COMPOSITIO MATHEMATICA

C. BRUCE HUGHES

# Spaces of approximate fibrations on Hilbert cube manifolds

*Compositio Mathematica*, tome 56, nº 2 (1985), p. 131-151

<http://www.numdam.org/item?id=CM_1985__56_2_131_0>

Compositio Mathematica 56 (1985) 131–151.


# SPACES OF APPROXIMATE FIBRATIONS ON HILBERT CUBE MANIFOLDS

C. Bruce Hughes

## Abstract

It is shown that the space of approximate fibrations from a compact Hilbert cube manifold to a compact polyhedron is locally $n$-connected for every non-negative integer $n$.

## 1. Introduction

This paper is concerned with deforming certain parameterized families of maps between a Hilbert cube manifold (i.e., a $Q$-manifold) $M$ and a polyhedron $B$ to parameterized families of approximate fibrations. When $M$ and $B$ are compact this results in showing that the space of approximate fibrations from $M$ to $B$ is locally $n$-connected for each non-negative integer $n$.

Approximate fibrations were introduced by Coram and Duvall [8] as a generalization of both Hurewicz fibrations and cell-like maps. Since then approximate fibrations have been studied by several authors (see [5], [9], [13], [19]) and have found numerous applications in geometric topology.

Here is our main result (see Theorem 7.1).

THEOREM: *Let $B$ be a polyhedron, let $n \geqslant 0$ be an integer, and let $\alpha$ be an open cover of $B$. There exists an open cover $\beta$ of $B$ so that if $M$ is a $Q$-manifold and $f$: $M \times I^n \to B \times I^n$ is a fiber preserving map such that $f_t$: $M \to B$ is a $\beta$-fibration for $t$ in $I^n$ and an approximate fibration for $t$ in $\partial I^n$, then there is a fiber preserving map $\tilde{f}$: $M \times I^n \to B \times I^n$ such that $\tilde{f}_t$ is an approximate fibration $\alpha$-close to $f_t$ for $t$ in $I^n$ and $\tilde{f} \mid M \times \partial I^n = f \mid M \times \partial I^n$.*

The $n = 0$ case of this theorem has previously been proved by Chapman [2, Theorem 1]. The shell of our proof is the same as Chapman's and we refer the reader to [2] for most of the common details. In order to make Chapman's program work in our parameterized setting we have developed a parameterized form of engulfing (Section 4). The key ingredient for this engulfing is a sliced lifting property for parameterized families of $\delta$-fibrations (Section 2). Another difference with [2] is that in the wrapping up construction (Section 5) we encounter non-compact $Q$-manifolds parameterized by submersions to $I^n$. Hence, we are forced to recast some basic fibered $Q$-manifold theory in a new setting (Section 3). A relative version of our main result is also obtained (see Section 7). This relative version has been useful in the study of controlled simple-homotopy theory [18].

Many authors have studied local properties of spaces of certain types of maps. Of particular relevance here are the theorems of Ferry [14], Torunczyk [21], and Haver [15]. Ferry and Torunczyk proved that the homeomorphism group of a compact $Q$-manifold is an ANR, while Haver proved a theorem which implies that the space of cell-like maps from a compact $Q$-manifold to itself is weakly locally contractible (and therefore, locally $n$-connected for each $n \geqslant 0$). Our main result implies the following (see Section 7):

COROLLARY: *Let $M$ be a compact $Q$-manifold and let $B$ be a compact polyhedron. Then the space of approximate fibrations from $M$ to $B$ endowed with the compact-open topology is locally $n$-connected for each $n \geqslant 0$.*

It is shown in Section 7 that the same result holds for the space of cell-like maps and the space of monotone approximate fibrations. The results of this paper have recently been used in [17] to show that the space of Hurewicz fibrations and the space of bundle projections from a compact Q-manifold to a compact polyhedron are locally $n$-connected for every $n \geqslant 0$. Whether any of these spaces of maps are locally contractible remains an open question.

This paper is organized as follows. Section 2 consists of preliminaries on approximate fibrations. A key result there shows that families of $\epsilon$-fibrations parameterized by finite dimensional polyhedra have a certain sliced lifting property. The restriction to finite dimensional parameter spaces is the main reason we are unable to prove stronger results on spaces of approximate fibrations (for example, local contractibility). The sliced lifting property has been useful in [17] and [18]. Section 3 mentions the basic $Q$-manifold theory for submersions. It may be of interest to note that our techniques prove that if $M$ is a foliated $Q$-manifold with model $I^n$ whose leaves (in the leaf topology) are also $Q$-manifolds, then there is a leaf-preserving homeomorphism $h$: $M \times Q \to M$ arbitrarily close to projection.

Section 4 contains the parameterized engulfing results. These have been key ingredients in [17] and [18]. Once the engulfing technique is established, the rest of the proof of the main result follows rather mechanically from the proof of Chapman's theorem [2, Theorem 1]. This procedure is outlined in Sections 5 and 6. The main result and its corollaries are stated and proved in Section 7.

Most of our notation and definitions are standard. Except for the various function spaces which we consider, all spaces are locally compact, separable and metric. We use $R^n$ to denote euclidean $n$-space and $B_r^n$ to denote the $n$-cell $[-r, r]^n \subset R^n$. The circle is denoted by $S^1$ and the $n$-torus is $T^n = S^1 \times \ldots \times S^1$ ($n$ times). The standard $n$-cell is $I^n = [0, 1]^n$ and its (combinatorial) boundary is $\partial I^n$. If $X$ is a space and $A \subset X$, then we use both $\mathring{A}$ and int($A$) to denote the topological interior of $A$ in $X$. The closure of $A$ in $X$ is denoted by cl($A$). If $X$ is a compact space, then $c(X)$ denotes the cone over $X$. That is, $c(X) = X \times [0, +\infty]/\sim$, where $\sim$ is the equivalence relation generated by $(x, 0) \sim (x', 0)$ for all $x$, $x'$ in $X$. Similarly, $\mathring{c}(X) = X \times [0, +\infty)/\sim$ denotes the open cone over $X$, and for any $r$ in $[0, +\infty]$ let $c_r(X) = X \times [0, r]/\sim$ and $\mathring{c}_r(X) = X \times [0, r)/\sim$.

This paper is a revision of part of the author's doctoral dissertation written at the University of Kentucky under the direction of T.A. Chapman [16].

## 2. Preliminaries on approximate fibrations

This section contains some basic facts about the various types of fibrations that will be needed in the sequel. A key result is Theorem 2.4 which says that a family of $\delta$-fibrations parameterized by a finite dimensional polyhedron has a certain sliced $\epsilon$-lifting property.

We begin with some definitions. A map $f$: $E \to B$ (i.e., a continuous function) is *proper* provided $f^{-1}(C)$ is compact for all compact subsets $C$ of $B$. If $\alpha$ is an open cover of $B$, then a proper map $f$: $E \to B$ is said to be an *$\alpha$-fibration* if for all maps $F$: $Z \times [0, 1] \to B$ and $g$: $Z \to E$ for which $fg = F_0$, there is a map $G$: $Z \times [0, 1] \to E$ such that $G_0 = g$ and $fG$ is $\alpha$-close to $F$ (that is, given any $(z, t) \in Z \times [0, 1]$ there is a $U \in \alpha$ containing both $fG(z, t)$ and $F(z, t)$). If $C \subset B$ and $\alpha$ is an open cover of $B$, then a proper map $f$: $E \to B$ is said to be an *$\alpha$-fibration over* $C$ provided the condition above is satisfied when the maps $F$: $Z \times [0, 1] \to B$ are required to satisfy $F(Z \times [0, 1]) \subset C$. If $\epsilon > 0$, then we also use $\epsilon$ to denote the open cover of $B$ by balls of diameter $\epsilon$. Thus, we speak of *$\epsilon$-fibrations*.

A proper map $f$: $E \to B$ is an *approximate fibration* provided it is an $\alpha$-fibration for each open cover $\alpha$ of $B$. We only consider approximate fibrations which are defined between ANRs, i.e., absolute neighborhood retracts for metric spaces. The following lemma is used repeatedly.

LEMMA 2.1: *Let $B$ be an* ANR *and let $C$ be a compact subset of $B$ with a compact neighborhood $\tilde{C}$. For every $\alpha > 0$ there exists a $\beta = \beta(\alpha, C, \tilde{C}, B) > 0$ such that if $\epsilon > 0$ and $f$: $E \to B$ is an $\epsilon$-fibration over $\tilde{C}$, then $f$ has the following lifting property: given maps $F$: $Z \times [0, 1] \to C \subset B$ and $g$: $Z \to E$ such that $fg$ is $\beta$-close to $F_0$, there exists a map $G$: $Z \times [0, 1] \to E$ such that $G_0 = g$ and $fG$ is $(\alpha + \epsilon)$-close to $F$.*

For a proof of Lemma 2.1, see [8, Proposition 1.2] or [9, Lemma 1.1].

Those two papers should also be consulted for other basic results on approximate fibrations.

We now define the key sliced lifting property that will be established in Theorem 2.4.

DEFINITION 2.2: Let $E$, $B$, and $X$ be spaces and let $C \subset B$. A map $f$: $E \times X \to B \times X$ is said to be *fiber preserving* (f.p.) if $p_X f = p_X$ where $p_X$ denotes projection to $X$. If $\epsilon > 0$ and $f$: $E \times X \to B \times X$ is a proper f.p. map, then we say $f$ is a *sliced $\epsilon$-fibration over $C \times X$* if $f$ satisfies the following *sliced $\epsilon$-lifting property over $C \times X$*:

if $f$: $Z \times [0, 1] \times X \to C \times X \subset B \times X$ and $g$: $Z \times X \to E \times X$ are f.p. maps such that $fg = F_0$, then there exists a f.p. map $G$: $Z \times [0, 1] \times X \to E \times X$ such that $G_0 = g$ and $fG$ is $\epsilon$-close to $F$.

If $C = B$, then $f$ is called a *sliced $\epsilon$-fibration.*

We will need the following lemma for the proof of Theorem 2.4. Recall from the introduction that all spaces in this paper are metric.

LEMMA 2.3: *Let $E \times X \to B \times X$ be a proper f.p. map where $E$ and $B$ are* ANRs. *Let $C$ be a compact subset of $B$ and let $\tilde{C}$ be a compact neighborhood of $C$ in $B$. Let $\epsilon > 0$ and suppose for each $x$ in $X$ $f_x = f \mid E \times \{x\} : E = E \times \{x\} \to B \times \{x\} = B$ is an $\epsilon$-fibration over $\tilde{C}$. For every $\alpha > 0$ there is an open cover $U$ of $X$ such that if $X_0$ is any subset of $X$ contained in some member of $U$, then $f \mid E \times X_0 : E \times X_0 \to B \times X_0$ is an $(\alpha + \epsilon)$-fibration over $C \times X_0$.*

PROOF: Given the hypothesis above, choose $\beta = \beta(\alpha/3, C, \tilde{C}, B)$ by Lemma 2.1. Choose the open cover $U$ so that the diameter of any member of $U$ is less than $\alpha/3$ and so that $f_x$ is $\beta$-close to $f_y$ over $\tilde{C}$ whenever $x$ and $y$ are elements of a common member of $U$.

