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DUALITY THEOREMS FOR Γ-EXTENSIONS OF ALGEBRAIC NUMBER FIELDS

Kay Wingberg

Introduction

There is a remarkable analogy between the theory of algebraic function fields in one variable over a finite field and the theory of algebraic number fields. Namely, on the one hand let C be a complete, non-singular curve of genus greater than zero defined over a function field F, let $\mathscr{J}$ be the Jacobian variety of C and let $\mathscr{J}_p$ be the p-primary subgroup of the group of points of $\mathscr{J}$ defined over the algebraic closure $\bar{F}$ of F (p an odd prime distinct from the characteristic of F). The Frobenius automorphism of $\bar{F}/F$ induces an endomorphism of $\mathscr{J}_p$, and a fundamental theorem of Weil asserts that the characteristic polynomial of this endomorphism is essentially the zeta function of the curve C. On the other hand let $\mathscr{K}_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathscr{K} = K(\mu_p)$, where K is a totally real number field and μ_{p^n} denotes the $(p^n)^{\text{th}}$ roots of unity. Let A_n be the p-primary subgroup of the ideal class group of $\mathscr{K}_n = \mathscr{K}(\mu_{p^{n+1}})$ and let A be the inductive limit of the groups A_n and $A^- = (1-\sigma)A$, where σ is the automorphism of the CM-field $\mathscr{K}$ induced by the complex conjugation. Iwasawa has proposed that the $\Gamma = G(\mathscr{K}_\infty/\mathscr{K})$-module A^- should provide an analogue of $\mathscr{J}_p$ and the "main-conjecture" of this theory is that the characteristic polynomial of $(A^-)^* = \mathrm{Hom}(A^-, \mathbb{Q}_p/\mathbb{Z}_p)$ should be very closely related to the p-adic zeta function of K. To prove this conjecture is extremely difficult and has been fully established for abelian number fields by Mazur/Wiles.

This idea in mind, we might expect a functional equation for the characteristic polynomial of $(A^-)^*$. But there are fields $\mathscr{K}$ with $A^- = \mathbb{Q}_p/\mathbb{Z}_p$ (for example $\mathscr{K} = \mathbb{Q}(\zeta_{37})$) and consequently there cannot in general exist a non-degenerating, skew-symmetric pairing on $(A^-)^*$ analogous to the Weil scalar product on $\mathscr{J}_p$. One aim of this paper is to show how to obtain a functional equation in the case of CM-fields by adding certain "local factors" to A^-.

Namely, in §9 we defined two Γ-modules $\mathscr{L}_+$ and $\mathscr{L}_-$ which have many of the desired properties. The most important result is the following: Tensoring these modules with $\mathbb{Q}_p$ we get two non-degenrating, skew-symmetric, Γ-invariant pairings and therefore a functional equation for the characteristic polynomial $\mathscr{h}_\pm(T)$ of $\mathscr{L}_\pm$, namely:

$$\kappa(\gamma_0)^{\lambda_3^\ddagger} \cdot \mathscr{h}_\pm(T-1) = T^{2\lambda_3^\ddagger} \cdot \mathscr{h}_\pm\left(\kappa(\gamma_0)T^{-1}-1\right),$$

where $2\lambda_3^{\pm}$ is the $\mathbb{Z}_p$-rank of $\mathscr{L}_{\pm}$, γ_0 a generator of Γ and κ the cyclotomic character $\kappa\colon \Gamma \to \mathbb{Z}_p^{\times}$. In the function field case we obtain the well known function equation for the zeta function of $\mathscr{K}$.

We will see that assuming the Leopoldt-conjecture for all the fields in the cyclotomic tower $\mathscr{L}_+ \otimes \mathbb{Q}_p$ is the symplectic $\mathbb{Q}_p[|\Gamma|]$-module studied by Iwasawa in [9] §11. Conjecturally this module is trivial (see Greenberg [5]).

The module $\mathscr{L}_-$, however, is in general non-trivial and turns out to be an interesting object of consideration. We obtain the following remarkable properties: In a finite p-extension E/K the $\mathbb{Z}_p$-ranks of $\mathscr{L}_-(E)$ and $\mathscr{L}_-(K)$ are connected by a relation analogue to the Riemann-Hurwitz formula for the genus of a curve. Further, $\mathscr{L}_-$ does not contain any finite non-trivial Γ-submodule. The module $\mathscr{L}_-$ is defined as the Galois group $G(M'/\mathscr{K}_\infty) = G(L'(\mathscr{K})M(K^+)/\mathscr{K}_\infty)/U$, where $L'(\mathscr{K})$ is the maximal completely decomposed abelian p-extension of $\mathscr{K}_\infty$, $M(K^+)$ the maximal abelian p-extension unramified outside p of the maximal totally real subfield K^+ of $\mathscr{K}$ and U is generated by the projective limits of the roots of unity of $(K_n^+)_{\mathfrak{p}}$, $\mathfrak{p} \mid p$, considered via the reciprocity homomorphism in $G(M(K^+)_{\mathfrak{p}}/(K_\infty^+)_{\mathfrak{p}}) \leqslant G(M(K^+)/K_\infty^+)$. In the case when none of the prime divisors of p splits in the extension $\mathscr{K}/K^+$ this field M' is simply the compositum of $M(K^+)$ and the maximal unramified abelian p-extension $L(\mathscr{K})$ of $\mathscr{K}_\infty$. If there is only one prime $\mathfrak{p}$ above p in $\mathscr{K}_\infty$ we shall see that $\mathscr{h}_-(T)$ is the product of the characteristic polynomials of the global group $(A^-)^*$ and the local galois group $G(M(K^+)_{\mathfrak{p}}/(K_\infty^+)_{\mathfrak{p}})$.

To obtain these results we first need a description of the $\mathbb{Z}_p[|\Gamma \times G(\mathscr{K}/K)|]$-module structure of the local Galois groups $G(\mathscr{K}(p)_{\mathfrak{p}}/\mathscr{K}_{\infty_{\mathfrak{p}}})^{ab}$, $\mathfrak{p} \mid p$, where $\mathscr{K}(p)_{\mathfrak{p}}$ denotes he maximal p-extension of the completion $\mathscr{K}_{\mathfrak{p}}$ of $\mathscr{K}$ (§6). Secondly we strongly used the global duality theorem, whereas most of the fundamental results obtained by Iwasawa in [9] are proved by means of Kummer-theory. We want to show that the application of the duality theorem of Tate/Poitou not only gives some finer results (e.g. formulas for the λ-invariants of the three Iwasawa-modules usually considered (§§7, 8)), but also leads to a duality theorem for certain other $G(\mathscr{K}_\infty/\mathscr{K})$-modules in a natural way (§7).

The Riemann-Hurwitz formula for $\mathscr{L}_-$ relies heavily on a Riemann-Hurwitz formula for the $\mathbb{Z}_p$-ranks $\lambda_1(E)$ and $\lambda_1(K)$ of the modules $G(M(E)/E_\infty)$ and $G(M(K)/K_\infty)$, where E/K is a finite p-extension of a totally real ground field K for which $\mu(K)=0$:

$$\lambda_1(E) - 1 = (\lambda_1(K) - 1)[E_\infty : K_\infty] + \sum_{\mathfrak{p} \nmid p} (e_{\mathfrak{p}} - 1),$$

where $e_{\mathfrak{p}}$ denote the ramification indices for the extension E_∞/K_∞. From this easily follows the result of Kida [16] and Iwasawa [11] for the $\mathbb{Z}_p$-rank of $(A^-)^*$. The proof in §7 is using the analogue of Riemann's existence theorem for p-closed number fields [20]. In particular, it is necessary to consider non abelian extensions of K_∞ resp. E_∞.

The facts about finite Γ-submodules are based on certain cohomological criterions for arbitrary noetherian Γ-modules and group extensions. By these general methods we will obtain the following. Let K_∞/K be an arbitrary Γ-extension, let $K_S(p)$ be the maximal p-extension of K, which is unramified outside the finite set S of primes containing all divisors of p and ∞ and let X_S be the galois group $G(K_S(p)/K_\infty)^{ab}$. Assume the weak Leopoldt-conjecture is true, then the $\mathbb{Z}_p[\,|\Gamma|\,]$-rank of X_S is equal to the number of complex primes of K and X_S contains not any finite non-trivial Γ-submodule. If K_∞/K is the cyclotomic extension or if the Leopoldt-conjecture is valid for K, this is the well known result of Iwasawa [9] resp. Greenberg [6].

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I. Λ-modules

§1 *Notations and conventions*

Let Γ denote a compact abelian group isomorphic to the additive group of $\mathbb{Z}_p$ and $\Lambda = \mathbb{Z}_p[[T]]$ the ring of all formal power series in an indeterminate T. The ring Λ is local, noetherian, regular, of dimension 2 and compact in the $\mathfrak{m}$-adic topology, where $\mathfrak{m} = (p, T)$ is the maximal ideal of Λ. By a theorem of Lazard we have a homeomorphism

$$\Lambda \xrightarrow{\sim} \mathbb{Z}_p[\,|\Gamma|\,] = \text{proj lim} \mathbb{Z}_p\left[\Gamma/\Gamma^{p^n}\right]$$

$$T \mapsto \gamma_0 - 1$$

γ_0 a generator of Γ, and we obtain an equivalence of category between the category of compact (discrete) Λ-modules and the category of compact (discrete) Γ-modules.

For a compact (discrete) Γ-module M we have the (co-) homology groups

$$\left(H^0(\Gamma, M) = \right) H_1(\Gamma, M) = M^\Gamma,$$

$$\left(H^1(\Gamma, M) = \right) H_0(\Gamma, M) = M_\Gamma.$$

If $0 \to M' \to M \to M'' \to 0$ is a exact sequence of compact (discrete) Γ-modules, then there exists an exact (co-) homology sequence

$$0 \to M'^\Gamma \to M^\Gamma \to M''^\Gamma \to M'_\Gamma \to M_\Gamma \to M''_\Gamma \to 0, \tag{1.1}$$

since Γ is of cohomological dimension 1. By the lemma of Nakayama we see that a compact (discrete) Γ-module M is zero if and only if $M_\Gamma = 0$ ($M^\Gamma = 0$).

Since Λ is a local ring a compact Λ-module M is Λ-free, if and only if M is Λ-projective. By the above remark the homological dimension $hd_\Lambda M$ is equal or less than 2; consequently there is a free resolution of M

$$0 \to \Lambda^{I_2} \to \Lambda^{I_1} \to \Lambda^{I_0} \to M \to 0,$$

where I_j, $j = 0, 1, 2$, are sets of indices. In particular we will be interested in compact Λ-modules M of homological dimension $\leqslant 1$.

The following criterian for Λ-freeness of a noetherian Λ-module turns out to be useful:

$$M \textit{ is } \Lambda\textit{-free} \Leftrightarrow M^\Gamma = 0 \textit{ and } M_\Gamma \textit{ is } \mathbb{Z}_p\textit{-free}. \tag{1.2}$$

For the proof of the non trivial implication consider the exact sequence

$$0 \to N \to \Lambda^m \xrightarrow{\varphi} M \to 0$$

where m is the $\mathbb{Z}_p$-rank of M_Γ and $N = \operatorname{Ker} \varphi$ (the surjection φ exists by Nakayama's lemma). Now, the exact sequence

$$0 \to N_\Gamma \to \mathbb{Z}_p^m \to M_\Gamma \to 0$$

shows that $N_\Gamma = 0$ and consequently that $N = 0$.

We will need the following notations:
For an abelian local-compact group A let A^* be the Pontryagindual, $A(p)$ the p-primary part of A and for $m \in \mathbb{N}$ let the groups ${}_mA$ and A_m are defined by the exact sequence

$$0 \to {}_mA \to A \xrightarrow{m} A \to A_m \to 0.$$

As usual we put $H^i(U)$ for the cohomology group $H^i(U, \mathbb{F}_p)$ of a pro-finite group U.

In the following M will be a noetherian Λ-module. We define the generator-rank by:

$$d(M) = \dim_{\mathbb{F}_p} H_0(\Gamma, M)_p$$

and the relation-rank by:

$$r(M) = \dim_{\mathbb{F}_p} H_1(\Gamma, M)_p + \dim_{\mathbb{F}_p} {}_pH_0(\Gamma, M).$$

From Nakayama's lemma follows the existence of a surjective homomorphism $\Lambda^{d(M)} \twoheadrightarrow M$ with kernel N; considering the exact sequence (1.1) we obtain easily the equality $d(N)=r(M)$. In particular, if $hd_\Lambda M \leqslant 1$, we get an exact sequence

$$0 \to \Lambda^{r(M)} \to \Lambda^{d(M)} \to M \to 0. \tag{1.3}$$

According to the general structure theory for noetherian Λ-modules [1] VII §4.4, theorem 4 and 5, we have a quasi-isomorphism

$$M \sim E(e_0;\ \mathfrak{p}_1^{e_1},\dots,\mathfrak{p}_s^{e_s}) := \Lambda^{e_0} \oplus \Lambda/\mathfrak{p}_1^{e_1} \oplus \dots \oplus \Lambda/\mathfrak{p}_s^{e_s},$$

where $e_i \geqslant 0$ and the $\mathfrak{p}_i = f_i(T)\cdot\Lambda$, $i=1,\dots,s$, are prime ideals of height 1 in Λ, that means: $f_i(T)=p$ or $f_i(T)$ is a Weierstrass-polynomial ($f_i(T) \equiv T^{\lambda_i} \bmod p$). We say that $E(e_0;\ \mathfrak{p}_1^{e_1},\dots,\mathfrak{p}_s^{e_s})$ is the elementary Λ-module associated to M.

If $T_\Lambda(M)$ is the Λ-torsion module of M and $F_\Lambda(M)$ the quotient $M/T_\Lambda(M)$, then we have the quasi-isomorphism $M \sim F_\Lambda(M) \oplus T_\Lambda(M)$ with

$$T_\Lambda(M) \sim E(\mathfrak{p}_1^{e_1},\dots,\mathfrak{p}_s^{e_s}) \text{ and } F_\Lambda(M) \sim E(e_0) = \Lambda_0^e,$$

where e_0 denotes the Λ-rank $rk_\Lambda M$ of M. Since $rk_{\mathbb{Z}_p} T_\Lambda(M)_\Gamma = rk_{\mathbb{Z}_p} T_\Lambda(M)^\Gamma$ we obtain

$$rk_\Lambda M = rk_{\mathbb{Z}_p}(M_\Gamma) - rk_{\mathbb{Z}_p}(M^\Gamma) \tag{1.4}$$

and in particular, if $hd_\Lambda M \leqslant 1$ holds, then by (1.3) we have:

$$rk_\Lambda M = d(M) - r(M). \tag{1.5}$$

The polynomial

$$f_M(T) = \prod_{i=1}^{s} f_i(T)^{e_i}$$

of degree

$$\lambda(M) = \deg f_M(T)$$

is called the characteristic polynomial of M; we have the alternative description

$$f_M(T) = p^{\mu(M)} \cdot \det\big(T - (\gamma_0 - 1);\ M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\big),$$

where $\mu(M) = \Sigma_\nu e_\nu$ is given by the $\mathbb{Z}_p$-torsion module $T_{\mathbb{Z}_p}(M)$ of M:

$$T_{\mathbb{Z}_p}(M) \sim \bigoplus_\nu \Lambda/p^{e_\nu}. \tag{1.6}$$

We say a sequence

$$M' \overset{\varphi}{\rightarrow} M \overset{\psi}{\rightarrow} M''$$

of noetherian Λ-modules is quasi-exact, if there is a quasi-isomorphism

$$\operatorname{Ker} \psi \sim \operatorname{im} \varphi.$$

If M is a Λ-torsion module quasi-isomorph to the Λ-module N, then it is easy to see, that there exist Λ-homomorphism φ' and ψ' such that the sequence

$$M' \overset{\varphi'}{\rightarrow} N \overset{\psi'}{\rightarrow} M''$$

is quasi-exact (see the proof of theorem 5. [1] VII §4.4).

For two Γ-modules M and N we give $\operatorname{Hom}_{\mathbb{Z}_p}(M, N)$ a Λ-module structure by defining

$$(\gamma f)(m) = \gamma f(\gamma^{-1} m), \quad \gamma \in \Gamma, \quad f \in \operatorname{Hom}_{\mathbb{Z}_p}(M, N), \quad m \in M.$$

Let $\alpha(M)$ the adjoint of a Λ-torsion module M, then we have [9] (1.3):

$$\alpha(M) \cong \operatorname{proj}\lim_{n} \operatorname{Hom}_{\mathbb{Z}_p}(M/\pi_n M, \mathbb{Q}_p/\mathbb{Z}_p).\text{[1]} \tag{1.7}$$

Here the inverse limit is taken with respect to the homomorphism induced by

$$\begin{matrix} M/\pi_n M & \rightarrow M/\pi_m M \\ a \bmod \pi_n M \mapsto & (\pi_m/\pi_n) a \bmod \pi_m M \end{matrix}, \quad m \geqslant n \geqslant 0,$$

where $\{\pi_n\}$ is a sequence of non-zero elements in Λ such that $\pi_0 \in \mathfrak{m}$, $\pi_{n+1} \in \pi_n \mathfrak{m}$, and such that the principal divisors $\pi_n \Lambda$, $n \geqslant 0$, are disjoint from the annihillator of M. We get a contravariant, quasi-exact functor $M \rightarrow \alpha(M)$ on the category of noetherian Λ-torsion modules. If $M \sim M'$, then $\alpha(M) \sim \alpha(M')$.

