)$ associated to b in §4. From 4.4 we know that the $G(L)$-conjugacy class of ν is fixed by σ. Since G is quasisplit, there exists $g \in G(L)$ such that $\mathrm{Int}(g) \circ \nu$ is defined over F (this is a variant of Lemma 1.1.3. of [K2]). Replacing b by $gb\sigma(g)^{-1}$, we may assume that ν is defined over F. Let $M = \mathrm{Cent}_G(\nu)$. The equation (4.4.3) shows that $b \in M(L)$. It is immediate that b is basic and G-regular in $M(L)$.

6.3. PROPOSITION: *Let M_1, M_2 be Levi subgroups of G and let $b_i \in B(M_i)_{br}$ $(i = 1, 2)$. Then b_1, b_2 have the same image in $B(G)$ if and only if there exists $g \in G(F)$ such that* $\mathrm{Int}(g)(M_1) = M_2$ *and* $\mathrm{Int}(g)(b_1) = b_2$ *(since* $\mathrm{Int}(g)$ *is defined over F, it induces a map $B(M_1) \to B(M_2)$).*

For $i = 1, 2$ let m_i be an element of $M_i(L)$ in the σ-conjugacy class b_i. Suppose that there exists an element $g \in G(F)$ such that $\mathrm{Int}(g)(M_1) = M_2$ and $\mathrm{Int}(g)(b_1) = b_2$. Then $gm_1g^{-1} = cm_2\sigma(c)^{-1}$ for some $c \in M_2(L)$. Since $g = \sigma(g)$, the elements m_1, m_2 are σ-conjugate in $G(L)$.

Conversely, assume that there exists $h \in G(L)$ such that $hm_1\sigma(h)^{-1} = m_2$. For $i = 1, 2$ let ν_i be the homomorphism $\mathbb{D} \to Z(M_i)$ over F associated to b_i. Then 4.4.2 shows that $\mathrm{Int}(h) \circ \nu_1 = \nu_2$. Consequently, there exists $g \in G(F)$ such that $\mathrm{Int}(g) \circ \nu_1 = \nu_2$ (see Lemma 1.1.3 of [K2]). Using that $b_i \in B(M)_{br}$, we see that $\mathrm{Int}(g)(M_1) = M_2$ and that $hg^{-1} \in M_2(L)$. Finally, the equality $\mathrm{Int}(g)(b_1) = b_2$ follows from

$$m_2 = hm_1\sigma(h)^{-1} = (hg^{-1})\cdot(gm_1g^{-1})\cdot\sigma(hg^{-1})^{-1}.$$

6.4. REMARK: We can summarize our results as follows. Let X be the set of pairs (M, b), where M is a Levi subgroup of G and b is an element of $B(M)_{br}$. Then $G(F)$ acts on X, and the quotient set is $B(G)$.

6.5. REMARK: Let M be a Levi subgroup of G and let b be a basic G-regular element of $M(L)$. Applying (5.2) to M and b, we get a pair (J, u) saisfying (5.2.1)–(5.2.3) (with G replaced by M). In particular, J is an inner twist of M. Let n be a positive integer and let E denote the fixed field of σ^n on L. We claim that u identifies $J(E)$ with

$$\{g \in G(L) \mid g \text{ commutes with } (b\sigma)^n\}. \tag{6.5.1}$$

In view of (5.2.3) it is enough to show that the set (6.5.1) is contained in

$M(L)$. We may assume that $n = 1$ (replace F by E and b by $b\sigma(b)\dots\sigma^{n-1}(b)$). Let g be an element of the group (6.5.1). Then $gb\sigma(g)^{-1} = b$. From (4.4.2) we see that g centralizes the homomorphism ν: $\mathbb{D} \to G$ associated to b. Since b is G-regular, we conclude that $g \in M(L)$.

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(Oblatum 27-II-1984)

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