Let $X_0$ be a subset of $X$ contained in some member of $U$. Let maps $F$: $Z \times [0, 1] \to C \times X_0$ and $g$: $Z \to E \times X_0$ be given such that $fg = F_0$. Since the diagram

$$\begin{array}{ccc} E \times X_0 & \overset{p_E = \text{proj}}{\to} & E \\ f \downarrow & & \downarrow f_x \\ B \times X_0 & \overset{p_B = \text{proj}}{\to} & B \end{array}$$

$\beta$-commutes over $\tilde{C}$ where $x \in X_0$ is fixed, the choice of $\beta$ implies the existence of a map $G$: $Z \times [0, 1] \to E$ such that $G_0 = p_E g$ and $f_x G$ is $((\alpha/3) + \epsilon)$-close to $p_B F$. Define $H$: $Z \times [0, 1] \to E \times X_0$ by $H(z, t) = (G(z, t), p_X g(z))$. Then $H$ is seen to be an $(\alpha + \epsilon)$-lift of $F$.

THEOREM 2.4: *Let $C$ be a compact subset of the* ANR *$B$ and let $\tilde{C}$ be a compact neighborhood of $C$ in $B$. Let $n$ be an integer. For every $\epsilon > 0$ there* *exists a* $\delta > 0$, $\delta = \delta(\epsilon, n, C, \tilde{C}, B)$, *such that if* $E$ *is an* ANR, $X$ *is an* $n$*-dimensional polyhedron, and* $f\colon E \times X \to B \times X$ *is a proper f.p. map such that* $f_x$ *is a* $\delta$*-fibration over* $\tilde{C}$ *for each* $x$ *in* $X$, *then* $f$ *is a sliced* $\epsilon$*-fibration over* $C \times X$.

PROOF: The proof is by induction on $n$. The theorem is clearly true for $n = 0$ by taking $\delta(\epsilon, 0, C, \tilde{C}, B) = \epsilon$. Assume $n > 0$ and that the theorem is true for $n - 1$. Let $\epsilon > 0$ be given and choose $\beta = \beta(\epsilon/4, C, \tilde{C}, B)$ by Lemma 2.1. Let $\delta = \delta(\epsilon, n, C, \tilde{C}, B) = \min\{\epsilon/8, \delta(\beta, n - 1, C, \tilde{C}, B)\}$. Let $f\colon E \times X \to B \times X$ be given as in the hypothesis. By Lemma 2.3 we can consider $X$ to have such a fine triangulation that if $\sigma$ is any simplex of $X$, then $f \mid E \times \sigma\colon E \times \sigma \to B \times \sigma$ is an $(\epsilon/4)$-fibration over $C \times \sigma$. Note that $p_B f \mid E \times \sigma\colon E \times \sigma \to B$ is also an $(\epsilon/4)$-fibration over $C$. We also assume that $f_x$ is $(\epsilon/2)$-close to $f_y$ over $\tilde{C}$ whenever $x$ and $y$ are in $\sigma$.

Given f.p. maps $F\colon Z \times [0, 1] \times X \to C \times X \subset B \times X$ and $g\colon Z \times X \to E \times X$ such that $fg = F_0$, the inductive assumption implies the existence of a f.p. map $G\colon Z \times [0, 1] \times X^{n-1} \to E \times X^{n-1}$ such that $G_0 = g \mid Z \times X^{n-1}$ and $fG$ is $\beta$-close to $F \mid Z \times [0, 1] \times X^{n-1}$ (here $X^{n-1}$ denotes the denotes the $(n - 1)$-skeleton of $X$). Define $\tilde{g}\colon [Z \times [0, 1] \times X^{n-1}] \cup [Z \times \{0\} \times X] \to E \times X$ by $\tilde{g} \mid Z \times [0, 1] \times X^{n-1} = G$ and $\tilde{g}(z, 0, x) = g(z, x)$. Let $\sigma$ be an $n$-simplex of $X$ and note that the pair $(Z \times [0, 1] \times \sigma, [Z \times [0, 1] \times \partial\sigma] \cup [X \times \{0\} \times \sigma])$ is homeomorphic to the pair $(Z \times [0, 1] \times I^n, Z \times \{0\} \times I^n)$. Hence, there exists a map $\tilde{F}_\sigma\colon Z \times [0, 1] \times \sigma \to E \times \sigma$ such that $\tilde{F}_\sigma \mid [Z \times [0, 1] \times \partial\sigma] \cup [Z \times \{0\} \times \sigma] = \tilde{g} \mid$ and $p_B f \tilde{F}_\sigma$ is $(\epsilon/2)$-close to $p_B F \mid$. ($\tilde{F}_0$ is not assumed to be f.p..) Define $\tilde{F}\colon Z \times [0, 1] \times X \to E \times X$ by $\tilde{F} \mid Z \times [0, 1] \times \sigma = \tilde{F}_\sigma$ whenever $\sigma$ is an $n$-simplex of $X$ and $\tilde{F} \mid Z \times [0, 1] \times X^{n-1} = G$. Finally, define $H\colon Z \times [0, 1] \times X \to E \times X$ by $H(z, t, x) = (p_E \tilde{F}(z, t, x), x)$. Then $H$ is seen to be the desired sliced $\epsilon$-lift of $F$.

Next we discuss a technical variation of the definition of an $\epsilon$-fibration which we call an $(\epsilon, \mu)$-fibration. A proof of Proposition 2.6 below can easily be given using as a model the proof of Proposition 2.2 in [2]. Therefore, the details of this argument are omitted.

DEFINITION 2.5: Let $C$ and $K$ be subsets of the ANR $B$ with $K \subset C$ and let $\epsilon > 0$ and $\mu > 0$. A proper map $f\colon E \to B$ is said to be an $(\epsilon, \mu)$*-fibration over* $(C, K)$ if given maps $F\colon Z \times [0, 1] \to C \subset B$ and $g\colon Z \to E$ with $fg = F_0$, then there is a map $G\colon Z \times [0, 1] \to E$ such that $G_0 = g$, $fG$ is $\epsilon$-close to $F$, and $fG \mid F^{-1}(K)$ is $\mu$-close to $F \mid F^{-1}(K)$.

PROPOSITION 2.6: *Let* $B$ *be an* ANR *and let* $K \subset V \subset C \subset U \subset B$ *where* $C$ *and* $K$ *are compact and* $U$ *and* $V$ *are open in* $B$. *For every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for every* $\mu > 0$ *there exists a* $\nu > 0$ *so that the following statement is true*:

*if $f$: $E \to B$ is a $\delta$-fibration over $U$ and a $\nu$-fibration over $V$, then $f$ is an $(\epsilon, \mu)$-fibration over $(C, K)$.*

We leave it to the reader to formulate and prove a version of Theorem 2.4 for $(\epsilon, \mu)$-fibrations. The reader should also have no problem in showing that $\epsilon$-fibrations, sliced $\epsilon$-fibrations, $(\epsilon, \mu)$-fibrations, and sliced $(\epsilon, \mu)$-fibrations all have an appropriate stationary (or regular) lifting property (see [8] where this is done for approximate fibrations).

## 3. Basic $Q$-manifold theory in a submersive setting

The Hilbert cube $Q$ is represented by the countable infinite product of closed intervals $[-1, 1]$. A space $M$ is a *Hilbert cube manifold* or *$Q$-manifold* if it is locally homeomorphic to open subsets of $Q$. Our reference for $Q$-manifold theory is Chapman's book [1] which should be consulted by the reader unfamiliar with the basic machinery of $Q$-manifolds including the notion of $Z$-sets.

We will need some basic results from $Q$-manifold theory parameterized by submersions. If $\pi$: $E \to B$ is a proper submersion for which the fibers $\pi^{-1}(b)$, $b \in B$, are $Q$-manifolds, then $\pi$ is actually a bundle projection. This is proved in [20] with [12] supplying the necessary deformation theorem (see also [7]). In this case adequate parameterized theories can be found in [3, Section 2], [6], [14, Section 4], and [21, Appendix 2]. Unfortunately, we will encounter in the sequel submersions whose fibers are non-compact $Q$-manifolds, and it is that fact which makes this section necessary. However, we will be working on compact pieces of the submersion and the following theorem due to Siebenmann is the main tool which allows us to deal with this situation (again, see [12] for a major ingredient).

PROPOSITION 3.1: ([20, *Corollary* 6.15]). *Let $\pi$: $E \to B$ be a submersion such that $\pi^{-1}(b)$ is a $Q$-manifold for each $b$ in $B$ and let $C \subset E$ be a compactum such that $\pi(C)$ is a point. Then there exist an open neighborhood $F$ of $C$ in $\pi^{-1}(\pi(C))$, an open neighborhood $N$ of $\pi(C)$ in $B$, and a product chart $\phi$: $F \times N \to E$ about $F$ for $\pi$.*

The following definition is a slight generalization of that given in [14] for sliced $Z$-sets in products.

DEFINITION 3.2: Let $\pi$: $E \to B$ be a submersion and let $K$ be a closed subset of $E$. Then $K$ is said to be a *sliced $Z$-set* if for every open cover $U$ of $E$ there is a map $f$: $E \to E \setminus K$ such that $f$ is $U$-close to id and $\pi f = \pi$.

The following proposition characterizes compact sliced $Z$-sets in certain submersions. The reader can prove this by using Proposition 3.1 and the characterization result from [6].

Proposition 3.3: *Let $\pi$: $M \to B$ be a submersion where $B$ is a polyhedron and $\pi^{-1}(b)$ is a $Q$-manifold for each $b$ in $B$ and let $K \subset M$ be compact. Then $K$ is a sliced $Z$-set if and only if $K \cap \pi^{-1}(b)$ is a $Z$-set in $\pi^{-1}(b)$ for each $b$ in $B$.*

The experienced reader should have no trouble in formulating and proving versions (parameterized by a submersion) of a mapping replacement theorem, a sliced $Z$-set unknotting theorem, a collaring theorem for sliced $Z$-set submanifolds, and a stability theorem. In each case the idea is to use the analogous result from one of the bundle versions mentioned above. Those results can be used locally by Proposition 3.1.

## 4. Parameterized engulfing

In this section we establish the key engulfing results used in the sequel. These results are stated as Theorems 4.3 and 4.4. Lemma 4.1 contains the basic geometric engulfing move used in the proof of Theorem 4.3.