For a compact Γ-module M we define a new Γ-structure by $\gamma \circ m = \gamma^{-1} m$, $\gamma \in \Gamma$, $m \in M$, and denote this new module by $\mathring{M}$. Obviously

$$\mathring{\mathring{M}} = M$$

and if we consider M as Λ-module we see that for $\xi(T) \in \Lambda$ and $m \in \mathring{M}$

$$\xi(T) \circ m = \mathring{\xi}(T) m := \xi(\mathring{T}) m,$$

[1] The definition of $\alpha(M)$ in [9] is slightly different because another Γ-structure for $\alpha(M)$ is considered.

where $\mathring{T} \in m$ is defined by

$$(1+T)(1+\mathring{T})=1.$$

In particular we obtain for a Λ-torsion module M a quasi-isomorphism

$$\alpha(M) \sim \mathring{M}, \tag{1.8}$$

and an isomorphism for an elmentary Λ-torsion module E

$$\alpha(E) \cong \mathring{E}.$$

We shall frequently consider the following elements of Λ

$$\omega_n = (1+T)^{p^n} - 1 \quad , n \geqslant 0$$

$$\xi_n = \omega_n/\omega_{n-1} = \sum_{i=0}^{p-1} (1+T)^{ip^{n-1}} \quad , n \geqslant 1,$$

$$\xi_0 = \omega_0 = T.$$

For each $n \geqslant 0$, let Γ^n the unique closed subgroup of Γ with index p^n.
Finally we prove the following

LEMMA 1.9: *Let M be a noetherian Λ-module, then*:

i) $\quad \operatorname{proj}\lim_n \operatorname{proj}\lim_m ({}_{p^m}M)^*_{\Gamma^n} \cong \alpha\left({}_{p^m}T_{\mathbf{Z}_p}(M)\right)_{\Gamma^n}$

ii) $\quad \operatorname{proj}\lim_n \operatorname{proj}\lim_m (M_{p^m})^{\Gamma^n *} \cong \alpha\left(T_\Lambda(M)/T_{\mathbf{Z}_p}(M)\right).$

PROOF: First let $M \cong \Lambda^{e_0} \oplus T_\Lambda(M)/T_{\mathbf{Z}_p}(M) \oplus T_{\mathbf{Z}_p}(M)$ be an elementary Λ-module. We then have

$$({}_{p^m}M)_{\Gamma^n} \cong \left({}_{p^m}T_{\mathbf{Z}_p}(M)\right)_{\Gamma^n}$$

and

$$(M_{p^m})^{\Gamma^n} \cong \left(\left(T_\Lambda(M)/T_{\mathbf{Z}_p}(M)\right)_{p^m}\right)^{\Gamma^n};$$

Since the prime divisors $\xi_n\Lambda$, $n \geqslant 0$, (resp. $p\Lambda$) are disjoint from the annihilator of $T_{\mathbf{Z}_p}(M)$ (resp. $T_\Lambda(M)/T_{\mathbf{Z}_p}(M)$), we take in (1.7) for π_n the element ω_n resp. p^n and obtain the assertions of the lemma in this case. To conclude in the general case we observe that

$$\alpha(\mathscr{E}) = \operatorname{proj}\lim_{n,m}({}_{p^m}\mathscr{E})_{\Gamma^n *} = \operatorname{proj}\lim_{n,m}(\mathscr{E}_{p^m})^{\Gamma^n *}$$

is zero, if $\mathscr{E}$ is a finite Λ-module.

§2 *Λ-modules of homological dimension ⩽ 1*

The following proposition characterizes noetherian Λ-modules of homological dimension equal or less than one.

PROPOSITION 2.1: *Let M be a noetherian Λ-module, then the following five assertions are equivalent*:

i) $hd_\Lambda M \leqslant 1$.

ii) *There exists an exact sequence* $0 \to M \to E \to \mathscr{E} \to 0$, *where* E (*resp.* $\mathscr{E}$) *is an elementary* (*resp. finite*) Λ*-module.*

iii) M^{Γ^n} *is* $\mathbb{Z}_p$*-free for all* $n \geqslant 0$.

iv) M^{Γ^n} *is* $\mathbb{Z}_p$*-free for an integer* $n \geqslant 0$.

v) M *does not contain any nontrivial finite* Λ*-submodule.*

PROOF: If $hd_\Lambda M \leqslant 1$ we obtain the following commutative exact diagram

$$\begin{array}{ccccccccc} & & & & & & \mathscr{E}' & & \\ & & & & & & \downarrow & & \\ 0 & \to & \Lambda^{r(M)} & \to & \Lambda^{d(M)} & \to & M & \to & 0 \\ & & & & \downarrow \varphi & & \downarrow & & \\ & & & & E & = & E & & \\ & & & & & & \downarrow & & \\ & & & & & & \mathscr{E} & & \end{array}$$

with finite Λ-modules $\mathscr{E}$ and $\mathscr{E}'$ and an elementary Λ-module E. By the snake lemma we get an exact sequence

$$0 \to \Lambda^{r(M)} \to \operatorname{Ker} \varphi \to \mathscr{E}' \to 0$$

and thereore, since $(\operatorname{Ker} \varphi)^\Gamma \leqslant (\Lambda^{d(M)})^\Gamma = 0$, the inclusion

$$(\mathscr{E}')^\Gamma \hookrightarrow (\Lambda^{r(M)})_\Gamma \cong \mathbb{Z}_p^{r(M)}.$$

This shows that $(\mathscr{E}')^\Gamma = 0$ and consequently that $\mathscr{E}' = 0$. Since E^{Γ^n} is $\mathbb{Z}_p$-free for all $n \geqslant 0$, ii. implies iii. The assertion iv. is obviously equivalent to v., because a finite Λ-module $\mathscr{E}$ is zero if and only if $\mathscr{E}^{\Gamma^n} = 0$ for a number $n \geqslant 0$. Now assume v. There exists an exact sequence

$$0 \to N \to \Lambda^{d(M)} \to M \to 0,$$

where N is a noetherian Λ-module with $N^\Gamma = 0$; the exact sequence

$$0 \to M^\Gamma \to N_\Gamma \to \mathbb{Z}_p^{d(M)}$$

shows that N_Γ is $\mathbb{Z}_p$-free, since M^Γ is $\mathbb{Z}_p$-free, and therefore by (1.2.) the Λ-freeness of N.

COROLLARY 2.2: *Let M be a noetherian Λ-mocule with $hd_\Lambda M \leqslant 1$. Then all the Λ-submodules of M have homological dimension smaller or equal to 1.*

Following Iwasawa, we shall call compact Γ-modules M n-regular, if $M^{\Gamma^n} = 0$, and regular, if M is n-regular for all $n \geqslant 0$.

PROPOSITION 2.3: *Let M be a noetherian Λ-module. Then the following assertions are equivalent*
i) *M is regular.*
ii) *M is n_0-regular for an integer $n_0 \geqslant \lambda(M)$.*

PROOF: By 2.1 the assertion ii. follows immediately from i. Now let

$$0 \to M \to E(e_0;\ \mathfrak{p}_1^{e_1}, \dots, \mathfrak{p}_s^{e_s}) \to \mathscr{E} \to 0$$

an exact sequence of Λ-modules, where $\mathscr{E}$ is finite, we see that the n-regularity of M is equivalent to

$$(\Lambda/\mathfrak{p}_i^{e_i})^{\Gamma^n} = 0 \qquad \text{for} \quad i = 1, \dots, s.$$

respectively to:

$$\mathfrak{p}_i \neq \xi_\nu \Lambda \qquad \text{for} \quad i = 1, \dots, s \text{ and all } \nu = n.$$

Consequently M is regular if and only if none of the prime divisors $\mathfrak{p}_i$, $i = 1, \dots, s$, are equal to $\xi_\nu \Lambda$, $\nu \geqslant 0$. But for $\nu = n_0$ this is true by ii) and for all $\nu > n_0 \geqslant \lambda(M) = \deg f_M(T)$ this follows from the fact that $\deg \xi_\nu = (p-1)p^{\nu-1} \geqslant \nu$.

§3 *Galois theoretical Λ-modules*

We shall consider in this section Λ-modules of the following type: Let G be a finitely generated pro-p-group with an exact sequence

$$1 \to H \to G \to \Gamma \to 1,$$

then the conjugation of G on H defines on $H^{ab} = H/[H, H]$ a Γ-structure. First we prove a few simple lemma's.

LEMMA 3.1: *Let A be a discrete p-primary G-torsion module. Then the following sequence is exact*

$$0 \to H^1(\Gamma, H^1(H, A)) \to H^2(G, A) \overset{\text{res}}{\to} H^2(H, A)^\Gamma \to 0.$$

This is a part of the Hochschild-Serre-spectralsequence, since we have $cd_p(\Gamma) = 1$.
Since $(H^{ab})^{\Gamma} = H^1(\Gamma, H^1(H, \mathbb{Q}_p/\mathbb{Z}_p))^*$ we get by 2.1

LEMMA 3.2: *The following assertions are equivalent:*
i) $hd_{\Lambda} H^{ab} \leqslant 1$.
ii) $H^1(\Gamma, H^1(H, \mathbb{Q}_p/\mathbb{Z}_p))$ *is a divisible group.*

LEMMA 3.3: *Let U be a pro-p-group. Then we have the equivalent assertions*
i) *U is a free pro-p-group.*
ii) $H^1(U, \mathbb{Q}_p/\mathbb{Z}_p)$ *is divisible (this means:* U^{ab} *is* $\mathbb{Z}_p$*-torsionfree) and* $H^2(U, \mathbb{Q}_p/\mathbb{Z}_p) = 0$.

For the proof consider the following exact cohomology sequence

$$0 \to \left({}_pU^{ab}\right)^* \to H^2(U) \to {}_pH^2\left(U, \mathbb{Q}_p/\mathbb{Z}_p\right) \to 0 \tag{3.4}$$

induced by

$$0 \to \mathbb{Z}/p \to \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$

We want now to determine the generator- and relation-rank of H^{ab}. For this we set

$$\ell_i = \dim_{\mathbb{F}_p} H^i(G),\ i \geqslant 0,$$

$$\chi_2(G) = \ell_0 - \ell_1 + \ell_2,$$

$$t = \dim_{\mathbb{F}_p} {}_pH^2\left(H, \mathbb{Q}_p/\mathbb{Z}_p\right)^{\Gamma}.$$

PROPOSITION 3.5: *Let* $hd_{\Lambda} H^{ab} \leqslant 1$; *then there exists an exact sequence of* Λ*-modules*

$$0 \to \Lambda^{\ell_2 - t} \to \Lambda^{\ell_1 - 1} \to H^{ab} \to 0,$$

and therefore

$$rg_{\Lambda} H^{ab} = -\chi_2(G) + t.$$

PROOF: The generator-rank is calculated in the following way:

$$d(H^{ab}) = \dim H_0(\Gamma, H^{ab})_p = \dim H_0\left(\Gamma, H_p^{ab}\right)$$
$$= \dim H^0\left(\Gamma, H^1(H)\right) = \dim H^1(G) - 1;$$

since $H^1(\Gamma, H^1(H, \mathbb{Q}_p/\mathbb{Z}_p))$ is divisible, we get because of 3.1

$$r(H^{ab}) = \dim H_1(\Gamma, H^{ab})_p + \dim_p H_0(\Gamma, H^{ab})$$
$$= \dim_p H^1\big(\Gamma, H^1(H, \mathbb{Q}_p/\mathbb{Z}_p)\big) + \dim_p\big(H_\Gamma^{ab}\big)$$
$$= \dim_p H^2(G, \mathbb{Q}_p/\mathbb{Z}_p) - t + \dim_p\big(H_\Gamma^{ab}\big).$$

Finally the exact sequence

$$0 \to H_\Gamma^{ab} \to G^{ab} \to \Gamma \to 0$$

combined with 3.4 shows that:

$$r(H^{ab}) = \dim_p H^2(G, \mathbb{Q}_p/\mathbb{Z}_p) - t + \dim_p G^{ab} = \ell_2 - t.$$

The assertions 1.3 and 1.5 complete the proof.

THEOREM 3.6: *The following properties are equivalent*:
i) H^{ab} *does not contain any nontrivial finite* Λ*-submodule and* $rk_\Lambda H^{ab} = -\chi_2(G)$.
ii) $H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)$ *is divisible and* $H^2(H, \mathbb{Q}_p/\mathbb{Z}_p) = 0$.

PROOF. Because of 2.1 and 3.5 the assertion i. is equivalent to $hd_\Lambda H^{ab} \leqslant 1$ and $t = 0$. Now, $t = 0$ is valid if and only if $H^2(H, \mathbb{Q}_p/\mathbb{Z}_p)$ is trivial and we obtain the theorem using 3.1 and 3.2.

For the pro-p-group H we get the following

THEOREM 3.7: *The following assertions are equivalent*:
i) *H is a free pro-p-group.*
ii) $H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)$ *is divisible*, $H^2(H, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ *and* $\mu(H^{ab}) = 0$.

PROOF: Using 3.3 and 3.1 we must show:
H^{ab} is $\mathbb{Z}_p$-torsionfree if and only if $(H^{ab})^\Gamma$ is $\mathbb{Z}_p$-free and $\mu(H^{ab}) = 0$. But, if $(H^{ab})^\Gamma$ is $\mathbb{Z}_p$-free, it follows from 2.1 ii. that the group ${}_pH^{ab}$ is zero if and only if $\mu(H^{ab}) = 0$.

COROLLARY 3.8: *Suppose* $cd_p(G) \leqslant 2$ *and let U be an open subgroup of G,* $V = H \cap U$ *and* $\Gamma' = U/V \leqslant \Gamma$. *Then* V^{ab} *is a noetherian* Γ'*-module and the following assertions are equivalent*:
i) $\mu(H^{ab}) = 0$ and $H^2(H, \mathbb{Q}_p/\mathbb{Z}_p) = 0$.
ii) $\mu(V^{ab}) = 0$ and $H^2(V, \mathbb{Q}_p/\mathbb{Z}_p) = 0$.

PROOF: Since $cd_p(U) = cd_p(G) \leqslant 2$ the cohomology groups $H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)$ and $H^2(U, \mathbb{Q}_p/\mathbb{Z}_p)$ are divisible. Since $cd_p(H) = cd_p(V)$ ([23] I.33 prop. 14) the corollary follows from theorem 3.7.

Next we want to study the connection between the characteristic polynomial $f_{H^{ab}}(T)$ of H^{ab} and the defining relations of the group G. We assume that

$$H^2\big(G, \mathbb{Q}_p/\mathbb{Z}_0\big) \text{ is divisible and } H^2\big(H, \mathbb{Q}_p/\mathbb{Z}_p\big) = 0$$

(and therefore: $hd_\Lambda H^{ab} \leqslant 1$).

Let F be a free pro-p-group with free generators $x_0, x_1, \ldots, x_n$, $n+1 = \ell_1$ and

$$1 \to R \to F \to G \to 1$$

a free (minimal) presentation of G such that x_0 is mapped on a lifting of $\gamma_0 \in \Gamma$ and the images of $x_1, \ldots, x_n$ are elements of H. We get the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & & & \langle x_0\rangle & \xrightarrow{\sim} & \Gamma & & \\
 & & & & \uparrow & & \uparrow & & \\
1 & \to & R & \to & F & \to & G & \to & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \to & R & \to & E & \to & H & \to & 1,
\end{array}$$

where E is a free operator-pro-p-group over Γ and

$$E^{ab} = \bigoplus_{i=1}^{n} \Lambda \bar{x}_i, \quad \bar{x}_i = x_i \bmod [E, E];$$

(see Jannsen [14], Satz 3.4). We obtain the following exact sequence of Λ-modules

$$0 \to R[E, E]/[E, E] \to E^{ab} \to H^{ab} \to 0;$$

because $hd_\Lambda H^{ab} \leqslant 1$ we get that

$$R[E, E]/[E, E] \cong \bigoplus_{i=0}^{m} \Lambda r_i$$

is a free Λ-module of rank $m = \ell_2$ (3.5) with generators

$$r_i \equiv w_i \bmod [E, E], \quad i = 1, \ldots, m,$$

where the elements $w_i = w_i(x_0, \ldots, x_n) \in R$ are defining relations for G.

Since $w_i \in E$, $i = 1, \ldots, m$, we have

$$w_i \equiv x^{\alpha_{i1}} \cdot \ldots \cdot x_n^{\alpha_{in}} \bmod [E, E], \quad \alpha_{ij} \in \Lambda, \quad i = 1 \ldots m.$$

We call the matrix

$$\mathscr{R}_G = (\alpha_{ij}) \in M_{nm}(\Lambda)$$

the relation-matrix for G (relative to (x_i) and (r_i)). By [1] VII §4.6 Cor we obtain the following

THEOREM 3.9: *If $H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)$ is divisible, $H^2(H, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ and $\chi_2(G) = 0$, then we have for the characteritic polynomial of the Λ-torsion module H^{ab}*

$$f_{H^{ab}} = \det(\mathscr{R}_G) \cdot \epsilon, \quad \epsilon \in \Lambda^{\times}.$$

If H^{ab} is not a Λ-torsion module, it is easy to prove the following more general

THEOREM 3.10: *Suppose that $H^2(G, \mathbb{Q}_p/\mathbb{Z}_p)$ is divisible and $H^2(H, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ and let $S = \bigcap_i (\Lambda/\mathfrak{p}_i)$, where $\{\mathfrak{p}_i\}$ is the finiteset of prime ideals of height 1 containing the annihilator of H^{ab}, and Λ_S the (semi-local, principal) ring of fractions. Then there exists matrices $P \in Gl_m(\Lambda_S)$ and $Q \in Gl_n(\Lambda_S)$ such that*

$$P\mathscr{R}_G Q = \begin{pmatrix} \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \lambda_m & 0 & \cdots & 0 \end{pmatrix} \in M_{nm}(\Lambda),$$

and

$$H^{ab} \sim \Lambda^{n-m} \oplus \bigoplus_{i=1}^{m} \Lambda/(\lambda_i);$$

consequently we have

$$f_{H^{ab}} = \prod_{i=1}^{m} \lambda_i \cdot \epsilon, \quad \epsilon \in \lambda^{\times}.$$

II. Λ-modules in algebraic number theory

§4 Class field theory and the global duality theorem

In this section we summarize the topics from class field theory that we will use in this paper and introduce notations that will be needed in chapter II.

Let $p \geqslant 2$ be a prime number,

μ_{p^m} the group of the $(p^m)^{\text{th}}$ roots of unity,

$\mu_{p^\infty} = \text{inj lim } \mu_{p^m} = \mathbb{Q}_p/\mathbb{Z}_p(1)$, proj lim $\mu_{p^m} = \mathbb{Z}_p(1)$,

$\mu_F = \mu_{p^\infty} \cap F$, F a field,

$F(p)$ the maximal p-extension of F.

$\mathbb{Q}_\infty$ will be the unique subfield of $\mathbb{Q}(\mu_{p^\infty})$ with $G(\mathbb{Q}_\infty | \mathbb{Q}) \cong \mathbb{Z}_p$. Let K be a finite algebraic number field,

$K_\infty = \bigcup_{n=0}^{\infty} K_n$ a Γ-extension of K – K_∞ is called cyclotomic, if $K_\infty = K \cdot \mathbb{Q}_\infty$ –,

$S_p = S_p(K)$ (resp. $\Sigma = \Sigma(K)$) the set of primes of K above p (resp. p or ∞),

$S = S(K)$ a finite set of primes of K containing Σ,

K_S(resp. $K_S(p)$) the maximal (p-) extension unramified ouside S,

$\mathbb{G}_S = \mathbb{G}_S(K)$ ($G_S = G_S(K)$) the Galois group of K_S/K ($K_S(p)/K$),

$\mathbb{H}_S = \mathbb{H}_S(K)$ ($H_S = H_S(K)$) the Galois group of K_S/K_∞ ($K_S(p)/K_\infty$)

$\mathbb{G} = \mathbb{G}_\mathfrak{p}(K)$ ($G_\mathfrak{p} = G_\mathfrak{p}(K)$) the Galois group of the algebraic closutre of the completion $K_\mathfrak{q}$ of K under the valuation corresponding to $\mathfrak{p}$(resp. $G(K_\mathfrak{p}(p)/K_\mathfrak{p})$);

$M = M(K)$ (resp. $L = L(K)$; resp. $L' = L'(K)$) the maximal abelian p-extension of K_∞ which is unramified ouside $\Sigma(K_\infty)$ (resp. unramified; resp. unramified and all primes $\mathfrak{p} \in \Sigma(K_\infty)$ split completely).

