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**BOREL'S THEOREM FOR  $C^\infty$ -FUNCTIONS ON A  
 NON-ARCHIMEDEAN VALUED FIELD**

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In this note we prove the following theorem (for explanations see below).

**THEOREM 1:** *Let  $K$  be a non-archimedean nontrivially valued field. Let  $\lambda_0, \lambda_1, \lambda_2, \dots$  be any sequence in  $K$ . Then there exists a  $C^\infty$ -function  $f: K \rightarrow K$  such that  $D_n f(0) = \lambda_n$  for all  $n \in \{0, 1, 2, \dots\}$ .*

**DEFINITION** ([1], [2]): For each  $n \in \mathbb{N}$  let  $\nabla^n K := \{(\xi_1, \xi_2, \dots, \xi_n) \in K^n: \text{if } i \neq j \text{ then } \xi_i \neq \xi_j\}$ . Let  $f: K \rightarrow K$ .

Define  $\Phi_n f: \nabla^{n+1} K \rightarrow K$  inductively as follows.  $\Phi_0 f := f$  and, for  $n \in \mathbb{N}$ ,

$$\begin{aligned} \Phi_n f(\xi_1, \xi_2, \dots, \xi_{n+1}) := & -(\xi_1 - \xi_2)^{-1}(\Phi_{n-1} f(\xi_2, \xi_3, \xi_4, \dots, \xi_{n+1}) \\ & - \Phi_{n-1} f(\xi_1, \xi_3, \xi_4, \dots, \xi_{n+1})) \\ & ((\xi_1, \xi_2, \dots, \xi_{n+1}) \in \nabla^{n+1} K). \end{aligned}$$

Let  $n \in \mathbb{N} \cup \{0\}$ .  $f$  is a  $C^n$ -function if  $\Phi_n f$  can be extended to a continuous function  $\bar{\Phi}_n f$  on  $K^{n+1}$ . For such a  $C^n$ -function we set (for  $n \in \mathbb{N} \cup \{0\}$ ,  $x, y \in K$ )

$$\begin{aligned} D_n f(x) &:= \bar{\Phi}_n f(x, x, \dots, x) \\ T_{n+1} f(x, y) &:= \sum_{j=0}^n (x-y)^j D_j f(y) \\ R_{n+1} f(x, y) &:= f(x) - T_{n+1} f(x, y). \end{aligned}$$

$f$  is a  $C^\infty$ -function if  $f$  is a  $C^n$ -function for each  $n \in \mathbb{N} \cup \{0\}$ .

**REMARK 1:** The “ordinary” definition of a  $C^n$ -function ( $f$  is  $n$  times differentiable and  $f^{(n)}$  is continuous) does not lead to nice properties. The stronger definition of above restores somewhat the damage caused by the absence of the Mean Value Theorem.

REMARK 2: The following statements can be obtained by elementary and straightforward arguments (see [2]).

- (i) A  $C^{n+1}$ -function is also a  $C^n$ -function. (Locally) analytic functions are  $C^\infty$ -functions, “ $f$  is a  $C^n$ -function” is a local property.
- (ii) (Taylor) if  $f$  is a  $C^n$ -function then

$$R_n f(x, y) = (x - y)^n \overline{\Phi}_n f(x, y, y, \dots, y) \quad (x, y \in K)$$

so that

$$\lim_{(x, y) \rightarrow (a, a)} \frac{R_n f(x, y)}{(x - y)^n} = D_n f(a) \quad (a \in K)$$

Further, we have  $n! D_n f = f^{(n)}$ .

- (iii) (Polynomials). Let  $\varkappa$  denote the function  $x \mapsto x$  ( $x \in K$ ). Then  $D_j \varkappa^n = \binom{n}{j} \varkappa^{n-j}$  ( $0 \leq j \leq n$ ). For a polynomial function  $f$  defined by  $f(x) = \lambda_0 + \lambda_1 x + \dots + \lambda_m x^m$  ( $x \in K$ ) we have  $D_j f(0) = \lambda_j$  ( $0 \leq j \leq m$ ) and  $\Phi_{m+1} f = 0$ .

REMARK 3: Observe that the characteristic of  $K$  is allowed to be  $\neq 0$ . For this reason we prefer to work with  $D_n$  rather than the less informative  $n$ -th derivative.

The next theorem reduces the number of variables involved.

THEOREM 2. ([2], 10.7): Let  $K$  be as in Theorem 1, let  $n \in \mathbb{N}$ . The following conditions on a  $C^{n-1}$ -function  $f: K \rightarrow K$  are equivalent.

- ( $\alpha$ )  $f$  is a  $C^n$ -function.
- ( $\beta$ ) For each  $a \in K$ ,  $\lim_{(x, y) \rightarrow (a, a)} (x - y)^{-n} R_n f(x, y)$  exists.