Throughout this section $B$ and $Z$ denote ANRs where $Z \times R$ is an open subset of $B$. Projection onto $Z$ is denoted by $p_1$ and projection onto $R$ by $p_2$. Let $n \geqslant 0$ be a fixed integer and let $C$ be a (possibly empty) closed subset of $\partial I^n$.

Data for Lemma 4.1: Let $Z$ be compact. Let $\alpha_+$: $I^n \to [0, 1]$, $\alpha_-$: $I^n \to [-1, 0]$, and $\rho$: $I^n \to [-1, 0]$ be maps satisfying the following conditions:

(i) $\alpha_+^{-1}(0) \cup \alpha_-^{-1}(0) \subset \partial \mathrm{I}^n$,

(ii) $\rho = 0$ on a neighborhood of $\alpha_+^{-1}(0) \cup \alpha_-^{-1}(0)$,

(iii) $\alpha_-(t) < \rho(t)$ for each $t \in I^n \setminus \alpha_-^{-1}(0)$.

We now define four subsets of $Z \times R \times I^n \subset B \times I^n$. Let $E = \{(z, x, t) \mid \alpha_-(t) \leqslant x \leqslant \alpha_+(t)\}$, $E_- = \{(z, x, t) \mid \alpha_-(t) = x\}$, $E_+ = \{(z, x, t,) \mid \alpha_+(t) = x\}$, and $X = \{(z, x, t) \mid \alpha_-(t) \leqslant x \leqslant \rho(t)\}$. Note that $X \subset E$ and $BdE = E_- \cup E_+$.

Finally, let $Y$ be a compact subset of $E$ which misses $E_+$ and $Z \times R \times [\alpha_-^{-1}(0) \cup \alpha_+^{-1}(0)]$.

Lemma 4.1: *For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $M$ is a $Q$-manifold and $f$: $M \times [0, 1] \times I^n \to B \times I^n$ is a f.p. map which is a sliced $\delta$-fibration over $Z \times [-2, 2] \times I^n$, then there is a f.p. homeomorphism $u$: $M \times [0, 1] \times I^n \to M \times [0, 1] \times I^n$ such that*

(i) $f^{-1}(Y) \cap (M \times \{0\} \times I^n) \subset uf^{-1}(X)$,

(ii) *$u$ is supported on* $f^{-1}(\text{int } E)$,

(iii) *there is a f.p. homotopy $u_s$:* $\mathrm{id} \simeq u$, $0 \leqslant s \leqslant 1$, *which is supported on $f^{-1}(\text{int } E)$ and which is a $(p_1 f)^{-1}(\epsilon)$-homotopy over $Z \times R \times I^n$.*

Proof: Let $N_1$ be a compact neighborhood of $\alpha_-^{-1}(0) \cup \alpha_+^{-1}(0)$ in $I^n$ such that $Y$ misses $Z \times R \times N_1$ and such that $\rho = 0$ on $N_1$. Choose a

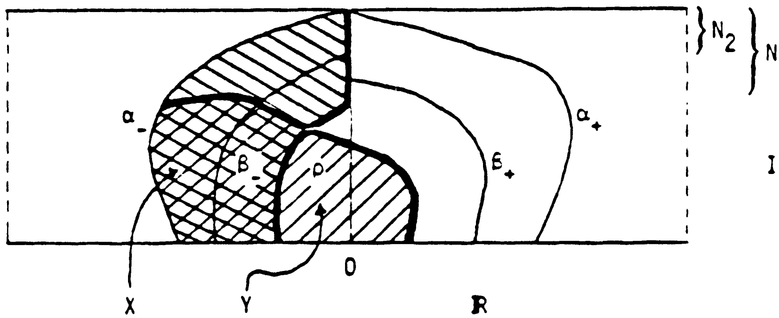


Figure 1.

compact neighborhood $N_2$ of $\alpha_-^{-1}(0) \cup \alpha_+^{-1}(0)$ in $I^n$ such that $N_2 \subset \operatorname{int} N_1$. Choose maps $\beta_+$: $I^n \to [0, 1]$ and $\beta_-$: $I^n \to [-1, 0]$ with the following properties:

(i) $\beta_+^{-1}(0) = N_1$, $\beta_-^{-1}(0) = N_2$;

(ii) $\beta_+(t) < \alpha_+(t)$ for $t \in I^n \setminus \alpha_+^{-1}(0)$, $\alpha_-(t) < \beta_-(t)$ for $t \in I^n \setminus \alpha_-^{-1}(0)$;

(iii) $Y \subset \{(z, x, t) \in Z \times R \times I^n \mid x < \beta_+(t)\}$;

(iv) $\beta_-(t) < \rho(t)$ for $t \in I^n \setminus N_2$.

See Figure 1 for a picture of the situation when $Z = \{\text{point}\}$ and $n = 1$.

Choose a f.p. isotopy $g_s$: $Z \times R \times I^n \to R \times I^n$, $0 \leqslant s \leqslant 1$, which slides the graph of $\beta_+$ over to the graph of $\beta_-$. More specifically, we require $g_s$ to satisfy the following properties:

(i) $g_0 = \mathrm{id}$;

(ii) $g_s$ affects only the $R$-coordinate of any point;

(iii) $g_s \mid Z \times R \times N_2 = \mathrm{id}$;

(iv) $g_s$ is supported on a compact subset $K$ of int $E$;

(v) $g_1(z, \beta_+(t), t) = (z, \beta_-(t), t)$ for each $(z, t) \in Z \times I^n$.

Now given a $Q$-manifold $M$ and a f.p. map $f$: $M \times [0, 1] \times I^n \to B \times I^n$ which is a sliced $\delta$-fibration over $Z \times [-2, 2] \times I^n$, define a f.p. homotopy $\tilde{g}_s$: $M \times [0, 1] \times I^n \to B \times I^n$, $0 \leqslant s \leqslant 1$, so that $\tilde{g}_s = g_s \circ f$ on $f^{-1}(Z \times R \times I^n)$ and $g_s = f$ on $(M \times [0, 1] \times I^n) \setminus f^{-1}(Z \times R \times I^n)$. Since $f$ is a sliced $\delta$-fibration over $Z \times [-2, 2] \times I^n$, there is a f.p. homotopy $G_s$: $f^{-1}(Z \times [-2, 2] \times I^n) \to M \times [0, 1] \times I^n$ such that $G_0 = \mathrm{id}$, $fG_s$ is $\delta$-close to $\tilde{g}_s \mid$ for each $s$, and $G_s$ is stationary with respect to $\tilde{g}_s \mid f^{-1}(Z \times [-2, 2] \times I^n)$

Observe that $G_s \mid [f^{-1}(Z \times [-2, 2] \times I^n) \cap (M \times \{0\} \times I^n)]$ extends via the identity to a map $G_s'$: $M \times \{0\} \times I^n \to M \times [0, 1] \times I^n$. Using [6, Theorem 4.1] we can approximate $G_1'$ by a f.p. embedding $\tilde{G}_1$: $M \times \{0\} \times I^n \to M \times [0, 1] \times I^n$ possessing the following properties:

(i) $\tilde{G}_1(M \times \{0\} \times I^n)$ is sliced $Z$-set;

(ii) $\tilde{G}_1$ is supported on $f^{-1}(K) \cap (M \times \{0\} \times I^n)$;

(iii) $\tilde{G}_1(f^{-1}(\operatorname{int} E) \cap (M \times \{0\} \times I^n)) \subset f^{-1}(\operatorname{int} E)$.

We can further assume that $\tilde{G}_1$ is so close to $G_1|$ that $\tilde{G}_1$ is f.p. homotopic to $G_1|$ via maps satisfying conditions (ii) and (iii).

Using sliced $Z$-set unknotting [6, Theorem 5.1] we can find a f.p. isotopy $u_s$: $M \times [0, 1] \times I^n \to M \times [0, 1] \times I^n$, $0 \leqslant s \leqslant 1$, so that $u_0 = \mathrm{id}$, $u_1 | \tilde{G}_1(M \times \{0\} \times I^n) = \tilde{G}_1^{-1}$, and $u_s | (M \times [0, 1] \times I^n) \setminus f^{-1}(\mathrm{int}\ E) = \mathrm{id}$. It is now easy to see that $u_1$ satisfies the conclusion of the lemma. It only remains to observe that we can assume $u_t$ is a $(p_1 f)^{-1}(\epsilon)$-homotopy over $Z \times R \times I^n$. This is because of the control on the f.p. isotopy $u_s$ given by [6, Theorem 5.1].

ADDENDUM TO LEMMA 4.1: *Let $F$ be a compact $Q$-manifold such that $F \times B \subset M$ and $f | F \times B \times \{0\} \times I^n$: $F \times B \times \{0\} \times I^n \to B \times I^n$ is projection. Extend $g_s$: $Z \times R \times I^n \to Z \times R \times I^n$ via the identity to a f.p. homeomorphism $\hat{g}_s$: $B \times I^n \to B \times I^n$, $0 \leqslant s \leqslant 1$. Then the f.p. homotopy $u_s$: $M \times [0, 1] \times I^n \to M \times [0, 1] \times I^n$, $0 \leqslant s \leqslant 1$, can be chosen to additionally satisfy $u_s | F \times B \times \{0\} \times I^n = \mathrm{id}_F \times \hat{g}_s^{-1}$, $0 \leqslant s \leqslant 1$.*

PROOF: We indicate here how to modify the proof of Lemma 4.1 in order to attain the added condition on $u_s$. Since $\tilde{g}_s | F \times B \times \{0\} \times I^n = f \circ (\mathrm{id}_F \times \hat{g}_s)$ for $0 \leqslant s \leqslant 1$ and $F \times B \times \{0\}$ is collared in $M \times [0, 1]$, it can be assumed that $G_s' | F \times B \times \{0\} \times I^n = (\mathrm{id}_F \times \hat{g}_s)$ for $0 \leqslant s \leqslant 1$. Using the full strength of [6, Theorem 4.1], the f.p. embedding $\tilde{G}_1$: $M \times \{0\} \times I^n \to M \times [0, 1] \times I^n$ can be chosen so that $\tilde{G}_1 | F \times B \times \{0\} \times I^n = (\mathrm{id}_F \times \hat{g}_1)$ and the homotopy from $G_1'$ to $\tilde{G}_1$ is rel $F \times B \times \{0\} \times I^n$. After a reparameterization of the homotopy from $G_0'$ to $\tilde{G}_1$, one simply uses a strong relative version of sliced $Z$-set unknotting to produce $u_s$: $M \times [0, 1] \times I^n \to M \times [0, 1] \times I^n$, $0 \leqslant s \leqslant 1$, with the desired properties.