The abelian groups $X_S(K) = H_S^{ab}$, $X_1(K) = H_\Sigma^{ab} = G(M \mid K_\infty)$, $X_2(K) = G(L \mid K_\infty)$, $X_3 = G(L' \mid K_\infty)$ are in a caonical way noetherian Λ-modules (X_2, X_3 are Λ-torsion modules), [9] th. 4, 5, 8, with invariants

$$\mu_S = \mu_S(K) = \mu(X_S(K)),$$
$$\mu_r = \mu_r(K) = \mu(X_r(K)),$$
$$\lambda_S = \lambda_S(K) = \lambda(X_S(K)),$$
$$\lambda_r = \lambda_r(K) = \lambda(X_r(K)), \qquad r = 1, 2, 3.$$

The global duality theorem of Tate/Poitou is essential in the sequel (see [8] theorem 1): let A be a finite p-primary $\mathbb{G}_S$-module, then the sequences

$$0 \to \ker^1(\mathbb{G}_S, A) \to H^1(\mathbb{G}_S, A) \to \prod_{\mathfrak{p} \in S} H^1(\mathbb{G}_\mathfrak{p}, A) \tag{4.1}$$

$$0 \to \ker^2(\mathbb{G}_S, A) \to H^2(\mathbb{G}_S, A)$$
$$\to \prod_{\mathfrak{p} \in S} H^2(\mathbb{G}_\mathfrak{p}, A) \to H^0(\mathbb{G}_S, A')^* \to 0 \tag{4.2}$$

are exact, where $A' = \text{Hom}(A, \mu_{p^\infty})$, and there is a canonical non-degenerating pairing

$$\ker^1(\mathbb{G}_S, A) \times \ker^2(\mathbb{G}_S, A') \to \mathbb{Q}_p/\mathbb{Z}_p \tag{4.3}$$

which has the following functorial behaviour, [8] 5.6, 5.3: let $K'|K$ be a finite extension in K_S and $m, m' \in \mathbb{N}$, $m' \geqslant m$, then the following diagrams are commutative

$$\begin{array}{ccccc} \ker^1(\mathbb{G}_S(K), A) & \times & \ker^2(\mathbb{G}_S(K), A') & \to & \mathbb{Q}_p/\mathbb{Z}_p \\ \mathrm{cor}\uparrow \quad \downarrow\mathrm{res} & & \mathrm{cor}\uparrow \quad \downarrow\mathrm{res} & & \| \\ \ker^1(\mathbb{G}_S(K'), A) & \times & \ker^2(\mathbb{G}_S(K'), A') & \to & \mathbb{Q}_p/\mathbb{Z}_p \end{array} \tag{4.4}$$

$$\begin{array}{ccccc} \ker^1(\mathbb{G}_S(K), \mu_{p^m}) & \times & \ker^2(\mathbb{G}_S(K), \mathbb{Z}/p^m) & \to & \mathbb{Q}_p/\mathbb{Z}_p \\ \theta_1\downarrow & & \uparrow\theta_2 & & \| \\ \ker^1(\mathbb{G}_S(K), \mu_{p^{m'}}) & \times & \ker^2(\mathbb{G}_S(K), \mathbb{Z}/p^{m'}) & \to & \mathbb{Q}_p/\mathbb{Z}_p \end{array} \tag{4.5}$$

where θ_1 (resp. θ_2) denotes the morphism induced by the p-power map (resp. the canonical projection).

Finally let

$$\mathscr{A}(K_{n\mathfrak{p}}) = \operatorname{proj\,lim}_m K_{n\mathfrak{p}}^\times / K_{n\mathfrak{p}}^{\times p^m}$$

be the p-completion of the multiplicative group $K_{n\mathfrak{p}}^\times$ of $K_{n\mathfrak{p}}$, $\mathfrak{p} \in S_p(K_n)$, and $U^1(K_{n\mathfrak{p}})$ the principal units of $K_{n\mathfrak{p}}^\times$; for a prime $\mathfrak{P}|\mathfrak{p}$ we put

$$\mathscr{A}_{\mathfrak{P}} = \operatorname{proj\,lim}_n \mathscr{A}(K_{n\mathfrak{P}}) \cong \operatorname{proj\,lim}_n G_{\mathfrak{P}}(K_n)^{ab} = G_{\mathfrak{P}}(K_\infty)^{ab},$$

$$U_{\mathfrak{P}}^1 = \operatorname{proj\,lim}_n U^1(K_{n\mathfrak{P}}),$$

and

$$\mathscr{A} = \bigoplus_{\mathfrak{p} \in S_p(K)} \bigoplus_{\mathfrak{P}|\mathfrak{p}} \mathscr{A}_{\mathfrak{P}}, \quad U^1 = \bigoplus_{\mathfrak{p} \in S_p(K)} \bigoplus_{\mathfrak{P}|\mathfrak{p}} U_{\mathfrak{P}}^1.$$

$E(K_n)$ (resp. $E'(K_n)$) denotes the units (resp. the $S_p(K_n)$-units) of K_n,

$$E_\infty = E(K_\infty) = \operatorname{inj\,lim}_n E(K_n), \qquad E'_\infty = E'(K_\infty) = \operatorname{inj\,lim}_n E'(K_n)$$

and $\overline{E(K_n)}$ (resp. $\overline{E'(K_n)}$) the topological closure of the images of

$E(K_n)$ (resp. $E'(K_n)$) by

$$\begin{array}{ccccc}
E(K_n) & \overset{\text{diag}}{\rightarrow} & \prod\limits_{\mathfrak{p} \in S_p(K_n)} U^0(K_{n\mathfrak{p}}) & \rightarrow & \prod\limits_{\mathfrak{p} \in S_p(K_n)} U^1(K_{n\mathfrak{p}}) \\
\downarrow & & \downarrow & & \downarrow \\
E'(K_n) & \overset{\text{diag}}{\rightarrow} & \prod\limits_{\mathfrak{p} \in S_p(K_n)}^{K_{n\mathfrak{p}}} K_{n\mathfrak{p}} & \rightarrow & \prod\limits_{\mathfrak{p} \in S_p(K_n)} \mathscr{A}(K_{n\mathfrak{p}})
\end{array}$$

and

$$\overline{E} = \overline{E}(K_\infty) = \operatorname*{proj\,lim}_n \overline{E(K_n)}, \qquad \overline{E'} = \overline{E'}(K_\infty) = \operatorname*{proj\,lim}_n \overline{E'(K_n)}.$$

By the global class field theory we have the following commutative exact diagram:

$$\begin{array}{ccccccccc}
0 & \rightarrow & \overline{E} & \rightarrow & U^1 & \rightarrow & X_1 & \rightarrow & X_2 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \| & & \twoheaddownarrow \\
0 & \rightarrow & \overline{E'} & \rightarrow & \mathscr{A} & \rightarrow & X_1 & \rightarrow & X_3 \rightarrow 0
\end{array} \tag{4.6}$$

(see for instance [18, Theorem 5.1], where the exactness of the upper row is proved on a finite level; for the other row the exactness can be proved in a similar way).

5. Application of the results of Sections 2 and 3

In the sequel the "weak Leopoldt-conjecture" will play an essential role. This conjecture states that for a Γ-extension $K_\infty \mid K$ the cohomology group $H^2(H_\Sigma(K), \mathbb{Q}_p/\mathbb{Z}_p)$ is zero. It is well known that this is true for the cyclotomic Γ-extension if $\mu_p \subset K$, $p > 2$ [22]. Here we will give a short proof for $p \geqslant 2$ and an arbitrary number field K.

PROPOSITION 5.1: *Let $K_\infty \mid K$ be the cyclotomic Γ-extension. Then* $H^2(H_S, \mathbb{Q}_p/\mathbb{Z}_p) = 0$.

PROOF: Let $\mathscr{K} = K(\mu_p)$; passing to the direct limit we get from (4.2) the exact sequence

$$0 \rightarrow \operatorname*{inj\,lim}_{n,m} \ker^2(\mathbb{G}_S(\mathscr{K}_n), \mu_{p^m}) \rightarrow H^2(\mathbb{H}_S(\mathscr{K}), \mu_{p^\infty})$$

$$\rightarrow \bigoplus_{\mathfrak{p} \in S_p(\mathscr{K}_\infty)} H^2(\mathbb{G}_\mathfrak{p}(\mathscr{K}_\infty), \mu_{p^\infty}).$$

On the one hand, we have by (4.3)

$$\operatorname*{inj\,lim}_m \ker^2\big(\mathbb{G}_S(\mathscr{K}_n),\ \mu_{p^m}\big) \cong \Big(\operatorname*{proj\,lim}_m \ker^1(\mathbb{G}_S(\mathscr{K}_n),\ \mathbb{Z}/p^m)\Big)^*$$

$$= \operatorname*{inj\,lim}_m \operatorname{Cl}_S(\mathscr{K}_n)_{p^m} = 0,$$

because of the finiteness of the S-ideal class group $\operatorname{Cl}_S(\mathscr{K}_n)$. Since the strict cohomological p-dimension of $G_{\mathfrak{p}}(K_\infty)$ is 2 for a finite prime $\mathfrak{p}$, on the other hand we have $H^2(\mathbb{G}_{\mathfrak{p}}(\mathscr{K}_\infty),\ \mu_{p^\infty}) = H^2(\mathbb{G}_{\mathfrak{p}}(\mathscr{K}_\infty),\ \mathbb{Q}_p/\mathbb{Z}_p) = 0$. This is also true for $\mathfrak{p}\,|\,\infty$: Let $\mathfrak{p}$ be real and $p = 2$, then the group $H^2(\mathbb{G}_{\mathfrak{p}}(\mathscr{K}_\infty),\ \mu_{2^\infty}) = H^2(\mathbb{Z}/2,\ \mathbb{Q}_2/\mathbb{Z}_2)$ is zero. Consequently, we obtain

$$H^2\big(\mathbb{H}_S(\mathscr{K}),\ \mu_{p^\infty}\big) = H^2\big(\mathbb{H}_S(\mathscr{K}),\ \mathbb{Q}_p/\mathbb{Z}_p\big) = 0.$$

Let Δ be the Galois group of the extension $\mathscr{K}\,|\,K$, then the order of Δ is 1 or prime to p. Hence, by the Hochschild–Serre spectral sequence,

$$0 = H^1\big(G(K_S/K_S(p)),\ \mathbb{Q}_p/\mathbb{Z}_p\big)^{H_S}$$

$$\to H^2\big(H_S(K),\ \mathbb{Q}_p/\mathbb{Z}_p\big) \overset{\text{inf}}{\to} H^2\big(\mathbb{H}_S(K),\ \mathbb{Q}_p/\mathbb{Z}_p\big)$$

and

$$H^2\big(\mathbb{H}_S(K),\ \mathbb{Q}_p/\mathbb{Z}_p\big) \underset{\sim}{\overset{\text{res}}{\to}} H^2\big(\mathbb{H}_S(\mathscr{K}),\ \mathbb{Q}_p/\mathbb{Z}_p\big)^{\Delta} = 0$$

the proposition is proved. □

Lemma 5.2: *The cohomology group $H^2(G_S,\ \mathbb{Q}_p/\mathbb{Z}_p)$ is divisible.*

Proof: Since the group G_S is of cohomological dimension 2 for $p \neq 2$ the assertion is clear in this case. Now let $p = 2$; by the global duality theorem [8, Theorem 1c.] and [8, Cor. 1 of Prop. 22], there are isomorphisms

$$H^3(G_S,\ \mathbb{Q}_2/\mathbb{Z}_2) \tilde{\to} \bigoplus_{\mathfrak{p}\text{ real}} H^3\big(G_{\mathfrak{p}},\ \mathbb{Q}_2/\mathbb{Z}_2\big)$$

$$= \bigoplus_{\mathfrak{p}\text{ real}} H^3\big(G_{\mathfrak{p}},\ \mathbb{Z}/2\big) \tilde{\leftarrow} H^3(G_S,\ \mathbb{Z}/2).$$

Considering the exact cohomology sequence

$$H^2(G_S,\ \mathbb{Q}_2/\mathbb{Z}_2) \overset{2}{\to} H^2(G_S,\ \mathbb{Q}_2/\mathbb{Z}_2) \to H^3(G_S)$$

$$\tilde{\to} H^3(G_S,\ \mathbb{Q}_2/\mathbb{Z}_2) \overset{2}{\to} 0,$$

induced by $0 \to \mathbb{Z}/2 \to \mathbb{Q}_2/\mathbb{Z}_2 \xrightarrow{2} \mathbb{Q}_2/\mathbb{Z}_2 \to 0$, finished the proof. □

Using Theorem 3.7 we now obtain the following result (see also [11, theorem 2]):

THEOREM 5.3: *Suppose $H^2(H_S, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ (for instance, let $K_\infty \mid K$ be cyclotomic). Then the following are equivalent:*
(i) $\mu_S = 0$.
(ii) $H_S = G(K_S(p)/K_\infty)$ *is a free pro-p-group.*

By [8, Prop. 22, Cor. 5], and Theorem 3.6 and Lemma 5.2 we get the next theorem:

THEOREM 5.4: *Suppose $H^2(H_S, \mathbb{Q}_p/\mathbb{Z}_p) = 0$; then*
(i) $\mathrm{rg}_\Lambda X_S = r_2$, *$r_2$ the number of complex primes of K*
(ii) *X_S does not contain any finite non-trivial Λ-submodule.*

REMARK: By Theorem 3.6 we see that the condition $H^2(H_S, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ is necessary for the validity of (i) and (ii). The above result is due to Iwasawa [9], if $K_\infty \mid K$ is the cyclotomic Γ-extension, and to Greenberg [6], if the Leopoldt-conjecture holds for K, that is if $H^2(G_S, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ (which is, by Lemma 3.1, a stronger condition than ours).

Applying proposition 2.3 we obtain:

PROPOSITION 5.5: *Let $K_\infty = \cup K_n$ be a Γ-extension of K; the following assertions are equivalent:*
(i) *The Leopoldt-conjecture is true for all intermediate fields K_n.*
(ii) *There exists a number $n_0 \geqslant \lambda_1$ such that the Leopoldt-conjecture holds for K_{n_0} and the weak Leopoldt-conjecture $H^2(H_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p) = 0$ is valid.*

PROOF: The non-trivial implication is proved as follows: By Lemma 3.1 we have that H_Σ^{ab} is n_0-regular,

$$\left(H_\Sigma^{ab}\right)^{\Gamma^{n_0}} = H^1\left(\Gamma^{n_0}, H^1\left(H_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p\right)\right)^*$$
$$= H^2\left(G_\Sigma\left(K_{n_0}\right), \mathbb{Q}_p/\mathbb{Z}_p\right)^* = 0,$$

and by Proposition 2.1 that $\mathrm{hd}_\Lambda H_\Sigma^{ab}$ is equal or less than 1. Therefore H_Σ^{ab} is a regular Λ-module by Proposition 2.3 and consequently $H^2(G_\Sigma(K_n), \mathbb{Q}_p/\mathbb{Z}_p)$ is zero for all $n \geqslant 0$.

In the following sections we will only consider the prime set $S = \Sigma$, because we have:

PROPOSITION 5.6: *Let $S \supseteq \Sigma$ be a finite set of primes, then*

$$\mu_S = \mu_1$$

and, if the weak Leopoldt-conjecture is verified for $K_\infty \mid K$,

$$\lambda_S = \lambda_1 + \#\{\mathfrak{p} \in S(K_\infty) \backslash \Sigma(K_\infty) \colon \mu_p \subset (K_\infty)_\mathfrak{p}\}.$$

PROOF: The analogue of Riemann's existence theorem [20, Theorem 2] tells us that the Galois group $G(K_S(p)/K_\Sigma(p))$ is a free pro-p-product

$$G_{S\backslash\Sigma} = G(K_S(p)/K_\Sigma(p)) \cong \mathop{*}_{\mathfrak{p} \in S(K_\infty)\backslash\Sigma(K_\infty)} T_\mathfrak{p},$$

where $T_\mathfrak{p}$ is the subgroup of $G_{S\backslash\Sigma}$ generated by all inertia groups $T_{\mathfrak{P}'}$, $\mathfrak{P}'$ a fixed prime lying over $\mathfrak{P}$ and $\mathfrak{P}$ varies over all primes of $K_\Sigma(p)$ dividing $\mathfrak{p}$. The inertia group $T_{\mathfrak{P}'}$ is isomorphic to $\mathbb{Z}_p$, if $(K_\Sigma)_\mathfrak{P}$ contains the group μ_p, and otherwise equal to $\{1\}$. The exact sequence

$$0 \to G_{S\backslash\Sigma}[H_S, H_S]/[H_S, H_S] \to H_S^{\mathrm{ab}} \to H_\Sigma^{\mathrm{ab}} \to 0$$

shows the equality $\mu_S = \mu_\Sigma$, since the kernel is finitely generated. Now, if $H^2(H_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p)$ is zero, we get

$$G_{S\backslash\Sigma}[H_S, H_S]/[H_S, H_S] = G_{S\backslash\Sigma}/[G_{S/\Sigma}, H_S].$$

If μ_p is contained in $(K_\infty)_\mathfrak{p}$ we have dim $H^1(T_\mathfrak{p})^{H_\Sigma} = 1$ and, since H_S^{ab} contains no finite nontrivial Λ-submodule, $H^1(T_\mathfrak{p}, \mathbb{Q}_p/\mathbb{Z}_p)^{H_\Sigma} \cong \mathbb{Z}_p$. This proves the second part of the proposition. □

COROLLARY 5.7: *Let $S \supseteq \Sigma$ be a finite set and $H^2(H_\Sigma, \mathbb{Q}_p/\mathbb{Z}_p) = H^2(H_S, \mathbb{Q}_p/\mathbb{Z}_p) = 0$. Then H_S is a free pro-p-group if and only if H_Σ is free.*

6. The $\Lambda[\Delta]$-module structure of the local groups $\mathscr{A}$ and U^1

In this section p will be an odd prime, K_∞/K the cyclotomic Γ-extension, $\mathscr{K} = K(\mu_p)$, $\mathscr{K}_\infty = K(\mu_{p^\infty})$, $d = [\mathscr{K} : K]$,

$$\Delta = G(\mathscr{K}/K), \qquad G_\infty = \Delta \times \Gamma = G(\mathscr{K}_\infty/K)$$

and

$$\psi : G_\infty \to \mathbb{Z}_p^\times$$

the cyclotomic character given by the operation of G_∞ on μ_{p^∞}. The restrictions of ψ on Δ resp. Γ are denoted by (resp.)

$$\theta: \Delta \to \mathbb{Z}/p-1, \qquad \kappa: \Gamma \to \left(1+p\mathbb{Z}_p\right).$$

Furthermore $A(i)$, $i \in \mathbb{Z}$, is the usual i-fold Tate twist of a p-torsion G_∞-module A: $A(i) = A \otimes \mathbb{Z}_p(i)$ with

$$\mathbb{Z}_p(0) = \mathbb{Z}_p,$$

$$\mathbb{Z}_p(i) = \mathbb{Z}_p(i-1) \otimes \mathbb{Z}_p(1), \quad i \geqslant 1,$$

$$\mathbb{Z}_p(i) = \mathrm{Hom}_{\mathbb{Z}_p}\left(\mathbb{Z}_p(-i), \mathbb{Z}_p\right), \quad i < 0.$$

There are canonical Γ-isomorphism

$$A(i)^{\Delta} \cong A(i)_{\Delta} \cong (e_{-i}A)(i), \quad i \in \mathbb{Z},$$

where $e_j A$ is the maximal submodule (or factor module) of A on which $\sigma \in \Delta$ acts as multiplication by $\theta(\sigma)^j$. Finally

$$G_{\infty\mathfrak{P}} = \Delta_{\mathfrak{P}} \times \Gamma_{\mathfrak{P}} \leqslant G_\infty$$

denotes the local galois group of the extension $\mathscr{K}_{\infty\mathfrak{P}} \mid K_{\mathfrak{p}}$, $\mathfrak{p} \in S_p(K)$ and $\mathfrak{P} \mid \mathfrak{p}$ an extension on $\mathfrak{K}_{\infty\mathfrak{P}}$.