For the proof of Theorem 1 we need two estimates on polynomial functions  $K \rightarrow K$ .

LEMMA 1: Let  $Q: x \mapsto \lambda_0 + \lambda_1 x + \dots + \lambda_s x^s$  be a polynomial function. Then

$$|D_j Q(x)| \leq \max(|\lambda_0|, |\lambda_1|, \dots, |\lambda_s|)$$

$$(j \in \mathbb{N} \cup \{0\}, x \in K, |x| \leq 1).$$

If, in addition,  $\lambda_0 = \lambda_1 = \dots = \lambda_m = 0$  for some  $m \leq s$  then

$$|D_j Q(x)| \leq |x|^{m+1-j} \max(|\lambda_{m+1}|, \dots, |\lambda_s|)$$

$$(j \in \mathbb{N} \cup \{0\}, j \leq m + 1, x \in K, |x| \leq 1).$$

PROOF: Everything follows from the formula

$$D_j Q(x) = \sum_{i=j}^s \lambda_i \binom{i}{j} x^{i-j}.$$

LEMMA 2: Let  $P: x \mapsto \lambda_0 + \lambda_1 x + \dots + \lambda_m x^m$  be a polynomial function and let  $n \in \mathbb{N} \cup \{0\}$ ,  $n \leq m$ . Then

$$\begin{aligned} |(x-y)^{-n} R_n P(x, y) - \lambda_n| &\leq \max(|x|, |y|) \cdot \\ &\max(|\lambda_0|, \dots, |\lambda_m|) (x, y \in K, |x| \leq 1, |y| \leq 1, x \neq y). \end{aligned}$$

PROOF: We have  $\Phi_{m+1} P = 0$  so that

$$R_{m+1} P(x, y) = (x-y)^{m+1} \bar{\Phi}_{m+1} P(x, y, y, \dots, y) = 0$$

for all  $x, y \in K$ . Hence

$$P(x) = T_{m+1} P(x, y)$$

and

$$\begin{aligned} R_n P(x, y) &= P(x) - T_n P(x, y) = T_{m+1} P(x, y) - T_n P(x, y) \\ &= \sum_{j=n}^m (x-y)^j D_j P(y). \end{aligned}$$

We find for  $x, y \in K$ ,  $|x| \leq 1$ ,  $|y| \leq 1$ ,  $x \neq y$  that

$$\begin{aligned} (*) \quad (x-y)^{-n} R_n P(x, y) - \lambda_n &= D_n P(y) - \lambda_n \\ &\quad + \sum_{j=n+1}^m (x-y)^{j-n} D_j P(y) \end{aligned}$$

Now

$$D_n P(y) = \lambda_n + \lambda_{n+1} \binom{n+1}{n} y + \dots + \lambda_m \binom{m}{n} y^{m-n}$$

so that

$$|D_n P(y) - \lambda_n| \leq |y| (\max|\lambda_0|, \dots, |\lambda_m|).$$

Further, by Lemma 1 we have

$$|D_j P(y)| \leq \max(|\lambda_0|, \dots, |\lambda_m|)$$

so

$$\left| \sum_{j=n+1}^m (x-y)^{j-n} D_j P(y) \right| \leq |x-y| \cdot \max(|\lambda_0|, \dots, |\lambda_m|).$$

Together with (\*) this proves Lemma 2.

PROOF OF THEOREM 1: Let

$$r_n := \{n(\max(|\lambda_0|, |\lambda_1|, \dots, |\lambda_n|) + 1)\}^{-1} \quad (n \in \mathbb{N}).$$

Then

$$r_1 > r_2 > \dots, \quad \lim_{n \rightarrow \infty} r_n = 0$$

and

$$r_n \max(|\lambda_0|, |\lambda_1|, \dots, |\lambda_n|) \leq n^{-1}$$

for each  $n$ . Define  $f: K \rightarrow K$  as follows.

$$f(x) := \begin{cases} 0 & \text{if } |x| > r_1 \\ P_n(x) := \lambda_0 + \lambda_1 x + \dots + \lambda_n x^n & \text{if } n \in \mathbb{N}, r_{n+1} < |x| \leq r_n \\ \lambda_0 & \text{if } x = 0 \end{cases}$$

Clearly  $f$  is a  $C^\infty$ -function at  $a$  for each  $a \in K$ ,  $a \neq 0$ . If  $r_{m+1} < |x| \leq r_m$  for some  $m \in \mathbb{N}$  then

$$\begin{aligned} |f(x) - \lambda_0| &= |P_m(x) - \lambda_0| = |\lambda_1 x + \lambda_2 x^2 + \dots + \lambda_m x^m| \\ &\leq |x| \max(|\lambda_1|, |\lambda_2|, \dots, |\lambda_m|) \\ &\leq r_m \max(|\lambda_0|, |\lambda_1|, \dots, |\lambda_m|) \leq m^{-1}. \end{aligned}$$