DATA FOR LEMMA 4.2: Let $Z$ be compact. Let $\alpha_1$: $I^n \to [-1, 1]$ and $\alpha_2$: $I^n \to [0, 1]$ denote maps such that $\alpha_1(t) < \alpha_2(t)$ for each $t \in I^n$ and $\alpha_1^{-1}(-1) = \alpha_2^{-1}(0) = C$. Let $\Gamma(\alpha_1) = \{(z, x, t) \in Z \times R \times I^n \mid x \leqslant \alpha_1(t)\}$. See Figure 2.

LEMMA 4.2. *For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $M$ is a $Q$-manifold and $f$: $M \times I^n \to B \times I^n$ is a f.p. map which is a sliced $\delta$-fibration over $Z \times [-2, 2] \times I^n$, then there is a f.p. homeomorphism $h$: $M \times I^n \to M \times I^n$ such that*

(i) *$h | M \times C$ is the identity,*
(ii) *$f^{-1}(\Gamma(\alpha_1)) \subset hf^{-1}(Z \times (-\infty, 0] \times I^n)$,*
(iii) *there is a f.p. homotopy $h_s$: $\mathrm{id} \simeq h$, $0 \leqslant s \leqslant 1$, which is a $(p_1 f)^{-1}(\epsilon)$-homotopy over $Z \times R \times I^n$,*
(iv) *$h_s$ is supported on $f^{-1}\{(z, x, t) \in Z \times R \times I^n \mid -0.9 \leqslant x \leqslant \alpha_2(t), t \in I^n \setminus C\}$ for each $0 \leqslant s \leqslant 1$.*

PROOF. Given $\epsilon > 0$, $\delta > 0$ is chosen by Lemma 4.1 so that the two basic engulfing moves described below can be made. Given a f.p. map $f$:

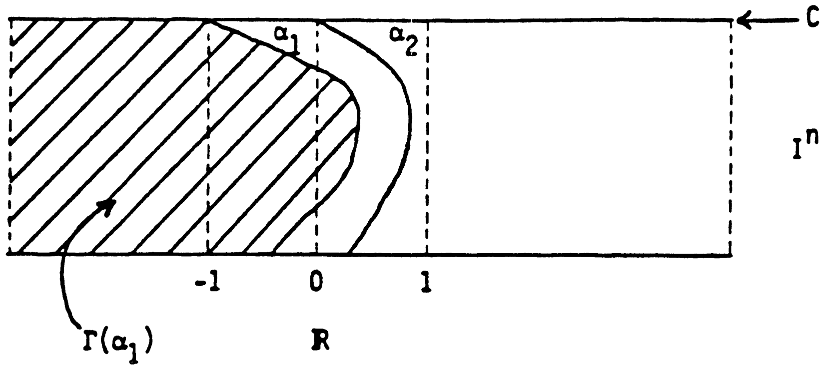


Figure 2.

$M \times I^n \to B \times I^n$ as in the hypothesis, choose a f.p. map $k$: $M \times [0, 1] \times I^n \to M \times I^n$ close to projection such that $k \mid M \times [0, 1] \times C$ is projection and $k \mid M \times [0, 1] \times (I^n \setminus C)$: $M \times [0, 1] \times (I^n \setminus C) \to M \times (I^n \setminus C)$ is a homeomorphism (see [14, Theorem 4.6]). It then follows that $fk$: $M \times [0, 1] \times I^n \to B \times I^n$ is also a sliced $\delta$-fibration over $Z \times [-2, 2] \times I^n$. The homeomorphism $h$ will be defined as a composition $h = k \circ v \circ w \circ u \circ k^{-1}$ where $v$, $w$, and $u$ are constructed below.

Use Lemma 4.1 to produce a f.p. homeomorphism $u$: $M \times [0, 1] \times I^n \to M \times [0, 1] \times I^n$ such that $(fk)^{-1}(\Gamma(\alpha_1)) \cap (M \times \{0\} \times I^n) \subset u(fk)^{-1}(Z \times (-\infty, 0) \times I^n)$ and $u$ is supported on $(fk)^{-1}\{(z, x, t) \in Z \times R \times I^n \mid -0.5 \leqslant x \leqslant \alpha_2(t),\ t \in I^n \setminus C\}$.

Let $S_1 = (fk)^{-1}\{(z, x, t) \in Z \times R \times I^n \mid -0.6 \leqslant x \leqslant \alpha_1(t)\}$ and let $S_2 = (fk)^{-1}\{(z, x, t) \in Z \times R \times I^n \mid \max(0, \alpha_1(t)) < \alpha_2(t)\}$. Use Lemma 4.1 again to produce a f.p. homeomorphism $v$: $M \times [0, 1] \times I^n \to M \times [0, 1] \times I^n$ such that $S_1 \cap (M \times \{1\} \times I^n) \subset v^{-1}(S_2)$ and $v$ is supported on $(fk)^{-1}\{(z, x, t) Z \times R \times I^n \mid -0.7 \leqslant x < \alpha_2(t),\ t \in I^n \setminus C\}$.

Let $U = (fkv)^{-1}\{(z, x, t) \in Z \times R \times I^n \mid \max(0, \alpha_1(t)) < x\}$ and observe that if $k$ is close enough to projection, then $S_1 \subset [\pi(U \cap (M \times \{1\} \times I^n))] \times [0, 1]$ where $\pi$: $M \times [0, 1] \times I^n \to M \times I^n$ is projection. Then $w$: $M \times [0, 1] \times I^n \to M \times [0, 1] \times I^n$ is a f.p. homeomorphism affecting only the $[0, 1]$-coordinate of any point such that $w[S_1 \setminus u(fk)^{-1}(Z \times (-\infty, 0) \times I^n)] \subset U$. The support of $w$ is on $(fk)^{-1}\{(z, x, t) \in Z \times R \times I^n \mid -0.7 \leqslant x \leqslant \alpha_2(t)\}$.

It is easily verified that $h = k \circ v \circ w \circ u \circ k^{-1}$ satisfies the conclusions of the lemma. The homotopy of the identity to $h$ comes from composing three homotopies of $u$, $v$, and $w$ to the identity. The homotopies for $u$ and $v$ are provided by Lemma 4.1; the homotopy needed for $w$ comes from pushing along the $[0, 1]$-factor in $M \times [0, 1] \times I^n$. □

ADDENDUM TO LEMMA 4.2: *Let $N_1$ and $N_2$ be compact neighborhoods of $C$ in $I^n$ such that $N_2 \subset \operatorname{int} N_1$ and $\Gamma(\alpha_1)$ misses $Z \times [-0.5, +\infty) \times N_1$.*

*Choose maps* $\beta_+\colon I^n \to [0, 1]$ *and* $\beta_-\colon I^n \to (-0.5, 0]$ *with the following properties*:

(i) $\beta_+^{-1}(0) = N_1$, $\beta_-^{-1}(0) = N_2$;

(ii) $\alpha_1(t) < \beta_+(t)$ *for each* $t \in I^n$;

(iii) $\beta_+(t) < \alpha_2(t)$ *for each* $t \in I^n \setminus C$.

*Suppose we are given a f.p. isotopy* $g_s\colon B \times I^n \to B \times I^n$, $0 \leqslant s \leqslant 1$, *such that* $g_s \mid Z \times R \times I^n$ *satisfies the properties listed for* $g_s$ *in the proof of Lemma* 4.1 *where now* $E = \{(z, x, t) \in Z \times R \times I^n \mid -0.5 \leqslant x \leqslant \alpha_2(t)\}$. *Let F be a compact Q-manifold such that* $F \times B$ *is a Z-set in M and* $f \mid F \times B \times I^n\colon F \times B \times I^n \to B \times I^n$ *is projection. Then the f.p. homeomorphism* $h\colon M \times I^n \to M \times I^n$ *can be chosen to additionally satisfy* $h \mid F \times B \times I^n = \mathrm{id}_F \times g_1^{-1}$ *and the f.p. homotopy* $h_s\colon \mathrm{id} \simeq h$, $0 \leqslant s \leqslant 1$, *can be chosen to additionally satisfy* $h_s \mid F \times B \times I^n = \mathrm{id}_F \times g_s^{-1}$, $0 \leqslant s \leqslant 1$.

PROOF: Just three modifications need to be made in the proof of Lemma 4.2. First, choose the f.p. map $k\colon M \times [0, 1] \times I^n \to M \times I^n$ to additionally satisfy $k \mid F \times B \times \{0\} \times I^n\colon F \times B \times \{0\} \times I^n \to F \times B \times I^n$ is the identity. This is possible by [6, Theorem 4.1]. Secondly, choose the f.p. homeomorphism $u\colon M \times [0, 1] \times I^n \to M \times I^n$ and the f.p. homotopy $u_s$: $\mathrm{id} \simeq u$, $0 \leqslant s \leqslant 1$, so that $u_s \mid F \times B \times \{0\} \times I^n = \mathrm{id} \times g_s^{-1}$. This is possible by the Addendum to Lemma 4.1. Finally, choose the f.p. homeomorphism $v\colon M \times [0, 1] \times I^n \to M \times [0, 1] \times I^n$ and the f.p. homotopy $v_s$: $\mathrm{id} \simeq v$, $0 \leqslant s \leqslant 1$, so that $v_s \mid M \times \{0\} \times I^n$ is the identity. To see that this is possible, recall that $v$ is provided by Lemma 4.1 and reexamine its proof.

With these modifications it is now easy to see that $h = k \circ v \circ w \circ u \circ k^{-1}$ satisfies the conclusion of the addendum. □

DATA FOR THEOREM 4.3: Let $Z$ be compact and let $\theta\colon R \times I^n \to R \times I^n$ be a f.p. homeomorphism with the following properties:

(i) $\theta \mid R \times C$ is the identity;

(ii) $x \leqslant p_2\theta(x, t)$ for each $x \in R$ and $t \in I^n$;

(iii) $\theta$ is supported on $[-1, 1] \times I^n$.

Let $\theta'\colon B \times I^n \to B \times I^n$ denote the f.p. homeomorphism which extends $\mathrm{id}_Z \times \theta$ via the identity.

For each $\bar{x} \in R$, let $\Gamma(\theta, \bar{x}) = \{(z, x, t) \in Z \times R \times I^n \mid x \leqslant p_2\theta(\bar{x}, t)\}$.

THEOREM 4.3: *For every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that if M is a Q-manifold and* $f\colon M \times I^n \to B \times I^n$ *is a f.p. map which is a sliced* $\delta$*-fibration over* $Z \times [-2, 2] \times I^n$, *then there is a f.p. homeomorphism* $\tilde{\theta}$: $M \times I^n \to M \times I^n$ *such that*

(i) $\tilde{\theta} \mid M \times C$ *is the identity*,

(ii) $f\tilde{\theta}$ *is* $\epsilon$*-close to* $\theta' f$,

(iii) $\tilde{\theta}$ *is supported on* $f^{-1}(Z \times [-1, 1] \times I^n)$,

(iv) *there is a f.p. homotopy* $\tilde{\theta}_s$: $\mathrm{id} \simeq \tilde{\theta}$, $0 \leqslant s \leqslant 1$, *which is a* $(p_1 f)^{-1}(\epsilon)$*-homotopy over* $Z \times R \times I^n$ *and is supported on* $f^{-1}(Z \times [-1, 1] \times (I^n \setminus C))$.