Let us analyse now the $G_{\infty\mathfrak{P}}$-module structure of

$$\mathscr{A}_{\mathfrak{P}} = \operatorname*{proj\,lim}_{n,\,m} \mathscr{K}_{n\mathfrak{P}}^{\times}/\mathscr{K}_{n\mathfrak{P}}^{\times p^m}.$$

If the group $\Delta_{\mathfrak{P}}$ is trivial, this is completely done in [9]. More generally, in [26], $\mathbb{Z}_p^r \times \Delta_{\mathfrak{P}}$-extensions, $r \geqslant 1$, are considered but no explicit description of the module structure of $\mathscr{A}_{\mathfrak{P}}$ is given. The reason for the difficulties to do this for $\Delta_{\mathfrak{P}} \neq 1$ lies in the fact that certain results of the representation theory are only valid for p-groups. Using the cohomological methods established (for p-groups) in [15] and especially their generalisation to arbitrary finite groups by Jannsen in [13], we get the wanted description of $\mathscr{A}_{\mathfrak{P}}$. The result below is easily obtained by the methods developed in [13], but for the convenience of the reader we shall prove it here. We need two lemma's:

LEMMA 6.1: *Let G be a finite group and A and C finitely generated $\mathbb{Z}_p[G]$-modules. if A is cohomological trivial and C $\mathbb{Z}_p$-free, then* $\mathrm{Ext}(C, A) = 0$.

PROOF: Let

$$0 \to P_1 \to P_2 \to A \to 0$$

be a presentation of A by projective finitely generated $\mathbb{Z}_p[G]$-modules P_2. This is possible by [7; 10.7, Theorem 3], since the kernel P_1 of a surjection of a projective module P_2 onto a cohomological trivial module A has to be cohomological trivial. Because of the $\mathbb{Z}_p$-freeness of C we obtain an exact sequence

$$0 \to \operatorname{Hom}_{\mathbb{Z}_p}(C, P_1) \to \operatorname{Hom}_{\mathbb{Z}_p}(C, P_2) \to \operatorname{Hom}_{\mathbb{Z}_p}(C, A) \to 0,$$

in which the middle and left module are projective [7; 10.1, Ex. 2]; hence $\operatorname{Hom}_{\mathbb{Z}_p}(C, A)$ is cohomological trivial and therefore $\operatorname{Ext}(C, A) = H^1(G, \operatorname{Hom}_{\mathbb{Z}_p}(C, A)) = 0$ [7; 10.1, Prop. 2]. □

LEMMA 6.2: *Let G be a finite group with a minimal number of generators $d = d(G)$ and $\mathcal{M}_G$ be the class of all finitely generated $\mathbb{Z}_p[G]$-modules M with*
(i) $H^1(G_p, M) = 0$,
(ii) $H^2(G_p, M) \cong \mathbb{Z}/(G_p : 1)$,
for a p-Sylow group G_p of G. Furthermore, let

$$1 \to R_d \to F_d \to G \to 1$$

be a minimal presentation of G by a free pro-finite group F_d of rank d and a closed normal subgroup R_d of F_d. Then the following assertions are valid for $M \in \mathcal{M}_G$ with a cohomological trivial $\mathbb{Z}_p$-torsion module $T_{\mathbb{Z}_p}(M)$:

(a) *There exists an $m \geqslant d$ and a $\mathbb{Z}_p[G]$-projective module N such that*

$$M \oplus N \cong \left(R_d^{\mathrm{ab}} \otimes \mathbb{Z}_p\right) \oplus \mathbb{Z}_p[G]^{m-d} \oplus T_{\mathbb{Z}_p}(M).$$

(b) *If there is an integer $n \geqslant 0$ and a $\mathbb{Q}_p[G]$-isomorphism*

$$M \otimes \mathbb{Q}_p \cong \mathbb{Q}_p \oplus \mathbb{Q}_p[G]^n,$$

then

$$M \cong \left(R_d^{\mathrm{ab}} \otimes \mathbb{Z}_p\right) \oplus \mathbb{Z}_p[G]^{n+1-d} \oplus T_{\mathbb{Z}_p}(M).$$

PROOF: Let χ_E be a generator of the cyclic group $H^2(G_p, M/T_{\mathbb{Z}_p}(M)) = H^2(G_p, M)$ corresponding to the group extension

$$1 \to M/T_{\mathbb{Z}_p}(M) \to E \to G_p \to 1$$

and let

$$1 \to R_m \to F_m \to G_p \to 1$$

be a presentation by a free pro-finite group F_m of rank $m \geqslant d$, for which there exists a surjection $\varphi: F_m \twoheadrightarrow E$, and a closed normal subgroup R_m of F_m. We get a commutative exact diagram:

$$\begin{array}{ccccccc} 1 \to & R_m^{\mathrm{ab}} & \to & F_m/[R_m, R_m] & \to G_p & \to 1 \\ & \downarrow \bar{\varphi} & & \downarrow & \| & \\ 1 \to & M/T_{\mathbb{Z}_p}(M) & \to & E & \to G_p & \to 1 \end{array}.$$

Because $\mathrm{scd}_p(F_m) = 2$ there existst a $\chi_F \in H^2(G_p, R_m^{\mathrm{ab}}) \cong \mathbb{Z}/(G_p:1)$ with

$$\bar{\varphi} * (\chi_F) = \chi_E.$$

Let N be defined by the exact sequence

$$0 \to N \to R_m^{\mathrm{ab}} \otimes \mathbb{Z}_p \to M/T_{\mathbb{Z}_p}(M) \to 0.$$

Since $H^1(G_p, M/T_{\mathbb{Z}_p}(M)) = 0 = H^3(G_p, R_m^{\mathrm{ab}} \otimes \mathbb{Z}_p)$ the exact cohomology sequence

$$0 \to H^2(G_p, N) \to H^2\big(G_p, R_m^{\mathrm{ab}} \otimes \mathbb{Z}_p\big)$$

$$\underset{\sim}{\overset{\bar{\varphi}*}{\to}} H^2\big(G_p, M/T_{\mathbb{Z}_p}(M)\big) \to H^3(G_p, N) \to 0$$

shows that

$$H^2(G_\ell, N) = H^3(G_\ell, N) = 0$$

for all the ℓ-Sylow groups G_ℓ of G. Now, a theorem of Nakayama [19] says that N is a cohomological trivial $\mathbb{Z}_p[G]$-module. Hence, by Lemma 6.1 the sequence

$$0 \to N \to R_m^{\mathrm{ab}} \otimes \mathbb{Z}_p \to M/T_{\mathbb{Z}_p}(M) \to 0$$

splits. By the lemma of Shanuel [7;8.10, Lemma 11] and the semi-local cancellation theorem [7;10.6, Theorem 1], we then get the $\mathbb{Z}_p[G]$-isomorphism

$$M/T_{\mathbb{Z}_p}(M) \oplus N \cong \big(R_d^{\mathrm{ab}} \otimes \mathbb{Z}_p\big) \oplus \mathbb{Z}_p[G]^{m-d}.$$

Again by Lemma 6.1 there is a splitting

$$M \cong M/T_{\mathbb{Z}_p}(M) \oplus T_{\mathbb{Z}_p}(M)$$

and therefore the assertion (a) is proved. Now, let $M \otimes \mathbb{Q}_p$ be isomorphic to $\mathbb{Q}_p \oplus \mathbb{Q}_p[G]^n$, then by the Krull–Schmidt theorem and since

$$R_d^{\mathrm{ab}} \otimes \mathbb{Q}_p \cong \mathbb{Q}_p \oplus \mathbb{Q}_p[G]^{d-1}$$

we get

$$N \otimes \mathbb{Q}_p \cong \mathbb{Q}_p[G]^{m-n-1}.$$

A theorem of Swan [24, Corr. 6.4] shows, that the projective module N is $\mathbb{Z}_p[G]$-free. Hence the cancellation theorem proves the assertion (b). □

THEOREM 6.3: *Let $K \mid k$ be a finite cyclotomic extension of $\mathfrak{p}$-adic number fields, $G = G(K \mid k)$, $n = [k : \mathbb{Q}_p]$, $p \neq 2$, and $|\ |_p$ the valuation of K. Then there is a commutative exact diagram of $\mathbb{Z}_p[G]$-modules*:

$$\begin{array}{ccccccc} 0 \to J \oplus \mathbb{Z}_p[G]^{n-1} \oplus \mu_K & \to & \mathbb{Z}_p[G]^n \oplus \mu_K \oplus \Gamma_K & \to & \mathbb{Z}_p & \to & 0 \\ \uparrow & & \uparrow\!\wr & & \| & & \\ 0 \longrightarrow U^1(K) & \longrightarrow & \mathscr{A}(K) & \xrightarrow{|\ |_p} & \mathbb{Z}_p & \to & 0 \end{array},$$

where Γ_K is the galois group $G(k(\mu_{p^\infty}) \mid K) \cong \mathbb{Z}_p$ and J is a $\mathbb{Z}_p[G]$-module with an exact sequence

$$0 \to J \to \mathbb{Z}_p[G] \oplus \Gamma_K \to \mathbb{Z}_p \to 0.$$

(*One can show*: *If $K \mid k$ is unramified* (*resp. totally ramified*) *then J is isomorphic to $\mathbb{Z}_p[G]$* (*resp. $I_G \oplus \Gamma_K$*), *where I_G is the augmentation ideal of $\mathbb{Z}_p[G]$*; *futhermore*, *the above theorem is also valid for $p = 2$ and $\mu_4 \subset k$.*)

PROOF: We have the following facts: by local class field theory $\mathscr{A}(K)$ is an element of $\mathscr{M}_G$; μ_K is cohomological trivial for the Galois group of the cyclotomic extension $K \mid k$; the group of principal units $U^1(K)$ contains a free $\mathbb{Z}_p[G]$-module of rank n with finite index (by the well-known argument using the $\mathfrak{p}$-adic exponential map and the normal basis theorem). Consequently we have $\mathbb{Q}_p[G]$-isomorphisms

$$\mathscr{A}(K) \otimes \mathbb{Q}_p \cong \left(U^1(K) \otimes \mathbb{Q}_p\right) \oplus \mathbb{Q}_p[G]^n \oplus \mathbb{Q}_p,$$

and therefore by Lemma 6.2(b) the $\mathbb{Z}_p[G]$-isomorphism

$$\mathscr{A}(K) \cong \left(R_1^{\mathrm{ab}} \otimes \mathbb{Z}_p\right) \oplus \mathbb{Z}_p[G]^n \oplus \mu_K.$$

The free presentation

$$0 \to R_1 \to \hat{\mathbb{Z}} \to G(K \mid k) \to 0,$$

$$R_1 \cong \Gamma_K \times \left(G\left(k(\mu_p) \mid k\right) : 1\right) \cdot \prod_{\ell \neq p} \mathbb{Z}_\ell \leqslant \prod_{\ell} \mathbb{Z}_\ell = \hat{\mathbb{Z}},$$

shows that $R_1 \times \mathbb{Z}_p \cong \Gamma_K$. Finally, let φ be the homomorphism $\mathbb{Z}_p[G]^n \oplus$

$\mu_K \oplus \Gamma_K \to \mathbb{Z}_p$ induced by $|\ |_p$, then ker φ contains μ_K and (after changing the basis) a free $\mathbb{Z}_p[G]$-module of rank $n-1$. Thus we get the structure of $U^1(K)$. □

COROLLARY 6.4: *There is a commutative exact diagram of $G_{\infty\mathfrak{P}}$-modules:*

$$\begin{array}{ccccccc} 0 \to \mathbb{Z}_p[\![\Gamma_{\mathfrak{P}}]\!][\Delta_{\mathfrak{P}}]^{n_{\mathfrak{p}}} \oplus \mathbb{Z}_p(1) & \to & \mathbb{Z}_p[\![\Gamma_{\mathfrak{P}}]\!][\Delta_{\mathfrak{P}}]^{n_{\mathfrak{p}}} \oplus \mathbb{Z}_p(1) & \to & \mathbb{Z}_p & \to 0 \\ \Uparrow\wr & & \Uparrow\wr & & \| & , \\ 0 \longrightarrow U^1_{\mathfrak{P}} & \longrightarrow & \mathscr{A}_{\mathfrak{P}} & \longrightarrow & \mathbb{Z}_p & \to 0 \end{array}$$

where $n_{\mathfrak{p}}$ is the degree $[K_{\mathfrak{p}}:\mathbb{Q}_p]$.

PROOF: This follows by Theorem 6.3 passing to the inverse limit; indeed, the exact sequence

$$0 \to J_\infty = \operatorname{proj\,lim}_n J(\mathscr{K}_{n\mathfrak{P}}) \to \mathbb{Z}_p[\![\Gamma_{\mathfrak{P}}]\!][\Delta_{\mathfrak{P}}] \to \mathbb{Z}_p \to 0$$

shows, on the one hand, that J_∞ is a free $\mathbb{Z}_p[\![\Gamma_{\mathfrak{P}}]\!]$-module of rank $\#\Delta_{\mathfrak{P}}$ (see (1.2)) and on the other hand, because $(p, \#\Delta_{\mathfrak{P}}) = 1$, that J_∞ is a cohomological trivial $\Delta_{\mathfrak{P}}$-module and hence $\mathbb{Z}_p[\![\Gamma_{\mathfrak{P}}]\!][\Delta_{\mathfrak{P}}]$-projective [7; 10.7, Theorem 3]. Since $\mathbb{Z}_p$ is a $\mathbb{Z}_p[\![\Gamma_{\mathfrak{P}}]\!]$-torsion module we have, by tensoring with quotient field of $\mathbb{Z}_p[\![\Gamma_{\mathfrak{P}}]\!]$,

$$\operatorname{quot}(\mathbb{Z}_p[\![\Gamma_{\mathfrak{P}}]\!]) \otimes J_\infty \cong \operatorname{quot}(\mathbb{Z}_p[\![\Gamma_{\mathfrak{P}}]\!])[\Delta_{\mathfrak{P}}]$$

and therefore by [24, Cor. 6.4]

$$J_\infty \cong \mathbb{Z}_p[\![\Gamma_{\mathfrak{P}}]\!][\Delta_{\mathfrak{P}}].$$

This proves the corollary. □

THEOREM 6.5: *We have the following commutative exact diagram for the $\Gamma[\Delta] = \oplus_{\mathfrak{P}|\mathfrak{p}} \mathbb{Z}_p[\![\Gamma_{\mathfrak{P}}]\!][\Delta_{\mathfrak{P}}]$-modules U^1 and $\mathscr{A}$:*

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \bigoplus_{\mathfrak{p}\in S_p(K)} (\Lambda[\Delta]^{n_{\mathfrak{p}}} \oplus \Lambda/(\omega_{r_{\mathfrak{p}}})[\Delta/\Delta_{\mathfrak{p}}](1)) & \xrightarrow{\ \sim\ } & U^1 \\ \downarrow & & \downarrow \\ \bigoplus_{\mathfrak{p}\in S_p(K)} (\Lambda[\Delta]^{n_{\mathfrak{p}}} \oplus \Lambda/(\omega_{r_{\mathfrak{p}}})[\Delta/\Delta_{\mathfrak{p}}](1)) & \xrightarrow{\ \sim\ } & \mathscr{A} \\ \downarrow & & \downarrow \\ \bigoplus_{\mathfrak{p}\in S_p(K)} \Lambda/(\omega_{r_{\mathfrak{p}}})[\Delta/\Delta_{\mathfrak{p}}] & = & \bigoplus_{\mathfrak{p}\in S_p(K)} \Lambda/(\omega_{r_{\mathfrak{p}}})[\Delta/\Delta_{\mathfrak{p}}] \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

where $r_{\mathfrak{p}} = (\Gamma : \Gamma_{\mathfrak{p}})$.

7. λ-invariants for cyclotomic Γ-extensions; a dulaity theorem.