We have proved the case  $n=0$  of the following statement. For each  $n \in \mathbb{N} \cup \{0\}$  the function  $f$  is  $C^n$  and  $D_j f(0) = \lambda_j$  ( $0 \leq j \leq n$ ). To prove the step from  $n-1$  to  $n$  it suffices, according to Theorem 2 and Remark 2 (ii), to show that

$$\lim_{(x,y) \rightarrow (0,0)} (x-y)^{-n} R_n f(x,y) = \lambda_n.$$

In its turn, this statement follows, by continuity of  $R_n$ , from "if

$$k \geq n, 0 < |x| \leq r_k, 0 < |y| \leq r_k, x \neq y$$

then

$$|(x - y)^{-n} R_n f(x, y) - \lambda_n| < k^{-1},$$

and it is the latter statement that we are intended to prove. So let

$$r_{m+1} < |x| \leq r_m, r_{s+1} < |y| \leq r_s$$

for some  $m, s \geq k$ . Then

$$R_n f(x, y) = P_m(x) - T_n P_s(x, y).$$

We consider two cases.

(i)  $s \geq m$ . Writing  $P_s = P_m + Q$  where  $Q(t) = \lambda_{m+1} t^{m+1} + \dots + \lambda_s t^s$  we obtain

$$\begin{aligned} |(x - y)^{-n} R_n f(x, y) - \lambda_n| &= |(x - y)^{-n} (P_m(x) - T_n P_m(x, y)) \\ &\quad - \lambda_n + (x - y)^{-n} (T_n P_m(x, y) - T_n P_s(x, y))| \\ &\leq |(x - y)^{-n} R_n P_m(x, y) - \lambda_n| \vee |x - y|^{-n} |T_n Q(x, y)|. \end{aligned}$$

By Lemma 2 the first part is  $\leq \max(|x|, |y|) \cdot \max(|\lambda_0|, |\lambda_1|, \dots, |\lambda_m|) \leq r_m \max(|\lambda_0|, |\lambda_1|, \dots, |\lambda_m|) \leq m^{-1} \leq k^{-1}$ .

To estimate the second part we may assume that  $s > m$ . Then  $|x - y| > |y|$ . Using this, the definition of  $T_n$ , and Lemma 1, we obtain

$$\begin{aligned} |x - y|^{-n} |T_n Q(x, y)| &= \left| \sum_{j=0}^{n-1} (x - y)^{j-n} D_j Q(y) \right| \\ &\leq \max_{0 \leq j < n} |x - y|^{j-n} |y|^{m+1-j} \cdot \max(|\lambda_0|, \dots, |\lambda_s|) \\ &\leq |y|^{m+1-n} \max(|\lambda_0|, \dots, |\lambda_s| \leq r_s \max(|\lambda_0|, \dots, |\lambda_s|) \leq s^{-1} \leq k^{-1}. \end{aligned}$$

(ii)  $s < m$ . Then set  $P_m = P_s + Q$  where  $Q(t) = \lambda_{s+1} t^{s+1} + \dots + \lambda_m t^m$ . We find

$$\begin{aligned} |(x - y)^{-n} R_n f(x, y) - \lambda_n| &= |(x - y)^{-n} (P_s(x) - T_n P_s(x, y)) \\ &\quad - \lambda_n + (x - y)^{-n} (P_m(x) - P_s(x))| \\ &\leq |(x - y)^{-n} R_n P_s(x, y) - \lambda_n| \vee |(x - y)^{-n} Q(x)|. \end{aligned}$$

By Lemma 2 the first part is  $\leq \max(|x|, |y|) \cdot \max(|\lambda_0|, \dots, |\lambda_s|) \leq r_s \max(|\lambda_0|, \dots, |\lambda_s|) \leq s^{-1} \leq k^{-1}$ . For the second part observe that  $|x - y| > |x|$  and  $|Q(x)| = |\lambda_{s+1}x^{s+1} + \dots + \lambda_m x^m| \leq |x|^{s+1} \max(|\lambda_0|, \dots, |\lambda_m|)$ . Then

$$\begin{aligned} |(x - y)^{-n} Q(x)| &\leq |x|^{s+1-n} \max(|\lambda_0|, \dots, |\lambda_m|) \\ &\leq |x| \max(|\lambda_0|, \dots, |\lambda_m|) \\ &\leq r_m \max(|\lambda_0|, \dots, |\lambda_m|) \leq m^{-1} \leq k^{-1}. \end{aligned}$$

### References

- [1] D. BARSKY: Fonctions  $k$ -lipschitziennes sur un anneau local et polynômes à valeurs entières. *Bull. Soc. Math. Fr.* 101, (1973) 397–411.
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