*Moreover, if we are additionally given a compact Q-manifold F such that* $F \times B$ *is a Z-set in M and* $f \mid F \times B \times I^n$: $F \times B \times I^n \to B \times I^n$ *is projection, then* $\tilde{\theta}$ *can be chosen so that* $\tilde{\theta} \mid F \times B \times I^n = \mathrm{id}_F \times \theta'$ *and the homotopy* $\tilde{\theta}_s$, $0 \leqslant s \leqslant 1$, *can be chosen so that* $p_1 \tilde{\theta}_s \mid F \times Z \times R \times I^n = p_1$ *for* $0 \leqslant s \leqslant 1$.

PROOF: Let $\epsilon > 0$ be given. Choose a partition $-1 = x_0 < x_1 < x_2 < \dots < x_{m-1} < x_m = 1$ of $[-1, 1]$ so fine that the interval $[p_2\theta(x_{i-2}, t), p_2\theta(x_i, t)]$ has diameter less than $\epsilon/2$ for each $i = 2, 3, \dots, m$ and $t \in I^n$. Then $\delta > 0$ is chosen according to Lemma 4.2 so that each of the $m-1$ engulfing moves described below can be performed.

Given a $Q$-manifold $M$ and a f.p. map $f$: $M \times I^n \to B \times I^n$ which is a sliced $\delta$-fibration over $Z \times [-2,2] \times I^n$, we proceed to define f.p. homeomorphisms $\tilde{\theta}^i$: $M \times I^n \to M \times I^n$ and $\theta^i$: $R \times I^n \to R \times I^n$. Choose compact neighborhoods $N_1$ and $N_2$ of $C$ in $I^n$ such that $N_2 \subset \mathrm{int} N_1$ and such that $\Gamma(\theta, x_{i-1})$ misses $Z \times [1/2(x_{i-1} + x_i), +\infty) \times N_1$ for $i = 1, \dots, m-1$. For $i = 1, \dots, m-1$ choose maps $\beta^i_+$: $I^n \to [x_i, +\infty)$ and $\beta^i_-$: $I^n \to (1/2(x_{i-1} + x_i), x_i]$ such that $(\beta^i_+)^{-1}(x_i) = N_1$, $(\beta^i_-)^{-1}(x_i) = N_2$, and $p_2\theta(x_{i-1}, t) < \beta^i_+(t) < p_2\theta(x_i, t)$ for each $t \in I^n \setminus C$. See Figure 3.

Now $\theta^i$: $R \times I^n \to R \times I^n$ is defined to be the f.p. homeomorphism which is supported on $\{(x, t) \in R \times I^n \mid 1/2(x_{i-1} + x_i) < x < p_2\theta(x_i, t), t \in I^n \setminus N_2\}$ and which slides the graph of $\beta^i_-$ over to the graph of $\beta^i_+$; that is, $\theta^i(\beta^i_-(t), t) = (\beta^i_+(t), t)$ for each $t \in I^n$. Let $\bar{\theta}^i$: $B \times I^n \to B \times I^n$ denote the f.p. homeomorphism which extends $\mathrm{id}_Z \times \theta^i$ via the identity. There is an obvious f.p. isotopy $\theta^i_s$: $\mathrm{id} \simeq \theta^i$, $0 \leqslant s \leqslant 1$, and $\mathrm{id}_Z \times \theta^i_s$ extends via the identity to $\bar{\theta}^i_s$: $\mathrm{id} \simeq \bar{\theta}^i$, $0 \leqslant s \leqslant 1$

According to Lemma 4.2 for each $i = 1, \dots, m-1$ there exists a f.p. homeomorphism $\tilde{\theta}^i$: $M \times I^n \to M \times I^n$ such that

(i) $\tilde{\theta}^i \mid M \times C$ is the identity,

(ii) $f^{-1}(\Gamma(\theta, x_{i-1})) \subset \tilde{\theta}^i f^{-1}(Z \times (-\infty, x_i] \times I^n)$,

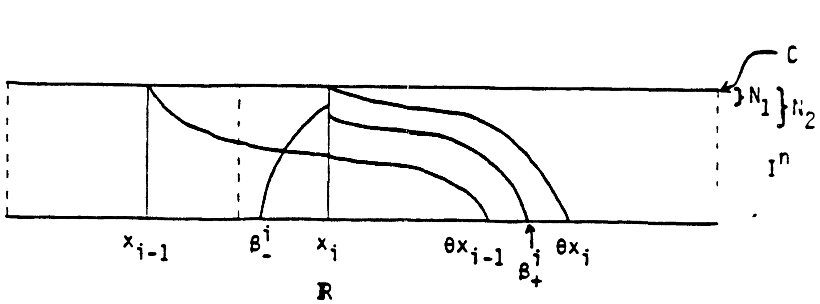


Figure 3.

(iii) $\tilde{\theta}^i$ is supported on $f^{-1}(\Gamma(\theta, x_i))\backslash f^{-1}(Z\times(-\infty, x_{i-1}]\times I^n)$,
(iv) there is a f.p. homotopy $\tilde{\theta}^i_s$: id $\simeq \tilde{\theta}^i$, $0 \leqslant s \leqslant 1$, which is a $(p_1 f)^{-1}$ $(\epsilon/2m)$-homotopy over $Z \times R \times I^n$ and which is supported on

$$\left[f^{-1}(\Gamma(\theta, x_i))\backslash f^{-1}(Z\times(-\infty, x_{i-1}]\times I^n)\right]\backslash(M\times C).$$

Moreover, if $F$ is a compact $Q$-manifold such that $F \times B$ is a $Z$-set in $M$ and $f \mid F\times B\times I^n$: $F\times B\times I^n \to B\times I^n$ is projection, then $\tilde{\theta}^i$ can be chosen so that $\tilde{\theta}^i \mid F\times B\times I^n = \mathrm{id}_F \times \bar{\theta}^i$ and $\tilde{\theta}^i_s$ can be chosen so that $\tilde{\theta}^i_s \mid F\times B\times I^n = \mathrm{id}_F\times\bar{\theta}^i_s$, $0\leqslant s\leqslant 1$.

Consider the following compositions:

$$\bar{\theta} = \bar{\theta}^1 \circ \bar{\theta}^2 \circ \ldots \circ \bar{\theta}^{m-1}, \quad \bar{\theta}_s = \bar{\theta}^1_s \circ \bar{\theta}^2_s \circ \ldots \circ \bar{\theta}^{m-1}_s$$

$$\tilde{\tilde{\theta}} = \tilde{\theta}^1 \circ \tilde{\theta}^2 \circ \ldots \circ \tilde{\theta}^{m-1} \quad \text{and} \quad \tilde{\tilde{\theta}}_s = \tilde{\theta}^1_s \circ \tilde{\theta}^2_s \circ \ldots \circ \tilde{\theta}^{m-1}_s$$

It follows from the construction that $\Gamma(\theta, x_{i-1}) \subset \bar{\theta}(Z\times(-\infty, x_i]\times I^n) \subset \Gamma(\theta, x_i)$ and $f^{-1}(\Gamma(\theta, x_{i1})) \subset \tilde{\tilde{\theta}} f^{-1}(Z\times(-\infty, x_i]\times I^n) \subset f^{-1}(\Gamma(\theta, x_i))$ for $i = 1, 2, \ldots, m$. From this it follows that $\bar{\theta}$ is $\epsilon$-close to $\theta$ and $f\tilde{\tilde{\theta}}$ is $\epsilon$-close to $\theta' f$. Also $\tilde{\tilde{\theta}}_s$: id $= \tilde{\tilde{\theta}}$, $0 \leqslant s \leqslant 1$, is a $(p_1 f)^{-1}(\epsilon)$-homotopy over $Z\times R\times I^n$. Since $\tilde{\tilde{\theta}} \mid F\times B\times I^n = \mathrm{id}_F \times \bar{\theta}$, we must modify $\tilde{\tilde{\theta}}$ to get the required $\tilde{\theta}$.

Since $F\times B$ is a $Z$-set in $M$, there is a collar about $F\times B$ in $M$. Thus, we can consider $F\times B\times[0, 2)$ as an open subset of $M$ with $F\times B$ and $F\times B\times\{0\}$ identified. Let $\psi$: $M\to M$ be an embedding which is supported on $F\times B\times[0, 1.5]$ and just pushes $M$ in along the collar so that $\psi(f, b, 0) = (f, b, 1)$ for $(f, b)\in F\times B$. There is a f.p. isotopy $H_s$: $\theta \simeq \theta^1 \circ \theta^2 \circ \ldots \circ \theta^{m-1}$, $0\leqslant s\leqslant 1$, which is supported on $[-1, 1]\times(I^n\backslash C)$. If the partition $-1 = x_0 < x_1 < \ldots < x_m = 1$ is fine, then $H_s$, $0\leqslant s\leqslant 1$, is a small isotopy. Let $\bar{H}_s$: $B\times I^n \to B\times I^n$, $0\leqslant s\leqslant 1$, denote the f.p. isotopy which extends $\mathrm{id}_Z\times H_s$ via the identity. Define the f.p. homeomorphism $\tilde{\theta}$: $M\times I^n \to M\times I^n$ as follows. First, let $\tilde{\theta} \mid [M\backslash(F\times B\times[0, 1))]\times I^n = (\psi\times\mathrm{id}_{I^n})\circ\tilde{\tilde{\theta}}\circ(\psi^{-1}\times\mathrm{id}_{I^n})$. Then, for $(f, b, u, t)\in F\times B\times[0, 1]\times I^n$, let $\tilde{\theta}(f, b, u, t) = (f, p_B\bar{H}_u(b, t), u, t)$ where $p_B$ denotes projection onto $B$. By making the collar on $F\times B$ short in $M$, it can be seen that $\tilde{\theta}$ is close to $\tilde{\tilde{\theta}}$ and satisfies the conclusions of the theorem. The appropriate f.p. isotopy $\tilde{\theta}_s$: id $\simeq \tilde{\theta}$, $0\leqslant s\leqslant 1$, comes by first using the isotopy $\tilde{\tilde{\theta}}_s$: id $\simeq \tilde{\tilde{\theta}}$, $0\leqslant s\leqslant 1$, and then using the collar coordinate and the definition of $\tilde{\theta}$ to get an isotopy from $\tilde{\tilde{\theta}}$ to $\tilde{\theta}$.