With the notations and conventions of section 6 we consider the $\Lambda[\Delta]$ modules $\mathscr{X}_r = X_r(\mathscr{K})$ and their decomposition by the action of Δ,

$$\mathscr{X}_r = \bigoplus_{i \bmod d} e_i \mathscr{X}_r, \quad r = 1, 2, 3.$$

We put

$$\mu_r^{(i)} = \mu(e_i \mathscr{X}_r) \quad \text{and} \quad \lambda_r^{(i)} = \lambda(e_i \mathscr{X}_r);$$

if K is totally real, in which case $\mathscr{K} = K(\mu_p)$ is a field of CM-type with maximal totally real subfield K^+, then we set

$$\mathscr{X}_r^+ = \bigoplus_{\substack{i \bmod d \\ i \text{ even}}} e_i \mathscr{X}_r \quad \text{and} \quad \mathscr{X}_r^- = \bigoplus_{\substack{i \bmod d \\ i \text{ odd}}} e_i \mathscr{X}_r,$$

$$\lambda_r^+ = \sum_{i \text{ even}} \lambda_r^{(i)} \quad \text{and} \quad \lambda_r^- = \sum_{i \text{ odd}} \lambda_r^{(i)}, \qquad r = 1, 2, 3.$$

Let $i, j \in \mathbb{Z}$ and $\mathfrak{p} \in S_p(K)$; we define

$$\delta_{i\mathfrak{p}} = \begin{cases} 1, & \theta^i|_{\Delta_\mathfrak{p}} = 1 \\ 0, & \text{otherwise} \end{cases}, \qquad \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Obviously we have

$$\sum_{i=0}^{d-1} \delta_{i_\mathfrak{p}} = (\Delta : \Delta_\mathfrak{p})$$

and

$$s_{\mathscr{K}_\infty} = \sum_{\mathfrak{p} \in S_p(K_\infty)} (\Delta : \Delta_\mathfrak{p}) = \sum_{i=0}^{d-1} \sum_{\mathfrak{p} \in S_p(K_\infty)} \delta_{i_\mathfrak{p}},$$

where s_F is the number of primes dividing p in an algebraic number field F.

Our first objective is to study the connections between the μ_1-invariants for finite p-extension of K. For the λ_1-invariant we obtain an analogue of the classical Riemann–Hurwitz formula for compact connected Rieman surfaces.

Proposition 7.1 ([10, Theorem 3): *Let $E \subset K(p)$ be a finite extension of K and $E_\infty = K_\infty E$, then*

$$\mu_1(E) = 0 \text{ if and only if } \mu_1(K) = 0.$$

PROOF: Let $S \supseteq \Sigma$ be finite with $E \subset K_S(p)$. Since $H_S(E)$ is an open subgroup of $H_S(K)$ and $\mathrm{cd}_p(H_S(K)) < \infty$ we obtain by [23; I, 3.3, Prop. 14]

$$\mathrm{cd}_p(H_S(K)) = \mathrm{cd}_p(H_S(E)).$$

The assertion now follows from Propositions 5.1 and 5.6, and Theorem 5.3. □

THEOREM 7.2: *Let $E \mid K$ be a finite extension of totally real number fields, $E \subset K(p)$, $E_\infty = K_\infty E$ and $\mu_1(K) = 0$. Then we have the equality*

$$\lambda_1(E) - 1 = (\lambda_1(K) - 1)[E_\infty : K_\infty] + \sum_{\mathfrak{p} \nmid p} (e_\mathfrak{p} - 1),$$

where $e_\mathfrak{p}$ is the ramification index of a prime $\mathfrak{p}$ of E_∞ for the extension $E_\infty \mid K_\infty$.

PROOF: Let $S(K_\infty) \supseteq S_p(K_\infty)$ be the set of primes which ramify in K_∞ or divide p, and $S(E_\infty)$ the set of primes lying above $S(K_\infty)$. Propositions 5.6 (and 5.1) yields the equalities

$$\lambda_S(E) = \lambda_1(E) + \#S(E_\infty) \backslash S_p(E_\infty),$$

$$\lambda_S(K) = \lambda_1(K) + \#S(K_\infty) \backslash S_p(K_\infty),$$

and by Propositions 7.1 and 5.6 we have

$$\mu_S(E) = \mu_S(K) = 0.$$

By Theorem 5.4 the $\mathbb{Z}_p$-modules $H_S(E)^{\mathrm{ab}}$ and $H_S(K)^{\mathrm{ab}}$ are free of rank $\lambda_S(E)$ (resp. $\lambda_S(K)$). For the Euler–Poincaré characteristics $\chi(E)$ and $\chi(K)$ of the free pro-p-groups $H_S(E)$ (resp. $H_S(K)$) we obtain (Theorem 5.3)

$$\begin{aligned} \lambda_S(E) - 1 = -\chi(E) &= -\chi(K)[E_\infty : K_\infty] \\ &= (\lambda_S(K) - 1)[E_\infty : K_\infty]. \end{aligned}$$

Since we have

$$[E_\infty : K_\infty] \cdot \#S(K_\infty) \backslash S_p(K_\infty) - \#S(E_\infty) \backslash S_p(E_\infty)$$
$$= \sum_{\mathfrak{p} \nmid p} (e_\mathfrak{p} - 1),$$

the theorem is proved. □

In [25] we proved the above theorem for $K=\mathbb{Q}$; it is possible in this case to give a more explicit description of $H_S(E)^{\mathrm{ab}}$ even for $p=2$.

Next we established a duality theorem, which gives us relations between the invariants $\lambda_1^{(1-i)}$ and $\lambda_3^{(i)}$. We need the following lemma's.

LEMMA 7.3: *Let $i \in \mathbb{Z}$, then*

$$\operatorname*{inj\,lim}_{n,\,\mathrm{res}}\ \operatorname*{inj\,lim}_{m}\ \ker^1\big(\mathbb{G}_{S_p}(K_n), \mathbb{Z}/p^m(i)\big) \cong (e_i\mathscr{X}_3)(-i)^*.$$

PROOF: Passing in (4.1) to the direct limit we obtain the exact sequence

$$0 \to \operatorname*{inj\,lim}_{n,\,m}\ \ker^1\big(\mathbb{G}_{S_p}(K_n), \mathbb{Z}/p^m(i)\big) \to H^1\big(\mathbb{H}_{S_p}(K), \mathbb{Q}_p/\mathbb{Z}_p(i)\big)$$

$$\to \prod_{\mathfrak{p}\in S_p(K_\infty)} H^1\big(\mathbb{G}_{\mathfrak{p}}(K_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i)\big).$$

Since $d \mid p-1$ and by [8; Prop. 22, Cor. 1] there are isomorphisms

$$H^1\big(\mathbb{H}_{S_p}(K), \mathbb{Q}_p/\mathbb{Z}_p(i)\big) \cong H^1\big(\mathbb{H}_{S_p}(\mathscr{K}), \mathbb{Q}_p/\mathbb{Z}_p(i)\big)^{\Delta}$$

$$\cong H^1\big(H_{S_p}(\mathscr{K}), \mathbb{Q}_p/\mathbb{Z}_p\big)(i)^{\Delta}$$

(and similiary for the cohomology groups of the local Galois groups). Hence

$$\operatorname*{inj\,lim}_{n,\,m}\ \ker^1\big(\mathbb{G}_{S_p}(K_n), \mathbb{Z}/p^m(i)\big) \cong \mathscr{X}_3^*(i)^{\Delta} \cong \big(\mathscr{X}_3(-i)_{\Delta}\big)^*$$

$$\cong (e_i\mathscr{X}_3)(-i)^*. \quad \square$$

LEMMA 7.4: *The following diagram commutes and is exact*:

$$\begin{array}{ccc} H^2(\mathbb{G}_{S_p}(\mathscr{K}_n), \mathbb{Q}_p/\mathbb{Z}_p(i)) & \xrightarrow{\mathrm{res}} & \bigoplus_{\mathfrak{J}\in S_p(\mathscr{K}_n)} H^2(\mathbb{G}_{\mathfrak{J}}(\mathscr{K}_n), \mathbb{Q}_p/\mathbb{Z}_p(i)) \\ \uparrow\wr & & \uparrow\wr \\ (T_\Lambda(\mathscr{X}_1)(-i)\Gamma^n)^* & \longrightarrow & (T_\Lambda(\mathscr{A})(-i)^{\Gamma^n})^* \end{array}$$

$$\begin{array}{cc} \to \operatorname*{inj\,lim}_{m} H^0(\mathbb{G}_{S_p}(\mathscr{K}_n), \mathbb{Z}/p^m(1-i))^* & \to 0 \\ \uparrow\wr & \\ \longrightarrow (T_\Lambda(\bar{E}')(-i))^{\Gamma^n *} & \longrightarrow 0 \end{array}$$

in particular,

$$T_\Lambda(\bar{E}') \cong \mathbb{Z}_p(1).$$

PROOF: By (4.6) and (4.2) (passing to the direct limit) the lines are exact. The vertical morphisms are defined as follows: By the spectral sequence, Lemma 3.1, we have a commutative diagram

$$\begin{array}{ccc} (\mathbb{G}_{S_p}(\mathcal{K}_\infty)^{\mathrm{ab}}_{p^m}(-i))^{\Gamma^n*} & \underset{\text{can}}{\rightarrow} & (\mathbb{G}_{\mathfrak{p}}(\mathcal{K}_\infty)^{\mathrm{ab}}_{p^m}(-i))^{\Gamma_{\mathfrak{p}}\cap\Gamma^n*} \\ \downarrow & & \downarrow\wr \\ H^2(\mathbb{G}_{S_p}(\mathcal{K}_n), \mathbb{Z}/p^m(i)) & \overset{\text{res}}{\rightarrow} & H^2(\mathbb{G}_{\mathfrak{p}}(\mathcal{K}_n), \mathbb{Z}/p^m(i)) \end{array}$$

where the right map is an isomorphism because of $\mathrm{cd}_p(\mathbb{G}_{\mathfrak{p}}(\mathcal{K}_\infty)) = 1$. Since we have

$$\operatorname*{inj\,lim}_m \left(\mathbb{G}_{S_p}(\mathcal{K}_\infty)^{\mathrm{ab}}_{p^m}(-i)\right)^{\Gamma^n*} = \left(\mathcal{X}_1(-i)^{\Gamma^n}\right)^* = \left(T_\Lambda(\mathcal{X}_1)(-i)^{\Gamma^n}\right)^*,$$

$$\operatorname*{inj\,lim}_m \bigoplus_{\mathfrak{p}\in S_p(\mathcal{K}_n)} \left(G_{\mathfrak{p}}(\mathcal{K}_\infty)^{\mathrm{ab}}_{p^m}(-i)\right)^{\Gamma^n*} = \left(\mathcal{A}(-i)^{\Gamma^n}\right)^*$$

$$= \left(T_\Lambda(\mathcal{A})(-i)^{\Gamma^n}\right)^*$$

and

$$\operatorname*{inj\,lim}_m H^2\left(\mathbb{H}_{S_p}, \mathbb{Z}/p^m(i)\right) = H^2\left(\mathbb{H}_{S_p}, \mathbb{Q}_p/\mathbb{Z}_p\right)(i) = 0,$$

by Proposition 5.1, we obtain the left square and hence the right. Now, because of

$$T_\Lambda(\bar{E}') \subset T_\Lambda(\mathcal{A}) \cong \bigoplus_{\mathfrak{p}\in S_p(K)} \Lambda/(\omega_{r_{\mathfrak{p}}})\left[\Delta/\Delta_{\mathfrak{p}}\right](1),$$

Thereom 6.5 and $T_\Lambda(\bar{E}')^{\Gamma^n} \cong \mu_{\mathcal{K}_n}$ we obtain the last assertion. □

LEMMA 7.5: *Let $i \in \mathbb{Z}$, then there is an exact sequence*

$$0 \rightarrow \alpha\left(T_\Lambda(e_i\mathcal{X}_1)(-i)/T_{\mathbb{Z}_p}(e_i\mathcal{X}_1)(-i)\right)$$

$$\rightarrow \operatorname*{proj\,lim}_{n,\,\mathrm{cor}} \operatorname*{proj\,lim}_m H^2\left(\mathbb{G}_{S_p}(K_n), \mathbb{Z}/p^m(i)\right)$$

$$\rightarrow \alpha\left(T_{\mathbb{Z}p}(e_i\mathcal{X}_1)(-i)\right) \rightarrow 0;$$

furtheremore, there exist a quasi-isomorphism ϕ_1 and two isomorphisms ϕ_2 and ϕ_3, such that the following diagram with quasiexact upper line and *exact bottom line commutes*:

$$
\begin{array}{ccl}
\alpha(T_\Lambda(e_\iota \mathscr{X}_1)(-i)) & \underset{\phi_1}{\to} & \operatorname*{proj\,lim}_{n,m} H^2(\mathbb{G}_{S_p}(K_n), \mathbb{Z}/p^m(i)) \\
\downarrow & & \downarrow \\
\alpha(T_\Lambda(e_\iota \mathscr{A})(-i)) & \underset{\phi_2}{\tilde{\to}} & \operatorname*{proj\,lim}_{n,m} \bigoplus_{\mathfrak{p}\in S_p(K_n)} H^2(\mathbb{G}_{\mathfrak{p}}(K_n), \mathbb{Z}/p^m(i)) \\
 & & \qquad \wr\Vert \\
 & & \qquad \bigoplus_{\mathfrak{p}\in S_p(K)} \Lambda/(\omega_{r_{\mathfrak{p}}})^{\delta_{1-\iota\,\mathfrak{p}}}(i-1) \\
\downarrow & & \downarrow \\
\alpha(T_\Lambda(e_\iota \bar{E}')(-i)) & \underset{\phi_3}{\tilde{\to}} & \operatorname*{proj\,lim}_{n,m} H^0(\mathbb{G}_{S_p}(K_n), \mathbb{Z}/p^m(1-i))^* \\
 & & \qquad \wr\Vert \\
 & & \qquad \mathbb{Z}_p(i-1)^{\delta_{1-\iota\,0}} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Proof: By [8; Prop. 22, Cor. 1] and because of Lemma 3.1 and $(p, d) = 1$ we obtain the exact sequence

$$
0 \to H^1\big(\Gamma^n, H^1\big(H_{S_p}(\mathscr{K}), \mathbb{Z}/p^m(i)\big)\big)^\Delta \to H^2\big(\mathbb{G}_{S_p}(K_n), \mathbb{Z}/p^m(i)\big)
$$

$$
\to H^2\big(H_{S_p}(\mathscr{K}), \mathbb{Z}/p^m(i)\big)^{\Gamma^n \times \Delta} \to 0.
$$

Since

$$
H^1\big(\Gamma^n, H^1\big(H_{S_p}(\mathscr{K}), \mathbb{Z}/p^m(i)\big)\big)^\Delta \cong \big(\mathscr{X}_{1_{p^m}}(-i)^{\Gamma^n}\big)_\Delta{}^*
$$

$$
\cong \big((e_\iota \mathscr{X}_1)(-i)_{p^m}\big)^{\Gamma^n *}
$$

and because of proposition 5.1

$$
H^2\big(H_{S_p}(\mathscr{K}), \mathbb{Z}/p^m(i)\big)^{\Gamma^n \times \Delta} \cong \big({}_{p^m}H_{S_p}(\mathscr{K})^{\mathrm{ab}}(-i)\big)^{*\Gamma^n \times \Delta}
$$

$$
\cong \big({}_{p^m}(e_\iota \mathscr{X}_1)(-i)\big)_{\Gamma^n}{}^*
$$

Lemma 1.9 proves the first assertion.

The lines of the diagram are quasi-exact resp. exact by (4.6) resp. (4.2). With the diagram in the proof of Lemma 7.4 (now passing to the inverse limit) and the structure theorem (Theorem 6.5) we obtain the commutative diagram

$$\begin{array}{ccc}
\alpha\Big(e_\iota\big(T_\Lambda\big(\mathscr{X}_1/T_{\mathbb{Z}_p}(\mathscr{X}_1)(-i)\big)\Big) & \to \alpha\big(e_\iota T_\Lambda(\mathscr{A})(-i)\big) \cong & \bigoplus_{\mathfrak{p}\in S_p(K)} \Lambda/(\omega_{r_\mathfrak{p}})^{\delta_{1-\iota,\mathfrak{p}}}(i-1) \\
\downarrow & & \downarrow\wr\ \phi_2 \\
\operatorname{proj\,lim}_{n,m} H^2\big(\mathbb{G}_{S_p}(K_n), \mathbb{Z}/p^m(i)\big) & \overset{\text{res}}{\to} & \operatorname{proj\,lim}_{n,m} \bigoplus_{\mathfrak{p}\in S_p(K_n)} H^2\big(\mathbb{G}_\mathfrak{p}(K_n), \mathbb{Z}/p^m(i)\big) \\
\updownarrow & & \\
\alpha\big(e_\iota T_{\mathbb{Z}_p}(\mathscr{X}_1)(-i)\big) & &
\end{array}$$

Since the vertical exact sequence is quasi-splitting and $\alpha(T_\Lambda(\mathscr{A}))$ is $\mathbb{Z}_p$-torsion free, we can define the quasi-isomorphism ϕ_1, such that the left square commutes. By Lemma 7.4 we have

$$\alpha\big(T_\Lambda(\bar{E}')(-i)\big) = T_\Lambda(\bar{E}')(i-2) = \operatorname{proj\,lim}_n T_\Lambda(\bar{E}')(i-2)^{\Gamma^n}$$

$$\tilde{\to} \operatorname{proj\,lim}_{n,m} H^0\big(\mathbb{G}_{S_p}(\mathscr{K}_n), \mathbb{Z}/p^m(i-1)\big).$$

Thus there is an isomorphism

$$\phi_3 : \alpha\big(e_\iota T_\Lambda(\bar{E}')(-i)\big) \tilde{\to} \operatorname{proj\,lim}_{n,m} H^0\big(\mathbb{G}_{S_p}(\mathscr{K}_n), \mathbb{Z}/p^m(1-i)\big)^*,$$

which because of Lemma 7.4 commutes with ϕ_2. □

LEMMA 7.6: *Let M and N be noetherian $\Lambda[\Delta]$-torsion modules, $i \in \mathbb{Z}$ and let $\tilde{\psi}$ be a $(\Gamma \times \Delta)$-invariant pairing with finite kernels; let $\tilde{\psi}_m$ th pairing induced by $\tilde{\psi}$.*

$$\begin{array}{llll}
\tilde{\psi}: & M \ \times & N^* & \to \mathbb{Q}_p/\mathbb{Z}_p(i) \\
 & \updownarrow & \uparrow & \uparrow \\
\tilde{\psi}_m: & M_{p^m} \times & {}_{p^m}(N^*) & \to p^{-m}\mathbb{Z}_p/\mathbb{Z}_p(i)
\end{array}$$

Then there exists a $(\Gamma \times \Delta)$-invariant pairing ψ,

$$\begin{array}{llll}
\psi: & M \ \times & \alpha(N/T_{\mathbb{Z}_p}(N)) & \to \mathbb{Z}_p(i) \\
 & \updownarrow & \updownarrow & \updownarrow \\
\psi_m: & M_{p^m} \times & \operatorname{Hom}(N_{p^m}, \mathbb{Q}_p/\mathbb{Z}_p) & \to \mathbb{Z}/p^m(i),
\end{array}$$

such that the pairing ψ_m, $m \geqslant 0$, induced by ψ, are equal to $\tilde{\psi}_m$ and the pairing

$$\psi \otimes \mathbb{Q}_p : (M \otimes \mathbb{Q}_p) \times (\alpha(N) \otimes \mathbb{Q}_p) \to \mathbb{Q}_p(i)$$

is non-degenerated.