We are now ready to state without proof a generalization of Theorem 4.3. Theorem 4.4 differs from Theorem 4.3 in two aspects. First, $Z$ is no longer required to be compact and second, we replace the product $M\times I^n$ by a submersion $\pi$: $M\to I^n$.

DATA FOR THEOREM 4.4: Let $\phi$: $Z \to [0, +\infty)$ be a proper map and for each $r \geqslant 0$ let $Z_r = \phi^{-1}([0, r])$ and $Z^r = \phi^{-1}([r, +\infty))$. Let $\bar{\theta}_r$: $R \times I^n \to R \times I^n$, $r \geqslant 0$, be a f.p. isotopy with the following properties:

(i) $\bar{\theta}_r$ is the identity for $r \geqslant 1$;

(ii) $\bar{\theta}_r | R \times C$ is the identity for $r \geqslant 0$;

(iii) $x \leqslant p_2\bar{\theta}_r(x, t)$ for each $(r, x, t)$ in $[0, +\infty) \times R \times I^n$;

(iv) $\bar{\theta}_r$ is supported on $[-1, 1] \times I^n$ for $r \geqslant 0$.

Define $\theta$: $Z \times R \times I^n \to Z \times R \times I^n$ by $\theta(z, x, t) = (z, \bar{\theta}_{\phi(z)}(x, t))$. Then $\theta$ is a f.p. homeomorphism supported on $Z_1 \times [-1, 1] \times (I^n \setminus C)$ which extends via the identity to a f.p. homeomorphism $\theta'$: $B \times I^n \to B \times I^n$.

THEOREM 4.4: *For every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for every* $\mu > 0$ *there exists a* $\nu > 0$ *so that the following statement is true*:

*If* $M$ *is a* $Q$*-manifold,* $\pi$: $M \to I^n$ *is a submersion with* $Q$*-manifold fibers,* $f$: $M \to B \times I^n$ *is a proper f.p. map such that* $f_t$: $\pi^{-1}(t) \to B$ *is a* $\delta$*-fibration over* $Z_3 \times [-3, 3]$ *and a* $\nu$*-fibration over* $(Z_3 \setminus Z_{1/3}) \times [-3, 3]$ *for each* $t$ *in* $I^n$*, then there is a f.p. homeomorphism* $\tilde{\theta}$: $M \to M$ *such that*

(i) $\tilde{\theta} | \pi^{-1}(C)$ *is the identity,*

(ii) $f\tilde{\theta}$ *is* $\epsilon$*-close to* $\theta' f$*,*

(iii) $f\tilde{\theta}$ *is* $\mu$*-close to* $\theta' f$ *over* $Z^{2/3} \times R \times I^n$*,*

(iv) $\tilde{\theta}$ *is supported on* $f^{-1}(Z_1 \times [-1, 1] \times I^n)$*,*

(v) *there is a f.p. isotopy* $\tilde{\theta}_s$: $\mathrm{id} \simeq \tilde{\theta}$, $0 \leqslant s \leqslant 1$, *which is a* $(p_1 f)^{-1}(\mu)$*-homotopy over* $Z \times R \times I^n$ *and a* $(p_1 f)^{-1}(\mu)$*-homotopy over* $Z \times R \times I^n$ *and which is supported on* $f^{-1}(Z_1 \times [-1, 1] \times (I^n \setminus C))$.

*Moreover, if we are additionally given a compact* $Q$*-manifold* $F$ *and a sliced* $Z$*-embedding* $g$: $F \times B \times I^n \to M$ *such that* $\pi g$ *is projection and* $fg$ *is projection, then* $\tilde{\theta}$ *can be chosen so that* $\tilde{\theta} g = \mathrm{id}_F \times \theta'$ *and the homotopy* $\theta_s$, $0 \leqslant s \leqslant 1$, *can be chosen so that* $p_1\tilde{\theta}_s g = p_1$ *for* $0 \leqslant s \leqslant 1$.

REMARKS ON PROOF: The proof proceeds along the general lines of the proof of Theorem 4.3. In proving the appropriate lemmas analogous to lemmas 4.1 and 4.2 only two significant changes need to be made besides the general extra care that must be taken in defining maps and homotopies in order to allow for the extra degree of freedom in the $Z$-direction. First, one must invoke the remarks made in Section 2 in order to conclude that $f$ has the appropriate sliced $(\delta, \nu)$-lifting property. Second, one must replace the standard $Q$-manifold apparatus by the submersion results developed in Section 3.

## 5. Parameterized wrapping

In this section we present a parameterized version of Chapman's construction for wrapping up $\delta$-fibrations around $S^1$. For notation let $B$ and $Z$ denote ANRs where $Z \times R$ is an open subset of $B$. Let $\phi$: $Z \to [0, +\infty)$ be a proper map and for $r$ in $[0, +\infty)$ define $Z_r = \phi^{-1}([0, r])$ and

$Z^r = \phi^{-1}([r, +\infty))$. Let $n \geqslant 0$ be an integer. The map $p_1$ denotes projection onto $Z$, $p_2$ projection onto $R$, and $p_3$ projection onto $I^n$. Finally, let $e$: $R \to S^1$ be the covering projection defined by $e(x) = \exp(\pi i x/4)$ (thus $e$ has period 8). This notation will be used throughout this section.

THEOREM 5.1: *For every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:*

*if $M$ is a Q-manifold and $f$: $M \times I^n \to B \times I^n$ is a proper f.p. map which is a sliced $\delta$-fibration over $Z_3 \times [-3, 3] \times I^n$ and a sliced $\nu$-fibration over $(Z_3 \setminus \mathring{Z}_{1/3}) \times [-3, 3] \times I^n$, then there is a Q-manifold $\tilde{M}$, a submersion $\pi$: $\tilde{M} \to I^n$ with Q-manifold fibers, a f.p. map $\tilde{f}$: $\tilde{M} \to \mathring{Z}_{2.5} \times S^1 \times I^n$ such that $\tilde{f}_t$: $\pi^{-1}(t) \to \mathring{Z}_{2.5} \times S^1$ is an $\epsilon$-fibration over $Z_2 \times S^1$ and a $\mu$-fibration over $(Z_2 \setminus \mathring{Z}_{2/3}) \times S^1$ for each $t$ in $I^n$, and a f.p. open embedding $\psi$: $f^{-1}(\mathring{Z}_1 \times (-1, 1) \times I^n) \to \tilde{M}$ for which the following diagram commutes:*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \mathring{Z}_{2.5} \times S^1 \times I^n \\ \psi \uparrow & & \mathrm{id} \times e \times \mathrm{id} \uparrow \\ f^{-1}(\mathring{Z}_1 \times (-1, 1) \times I^n) & \xrightarrow{f|} & \mathring{Z}_1 \times (-1, 1) \times I^n \end{array}$$

*Moreover, if we are additionally given a compact Q-manifold F such that $F \times B$ is a Z-set in $M$ and $f \mid F \times B \times I^n$: $F \times B \times I^n \to B \times I^n$ is projection, then we can additionally conclude that there is a sliced Z-embedding $g$: $F \times \mathring{Z}_{2.5} \times S^1 \times I^n \to \tilde{M}$ for which $\tilde{f}g$: $F \times \mathring{Z}_{2.5} \times S^1 \times I^n \to \mathring{Z}_{2.5} \times S^1 \times I^n$ is projection and for which the following diagram commutes:*

$$\begin{array}{ccc} F \times \mathring{Z}_{2.5} \times S^1 \times I^n & \xrightarrow{g} & \tilde{M} \\ \mathrm{id} \times e \times \mathrm{id} \uparrow & & \uparrow \psi \\ F \times \mathring{Z}_1 \times (-1, 1) \times I^n & \subset & f^{-1}(\mathring{Z}_1 \times (-1, 1) \times I^n) \end{array}$$

PROOF OF THEOREM 5.1: Let $\bar{\theta}_r$: $R \times I^n \to R \times I^n$, $0 \leqslant r < +\infty$, be the f.p. isotopy such that for $0 \leqslant r \leqslant 2.7$ $\bar{\theta}_r$ is the f.p. $PL$ homeomorphism supported on $[-2.4, 2.4] \times I^n$ with the property that $\bar{\theta}_r(x, t) = (x + 4, t)$ for $-2.2 \leqslant x \leqslant -1.8$ and $t \in I^n$. For $2.7 \leqslant r \leqslant 2.8$, $\bar{\theta}_r$ is phased out to the identity so that $\bar{\theta}_r = \mathrm{id}$ for $r \geqslant 2.8$. Define $\theta$: $Z \times R \times I^n \to Z \times R \times I^n$ by $\theta(z, x, t) = (z, \bar{\theta}_{\phi(z)}(x, t))$. By engulfing (Theorem 4.8) there is a f.p. homotopy $h_s$: $\mathrm{id} \simeq h_1$, $0 \leqslant s \leqslant 1$, on $M \times I^n$ where $h_1$: $M \times I^n \to M \times I^n$ is a f.p. homeomorphism such that $fh_1$ is $\delta'$-close to $\theta f$ over $Z \times R \times I^n$, and $fh_1$ is $\nu'$-close to $\theta f$ over $Z^{1/2} \times R \times I^n$, and the homotopy is supported on $f^{-1}(Z_3 \times [-3, 3] \times I^n)$ (Theorem 4.8 also gives some control on the size of the homotopy which we will need.) Here, $\delta'$ and $\nu'$ are small if $\delta$ and $\nu$ are small, respectively. Moreover, if we are given a compact $Q$-manifold $F$ as in the hypothesis, then we may assume that $h_1 \mid F \times Z \times R \times I^n = \mathrm{id}_F \times \theta$ and $p_1 h_s \mid F \times Z \times R \times I^n = p_1$ for $0 \leqslant s \leqslant 1$.

Let $Y = h_1 f^{-1}(Z_{2.7} \times (-\infty, -2] \times I^n) \setminus f^{-1}(Z \times (-\infty, -2) \times I^n)$, $E_- = Y \cap f^{-1}(Z \times \{-2\} \times I^n)$, and $E_+ = h_1 f^{-1}(Z_{2.7} \times \{-2\} \times I^n)$. Let $\sim$ be the equivalence relation on $Y$ generated by the rule: if $y$ is in $Y \cap f^{-1}(Z_{2.7} \times \{-2\} \times I^n)$, then $y \sim h_1(y)$. Let $\overline{M} = Y/\sim$ and let $q$: $Y \to \overline{M}$ denote the quotient map. The proof of the first assertion is straightforward.