PROOF: It is easy to see, that the diagram

$$\begin{array}{cccc} M_{p^n} \times_{p^n} (N^*) & \overset{\tilde{\psi}_n}{\to} \mathbb{Z}/p^n(i) \to & \mathbb{Q}_p/\mathbb{Z}_p(i) \\ \downarrow \quad \downarrow p^{n-m} & \downarrow \text{can} & \downarrow p^{n-m} \\ M_{p^m} \times_{p^m}(N^*) & \overset{\tilde{\psi}_m}{\to} \mathbb{Z}/p^m(i) \to & \mathbb{Q}_p/\mathbb{Z}_p(i) \end{array}$$

commutes. Since we can choose $\pi_m = p^m$ to obtain the adjoint of $N/T_{\mathbb{Z}_p}(N)$, we get

$$\operatorname{proj}\lim_m {}_{p^m}(N^*) = \operatorname{proj}\lim_m {}_{p^m}\left(N/T_{\mathbb{Z}_p}N\right)^* = \alpha\left(N/T_{\mathbb{Z}_p}(N)\right)$$

and hence a pairing $\psi := \operatorname{proj}\lim \tilde{\psi}_m$ with the required properties. □

Now we define the $\Lambda[\Delta]$-torsion module $\mathscr{Z}$ by the following exact sequence, which is induced by (4.6):

$$0 \to T_\Lambda(\bar{E}') \to T_\Lambda(\mathscr{A}) \to T_\Lambda(\mathscr{X}_1) \to \mathscr{Z} \to 0. \tag{7.7}$$

Lemma 7.5 implies a quasi-isomorphism

$$((e_i\mathscr{Z})(-i))^\circ \sim \operatorname{proj}\lim_{n,m} \ker^2\left(\mathbb{G}_{S_p}(K_n), \mathbb{Z}/p^m(i)\right), \tag{7.8}$$

and therefore we get by the global duality theorem, that is (4.3), (4.4), (4.5) and Lemma 7.6 the following duality assertion:

THEOREM 7.9: *There exists a pairing induced by the cupproduct*

$$(e_{1-i}\mathscr{X}_3)^* \times (e_i\mathscr{Z})^\circ \to \mathbb{Q}_p/\mathbb{Z}_p(-1)$$

with finite kernels and hence a quasi-isomorphism

$$((e_i\mathscr{Z})(-i))^\circ \to (e_{1-i}\mathscr{X}_3)(i-1).$$

This bilinear form induces a non-degenerated pairing of $\mathbb{Q}_p$*-vector spaces*

$$\psi_0^{(1-i)} : \left(e_{1-i}\mathscr{X}_3 \otimes \mathbb{Q}_p\right) \times \left(e_i\mathscr{Z} \otimes \mathbb{Q}_p\right) \to \mathbb{Q}_p(1),$$

for which we have

$$\psi_0^{(1-i)}(\gamma x, \gamma z) = \kappa(\gamma)\psi_0^{(1-i)}(x, z),$$

$\gamma \in \Gamma$, $x \in e_{1-i}\mathscr{X}_3 \otimes \mathbb{Q}_p$, $z \in e_i\mathscr{Z} \otimes \mathbb{Q}_p$, *and a Γ-invariant pairing with finite kernels*

$$T_{\mathbb{Z}_p}(e_{1-i}\mathscr{X}_3) \times T_{\mathbb{Z}_p}(e_i\mathscr{Z}) \to \mathbb{Z}/p^{\mu}(1),$$

with $\mu = \mu_1^{(i)} = \mu_3^{(1-i)}$.

COROLLARY 7.10: *Let $i \in \mathbb{Z}$; then the following sequence is quasi-exact*:

$$0 \to ((e_{1-i}\mathscr{X}_3)(i-1))^{\circ} \to T_{\Lambda}(e_i\mathscr{X}_1)(-i) \to \bigoplus_{\mathfrak{p} \in S_p(K)} \Lambda/(\omega_{r_{\mathfrak{p}}})^{\delta_{1-i,\mathfrak{p}}(1-i)}$$

$$\to \mathbb{Z}_p(1-i)^{\delta_{1-i,0}} \to 0;$$

As a consequence, we have

$$\mu_1^{(i)} = \mu_3^{(1-i)} \quad \textit{and} \quad \lambda_1^{(i)} = \lambda_3^{(1-i)} + n_{1-i} - \delta_{1-i,0},$$

where

$$n_j = \sum_{\mathfrak{p} \in S_p(K_\infty)} \delta_{j,\mathfrak{p}}, \quad j \in \mathbb{Z},$$

and in the case $\mu_p \subset K = \mathscr{K}$

$$\mu_1 = \mu_3 \quad \text{and} \quad \lambda_1 = \lambda_3 + s_{\mathscr{K}_\infty} - 1.$$

COROLLARY 7.11: *Let $\mathscr{K}$ be a number field of CM-type with maximal totally real subfield K^+, then*

$$\lambda_1^- = \lambda_3^+ + s_{K_\infty^-} - 1, \qquad \lambda_1^+ = \lambda_3^- + s_{\mathscr{K}_\infty} - s_{K_\infty^+}.$$

REMARK: The assertions of Corollary 7.10 and Theorem 5.4(i) in the case $\mu_p \subset K$ (without the exact description of the factors $\omega_{r,\mathfrak{p}}$ of the characteristic polynomial of $\mathscr{X}_1$) is due to Iwasawa [9, Theorems 15 and 16] (if $\dot{T} := \kappa(\gamma_0)(1+T)^{-1} - 1$ denotes the "Iwasawa-involution", we have for a compact Λ-module M the equality $\dot{M} = (M(-1))^{\circ} = \mathring{M}(1)$).

COROLLARY 7.12: *Let $E \mid K$ be a finite extension of totally real number fields, $E \subset K(p)$, $\mu_1(K) = 0$, $\mathscr{E} = E(\mu_p)$ and $\mathscr{K} = K(\mu_p)$, then*

$$\lambda_3^{(1)}(\mathscr{E}) - 1 = (\lambda_3^{(1)}(\mathscr{K}) - 1)[\mathscr{E}_\infty : \mathscr{K}_\infty] + \sum_{\substack{\mathfrak{p} \in S(E_\infty) \\ \mu_p \subset E_{\infty\mathfrak{p}}}} (n_{\mathfrak{p}} - 1),$$

where $S(E_\infty)$ is a finite set of primes containing $S_p(E_\infty)$ and all prime divisors, which ramify in the extension $E_\infty \mid K_\infty$, and $n_\mathfrak{p} = [E_{\infty\mathfrak{p}} : K_{\infty\mathfrak{p}}]$ (= ramification index if $\mathfrak{Z} \nmid p$).

PROOF: We have

$$n_1(K) = \sum_{\mathfrak{p} \in S_p(K_\infty)} \delta_{1\mathfrak{p}} = \#\left\{ \mathfrak{p} \in S_p(K_\infty) : \mu_p \subset K_{\infty\mathfrak{p}} \right\},$$

hence

$$n_1(K)[E_\infty : K_\infty] - n_1(E) = \sum_{\substack{\mathfrak{p} \in S_p(E_\infty) \\ \mu_p \subset E_{\infty\mathfrak{p}}}} (n_\mathfrak{p} - 1).$$

Since the condition $\mu_p \subset E_{\infty_\mathfrak{p}}$ is necessary for $n_\mathfrak{p} > 1$, $\mathfrak{p} \in S(E_\infty) \setminus S_p(E_\infty)$, and $\lambda_1^{(0)}(\mathscr{E}) = \lambda_1(E)$, $\lambda_1^{(0)}(\mathscr{K}) = \lambda_1(K)$ the corollary follows from Theorem 7.2 and Corollary 7.10. □

Finally, we give a description of the fixed part of $\mathscr{X}_1(-1)$ used in the next section. For this we define

$$B := T_\Lambda(\mathscr{A}) / T_\Lambda(\bar{E}'),$$

and obtain the following:

PROPOSITION 7.13: *Let* $n \geqslant 0$ *be sufficiently large, then*

$$B(-1) = \mathscr{X}_1(-1)^{\Gamma^n}.$$

PROOF: The assertion follows from the commutative exact diagram in Lemma 7.4, since we have, for large n (Theorem 6.5),

$$T_\Lambda(\mathscr{A})(-1)^{\Gamma^n} = T_\Lambda(\mathscr{A})(-1),$$

$T_\Lambda(\bar{E}')(-1)^{\Gamma^n} = T_\Lambda(\bar{E}')(-1)$ for all $n \geqslant 0$ (Lemma 7.4) and by (4.3)

$$\begin{aligned} &\operatorname*{inj\,lim}_m \ker^2\left(\mathbb{G}_{S_p}(\mathscr{K}_n), \mathbb{Z}/p^m(1)\right) \\ &\cong \left(\operatorname*{proj\,lim}_m \ker^1\left(\mathbb{G}_{S_p}(\mathscr{K}_n), \mathbb{Z}/p^m\right)\right)^* \\ &= \operatorname*{inj\,lim}_m \mathrm{Cl}_{S_p}(\mathscr{K}_n)_{p^m} = 0, \end{aligned}$$

where $\mathrm{Cl}_{S_p}(\mathscr{K}_n)$ denotes the finite S_p-ideal class group of $\mathscr{K}_n$.

8. The relations between λ-invariants for fields of CM-type

With the notations and conventions of section 7 we assume further that K is a totally real number field and hence $\mathscr{K} = K(\mu_p)$ is a CM-field with maximal totally real subfield K^+. In this case the subgroup generated by the units $E(K_n^+)$ and the roots of unity of $\mathscr{K}_n$, $n \geqslant 0$, has index 1 or 2 in the group $E(\mathscr{K}_n)$. since $p \neq 2$ we have

$$e_i(E_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p) = 0 \quad \text{for } i \text{ odd}. \tag{8.1}$$

To get the relations between the invariants λ_2 and λ_3 we use Kummer-theory.

We summarize some results of [9, §7]: Let I'_∞ be the free abelian group on the non-archimedean primes of $\mathscr{K}_\infty$ which do not lie above p. Define $\mathfrak{M}$ by the exactness of the sequence

$$0 \to \mathfrak{M} \to \mathscr{K}_\infty^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\varphi} I'_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p,$$

where $\varphi(a \otimes \alpha) = (a)' \otimes \alpha$, $(a)' = \sum_{\mathfrak{p} \nmid p} v_\mathfrak{p}(a)\mathfrak{p}$ with the $\mathfrak{p}$-adic valuation $v_\mathfrak{p}$. Then there exists the perfect Kummer-pairing

$$(\,,\,): \mathscr{X}_1 \times \mathfrak{M} \to \mathbb{Q}_p/\mathbb{Z}_p(1) \tag{8.2}$$

satisfying $(\sigma x, \sigma m) = \sigma(x, m)$ for all $\sigma \in \Gamma \times \Delta$. For the fields

$$N = \mathscr{K}_\infty\left(E_\infty^{1/p^\infty}\right) \quad \text{and} \quad N' = \mathscr{K}_\infty\left(E_\infty'^{1/p^\infty}\right)$$

exist quasi-exact sequences induced by the above pairing [9, Lemma 10 with the remark following it and Theorem 11]:

$$0 \to \mathring{\mathscr{X}}_2(1) \to \mathscr{X}_1 \to G(N/\mathscr{K}_\infty) \to 0, \tag{8.3}$$

$$0 \to \mathring{\mathscr{X}}_3(1) \to \mathscr{X}_1 \to G(N'/\mathscr{K}_\infty) \to 0. \tag{8.4}$$

Since the duals of $E_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p$ and $E'_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p$ are free $\mathbb{Z}_p$-modules we see by Proposition 2.1 that

$$\mathrm{hd}_\Lambda G(N/\mathscr{K}_\infty) \leqslant 1 \quad \text{and} \quad \mathrm{hd}_\Lambda G(N'/\mathscr{K}_\infty) \leqslant 1. \tag{8.5}$$

Now, from the exact sequence (4.6) we obtain by the structure theorem (Theorem 6.5)

$$\mathrm{rg}_\Lambda e_i \mathscr{X}_1 \leqslant \sum_{\mathfrak{p} \in S_p(K)} n_\mathfrak{p} = n = [K:\mathbb{Q}].$$

Since $\mathscr{X}_1^+$ is a Λ-torsion module and

$$\mathrm{rg}_\Lambda \mathscr{X}_1 = \mathrm{rg}_\Lambda \mathscr{X}_1^- = \sum_{i \text{ odd}} \mathrm{rg}_\Lambda e_i \mathscr{X}_1 = n \cdot d/2$$

(Theorems 5.4) the following proposition is proved:

PROPOSITION 8.6:

$$\mathrm{rg}_\Lambda e_i \mathscr{X}_1 = \begin{cases} n, & i \text{ odd}, \\ 0, & i \text{ even}. \end{cases}$$

Since a noetherian Λ-module M is quasi-isomorphic to $T_\Lambda(M) \oplus F_\Lambda(M)$, the quasi-exact sequence in Corollary 7.10 induces a quasi-exact sequence

$$0 \to G(N'/\mathscr{K}_\infty) \to (\Lambda[\Delta]^n)^- \oplus \bigoplus_{\mathfrak{p} \in S_p(K)} \Lambda/(\omega_{r_\mathfrak{p}})[\Delta/\Delta_\mathfrak{p}](1)$$

$$\to \mathbb{Z}_p(1) \to 0. \tag{8.7}$$

Especially we see that the μ-invariant of the Λ-torsion module $G(N'/N)$ is zero, hence

$$\mu_1 = \mu_2 = \mu_3. \tag{8.8}$$

Since $e_i(E_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p) = 0$ for an odd number i we get via Kummer pairing $e_{1-i}G(N/\mathscr{K}_\infty) = 0$ (i odd). The existence of a quasi-isomorphism

$$((e_i\mathscr{X}_2)(-i))^\circ \sim (e_{1-i}\mathscr{X}_1)(i-1) = T_\Lambda(e_{1-i}\mathscr{X}_1)(i-1), \quad i \text{ odd}, \tag{8.9}$$

follows from (8.3).

From this last assertion and Theorem 7.2 we obtain now a Riemann–Hurwitz formula for $\lambda_2^{(1)}$ which for $K = K^+$ is the result obtained by Kida [16] and Iwasawa [11].

COROLLARY 8.10: *Let $E \mid K$ be a finite extension of totally real number fields, $E \subset K(p)$, $\mu(K) = 0$, $\mathscr{E} = E(\mu_p)$ and $\mathscr{K} = K(\mu_p)$, then*

$$\lambda_2^{(1)}(\mathscr{E}) - 1 = (\lambda_2^{(1)}(\mathscr{K}) - 1)[\mathscr{E}_\infty : \mathscr{K}_\infty] + \sum_{\mathfrak{p} \nmid p} (e_\mathfrak{p} - 1),$$

where $e_\mathfrak{p}$ denotes the ramification index of $\mathfrak{p}$ for the extension $\mathscr{E}_\infty/\mathscr{K}_\infty$.

PROPOSITION 8.11: *Let $i \in \mathbb{Z}$ be odd, then the sequence*

$$0 \to \bigoplus_{\mathfrak{p} \in S_p(K)} \Lambda/(\omega_{r_\mathfrak{p}})^{\delta_{i\mathfrak{p}}} \to e_i \mathscr{X}_2 \underset{\text{can}}{\to} e_i \mathscr{X}_3 \to 0$$

is exact. Consequently

$$\lambda_1^{(1-i)} = \lambda_2^{(i)} = \lambda_3^{(i)} + n_i$$

and

$$s_{\mathscr{K}_\infty} - 1 \geqslant \lambda_2 - \lambda_3 \geqslant \lambda_2^- - \lambda_3^- = s_{\mathscr{K}\infty} - s_{K_\infty^+}.$$

PROOF: Since we have the equalities

$$\sum_{i \text{ odd}} n_i = s_{\mathscr{K}_\infty} - \sum_{\mathfrak{p} \in S_p(\mathscr{K}_\infty)} \sum_{i \text{ even}} \delta_{i,\mathfrak{p}} = s_{\mathscr{K}_\infty} - s_{K_\infty^+}$$

and

$$\lambda_1^{(1-i)} = \lambda_2^{(i)} = \lambda_3^{(i)} + n_i, \quad i \text{ odd},$$

(by (8.7) and Corollary 7.10) we get $\lambda_2^- - \lambda_2^- = s_{\mathscr{K}_\infty} - s_{K_\infty^+}$. The quasi-exact sequence (8.7) shows the inequality

$$\lambda_2 - \lambda_3 = \lambda(G(N'/N)) \leqslant \lambda(G(N'/\mathscr{K}_\infty)) = s_{\mathscr{K}_\infty} - 1.$$

By the snake lemma and the exact sequence (Theorem 6.5)

$$0 \to e_i U^1 \to e_i \mathscr{A} \to \bigoplus_{\mathfrak{p} \in S_p(K)} \Lambda/(\omega_{r_\mathfrak{p}})^{\delta_{i\,\mathfrak{p}}} \to 0,$$

we obtain from the commutative exact diagram (4.6) the exact sequence

$$\bigoplus_{\mathfrak{p} \in S_p(K)} \Lambda/(\omega_{i_\mathfrak{p}})^{\delta_{i,\mathfrak{p}}} e_i \to e_i \mathscr{X}_2 \to e_i \mathscr{X}_3 \to 0.$$

If i is odd, the equality $\lambda_2^{(i)} = \lambda_3^{(i)} + n_i$ yields the injectivity of the left morphism. □

Let

$$\nabla = T_\Lambda(\mathscr{X}_1)^\perp$$

be the orthogonal complement of $T_\Lambda(\mathscr{X}_1)$ in the Kummer pairing (8.2). Since the morphism

$$\operatorname{Hom}(E_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p(1)) = G(N/\mathscr{K}_\infty) \overset{\text{can}}{\twoheadrightarrow} F_\Lambda(\mathscr{X}_1)$$

$$= \operatorname{Hom}(\nabla, \mathbb{Q}_p/\mathbb{Z}_p(1))$$

is surjective, we have the inclusion

$$\nabla \subseteq E_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

In [2, Theorem 5], Coates showed that these groups are equal, if there is only one prime of $\mathscr{K}_\infty$ lying above p. But in general this is not true; we prove that this assertion is still valid under the weaker assumption $s_{K_\infty^+}=1$ and, if the Leopoldt-conjecture is true for all fields K_n^+, $n \geqslant 0$, that this condition is also necessary for the equality $\nabla = E_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p$.

PROPOSITION 8.12: *Let* $s_{K_\infty^+}=1$, *then*

(a) $\lambda_1^{(1-i)}=\lambda_2^{(i)}=\lambda_3^{(i)}+(s_{\mathscr{K}_\infty}-1)\cdot\delta_{i,\,d/2},\quad i=0,1,\dots d-1;$

in particular $\lambda_2^+=\lambda_3^+$.

(b) $G(N'/\mathscr{K}_\infty)\sim\Lambda^{r_2}\oplus\mathbb{Z}_p(1)^{s_{\mathscr{K}_\infty}-1}$.

(c) $G(N/\mathscr{K}_\infty)=F_\Lambda(\mathscr{X}_1)\sim\Lambda^{r_2}$ *and* $\nabla = E_\infty\otimes\mathbb{Q}_p/\mathbb{Z}$.

PROOF: With $S_p(K_\infty^+)=\{\mathfrak{p}\}$ and $\Delta^+=G(\mathscr{K}_\infty/K_\infty^+)$ we have $\Delta/\Delta^+=(\Delta/\Delta^+)_\mathfrak{p}=\Delta_\mathfrak{p}\Delta^+/\Delta^+$, hence $\Delta=\Delta_\mathfrak{p}\Delta^+$;

$$s_{\mathscr{K}_\infty}=1 \;\Rightarrow\; \Delta=\Delta_\mathfrak{p} \text{ and } n_i=\delta_{i,\,\mathfrak{p}}=\delta_{i,0},$$

$$s_{\mathscr{K}_\infty}=2 \;\Rightarrow\; \Delta=\Delta_\mathfrak{p}\oplus\Delta^+ \text{ and } n_i=\delta_{i,\,\mathfrak{p}}=\delta_{i,0}+\delta_{i,\,d/2}.$$

By Proposition 8.11 we obtain the assertion (a) for an odd number i; furthermore we see from the equality

$$s_{\mathscr{K}_\infty}-1=\lambda_2-\lambda_3=\lambda_2^- -\lambda_3^-$$

that

$$\lambda_2^{(i)}=\lambda_3^{(i)}\quad\text{for } i \text{ even},$$

and that $d/2$ is necessarily odd, if $s_{\mathscr{K}_\infty}$ is equal to 2. Hence, by Corollary 7.10 the assertion (a) is proved for all i. The quasi-exact sequence (8.7) proves (b). Because of the equality $\lambda(G(N'/N))=\lambda_2-\lambda_3=s_{\mathscr{K}_\infty}-1$ and the fact, that $G(N/\mathscr{K}_\infty)$ does not contain any nontrivial submodule (8.5), the kernel of the surjection

$$G(N/\mathscr{K}_\infty)\overset{\text{can}}{\twoheadrightarrow}F_\Lambda(\mathscr{X}_1)$$

has to be trivial. □

PROPOSITION 8.13: *The following assertions are equivalent*:

(*i*) $\lambda_2^+ = \lambda_3^+$.

(*ii*) $\lambda(G(N/\mathscr{K}_\infty)) = s_{K_\infty^+} - 1$.

PROOF: This is trivial because of

$$\lambda_2^+ - \lambda_3^+ = \lambda_2 - \lambda_3 - (\lambda_2^- - \lambda_3^-) = \lambda(G(N'/N)) - s_{\mathscr{K}_\infty} + s_{K_\infty^+}$$
$$= s_{K_\infty^+} - 1 - \lambda(G(N/\mathscr{K}_\infty)).$$

REMARK 8.14: We may conjecture that $\lambda_2^+ = \lambda_3^+$ is true in general and not only for $s_{K_\infty^+} = 1$ (Proposition 8.12) (Greenberg's conjecture asserts even more: $\lambda_2^+ = 0$ and hence $\lambda_3^+ = 0$). Proposition 8.13 shows that for fields K with $\lambda_2^+ = \lambda_3^+$ the equality $\nabla = E_\infty \otimes \mathbb{Q}_p/\mathbb{Z}_p$ holds if and only if $s_{K_\infty^+} = 1$. In analogy to the quasi-exact sequence (8.7) we may further expect the existence of a quasi-exact sequence

$$0 \to G(N/\mathscr{K}_\infty) \to (\Lambda[\Delta]^n)^- \oplus \bigoplus_{\mathfrak{p} \in S_p(K)} \Lambda/(\omega_{r\mathfrak{p}})[\Delta/\Delta_\mathfrak{p}\Delta^+](1)$$
$$\to \mathbb{Z}_p(1) \to 0.$$

We shall now prove the conjecture $\lambda_2^+ = \lambda_3^+$ assuming the Leopoldt-conjecture. For that purpose we consider the following fundamental exact sequence obtained by the snake lemma form the commutative exact diagram (4.6) and the exact sequence

$$0 \to e_\iota U^1 \to e_\iota \mathscr{A} \to \bigoplus_{\mathfrak{p} \in S_p(K)} \Lambda/(\omega_{r_\mathfrak{p}})^{\delta_{\iota,\mathfrak{p}}} e_\iota \to 0:$$

$$0 \to e_\iota \bar{E} \to e_\iota \bar{E}' \to \bigoplus_{\mathfrak{p} \in S_p(K)} \Lambda/(\omega_{r_\mathfrak{p}})^{\delta_{\iota,\mathfrak{p}}} e_\iota \to e_\iota \mathscr{X}_2 \to e_\iota \mathscr{X}_3 \to 0. \tag{8.15}$$

We show now:

PROPOSITION 8.16: *Let $n_0 \geq \lambda_1^+$ and let the Leopoldt-conjecture be true for the maximal totally real subfield $K_{n_0}^+$ of $\mathscr{K}_{n_0}$ (for example, if $K/\mathbb{Q}$ is an abelian extension). Then the equality*

$$\lambda_1^{(1-\iota)} - n_\iota + \delta_{\iota,0} = \lambda_2^{(\iota)} = \lambda_3^{(\iota)}, \quad \iota \text{ even},$$

holds, and in particular

$$\lambda_2^+ = \lambda_3^+.$$

PROOF: By Propositions 5.5 and 5.1 the Leopoldt-conjecture is true for all fields K_n^+, $n \geqslant 0$, and hence by Lemma 3.1

$$(e_i \mathscr{X}_1)^{\Gamma^n} = 0 \quad \text{for } i \text{ even and all } n.$$

Consequently, the characteristic polynomials $f_1^{(i)}(T)$ of $e_i\mathscr{X}_1$, i even, are prime to ω_n for all n. Since $\mathscr{X}_1^+$ is a Λ-torsion module, this also holds for the character polynomials $f_2^{(i)}(T)$ of $e_i\mathscr{X}_2$. Hence, because of the exact sequence (8.15) the kernel of $e_i\mathscr{X}_2 \to e_i\mathscr{X}_3$ is finite and therefore we obtain $\lambda_2^{(i)} = \lambda_3^{(i)}$, i even. □

In the following we describe the $\Lambda[\Delta]$-module structure of $\overline{E}$ and $\overline{E}'$.

THEOREM 8.17: *Let n be the degree* $[K:\mathbb{Q}]$, *then there are* $\Lambda[\Delta]-$ *isomorphisms*

$$\overline{E}(\mathscr{K}_\infty) \cong \overline{E}(K_\infty^+) \oplus \overline{E}(\mathscr{K}_\infty)^- \cong \Lambda[\Delta/\Delta^+]^n \oplus \mathbb{Z}_p(1),$$

$$\overline{E}'(\mathscr{K}_\infty) \cong \overline{E}'(K_\infty^+) \oplus \overline{E}'(\mathscr{K}_\infty)^- \cong \Lambda[\Delta/\Delta^+]^n \oplus \mathbb{Z}_p(1).$$

PROOF: Since the operation$^-$ commutes with the Δ-action, we get

$$\overline{E(\mathscr{K}_n)} = \overline{E(K_n^+)} \oplus \mu_{\mathscr{K}_n},$$

hence

$$\overline{E}(\mathscr{K}_\infty) = \overline{E}(K_\infty^+) \oplus \operatorname{proj}\lim_n \mu_{\mathscr{K}_n} = \overline{E}(\mathscr{K}_\infty)^+ \oplus \overline{E}(\mathscr{K}_\infty)^-.$$

The exact sequences (8.15) and (8.11) show, for an odd i, that

$$e_i\overline{E} = e_i\overline{E}',$$

hence by Lemma 7.4

$$\mathbb{Z}_p(1) = (\overline{E})^- = (\overline{E}')^- = T_\lambda(\overline{E}').$$

Consequently the module $(\overline{E}')^+$ is $\Lambda-$torsion free. Let C_1 and C_2 be defined by the commutative exact diagram

$$\begin{array}{ccccccccc}
0 \to & (\overline{E}')^+ & \to & F_\Lambda(\mathscr{A}) & \to & C_2 & \to 0 \\
 & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow & \\
0 \to & \overline{E}' & \to & \mathscr{A} & \to & C_1 & \to 0. \\
 & \hookuparrow & & \hookuparrow & & \hookuparrow & \\
0 \to & T_\Lambda(\overline{E}') & \to & T_\Lambda(\mathscr{A}) & \to & B & \to 0
\end{array}$$

Since $C_1^+ \subseteq \mathscr{X}_1^+$ is a Λ-torsion module, it follows from Theorem 6.5 for the Λ – torsion free module $(\overline{E}')^+$ the quasi-isomorphism

$$e_i\overline{E}' \sim \Lambda^n, \quad i \text{ even}.$$

We show now that $(\overline{E}')^+_\Gamma$ is $\mathbb{Z}_p$-free. Indeed, by Proposition 7.13 we get, for a large n,

$$B(-1) \leqslant C_1(-1)^{\Gamma^n} \leqslant \mathscr{X}_1(-1)^{\Gamma^n} = B(-1),$$

and hence the first map in the exact sequence

$$0 \to B(-1) \tilde{\to} C_1(-1)^{\Gamma^n} \to C_2(-1)^{\Gamma^n} \to B(-1)$$

is an isomorphism. Therefore $C_2(-1)^{\Gamma^n}$ is $\mathbb{Z}_p$-free ($B(-1) = \mathscr{X}_1(-1)^{\Gamma^n}$ is $\mathbb{Z}_p$-free, by Theorem 5.4 (ii)) and consequently $C_2(-1)$ (resp. C_2) does not contain any non-trivial finite Λ-submodule. Now, the exact sequence

$$0 \to C_2^\Gamma \to (\overline{E}')^+_\Gamma \to F_\Lambda(\mathscr{A})_\Gamma \cong \mathbb{Z}_p^{n \cdot \#\Delta}$$

proves that $(\overline{E}')^+_\Gamma$ is $\mathbb{Z}_p$-free. By (1.2) we see

$$e_i\overline{E}' \cong \Lambda^n, \quad i \text{ even},$$

and then the exact sequence (8.15) yields our assertion for $e_i\overline{E}$. □

For the homological dimensions of the Λ-modules $\mathscr{X}_2^-$ and $\mathscr{X}_3^-$ we obtain the following:

Proposition 8.18:

$$\operatorname{hd}_\Lambda \mathscr{X}_2^- \leqslant 1 \quad \textit{and} \quad \operatorname{hd}_\Lambda \mathscr{X}_3^- \leqslant 1.$$

Proof: Because $(\overline{E}')^- = T_\Lambda(\overline{E}')$ the diagram in the proof of Theorem 8.17 implies $T_\Lambda(C_1)^- = B^-$ and hence induces the following commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & & & F_\Lambda(\mathscr{A})^- & \tilde{\to} & F_\Lambda(C_1) & & \\
 & & & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow & & \\
0 & \to & (\overline{E}')^- & \to & \mathscr{A}^- & \to & C_1^- & \to & 0. \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \to & T_\Lambda(\overline{E}')^- & \to & T_\Lambda(\mathscr{A})^- & \to & B^- & \to & 0
\end{array}$$

By Proposition 7.13 we obtain for large n

$$T_\Lambda(C_1)^-(-1) = B^-(-1) = B(-1)_{\Delta^+} = \left(\mathscr{X}_1(-1)^{\Gamma^n}\right)_{\Delta^+}$$

$$= \left(\mathscr{X}_1(-1)_{\Delta^+}\right)^{\Gamma^n} = \mathscr{X}_1^-(-1)^{\Gamma^n}.$$

Since $B(-1)$ is $\mathbb{Z}_p$-free, the exact sequence

$$0 \to T_\Lambda(C_1)^-(-1) \to C_1^-(-1)_{\Gamma^n} \to F_\Lambda(C_1)_{\Gamma^n} \to 0$$

shows $C_1^-(-1)_{\Gamma^n}$ to be $\mathbb{Z}_p$-free too. Furthermore, we have

$$C_1^-(-1)^{\Gamma^n} = T_\Lambda(C_1^-)(-1) = \mathscr{X}_1^-(-1)^{\Gamma^n}$$

and hence an inclusion induced by the exact sequence $0 \to C_1^- \to \mathscr{X}_1^- \to \mathscr{X}_3^- \to 0$,

$$\mathscr{X}_3^-(-1)^{\Gamma^n} \to C_1^-(-1)_{\Gamma^n}.$$

This proves the proposition for $\mathscr{X}_3^-$ and the exact sequence (8.11) gives the assertion for $\mathscr{X}_2^-$. □

Remark: There are examples by Greenberg [5] showing that the module $\mathscr{X}_2^+$ is finite but not trivial, hence $\mathrm{hd}_\Lambda \mathscr{X}_2^+ > 1$.

Since $\mathscr{X}_3^-$ does not contain any non-trivial finite Λ-submodule we have as a consequence of the exact sequence (8.11) the following divisibility assertion which is connected with the famous Lichtenbaum-conjecture:

Proposition 8.19: *Let $i \in \mathbb{Z}$ be odd and the order of the group $(e_i\mathscr{X}_3)(-i)_\Gamma$ be finite. Then the order of $e_i\mathscr{X}_2(-i)_\Gamma$ is finite and divisible by*

$$\prod_{\mathfrak{p} \in S_p(K)} w_i(K_\mathfrak{p}),$$

where $w_i(K_\mathfrak{p}) = \max\{p^j : [K_\mathfrak{p}(\mu_{p^j}) : K_\mathfrak{p}] \mid i\}$.

Remark: (a) For $i \geqslant 1$ this is the well-known result of Lichtenbaum [3, Theorem 9; 4].

(b) For negative i the group $(e_i\mathscr{X}_3)(-i)_\Gamma$ is finite. This is proved in [22; §7, Satz 12] using results obtained by algebraic K-theory.

Proof: The exact sequence (8.11) induces the exactness of the following sequence

$$(e_i\mathscr{X}_3)(-i)^\Gamma \to \bigoplus_{\mathfrak{p} \in S_p(K)} \Lambda/(\omega_{r_\mathfrak{p}})^{\delta_{i,\mathfrak{p}}}(-i)_\Gamma$$

$$\to (e_i\mathscr{X}_2)(-i)_\Gamma \to (e_i\mathscr{X}_3)(-i)_\Gamma \to 0,$$

i odd. Because of our assumption $(e_\iota \mathscr{X}_3)(-i)_\Gamma$ and consequently $(e_\iota \mathscr{X}_3)(-i)^\Gamma$ are finite groups; hence, $(e_\iota \mathscr{X}_3)(-i)^\Gamma$ is zero (Proposition 8.18). Now with the equalities

$$\#\left(\bigoplus_{\mathfrak{p}\in S_p(K)} \Lambda/(\omega_{r_\mathfrak{p}})^{\delta_{\iota,\mathfrak{p}}}(-i)_\Gamma\right)$$

$$= \#\left(\prod_{\mathfrak{p}\in S_p(K)} \prod_{\mathfrak{P}\mid\mathfrak{p}} H^0\left(\mathbb{G}_{\mathfrak{P}}(K_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i)\right)^*\right)_\Gamma$$

$$= \#\left(\prod_{\mathfrak{p}\in S_p(K)} H^0\left(\mathbb{G}_{\mathfrak{p}}(K), \mathbb{Q}_p/\mathbb{Z}_p(i)\right)^*\right) = \prod_{\mathfrak{p}\in S_p(K)} w_\iota(K_\mathfrak{p}),$$

we obtain

$$\#(e_\iota \mathscr{X}_2)(-i)_\Gamma = \prod_{\mathfrak{p}\in S_p(K)} w_i(K_\mathfrak{p}) \cdot \#(e_\iota \mathscr{X}_3)(-i)_\Gamma.$$

Or expressed in another way: if $|\ |_p$ is the normalized valuation of $\mathbb{C}_p$ with $|p|_p = p^{-1}$ and if $f_r^{(\iota)}(T)$ denotes the characteristic polynomials of $e_\iota \mathscr{X}_r$, $r = 2, 3$, we have the equality

$$\left| f_2^{(\iota)}\left(\kappa(\gamma_0)^{-\iota} - 1\right)\right|_p^{-1} = \prod_{\mathfrak{p}\in S_p(K)} w_\iota(K_\mathfrak{p}) \cdot \left| f_3^{(\iota)}\left(\kappa(\gamma_0)^{-\iota} - 1\right)\right|_p^{-1}.$$

9. Symplectic paarings for CM-fields

We keep the notations of section 8. In the following we shall define two Λ-modules $\mathscr{L}_+$ and $\mathscr{L}_-$ with skew symmetric, Γ-invariant pairings ψ_+ resp. ψ_-, which are non degenerated after tensoring with $\mathbb{Q}_p$. Thus we obtain functional equations for their characteristic polynomials. In the function field case this will be the well-known functional equation for the zeta function of $\mathscr{K}$ (in this case we get $\mathscr{L}_+ = \mathscr{L}_- = \mathscr{X}_3$).