ASSERTION 1: *The relation $\sim$ induces an upper semi-continuous decomposition of $Y$.*

ASSERTION 2: *There exists a map $\alpha$: $Y \to Z$ such that*

(i) *$\alpha(y) = \alpha(y')$ if $y \sim y'$,*

(ii) *$\alpha \mid [f^{-1}(Z \times [-2, 1.99] \times I^n) \cap Y] = p_1 f \mid$,*

(iii) *$\alpha$ is $\delta'$-close to $p_1 f \mid Y$,*

(iv) *$\alpha$ is $\nu'$-close to $p_1 f \mid Y$ over $Z^{r_1}$ where $r_1$ is fixed so that $1/2 < r_1 < 2/3$.*

PROOF: Define a homotopy $g_s$:$[f^{-1}(Z \times [-2, 1.99] \times I^n) \cap Y] \cup E_+ \to Z$, $0 \leqslant s \leqslant 1$, by $g_s \mid [f^{-1}(Z \times [-2, 1.99] \times I^n) \cap Y] = p_1 f \mid$ and $g_s \mid E_+ = p_1 f h_{1-s} h_1^{-1} \mid E_+$. Note that $g_0$ extends to $p_1 f \mid$: $Y \to Z$. By the homotopy extension property there is an extension $\tilde{g}_s$: $Y \to Z$ of $g_s$ such that $\tilde{g}_0 = p_1 f \mid$. Using the estimated homotopy extension property (see [4]) and the control on the homotopy $h_s$, we may assume that the homotopy $\tilde{g}_s$ is controlled in the $p_1 f$ direction. Then define $\alpha = \tilde{g}_1$.

ASSERTION 3: *If we are given the compact $Q$-manifold $F$ as in the hypothesis, then the map $\alpha$ of Assertion 2 can be chosen so that $\alpha \mid (F \times Z \times R \times I^n) \cap Y = p_1 f \mid = p_1$.*

PROOF: In the proof of Assertion 2 extend $g_s$ to $(F \times Z \times R \times I^n) \cap Y$ by setting $g_s = p_1 f \mid = p_1$ on this set. This is well-defined because $p_1 f h_{1-s} h_1^{-1} \mid F \times Z \times R \times I^n = p_1$.

ASSERTION 4: *There is a map $\beta$: $Y \to [-2, 2]$ such that*

(i) *$\beta(E_-) = -2$,*

(ii) *$\beta(E_+) = +2$,*

(iii) *$\beta \mid [f^{-1}(Z \times [-2, 1.99] \times I^n) \cap Y] = p_2 f \mid$,*

(iv) *$\beta$ is $\delta'$-close to $p_2 f \mid$,*

(v) *$\beta$ is $\nu'$-close to $p_2 f \mid$ on $f^{-1}(Z^{r_1} \times R \times I^n) \cap Y$,*

(vi) *if $F$ is a compact $Q$-manifold given as in the hypothesis, then $\beta \mid (F \times Z \times R \times I^n) \cap Y = p_2 f \mid = p_2$.*

PROOF: Define a homotopy $g_s$: $([f^{-1}(Z\times[-2, 1.99]\times I^n)\cup(F\times Z\times R\times I^n)]\cap Y)\cup E_+\to R$, $0\leqslant s\leqslant 1$, as follows: first $g_s|[f^{-1}(Z\times[-2, 1.99]\times I^n)\cup(F\times Z\times R\times I^n)]\cap Y=p_2f|$. On $E_+$ define $g_s$ so that $g_0|E_+=p_2f|$ and as $s$ goes from 0 to 1, $g_s$ shrinks $p_2f(E_+)$ to $+2$ so that $g_1(E_+)=+2$. Note that this can be done so that it does not conflict with the definition of $g_s|(F\times Z\times R\times I^n)\cap Y$. Now $g_0$ extends to $p_2f|$: $Y\to R$ and so we may use the estimated homotopy extension property to extend $g_s$ to $\tilde{g}_s$: $Y\to R$. Let $r$: $R\to[-2, 2]$ be the retraction such that $r((-\infty, -2])=-2$ and $r([+2, +\infty))=+2$. Then define $\beta=r\tilde{g}_1$. This completes the proof of Assertion 4.

Identify $S^1$ with the quotient space $[-2, 2]/\{-2, 2\}$ and let $u$: $[-2, 2]\to S^1$ be the quotient map. Do this in such a way that $u|[-1, 1]=e|[-1, 1]$.

Define $\bar{f}$: $\overline{M}\to Z\times S^1\times I^n$ by $\bar{f}(q(y))=(\alpha(y), u\beta(y), p_3(y))$ for $y$ in $Y$. This map is well-defined. Let $\tilde{M}=\bar{f}^{-1}(\mathring{Z}_{2.5}\times S^1\times I^n)$ and let $\tilde{f}$: $\tilde{M}\to\mathring{Z}_{2.5}\times S^1\times I^n$ denote the restriction of $\bar{f}$ to $\tilde{M}$. Define $\pi$: $\tilde{M}\to I^n$ by $\pi(q(y))=p_3(y)$ for $y$ in $q^{-1}(\tilde{M})\subset Y$.

ASSERTION 5: *$\pi$: $\tilde{M}\to I^n$ is a submersion.*

PROOF: First let $y\in E_-$ such that $q(y)\in\tilde{M}$. Thus $\bar{f}q(y)\in\mathring{Z}_{2.5}\times S^1\times I^n$ and from this we may conclude that $y\in f^{-1}(\mathring{Z}_{2.5}\times\{-2\}\times I^n)$. Let $U=h_1f^{-1}(\mathring{Z}_{2.6}\times(-\infty, -1.8)\times I^n)\backslash f^{-1}(Z\times(-\infty, 1.8]\times I^n)$. Define $q'$: $U\to\overline{M}$ by $q'=q$ on $U\cap Y$ and $q'=qh_1^{-1}$ on $U\backslash Y$. Note that $q'$ is an open embedding and $q(y)\in q'(U)$. Let $\tilde{U}=U\cap(q')^{-1}(\tilde{M})$. Then $\tilde{U}$ is an open subset of $M\times I^n$ and $q'|\tilde{U}$: $\tilde{U}\to\tilde{M}$ is an open embedding onto a neighborhood of $q(y)$ such that $\pi q'|\tilde{U}=p_3$. It follows that there are product charts about $q(y)$ for $\pi$.

Next let $y\in Y$ such that $q(y)\in\tilde{M}\backslash q(E_-)$. Since $q(y)\in\tilde{M}$), we have $y\in f^{-1}(\mathring{Z}_{2.55}\times[-3, 3]\times I^n)$. Since $q(y)$ is not in $q(E_-)$, we have $y\in V=h_1f^{-1}(\mathring{Z}_{2.6}\times(-\infty, -2)\times I^n)\backslash f^{-1}(Z\times(-\infty, -2]\times I^n)$. And $q|V$: $V\to\overline{M}$ is an open embedding. By setting $\tilde{V}=V\cap q^{-1}(\tilde{M})$ we get product charts about $q(y)$ for $\pi$ by using $q|\tilde{V}$: $\tilde{V}\to\tilde{M}$. This completes the proof of Assertion 5.

Notice that the proof of Assertion 5 shows that $\tilde{M}$, as well as $\pi^{-1}(t)$ for each $t$ in $I^n$, is a $Q$-manifold.

It is left to the reader to show that $\tilde{f}$ has the appropriate fibration properties. For more details see [2, Section 4] or [16, Section 4].

## 6. Handle lemmas

In this section we state two handle lemmas needed for the results in Section 7. Since these lemmas are formally proved by applying our engulfing and wrapping results of Sections 4 and 5 in the same manner that Chapman establishes his handle lemmas in [2, Section 5], we give no further remarks here on their proofs. The reader is referred to [2] or [16] for more details.

For notation $B$ and $X$ will denote ANRs where $X$ is compact. Let $n \geqslant 0$ be an integer and let $C$ be a closed subset of $\partial I^n$ which is collared in $I^n$. (The possibility that $C$ is empty is not ruled out.) More generally, our results would hold true if it were only assumed that $C$ has a radial neighborhood in $I^n$ (i.e., an open neighborhood $U$ of $C$ such that $U \setminus C$ is homeomorphic to $K \times R$ in such a way that $C \cup K \times (-\infty, r]$ are closed neighborhoods for all $r$ in $R$).

PROPOSITION 6.1: *Suppose $m$ is a positive integer and $R^m \hookrightarrow B$ is an open embedding. For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\mu > 0$, $M$ is a $Q$-manifold, and $f$: $M \times I^n \to B \times I^n$ is a proper f.p. map such that $f$ is a sliced $\delta$-fibration over $B_3^m \times I^n$ and $f_t$ is an approximate fibration for each $t$ in $C$, then there is a proper f.p. map $\tilde{f}$: $M \times I^n \to B \times I^n$ which is a sliced $\mu$-fibration over $B_1^m \times I^n$ and which is f.p. $\epsilon$-homotopic to $f$* $\mathrm{rel}[(M \times I^n) \setminus f^{-1}(\mathring{B}_3^m \times I^n)] \cup [M \times C]$.

*Moreover, if we are additionally given a compact $Q$-manifold $F$ such that $F \times B$ is a $Z$-set in $M$ and $f \mid F \times B \times I^n$: $F \times B \times I^n \to B \times I^n$ is projection, then we can additionally conclude that $\tilde{f} \mid F \times B \times I^n$ is projection and that the homotopy from $f$ to $\tilde{f}$ is* rel $F \times B \times I^n$. □

THEOREM 6.2: *Suppose $m \geqslant 0$ is an integer and $\mathring{c}(X) \times R^m \hookrightarrow B$ is an open embedding. For every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $\mu > 0$ there exists a $\nu > 0$ so that the following statement is true:*

*if $M$ is a $Q$-manifold and $f$: $M \times I^n \to B \times I^n$ is a proper f.p. map such that $f$ is a sliced $\delta$-fibration over $c_3(X) \times B_3^m \times I^n$ and a sliced $\nu$-fibration over $[c_3(X) \setminus \mathring{c}_{1/3}(X)] \times B_3^m \times I^n$ and $f_t$ is an approximate fibration for each $t$ in $C$, then there is a proper f.p. map $\tilde{f}$: $M \times I^n \to B \times I^n$ which is a sliced $\mu$-fibration over $c_1(X) \times B_1^m \times I^n$ and which is f.p. $\epsilon$-homotopic to $f$* rel $[(M \times I^n) \setminus f^{-1}(\mathring{c}_{2/3}(X) \times \mathring{B}_3^m \times I^n)] \cup [M \times C]$.

*Moreover, if we are additionally given a compact $Q$-manifold $F$ such that $F \times B$ is a $Z$-set in $M$ and $f \mid F \times B \times I^n$: $F \times B \times I^n \to B \times I^n$ is projection, then we can additionally conclude that $\tilde{f} \mid F \times B \times I^n$ is projection and that the homotopy from $f$ to $\tilde{f}$ is* rel $F \times B \times I^n$. □

## 7. The main results

In this section we state our main result on deforming a parameterized family of $\epsilon$-fibrations to a parameterized family of approximate fibrations (Theorem 7.1). It will follow from this that the space of approximate fibrations from a compact $Q$-manifold to a compact polyhedron is uniformly $LC^n$ for every $n \geqslant 0$.