Because $T_\Lambda(\bar{E}')^- = (\bar{E}')^-$, $T_\Lambda(\bar{E}')^+ = 0$ and $T_\Lambda(\mathscr{X}_1^+) = \mathscr{X}_1^+$ the exact sequence $0 \to \bar{E}' \to \mathscr{A} \to \mathscr{X}_1 \to \mathscr{X}_3 \to 0$ gives us the following commutative exact diagrams:

$$\begin{array}{ccccccccc}
0 & \to & F_\Lambda(\mathscr{A})^- & \to & F_\Lambda(\mathscr{X}_1) & \to & \mathscr{Y}^- & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 \to (\bar{E}')^- & \to & \mathscr{A}^- & \to & \mathscr{X}_1^- & \to & \mathscr{X}_3^- & \to & 0 \\
\| & & \uparrow & & \uparrow & & \uparrow & & \\
0 \to (\bar{E}')^- & \to & T_\Lambda(\mathscr{A})^- & \to & T_\Lambda(\mathscr{X}_1)^- & \to & \mathscr{Z}^- & \to & 0
\end{array} \qquad 9.1$$

$$\begin{array}{ccccccccc}
 & & & 0 & & 0 & & & \\
 & & & & \searrow & \nearrow & & & \\
 & & & & \mathcal{Y}^+ & & & & \\
 & & & \nearrow & & \searrow & & & \\
0 \to (\bar{E}')^+ & \to & F_\Lambda(\mathcal{A})^+ & & \to & & \mathcal{Z}^+ \to & \mathcal{X}_3^+ & \to 0 \\
\| & & \uparrow & & & & \uparrow & \| & , \\
0 \to (\bar{E}')^+ & \to & \mathcal{A}^+ & & \to & & \mathcal{X}_1^+ \to & \mathcal{X}_3^+ & \to 0 \\
 & & \cup & & & & \cup & & \\
 & & T_\Lambda(\mathcal{A})^+ & = & = & = & T_\Lambda(\mathcal{A})^+ & &
\end{array} \tag{9.2}$$

where $\mathcal{Y}^- = F_\Lambda(\mathcal{X}_1)/F_\Lambda(\mathcal{A})^-$ and $\mathcal{Y}^+ = F_\Lambda(\mathcal{A})^+/(\bar{E}')^+$. The λ-invariant of the Λ-torsion modules $\mathcal{Y}$ and $\mathcal{Z}$ are (Theorem 7.9)

$$\begin{array}{ll}
\lambda(\mathcal{Z}^-) = \lambda_3^+, & \lambda(\mathcal{Z}^+) = \lambda_3^-, \\
\lambda(\mathcal{Y}^-) = \lambda_3^- - \lambda_3^+, & \lambda(\mathcal{Y}^+) = \lambda_3^- - \lambda_3^+;
\end{array} \tag{9.3}$$

further we obtain the exact sequences

$$\begin{array}{ccccccc}
0 \to \mathcal{Y}^+ \to & \mathcal{Z} & \longrightarrow & \mathcal{X}_3 & \to \mathcal{Y}^- \to 0. \\
 & \searrow & & \nearrow & \\
 & & \mathcal{X}_3^+ \oplus \mathcal{Z}^- & &
\end{array} \tag{9.4}$$

REMARK: If $\mathcal{K}$ is not necessarily a number field of CM-type, instead of (9.4) we get the exact sequence

$$0 \to T_\Lambda(F_\Lambda(\mathcal{A})/F_\Lambda(\bar{E}')) \to \mathcal{Z} \to \mathcal{X}_3$$

$$\to F_\Lambda(\mathcal{X}_1)/F_\Lambda(F_\Lambda(\mathcal{A})F_\Lambda(\bar{E}')) \to 0. \tag{9.5}$$

As before, let $B = T_\Lambda(\mathcal{A})/T_\Lambda(\bar{E}')$; we define the Λ-torsion modules $\mathcal{L}_+$ and $\mathcal{L}_-$ by the commutative exact diagrams

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & F_\Lambda(\mathcal{A})^- & \longrightarrow & F_\Lambda(\mathcal{X}_1) \to & \mathcal{Y}^- & \to 0 \\
 & & & & \uparrow & & \uparrow & \uparrow & \\
0 & \longrightarrow & \bar{E}' & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{X}_1 \to & \mathcal{X}_3 & \to 0 \\
 & & \| & & \cup & & \cup & \cup & \\
0 & \to & \bar{E}' & \to & T_\Lambda(\mathcal{A}) \oplus F_\Lambda(\mathcal{A})^+ & \to & T_\Lambda(\mathcal{X}_1) \to & \mathcal{L}_+ & \to 0
\end{array} \tag{9.6}$$

and

$$\begin{array}{ccccccccc}
 & & & & 0 \searrow & & \nearrow 0 & & \\
 & & & & & \mathscr{Y}^+ & & & \\
 & & & & \nearrow & & \searrow & & \\
0 \to & (\bar{E}')^+ & \longrightarrow & F_\Lambda(\mathscr{A})^+ & & \longrightarrow & \mathscr{L}_- & \longrightarrow & \mathscr{X}_3 \to 0 \\
 & \twoheaduparrow & & \twoheaduparrow & & & \twoheaduparrow & & \| \\
0 \to & \bar{E}' & \longrightarrow & \mathscr{A} & & \longrightarrow & \mathscr{X}_1 & \longrightarrow & \mathscr{X}_3 \to 0 \\
 & \uparrow & & \uparrow & & & \uparrow & & \\
0 \to & T_\Lambda(\bar{E}') & \to & T_\Lambda(\mathscr{A}) \oplus F_\Lambda(\mathscr{A})^- & & \longrightarrow & B \oplus F_\Lambda(\mathscr{A})^- & \to & 0
\end{array} \tag{9.7}$$

Obviously, we have

$$\begin{aligned}
\mathscr{L}_+ &= G(L'/L' \cap T), & T &:= M(\mathscr{K})^{T_\Lambda(\mathscr{X}_1)} \\
\mathscr{L}_- &= G(M'/\mathscr{K}_\infty), & M' &:= M(\mathscr{K})^{B \oplus F_\Lambda(\mathscr{A})^-} \subseteq M(K^+) \cdot L'.
\end{aligned} \tag{9.8}$$

By the exact sequence (9.4) we obtain

$$\mathscr{L}_+ = \mathscr{X}_3^+ \oplus \mathscr{Z}^- \quad \text{and} \quad \mathscr{L}_- = \mathscr{X}_3^- \oplus \mathscr{Z}^+, \tag{9.9}$$

hence

$$\lambda(\mathscr{L}_+) = 2\lambda_3^+ \quad \text{and} \quad \lambda(\mathscr{L}_-) = 2\lambda_3^-. \tag{9.10}$$

In the case when none of the prime divisors of p splits in the extension $\mathscr{K}/K^+$, we have $T_\Lambda(\mathscr{A})^+ = 0$, hence by (9.2) and (8.11)

$$\mathscr{L}_- = \mathscr{X}_2^- \oplus \mathscr{X}_1^+ = G(M(K^+)L(\mathscr{K})/\mathscr{K}_\infty).$$

Further, we have the following properties of the Λ-module $\mathscr{L}_-$: Let E^+/K^+ be a finite extension with $E^+ \subset K^+(p)$ and $\mathscr{E} = E^+(\mu_p)$. If we assume $\mu(k^+) = 0$, then the λ-invariants of $\ell_-(\mathscr{E})$ and $\mathscr{L}_-(\mathscr{K})$ are connected by a Riemann–Hurwitz formula; indeed by Corollary 7.12 we get

$$\begin{aligned}
(\lambda(\mathscr{L}_-(\mathscr{E})) - 2) &= (\lambda(\mathscr{L}_-(\mathscr{K})) - 2)[\mathscr{E}_\infty : \mathscr{K}_\infty] \\
&\quad + \sum_{\substack{\mathfrak{p} \\ \mu_p \subset (E_\infty^+)_\mathfrak{p}}} (n_\mathfrak{p} - 1).
\end{aligned} \tag{9.11}$$

Secondly we see that $\mathscr{L}_-$ does not contain any finite Λ-submodule:

PROPOSITION 9.12: *We have* $\mathrm{hd}_{\Lambda}(\mathscr{L}_-) \leqslant 1$.

PROOF: In the exact sequence induced by $0 \to B \oplus F_{\Lambda}(\mathscr{A})^- \to \mathscr{X}_1 \to \mathscr{L}_- \to 0$ (9.7),

$$0 \to B(-1) \tilde{\to} \mathscr{X}_1(-1)^{\Gamma^n} \to \mathscr{L}_-(-1)^{\Gamma^n} \to B(-1) \oplus F_{\Lambda}(\mathscr{A})^-_{\Gamma^n},$$

the first map is an isomorphism for sufficiently large n (Corollary 7.13); consequently, the $\mathbb{Z}_p$-module $\mathscr{L}(-1)^{\Gamma^n}$ is free and by Proposition 2.1 the proposition is proved. □

In order to prove the existence of a functional equation for the characteristic polynomial of $\mathscr{L}_-$ (resp. $\mathscr{L}_+$) we define two pairings,

$$\psi_{\pm} : (\mathscr{L}_{\pm} \otimes \mathbb{Q}_p) \times (\mathscr{L}_{\pm} \otimes \mathbb{Q}_p) \to \mathbb{Q}_p(1);$$

$$\psi_+(x, y) = \sum_{i \text{ even}} \psi_0^{(i)}(e_i x, e_{1-i} y) - \psi_0^{(i)}(e_i y, e_{1-i} x)$$

and

$$\psi_-(x, y) = \sum_{i \text{ odd}} \psi_0^{(i)}(e_i x, e_{1-i} y) - \psi_0^{(i)}(e_i y, e_{1-i} x),$$

where $\psi_0^{(i)}$ is the pairing introduced in Theorem 7.9.

Recall the following definitions: Let R be a commutative ring with 1, A an associative algebra over R and $a \mapsto a^*$ an involutory antiautomorphism of A. A symplectic A-space is a pair (M, ϕ) consisting of a A-module M and a nonsingular skew-symmetric A-invariant (that is, $\phi(xa, y) = \phi(x, ya^*)$, $x, y \in M$, $a \in A$) R-bilinear form ϕ on M. We say ϕ is hyperbolic, if there exists a direct decomposition of M in totally isotropic A-submodules. Two forms ϕ_1 and ϕ_2 on M are said to be equivalent if there exists an A-isometry, that is an A-isomorphism ρ such that $\phi_1(x, y) = \phi_2(\rho(x), \rho(y))$ for any $x, y \in M$.

By Theorem 7.9 and the decomposition (9.9) we obtain:

THEOREM 9.13: *The pairings*

$$\psi_{\pm} : (= \mathscr{L}_{\pm} \otimes \mathbb{Q}_p) \times (\mathscr{L}_{\pm} \otimes \mathbb{Q}_p) \to \mathbb{Q}_p(1)$$

are Γ*-invariant, that is,*

$$\psi_{\pm}(\gamma x, \gamma y) = \kappa(\gamma) \psi_{\pm}(x, y), \quad \gamma \in \Gamma, \ x, y \in \mathscr{L}_{\pm} \otimes \mathbb{Q}_p,$$

nonsingular, skew-symmetric and hyperbolic.

REMARKS: (i) If the divisors of the Λ-module $G(L'/L' \cap N')$ are disjoint from all principal divisors (ω_n), $n \geqslant 0$ (for instance, if the Leopoldt-conjecture holds for all intermediate fields $\mathscr{K}_n$, $n \geqslant 0$ [9, Lemma 21]), then there is a quasi-isomorphism $G(L'/L' \cap N') \sim G(L'/L' \cap T) = \mathscr{L}_+$ (see (9.10) and [9, Theorem 24]) and Iwasawa defined in [9, Theorem 23] a pairing ψ_I on the Λ-module $\mathscr{L}_+$ with the same properties as ψ_+. Now, a theorem of Jakovlev concerning symplectic hyperbolic forms [12, Theorem 1], tells that the forms ψ_I and ψ_+ are equivalent (the involutory antiautomorphism on $\mathbb{Q}_p[\![T]\!]$ is given by $T \mapsto \dot{T} = (T+1)^{-1}\kappa(\gamma_0) - 1)$. Remember that if the conjecture of Greenberg ($\lambda_2^+ = 0$) is true, the pairing ψ_+ is trivial.

(ii) In [17] Kuz'min obtained a pairing on $\mathscr{L}_- \otimes \mathbb{Q}_p$ with the properties of Theorem 9.13. The same argument as in (i) shows that this pairing is equivalent to ψ_-.

COROLLARY 9.14: *The characteristic polynomials $h_+(T)$ and $h_-(T)$ of $\mathscr{L}_+$ resp. $\mathscr{L}_-$ satisfy the following functional equations*

$$\kappa(\gamma_0)^{\lambda_3^+} \cdot h_+(T-1) = T^{2\lambda_3^+} \cdot h_+\big(\kappa(\gamma_0)T^{-1} - 1\big),$$

$$\kappa(\gamma_0)^{\gamma_3^+} \cdot h_-(T-1) = T^{2\lambda_3^-} \cdot h_-\big(\kappa(\gamma_0)T^{-1} - 1\big).$$

PROOF: Since the p^μ-factor is the same on both sides of the equations the corollary follows by a standard argument of linear algebra (see for instance [Hartshorne, Algebraic geometry, p. 456]. □

Considering the exact sequence

$$0 \to \mathscr{Y}^+ \to \mathscr{L}_- \to \mathscr{X}_3 \to 0$$

we may say, that $h_-(T)$ is the product of the "global" characteristic polynomial $f_3(T)$ of $\mathscr{X}_3$ and the "local" polynomial of $\mathscr{Y}^+$. In order to justify this terminology we consider the commutative exact diagram induced by (9.7),

$$\begin{array}{ccccccc}
0 \to G(M'/L') = \mathscr{Y}^+ \to & \mathscr{Z}^+ & \to \mathscr{X}_3^+ \to 0 \\
\twoheaduparrow & \twoheaduparrow & \| \\
0 \to G(M(K^+)L'/L') \to & \mathscr{X}_1^+ & \to \mathscr{X}_3^+ \to 0 \\
\hookuparrow & \hookuparrow & \\
T_\Lambda(\mathscr{A})^+ = & T_\Lambda(\mathscr{A})^+ &
\end{array}$$

Hence, the group $\mathscr{Y}^+$ is generated by the local galois groups

$$G\big(M'_{\mathfrak{g}}/\mathscr{K}_{\infty\mathfrak{p}}\big) = G\big(M'_{\mathfrak{p}}/L'_{\mathfrak{p}}\big), \quad \mathfrak{p} \mid p.$$

If $s_{K_\infty^+} = 1$ we obtain the following exact sequence:

$$\begin{array}{ccccc} 0 \to T_\Lambda(\mathscr{A})^+ & \longrightarrow & (\mathscr{X}_1^+)_\mathfrak{p} \longrightarrow & \mathscr{Y}_\mathfrak{g}^+ & = \mathscr{Y}^+ \to 0; \\ \| & & \| & \| & \\ \mathbb{Z}_p(1)^{\delta_{1\mathfrak{p}}} & & G(M(K^+)_\mathfrak{p}/K_{\infty\mathfrak{p}}^+) & G(M_\mathfrak{p}'/\mathscr{K}\infty_\mathfrak{p}) & \end{array}$$

and consequently an isomorphism

$$\mathscr{Y}^+ \cong G\big(M(K^+)_\mathfrak{p} \,|\, K_{\infty\mathfrak{p}}^+\big), \quad \text{if } \mu_p \not\subset K_\mathfrak{p}^+ .$$

Thus, by Corollary 9.14 (and Proposition 8.12(a)) we get:

THEOREM 9.15: *If there exists only one prime $\mathfrak{p}$ of $\mathscr{K}_\infty$ dividing p and if $f_\mathfrak{p}(T) \in \mathbb{Z}_p[T]$ denotes the characteristic polynomial of the local galois group*

$$G\big(M(K^+)_\mathfrak{p} \,|\, K_{\infty\mathfrak{p}}^+\big) = G\big(K_{S_p}^+(p)/K_\infty^+\big)_\mathfrak{p}^{\mathrm{ab}},$$

then the polynomial $\mathrm{f}_\mathfrak{p} \cdot \mathrm{f}_2$, $\mathrm{f}_2 = \mathrm{f}_{\mathfrak{X}_2}$, satisfies the functional equation

$$\kappa(\gamma_0)^{\lambda_2} \cdot f_\mathfrak{p}(T-1) f_2(T-1)$$
$$= T^{2\lambda_2} \cdot f_\mathfrak{p}\big(\kappa(\gamma_0)T^{-1} - 1\big) f_2\big(\kappa(\gamma_0)T^{-1} - 1\big).$$

Appendix: The analogy with the function field case [2]

Let $\mathscr{K}$ be an algebraic function field of one variable over the finite constant field F containing the pth roots of unity (hence $p \neq \mathrm{char}\, F$). Let A denote the subgroup of all points of p-power order in the jacobian variety associated with $\mathscr{K}$ and F_∞ the field generated over F by A. Replacing F by a finite extension we may assume that every point in ${}_pA$ is rational over F. Thus $F_\infty = F(\mu_{p^\infty})$ and $\mathscr{K}_\infty = \mathscr{K} \cdot F_\infty$ are Γ-extensions of F (resp. $\mathscr{K}$).

We fix a prime $\mathfrak{p}$ of $\mathscr{K}$; if $\mathfrak{P}_1, \dots, \mathfrak{P}_s$ denote the finitely many primes of $\mathscr{K}_\infty$ dividing $\mathfrak{p}$ and $\Sigma = \{\mathfrak{p}\}$, then with the notations of Chapter II (if it makes sense) we have $L = L'$ and $\mathscr{X}_2 = \mathscr{X}_3$ and by global class field theory we obtain an exact sequence of Λ-*torsion* modules

$$\begin{array}{ccc} 0 \to \bar{E}_\Sigma & \to \mathscr{A} & \to \mathscr{X}_1 \to \mathscr{X}_3 \to 0; \\ \|\wr & \|\wr & \\ \mathbb{Z}_p(1) & \mathbb{Z}_p[\Gamma/\Gamma_\mathfrak{p}](1) & \end{array}$$

[2] (see [9; 12.3].

hence

$$\lambda_1 = \lambda_3 + s - 1.$$

It is well known that as $\mathbb{Z}_p$-modules

$$\mathfrak{X}_3 \cong \mathbb{Z}_p^{2g},$$

where g denotes the genus of the function field $\mathscr{K}$. Consequently we get

$$\mu = 0, \qquad \lambda_3 = 2g.$$

Since $F_\Lambda(\mathscr{A}) = 0$ we obtain by the analog construction (9.6), (9.7) for $\mathscr{L}_\pm$ the module $\mathfrak{X}_3$ and, since the global duality theorem is also valid in the function field case, a symplectic paaring $\psi = \psi_\pm$,

$$\psi : (\mathfrak{X}_3 \otimes \mathbb{Q}_p) \times (\mathfrak{X}_3 \otimes \mathbb{Q}_p) \to \mathbb{Q}_p(1).$$

Let $q = \kappa(\gamma_0) = \#F$ then the functional equation

$$q^g \cdot f_3(T-1) = T^{2g} \cdot f_3(qT^{-1} - 1)$$

induces the functional equation for the zeta-function of the curve X associated with $\mathscr{K}$,

$$Z(X, 1/qT) = q^{1-g} \cdot T^{2-2g} \cdot Z(X, T),$$

where

$$Z(X, T) = \frac{\det\left(1 - \gamma_0^{-1}T,\ H^1_{\text{et}}(\overline{X}, \mathbb{Q}_p)\right)}{(1-T)(1-qT)} = \frac{T^{2g} f_3(T^{-1} - 1)}{(1-T)(1-qT)},$$

$$\overline{X} = X \times_F F_\infty.$$

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