The proof of Theorem 7.1 can be derived from the handle lemmas of Section 6 using the same procedure as in [2, Section 6].

THEOREM 7.1: *Let $B$ be a polyhedron, $n \geqslant 0$ an integer, and $C$ a closed subset of $\partial I^n$ which is collared in $I^n$. For every open cover $\alpha$ of $B$ there exists an open cover $\beta$ of $B$ so that if $M$ is a $Q$-manifold and $f$: $M \times I^n \to B \times I^n$ is a proper f.p. map such that $f_t$ is a $\beta$-fibration for each $t$ in $I^n$ and an approximate fibration for each $t$ in $C$, then there is a proper f.p. map $\tilde{f}$: $M \times I^n \to B \times I^n$ such that $\tilde{f}_t$ is an approximate fibration $\alpha$-close to $f_t$ for each $t$ in $I^n$ and $\tilde{f}_t = f_t$ for each $t$ in $C$.*

*Moreover, if we are additionally given a compact $Q$-manifold $F$ such that $F \times B$ is a $Z$-set in $M$ and $f \mid F \times B \times I^n$: $F \times B \times I^n \to B \times I^n$ is projection, then we can additionally conclude that $\tilde{f} \mid F \times B \times I^n$ is projection.* □

One should notice that Theorem 7.1 remains true when it is only assumed that $C$ has a radial neighborhood in $I^n$. (See the introduction to Section 6.)

If $X$ is a space (not necessarily locally compact) and $n \geqslant 0$ is an integer, then $X$ is said to be *locally n-connected* (written $LC^n$) if for each $x$ in $X$ and each subset $U$ of $X$ containing $x$, there exists an open subset $V$ of $X$ containing $x$ such that $V \subset U$ and any map $f$: $\partial I^{n+1} \to V$ extends to a map $\tilde{f}$: $I^{n+1} \to U$.

COROLLARY 7.2: *If $M$ is a compact $Q$-manifold and $B$ is a compact polyhedron, then the space of approximate fibrations from $M$ to $B$ endowed with the compact-open topology is $LC^n$ for each non-negative integer $n$.*

PROOF: Recall that since $M$ and $B$ are compact the compact-open topology coincides with the uniform topology. We consider $B$ to have a fixed metric. Let $\epsilon > 0$ and $n \geqslant 0$ be given and choose $\beta = \beta(\epsilon/3) > 0$ by Theorem 7.1 with $C = \partial I^{n+1}$ so that any f.p. map $f$: $M \times I^{n+1} + B \times I^{n+1}$ with $f_t$ an approximate fibration for each $t$ in $\partial I^{n+1}$ and $f_t$ a $\beta$-fibration for each $t$ in $I^{n+1}$ is f.p. $(\epsilon/3)$-homotopic rel $M \times \partial I^{n+1}$ to a f.p. map $\tilde{f}$: $M \times I^{n+1} \to B \times I^{n+1}$ such that $\tilde{f}_t$ is an approximate fibration for each $t$ in $I^{n+1}$. Now choose $0 < \gamma < \epsilon/3$ so that any map to $B$ which is $\gamma$-close to an approximate fibration is a $\beta$-fibration.

Choose $\delta > 0$ so that if $f$: $M \times \partial I^{n+1} \to B \times \partial I^{n+1}$ is any f.p. map with the property that for each $s$, $t$ in $\partial I^{n+1}$ $f_s$ is $\delta$-close to $f_t$, then there exists a f.p. extension $g$: $M \times I^{n+1} \to B \times I^{n+1}$ of $f$ such that $g_s$ is $\gamma$-close to $g_t$ for all $s$, $t$ in $I^{n+1}$.

To complete the proof we claim that if $f$: $M \times \partial I^{n+1} \to B \times \partial I^{n+1}$ is a f.p. map such that $f_s$ is $\delta$-close to $f_t$ for all $s$, $t$ in $\partial I^{n+1}$ and $f_t$ is an approximate fibration for each $t$ in $\partial I^{n+1}$, then there exists a f.p. extension $\tilde{f}$: $M \times I^{n+1} \to B \times I^{n+1}$ of $f$ with the property that $\tilde{f}_s$ is $\epsilon$-close to $\tilde{f}_t$ for all $s$, $t$ in $I^{n+1}$ and $f_t$ is an approximate fibration for each $t$ in $I^{n+1}$. This is obvious from the choices made above. □

REMARK 7.3: A (possibly non-locally compact) metric space $(X, d)$ is said to be *uniformly* $LC^n$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that every map $f$: $\partial I^{n+1} \to X$ with the diameter of $f(\partial I^{n+1})$ less than $\delta$ extends to a map $\tilde{f}$: $I^{n+1} \to X$ with the diameter of $\tilde{f}(I^{n+1})$ less than $\epsilon$. If in the statement of Corollary 7.2 we fix a metric for $B$, then the proof shows that the space of approximate fibrations from $M$ to $B$ endowed with the uniform topology is uniformly $LC^n$ for each non-negative integer $n$.

REMARK 7.4: If the proof of Theorem 7.1 is examined, it will be seen that we can replace $B$ by $R^m$ with the standard metric and replace the open covers by positive numbers so that the statement remains true. Then the proof of Corollary 7.2 shows that the space of approximate fibrations from a (noncompact) $Q$-manifold $M$ to $R^m$ endowed with the uniform topology (induced by the standard metric on $R^m$) is uniformly $LC^n$ for each non-negative integer $n$.

We now turn our attention to special types of approximate fibrations. A closed subset of an ANR $X$ is *cell-like* if it is contractible in any neighborhood of itself. A proper map $f$: $E \to B$ between ANRs is *cell-like* provided $f^{-1}(b)$ is cell-like for each $b$ in $B$. A cell-like map is also an approximate fibration. A map $f$: $E \to B$ is *monotone* provided $f^{-1}(b)$ is connected for each $b$ in $B$.

In [13] it is shown that if $f$: $E \to B$ is an approximate fibration between connected ANRs, then $f^{-1}(b)$ is shape equivalent to the homotopy fiber of $f$ such each $b$ in $B$. From this it follows that if $f$, $g$: $E \to B$ are homotopic approximate fibrations between (not necessarily connected) ANRs and $b$ is in $B$, then $f^{-1}(b)$ is shape equivalent to $g^{-1}(b)$. For example, if $f$, $g$: $E \to B$ are homotopic approximate fibrations and $f$ is cell-like, then $g$ is cell-like. Or, if $f$, $g$: $E \to B$ are homotopic approximate fibrations and $f$ is monotone, then $g$ is monotone. With these facts in mind the following two corollaries follow immediately from Corollary 7.2.

COROLLARY 7.5: *If $M$ is a compact $Q$-manifold and $B$ is a compact polyhedron, then the space of cell-like maps from $M$ to $B$ endowed with the compact-open topology is $LC^n$ for each non-negative integer $n$.*

COROLLARY 7.6: *If $M$ is a compact $Q$-manifold and $B$ is a compact polyhedron, then the space of monotone approximate fibrations from $M$ to $B$ endowed with the compact-open toplogy is $LC^n$ for each non-negative integer $n$.*

## References

[1] T.A. CHAPMAN: Lectures on Hilbert cube manifolds, *C.B.M.S. Regional Conference Series in Math.*, No. 28 (1976).

[2] T.A. CHAPMAN: Approximation results in Hilbert cube manifolds, *Trans. Amer. Math. Soc.* 262 (1980) 303–334.

[3] T.A. CHAPMAN and STEVE FERRY, Hurewicz fiber maps with ANR fibers, *Topology 16* (1977), 131–143.

[4] T.A. CHAPMAN and STEVE FERRY: Approximating homotopy equivalences by homeomorphisms, *Amer. J. Math.* 101 (1979) 583–607.

[5] T.A. CHAPMAN and STEVE FERRY: Constructing approximate fibrations, *Trans. Amer. Math. Soc.* 276 (1983) 757–774.

[6] T.A. CHAPMAN and R.Y.T. WONG: On homeomorphisms of infinite dimensional bundles. III. *Trans. Amer. Math. Soc.* 191 (1974) 269–276.

[7] J. CHEEGER and J. KISTER: Counting toplogical manifolds, *Topology* 9 (1970) 149–152.

[8] D.S. CORAM and P.F. DUVALL, Jr.: Approximate fibrations, *Rocky Mtn. J. Math.* 7 (1977) 275–288.

[9] D.S. CORAM and P.F. DUVALL, Jr.: Approximate fibrations and a movability condition for maps, *Pacific J. Math.* 72 (1977) 41–56.

[10] JAMES DUGUNDJI: *Topology*, Allyn and Bacon, Inc., Boston (1966).

[11] R.D. EDWARDS and R.C. KIRBY: Deformations of spaces of embeddings, *Ann. of Math.* 93 (1971) 63–88.

[12] A. FATHI and Y.M. VISETTI: Deformation of open embeddings of $Q$-manifolds, *Trans. Amer. Math. Soc.* 224 (1976) 427–436.

[13] STEVE FEERY: Approximate fibrations with nonfinite fibers, *Proc. Amer. Math. Soc.* 64 (1977) 335–345.

[14] STEVE FERRY: The homeomorphism group of a compact Hilbert cube manifold is an ANR, *Ann. of Math.* 106 (1977) 101–119.

[15] WILLIAM E. HAVER: The closure of the space of homeomorphisms on a manifold, *Trans. Amer. Math. Soc.* 195 (1974) 401–419.

[16] C. BRUCE HUGHES: *Local Homotopy Properties in Spaces of Approximate Fibrations*, Dissertation, University of Kentucky (1981).

[17] C. BRUCE HUGHES: Approximate fibrations and bundle maps on Hilbert cube manifolds, *Top. and its Applications* 15 (1983) 159–172.

[18] C. BRUCE HUGHES: Bounded homotopy equivalences of Hilbert cube manifolds, preprint.

[19] L.S. HUSCH: Approximating approximate fibrations by fibrations, *Canad. J. Math.* 29 (1977) 897–913.

[20] L.C. SIEBENMANN: Deformation of homeomorphisms on stratified sets, *Comment. Math. Helv.* 47 (1972) 123–163.

[21] H. TORUNCZYK: Homeomorphism groups of compact Hilbert cube manifolds which are manifolds, *Bull. Acad. Pol. Sci.* 25 (1977) 401–408.

[22] H. TORUNCZYK and J. WEST: Fibrations and bundles with Hilbert cube manifold fibers, preprint.


(Oblatum 4-X-1983 & 17-X-1984)

Department of Mathematics
Vanderbilt University
Nashville, TN 37235
USA